The University of Texas at Austin
Mechanical Engineering
Cockrell School of Engineering

ME397: Algorithms for Sensor-Based Robotics (ASBR)
Prof. Farshid Alambeigi, Spring 2021

Homework/Programming Assignment #1 **Homework Due: 02/23/2021- 5:00PM**

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Maxim Gurevich (Mar 4, 2021 16:51 CST)

Question	Points	Total
HA 1		
HA 2		
HA 3a		
HA 3b		
HA 4a		
HA 4b		
HA 4c		
HA 5a		
HA 5b		
HA 6		
HA 7		
HA 8		
HA 9		
HA 10(Bonus)		
PA 1		
PA 2		
PA 3		
PA4 (Bonus)		

Instruction:

1. Remember that this is a graded homework assignment. It is the equivalent of a **mini take-home exam**.
2. You are to work **alone** or **in teams of two** and are **not to discuss the problems with anyone** other than the TAs or the instructor.
3. It is open book, notes, and web. But you should cite any references you consult.
4. Unless I say otherwise in class, it is due before the start of class on the due date mentioned in the P/H Assignment.
6. **Sign and append** this score sheet as the first sheet of your assignment.
7. Remember to submit your assignment in Canvas.

➢ Homework Assignment (HA)

1. Prove that a rotation matrix $\boldsymbol{R} \in \boldsymbol{SO}(\boldsymbol{3})$ is a rigid body transformation.
2. Prove that rigid transformation **g = (p,R)** on a vector **v = s − r** is $\mathbf{g_*(v)} := \mathbf{Rv}$.

3. **(MLS Book [1])** *Properties of rotation matrices*

   Let $R \in SO(3)$ be a rotation matrix generated by rotating about a unit vector ω by θ radians. That is, R satisfies $R = \exp(\widehat{\omega}\theta)$.

   a. Show that the eigenvalues of $\widehat{\omega}$ are 0, i, and −i, where i = √−1. What are the corresponding eigenvectors?
   b. Show that the eigenvalues of R are 1, $e^{i\theta}$, and $e^{-i\theta}$. What is the eigenvector whose eigenvalue is 1? What is the physical interpretation of this eigenvector?

4. **(MLS Book [1])** *Properties of skew-symmetric matrices*
   Show That the following properties of skew- symmetric matrices are true:
   a. If $R \in SO(3)$ and $\omega \in \mathbb{R}^3$, then $R\widehat{\omega}R^T = \widehat{R\omega}$.
   b. If $R \in SO(3)$ and $v, \omega \in \mathbb{R}^3$, then $R(v \times \omega) = (Rv) \times (R\omega)$.
   c. Verify the following formula given $x \in \mathbb{R}^3$ $(\|x\| \neq 1)$,

   $$e^{\hat{x}} = \mathbb{I} + \frac{\sin\|x\|}{\|x\|}\hat{x} + \frac{1 - \cos\|x\|}{\|x\|^2}\hat{x}^2$$

5. **(MLS Book [1])** *Unit quaternions*
   Let $Q = (q_o, \vec{q})$ and $P = (p_o, \vec{p})$ be quaternions, where $q_o, p_o \in \mathbb{R}$ are the scalar parts of Q and P and $\vec{q}, \vec{p}$ are the vector parts.
   a. Show that the set of *unit* quaternions satisfies the axioms of the group.
   b. Let x be a point and let X be a quaternion whose scalar part is zero and whose vector part is equal to x (such a quaternion is called a *pure* quaternion). Show that if Q is a unit quaternion, the product QXQ^* is a pure quaternion and the vector part of QXQ^* satisfies

   $$(q_0^2 - \vec{q} \cdot \vec{q})\vec{x} + 2(q_0\,(\vec{q} \times \vec{x}) + (x \cdot \vec{q})\vec{q}$$

6. Recall from notes or the book that the explicit matrix representations for the rigid body rotations:
   - $R_x(\phi) : \mathbb{R} \mapsto SO(3)$ corresponding to a rotation of ϕ radians about the x-axis — i.e. a 3 × 3 matrix whose elements contain expressions such as $\sin(\phi)$ and $\cos(\phi)$

- $R_y(\theta) : \mathbb{R} \longmapsto SO(3)$ corresponding to a rotation of θ radians about the y-axis.
- $R_z(\psi) : \mathbb{R} \longmapsto SO(3)$ corresponding to a rotation of ψ radians about the z-axis.

