Introduction

Book: Not mandatory
- → Uploaded relevant excersises
- → Use another book or Google

Prerequisites: Recap today
Expect you to fill in the blanks

Exam: 3 hour in Wiseflow
Documentation must by uploaded
Must be .ipynb format

Tools: Python 3
Jupyter Notebook
- ↳ VScode
- ↳ Jupyter Lab
- ↳ DataSpell (Jetbrains)
- ↳ Google Colab

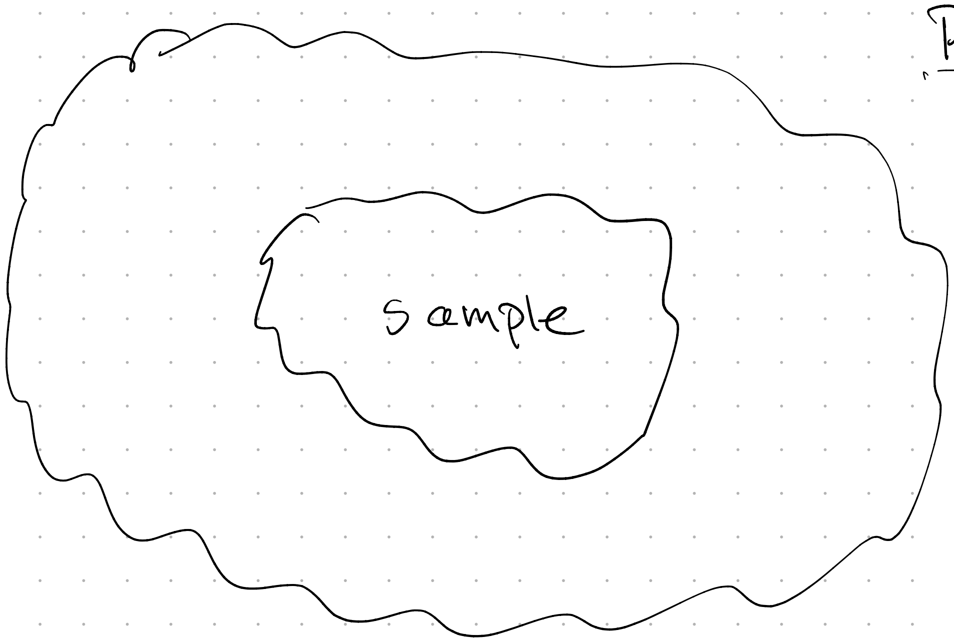
Itslearning: I use in list mode
Session 0 will contain all material not associated with a specific session!

Wiseflow: You will receive multiple Wiseflow assignment

Why Probability and statistics?

- Stochastic = Random
- Natural Proces → Variability → Uncertainty
  - ↳ statistics is the study of how to deal with uncertainty
  - ↳ Probability is the unit in which we measure uncertainty

Samples and Populations:



Population: all Android Devices

Sample = Representative subset of Population

Sample size → Big Data: $E = \frac{Z \cdot \sigma}{\sqrt{n}}$

Scales and Measurements:

Nominal: Values have no quantitative significance: 0 = male, 1 = female
↳ 1 > 0 ?

↳ Groups, classes, categories
e.g. gender, color, Jobs, et.

Ordinal: Values are comparable, but difference is not known

↳ Order matters: Ranking.

Interval: Values are comparable and diffence and distance matter
↳ Zero is assigned arbitrarily

* Temperature
* Time of day
* Dates
* Likert scale

Ratio: Same as interval but with a "natural" zero:

* Height
* Weight
* Income
* Number of children

Random Experiments

- Process where something uncertain is Observed
- An outcome is the result of a random experiment
- A sample space S is the set of all possible outcomes.

Toss a coin:	$S = \{H, T\}$
Guess a bit string:	$S = \{0, 1\}^n$
Roll a die :	$S = \{1, 2 \ldots 6\}$
Price of item :	$S = \{0 \ldots \infty\}$
Goals scored in football match :	$S = \{0, 1, \ldots \infty\}$
Goals scored in Italian football :	$S = \{0, 1\}$

- An event A is a subset of the sample space.

Probability

Objective/Classical	Emperical	Subjective
- Based on <u>equally</u> likely events - Long run relative frequency - Same for all observers	- Based on <u>observation</u> - Relative frequency of large amount of observations	- Based on <u>personal</u> belief, experience, prejudice, etc. - Different for all observers

Axioms of Probability

1. For any event A, $P(A) \geq 0$
2. $P(S) = 1$
3. If $A_1, A_2, A_3 \ldots$ are disjoint

$$P(A_1 \cup A_2 \cup A_3 \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$$

<u>Notation:</u>

$$P(A \cap B) = P(A \text{ and } B) = P(A, B)$$

$$P(A \cup B) = P(A \text{ or } B)$$

$$P(\bar{A}) = P(A') = P(A^c) = P(\text{not } A)$$

Rules of Probability

a) $P(A^c) = 1 - P(A)$

b) $P(\emptyset) = 0$

c) $P(A) \leq 1$

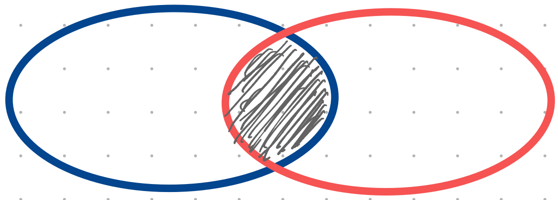
d) $P(A-B) = P(A) - P(A \cap B)$

e) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

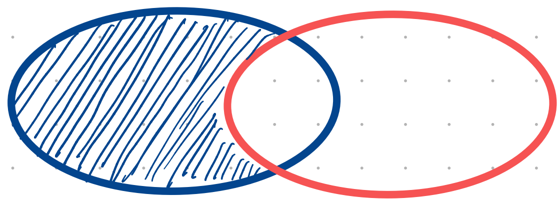
d) $A \subset B \rightarrow P(A) \leq P(B)$

Is a subset of

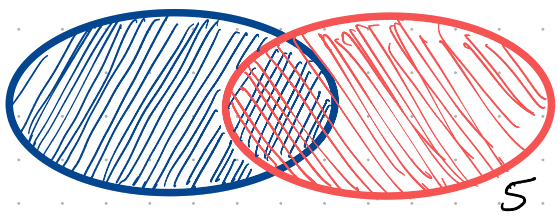
$A \cap B$



$A - B$



$A \cup B$



Example:

1. 60% Chance of rain today: $P(A) = 0.6$
2. 50% Chance of rain tomorrow: $P(B) = 0.5$
3. 30% Chance of no rain either day: $P(A^c \cap B^c) = 0.3$

a. Probability of rain either day:

$$\left.\begin{aligned} P(A \cup B) &= 1 - P(A \cup B)^c \\ &= 1 - P(A^c \cap B^c) \end{