In the class, we have learned that the ZXZ and ZYZ are the most common (the explicit form of the ZYZ Euler angles is in MLS). Here, construct the explicit representation for $R_{xyz}(\psi) : \mathbb{R} \longmapsto SO(3)$ given by

$$R_{xyz}(\psi, \theta, \phi) = R_x(\phi) R_y(\theta) R_z(\psi)$$

Construct the inverse function $R_{xyz}^{-1} : SO(3) \longmapsto \mathbb{R}^3$ such that $\forall R \in SO(3)$ if $y = R_{xyz}^{-1}(R)$ then $R = R_{xyz}(y)$, i.e., derive the formulas such as in (2.20) in MLS.

7. **Exercise 3.16** (pg.121) in *Modern Robotics: Mechanics, Planning, and Control* (Lynch et al.) [2].
8. **Exercise 3.18 (**pg.123) in Modern Robotics: Mechanics, Planning, and Control (Lynch et al.) [2].
9. Consider the pelvic osteotomy situation illustrated in **Fig. 1**. Here we assume that a three locating pins have been inserted into the patient's pelvis, and that a CT scan of the pelvis with the pins inserted has been produced. The patient has been placed onto the operating table. Also, a magnetic navigation system (here, the Northern Digital Aurora) is present in the room.

   Two surgical tools are available:
   - A probe/pointer device
   - An osteotome (essentially a fancy chisel) that will be used to cut the pelvis.
   6-DOF Aurora tracking sensors have been attached to the handle of each tool and an additional 6-DOF sensor has been affixed rigidly to the pelvis. The Aurora is capable of determining the position and orientation of each sensor relative to the Aurora base unit.

   We will define the following coordinate systems:
   $\mathbf{F}_B$ = Coordinate system of tracking system base unit
   $\mathbf{F}_D$ = Coordinate system of tracking device on pointer handle
   $\mathbf{F}_H$ = Coordinate system of tracking device on osteotome handle
   $\mathbf{F}_G$ = Coordinate system of tracking device attached to pelvis
   $\mathbf{F}_C$ = Coordinate system of CT image

   We also have the following relationships
   $\mathbf{F}_{Bx}$ =Measured 6 DOF pose of tracking device x relative to base unit
   $\mathbf{F}_{HK}$ = 6 DOF pose of osteotome blade relative to osteotome handle tracking device
   $\mathbf{F}_{DK}$ = 6 DOF pose of pointer tip relative to pointer handle tracking device

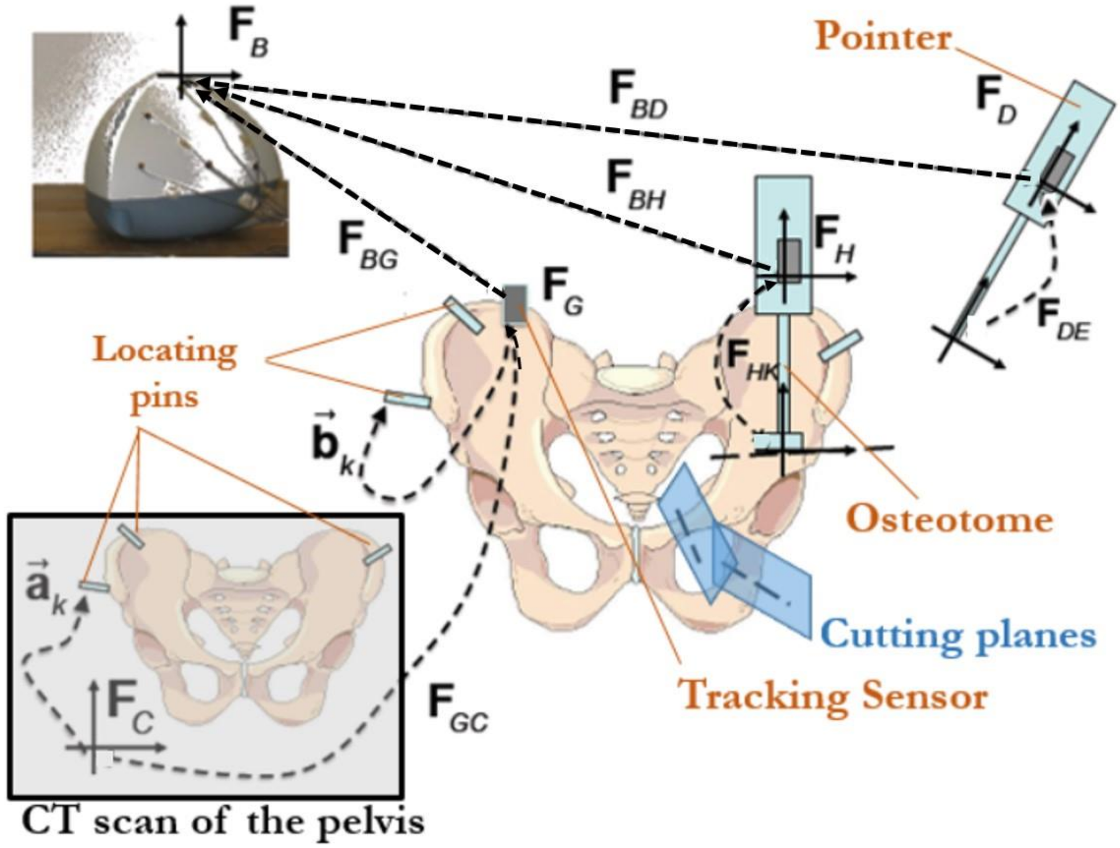

Fig. 1: Computer-Assisted Osteotomy

$\mathbf{a}_k$ =Position of the top of pin k in CT coordinates
$\mathbf{b}_k$ =Position of the top of pin k relative to tracking device G

Suppose that we have touched the tops of the three fiducial pins and used the results to compute a registration transformation $\mathbf{F}_{GC}$ such that $\mathbf{F}_{GC}\ \mathbf{a}_k = \mathbf{b}_k$. Give an expression for computing the position and orientation $\mathbf{F}_{CK}$ of the osteotome blade in CT coordinates, based on the available tracking system measurements $\mathbf{F}_{Bx}$ [3].

10. **(Bonus question [2])** Because arithmetic precision is only finite, the numerically obtained product of two rotation matrices is not necessarily a rotation matrix; that is, the resulting rotation A may not exactly satisfy $A^T A = I$ as desired. Devise an iterative numerical procedure that takes an arbitrary matrix $A \in \mathbb{R}^{3\times 3}$ and produces a matrix $R \in SO(3)$ that minimizes

$$\|A - R\|^2 = \text{tr}\,(A - R)(A - R)^T.$$

(Hint: See Appendix D of Lynch et al. for the relevant background on optimization.)

➢ Programming Assignment (PA)

1. Write functions that given a rotation matrix $R \in SO(3)$ returns:
   a. Its equivalent axis-angle representation.
   b. Quaternion representation.
   c. ZYZ and roll-pitch-yaw representation.

2. Write functions that:
   a. Given an axis-angle representation returns the equivalent rotation matrix.
   b. Given a quaternion representation returns the equivalent rotation matrix.

3. Using the functions you have written write a program that allows the user to specify an initial configuration of a rigid body by T, a screw axis specified by $\{q, \hat{s}, h\}$ in the fixed frame {s}, and the total distance traveled along the screw axis θ. The program should calculate the final configuration $T_1 = e^{[\mathcal{S}]\theta}T$ attained when the rigid body follows the screw $\mathcal{S}$ a distance θ, as well as the intermediate configurations at $\frac{\theta}{4}, \frac{\theta}{2}$, and $\frac{3\theta}{4}$. At the initial, intermediate, and final configurations, the program should plot the {b} axes of the rigid body. The program should also calculate the screw axis $\mathcal{S}_1$ and the distance θ_1 following $\mathcal{S}_1$ that takes the rigid body from T_1 to the origin and it should plot the screw axis $\mathcal{S}_1$. Test the program with q = (0,2,0), $\hat{s}$= (0,0,1), h = 2, θ = π, and
$$T = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. **(Bonus question [2])** Write a function the returns "true" if a given 3×3 within ϵ of being a rotation matrix and "false" otherwise. It is up to you how to define the "distance" between a random 3×3 real matrix and the closest member of SO(3). If the function returns "true," it should also return the "nearest" matrix in SO(3). Hint: you may use the result of HA 9.

➢ References:

1. **(MLS Book)** Murray, R.M., Li, Z., Sastry, S.S., "*A Mathematical Introduction to Robotic Manipulation.*", **Chapter 2**.
2. Lynch and Park, "***Modern Robotics***," Cambridge U. Press, 2017, **Chapter 3.**
3. Computer Integrated Surgery course, Russell H. Taylor, JHU

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THA1

Final Audit Report 2021-03-04

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ASBR Homework 1

3-4-21

Written Problems

1. Prove that a rotation matrix $\boldsymbol{R} \in \boldsymbol{SO(3)}$ is a rigid body transformation.

   Solution: Required properties:
   1. Length is preserved: $\|g(p) - g(q)\| = \|p - q\|$

   $\|R(p) - R(q)\|^2 = \|p - q\|^2$
   $(R(p - q))^T(R(p - q))$
   $(p - q)^T R^T R(p - q)$
   $(p - q)^T(p - q) = \|p - q\|^2 \therefore$

   2. Gross product (orientation) is preserved: $g(v \times w) = g(v) \times g(w)$

   Because $R \in SO(3) \rightarrow det(R) = 1 \rightarrow$ orthogonal transformation and vxw is orthogonal and invariant under rotation...
   Rotation matrix R commutates with cross product$\therefore$

2. Prove that rigid transformation $\mathbf{g} = \mathbf{(p,R)}$ on a vector $\mathbf{v} = \mathbf{s} - \mathbf{r}$ is $\mathbf{g_*(v)} := \mathbf{Rv}$.

   Solution: $g_*(v) = g_*(s - r) = g(s) - g(r) = p + Rs - p - Rr = Rs - Rr = R(s - r) = Rv$

3. Let$\in$ $SO(3)$ be a rotation matrix generated by rotating about a unit vector ω by θ radians. That is R statisfies $R = exp(\omega\theta)$.

   (a) Show that the eigenvalues of ω are 0,i, and -i, where i=$\sqrt{-1}$. What are the corresponding eigenvectors?

$$\hat{w} = \begin{bmatrix} -\lambda & -w_3 & w_2 \\ w_3 & -\lambda & -w_1 \\ -w_2 & w_1 & -\lambda \end{bmatrix} \rightarrow -\lambda(\lambda^2 + w_1^2) + w_3(-\lambda w_3 - w_1 w_2) + w_2(w_1 w_3 - w_2\lambda)$$

$\lambda(\lambda^2 + w_1^2 + w_2^2 + w_3^2)$

$\lambda = 0, \pm\sqrt{-1}$

Eigenvector for $\lambda = 0$:

$$\begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-w_1 w_3 v_{12} + w_2 w_3 = 0 \rightarrow v_{12} = w_2/w_1$

$w_3 - w_1 v_{13} = 0 \rightarrow v_{13} = w_3/w_1$

$v_{11} = 1$

$$v_1 = \begin{bmatrix} 1 \\ w_2/w_1 \\ w_3/w_1 \end{bmatrix}$$

Eigenvectors for $\lambda = -i, i$:

$$\begin{bmatrix} i & -w_3 & w_2 \\ w_3 & i & -w_1 \\ -w_2 & w_1 & i \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -(w_2^2 + w_3^2)/(w_1 w_3 + i w_2) \\ (w_1 w_2 - i w_3)/(w_1 w_3 + i w_2) \\ 1 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} -(w_2^2 + w_3^2)/(w_1 w_3 - i w_2) \\ (w_1 w_2 + i w_3)/(w_1 w_3 - i w_2) \\ 1 \end{bmatrix}$$

   (b) Show that the eigenvalues of R are 1,$e^{i\theta}$,$e^{-i\theta}$. What is the eigenvector whose eigenvalue is 1? What is the physical interpretation of this eigenvector?

$wv = \lambda v$

$eig(e^{w\theta}) = e^{eig(w\theta)}$ *due to the spectral mapping property

$eig(w\theta) = eig \begin{bmatrix} -\lambda & -w_3\theta & w_2\theta \\ w_3\theta & -\lambda & -w_1\theta \\ -w_2\theta & w_1\theta & -\lambda \end{bmatrix}$

$\lambda_{w\theta} = 0, \pm\theta i$

Therefore, the eigenvalues of R are $1, e^{i\theta}, e^{-i\theta}$ $\therefore$

The eigenvector for $\lambda = 1$ is the same as found in part a, $v_1 = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ and represents the axis of rotation.

4. **(MLS Book [1])** *Properties of skew-symmetric matrices*

Show that the following properties of skew- symmetric matrices are true:

(a) If $R \in SO(3)$ and $\omega \in \mathbb{R}^3$, then $R\hat{\omega}R^T = \widehat{R\omega}$.

(b) If $R \in SO(3)$ and $v, \omega \in \mathbb{R}^3$, then $R(v \times \omega) = (Rv) \times (R\omega)$.

(c) Verify the following formula given $x \in \mathbb{R}^3$ $(||x|| \neq 1)$,

$$e^x = \mathbb{I} + \frac{\sin ||x||}{||x||}\hat{x} + \frac{1 - \cos ||x||}{||x||^2}\hat{x}^2$$

Solution:

(a) Letting r_i^T be the ith row of R, we have

$$\begin{aligned} R\hat{\omega}R^T &= \begin{bmatrix} r_1^T(\omega \times r_1) & r_1^T(\omega \times r_2) & r_1^T(\omega \times r_3) \\ r_2^T(\omega \times r_1) & r_2^T(\omega \times r_2) & r_2^T(\omega \times r_3) \\ r_3^T(\omega \times r_1) & r_3^T(\omega \times r_2) & r_3^T(\omega \times r_3) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -r_3^T\omega & r_2^T\omega \\ r_3^T\omega & 0 & -r_1^T\omega \\ -r_2^T\omega & r_1^T\omega & 0 \end{bmatrix} \\ &= \widehat{R\omega} \end{aligned}$$

(b)

$$\text{Consider: } u \cdot (v \times w) = \det \begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix} = \det(A)$$

$$\begin{aligned}
(Ru) \cdot (Rv \times Rw) &= \det \begin{pmatrix} Ru_x & Rv_x & Rw_x \\ Ru_y & Rv_y & Rw_y \\ Ru_z & Rv_z & Rw_z \end{pmatrix} \\
&= \det(RA) \\
&= (\det R)(\det A) \\
&= \det A
\end{aligned}$$

$$\begin{aligned}
(Ru) \cdot (Rv \times Rw) &= u \cdot (v \times w) = (Ru) \cdot R(v \times w) \ \forall u \\
\therefore (Rv) \times (Rw) &= R(v \times w)
\end{aligned}$$

(c) Given Rodriguez's formula,

$$e^{\hat{\omega}\theta} = \mathbb{I} + \hat{\omega} \sin\theta + \hat{\omega}^2(1 - \cos\theta),$$

we can represent $\hat{\omega}$ and θ, unit vector and scalar angle, respectively, as features of a non-unit vector $\hat{x}$, where $\hat{\omega} = \frac{\hat{x}}{||x||}$ and $\theta = ||x||$. Applying these substitutions to Rodriguez's formula yields

$$e^{\hat{x}} = \mathbb{I} + \frac{\sin||x||}{||x||}\hat{x} + \frac{1 - \cos||x||}{||x||^2}\hat{x}^2.$$

5. Let $Q = (q_0, \vec{q})$ and $P = (p_0, \vec{q})$ be quaternions, where $q_o, p_o \in R$ are the scalar parts of Q and P and $\hat{p}, \hat{q}$ are the vector parts.

   (a) Show that the set of unit quaternions satisfies the axioms of the group.

   1. $g_1, g_2 \in G \rightarrow g_1 * g_2 \in G$:

   $Q \cdot P = (q_0 + q_1 i + q_2 j + q_3 k) \times (p_0 + p_1 i + p_2 j + p_3 k)$
   $q_0p_0 + q_0p_1 i + q_0p_2 j + q_0p_3 k + p_0q_1 i + -q_1p_1 + q_1p_2 k - q_1p_3 j + q_2p_0 j - q_2p_1 k - q_2p_2 + q_2p_3 i + q_3p_0 k + q_3p_1 j - q_3p_2 i - q_3p_3$
   $= (q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3) + (q_0p_1 + p_0q_1 + q_2p_3 - q_3p_2)i + (q_0p_2 - q_1p_3 + q_2p_0 + q_3p_1)j + (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)k$

   CLOSURE: SATISFIED

   2. $Q \times e = e \times Q$:

   $e = 1 + 0i + 0j + 0k$

   IDENTITY: SATISFIED

   3. Inverse of Q times Q and Q times inverse of Q must equal 1:

   $Q = (q_0 + q_1 i + q_2 j + q_3 k)$
   $Q^{-1} = (q_0 - q_1 i - q_2 j - q_3 k)$
   $QQ^{-1} = 1$ and $Q^{-1}Q = 1$

   INVERSE: SATISFIED

   4. Order of multiplication irrelevant:
   $(Q_{ab}Q_{bc})Q_{cd} = QacQcd = Qad$
   $Q_{ab}(Q_{bc}Q_{cd}) = QabQbd = Qad$

   ASSOCIATIVITY: SATISFIED

   (b) Let x be a point and let X be a quaternion whose scalar part is zero and whose vector part is equal to x (such a quaternion is called a pure quaternion). Show that is Q is a unit quaternion, the product QXQ* is a pure quaternion and the vector part of QXQ* satisfies $(q_o^2 - \vec{q} \cdot \vec{q})\vec{x} + 2(q_o(\vec{q} \times \vec{x}) + (x \cdot \vec{q})\vec{q})$

X $= 0 + x_1 i + x_2 j + x_3 k = (0, \vec{x})$
Q $= q_0 + q_1 i + q_2 j + q_3 k = (0, \vec{q})$
Q* $= q_0 - q_1 i - q_2 j - q_3 k = (0, -\vec{q})$

XQ* $= (\vec{x} \cdot \vec{q}, q_0 \vec{x} + \vec{q} \times \vec{x})$

Vector component of QXQ*$=q_0(q_0\vec{x} + \vec{q} \times \vec{x}) + (\vec{x} \cdot \vec{q}) \times \vec{q} + \vec{q}(\vec{q_0}\vec{x} + \vec{q} \times \vec{x})$
$q_0^2\vec{x} + 2q_0(\vec{q} \times \vec{x}) + (\vec{x} \cdot \vec{q})\vec{q} - (-\vec{q} \cdot \vec{x})(\vec{q}) + (-\vec{q} \cdot \vec{q})\vec{x} + (\vec{x} \cdot \vec{q})(\vec{q}) - (\vec{q} \cdot \vec{q})\vec{x}$
Vector component $= (q_0^2 - \vec{q} \cdot \vec{q})\vec{x} + 2(q_0(\vec{q} \times \vec{x}) + (\vec{x} \cdot \vec{q})\vec{q})$

Scalar component $= QXQ* = q_0 \cdot 0 \cdot q_0 = 0$ $\therefore$pure quaternion

6. In the class, we have learned that the ZXZ and ZYZ are the most common (the explicit form of the ZYZ Euler angles is in MLS). Here, construct the explicit representation for $R_{xyz}(\phi) : \mathbb{R} \mapsto SO(3)$ given by

$$R_{xyz}(\psi, \theta, \phi) = R_x(\phi)R_y(\theta)R_z(\psi)$$

Construct the inverse function $R_{xyz}^{-1} : SO(3) \mapsto \mathbb{R}^3$ such that $\forall R \in SO(3)$ if $y = R_{xyz}^{-1}(R)$ then $R = R_{xyz}(y)$, i.e., derive the formulas such as in (2.20) in MLS.

Solution:

$$R_x(\phi) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix},$$

$$R_y(\theta) := \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

$$R_z(\psi) := \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$
\begin{aligned}
R_{xyz}(\psi,\theta,\phi) &= R_x(\phi)R_y(\theta)R_z(\psi) \\
&= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ c_\phi s_\psi + c_\psi s_\phi s_\theta & c_\phi c_\psi - s_\phi s_\theta s_\psi & -c_\theta s_\phi \\ s_\phi s_\psi - c_\phi c_\psi s_\theta & c_\psi s_\phi + c_\phi s_\theta s_\psi & c_\phi c_\theta \end{bmatrix}
\end{aligned}
$$

Use Atan2 function to construct the inverse functions for finding ϕ, θ, and ψ given R_{xyz} when $\theta \in (-\pi/2, \pi/2)$:

$$
\begin{aligned}
\phi &= \text{Atan2}(-r_{23}, r_{33}) \\
\theta &= \text{Atan2}(r_{13}, \sqrt{r_{11}^2 + r_{12}^2}) \\
\psi &= \text{Atan2}(-r_{12}, r_{11})
\end{aligned}
$$

When $\theta \in (\pi/2, 3\pi/2)$, the terms in the Atan function are multiplied by -1:

$$
\begin{aligned}
\phi &= \text{Atan2}(r_{23}, -r_{33}) \\
\theta &= \text{Atan2}(-r_{13}, -\sqrt{r_{11}^2 + r_{12}^2}) \\
\psi &= \text{Atan2}(r_{12}, -r_{11})
\end{aligned}
$$

7. Exercise 3.16

(a) Draw by hand a diagram showing a and b relative to s.

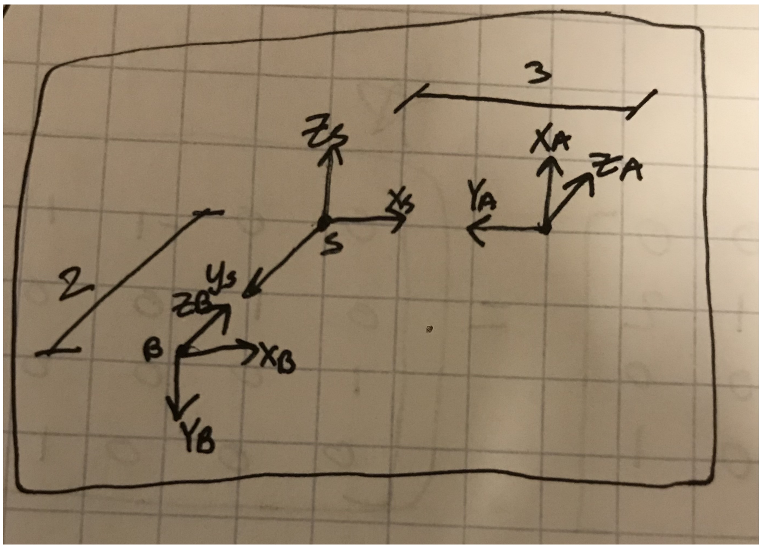

Figure 1: Frames

(b) Write down the rotation matrices R_{sa} and R_{sb} and the transformation matrices T_{sa} and T_{sb}.

$$R_{sa} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \quad R_{sb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

(c) Given T_{sb}, how do you calculate T_{sb}^{-1} without using a matrix inverse? Write T_{sb}^{-1} and verify its correctness using your drawing.

$$p = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \quad R = R_{sb}$$

$$T_{sb}^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ : Looks correct in drawing}$$

(d) Given T_{sa} and T_{sb}, how do you calculate T_{ab} (again without using matrix inverses)? Compute the answer and verify its correctness using your drawing.

$$T_{ab} = \begin{bmatrix} R_{sa} & -R_{sa}^T p \\ 0 & 1 \end{bmatrix} T_{sb}$$

$$T_{ab} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} : \text{Looks correct}$$

(e) Let $T = T_{sb}$ be considered as a transformation operator consisting of a rotation about $\hat{x}$ by -90 and a translation along $\hat{y}$ by 2 units. Calculate $T_1 = T_{sa}T$. Does T_1 correspond to a rotation and translation about $\hat{x_s}$ and $\hat{y_s}$, respectfully (a world fixed transformation of T_{sa}? Now calculate $T_2 = TT_{sa}$. Does T_2 correspond to a body-fixed frame or world-fixed transformation of T_{sa}?

$$T = T_{sb} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_1 = T_{sa}T = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T_1 is a body frame transformation of T_{sa}

$$T_2 = TT_{sa} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T_2 is a world fixed transformation of T_{sa}

(f) Use T_{sb} to change the representation of the point p_b=(1,2,3) in b coordinates to s coordinates.

$$p_s = T_{sb}p_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} \text{ coordinate form*}$$

(g) Choose a point p represented by p_s=(1,2,3) in s coordinates. Calculate $p' = T_{sb}p_s$ and $p" = T_{sb}^{-1}p_s$. For each operation, should the result be interpreted as changing coordinates (from the s frame to the b) without moving the point p, or as moving the location of the point without changing the reference frame of the representation?

$$p' = T_{sb}p_s = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix} \rightarrow \text{different point in same frame S}$$

$$p" = T_{sb}^{-1}p_s = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \rightarrow \text{same point but in frame B}$$

(h) A twist V is represented in s as $V_s = (3, 2, 1, -1, -2, -3)$. What is its representation V_a in the frame a?

$$p_{sa} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} R_{sa} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$V_a = [Ad_{T_{as}}]V_s = \begin{bmatrix} R_{sa}^T & 0 \\ -R_{sa}^T[p_{sa}] & R_{sa}^T \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ -1 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -2 \\ 6 \\ 4 \\ 2 \end{bmatrix}$$

(i) By hand, calculate the matrix logarithm $[S]\theta$ of T_{sa}. (You may verify your answer with software.) Extract the normalized screw axis S and rotation amount θ. Find the q,s,h representation of the screw axis. Redraw the fixed frames and in it draw S.

$2cos(\theta) + 1 = trace(R_{sa})$
$\theta = 2\pi/3$

$\hat{w} = (R_{sa} - R_{sa}^{-1})/(2sin(\theta))$

$$\hat{w} = \begin{bmatrix} 0 & -.5774 & -.5774 \\ .5774 & 0 & -.5774 \\ .5774 & .5774 & 0 \end{bmatrix}$$

$$A^{-1} = I/\theta - [w]/2 + (1/\theta - cot(\pi/3)/2)[w]^2 = \begin{bmatrix} .3516 & .2257 & .3516 \\ -.3516 & .3516 & .2257 \\ -.2257 & -.3516 & .3516 \end{bmatrix}$$

$$p_{sa} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$$S[\theta] = \begin{bmatrix} \hat{w}\theta & A^{-1}p\theta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1.2093 & -1.2093 & 1.4181 \\ 1.2093 & 0 & -1.2093 & 2.2092 \\ 1.2093 & 1.2093 & 0 & -2.2092 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} \hat{w} & A^{-1}p \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -.5774 & -.5774 & .6771 \\ .5774 & 0 & -.5774 & 1.0548 \\ .5774 & .5774 & 0 & -1.0548 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} .5774 \\ -.5774 \\ .5774 \\ .6771 \\ 1.0548 \\ -1.0548 \end{bmatrix}$$

$$\hat{s} = w/\|w\| = \begin{bmatrix} .5774 \\ -.5774 \\ .5774 \end{bmatrix} :: h = \|v\|/\|w\| = 1.6382 :: q = w \times v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

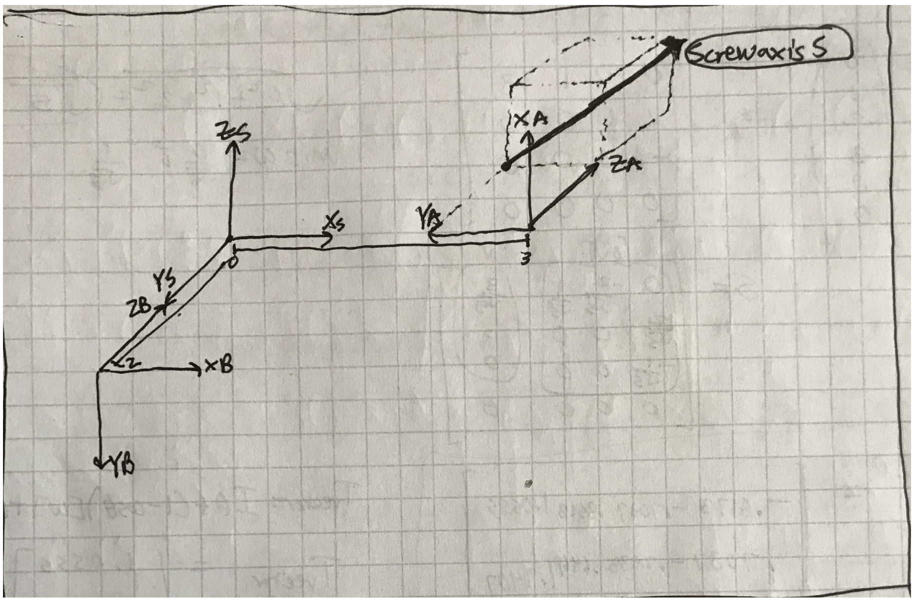

Figure 2: Frames

8. **Exercise 3.18** Consider a robot arm mounted on a spacecraft as shown in Figure 3.24, in which frames are attached to the Earth {e}, a satellite {s}, the spacecraft {a}, and the robot arm {r}, respectively.

   (a) Given T_{ea}, T_{ar}, and T_{es}, find T_{rs}.

   (b) Suppose that the frame {s} origin as seen from {e} is (1,1,1) and that

$$T_{er} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   Write down the coordinates of the frame {s} origin as seen from frame {r}.

   Solution:

   (a) $T_{rs} = (T_{ar})^{-1}(T_{ae})(T_{es})$

   (b) $T_{er}^{-1} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

   The coordinates of the frame {s} origin as seen from frame {r} are (0,0,0).

9. Pelvic Osteotomy Situation Question: Give an expression for computing the position and orientation F_{CK} of the osteotome blade in CT coordinates , based on the available tracking system measurements F_{Bx}

   Solution: $F_{CK} = F_{GC}^{-1} F_{BG}^{-1} F_{BH} F_{HK}$ with $F_{BX} = F_{BH} F_{HK}$

10. **(Bonus question [1])** Because arithmetic precision is only finite, the numerically obtained product of two rotation matrices is not necessarily a rotation matrix; that is, the resulting rotation A may not exactly satisfy $A^T A = I$ as desired. Devise an iterative numerical procedure that takes an arbitrary matrix $A \in \mathbb{R}^{3x3}$ and produces a matrix $R \in SO(3)$ that minimizes

$$||A - R||^2 = \text{tr}(A - R)(A - R)^T.$$

   (Hint: See Appendix D of Lynch et al. for the relevant background on optimization.)

Solution: A description of the mathematical approach and MATLAB code for this problem is included in the programming assignment report document.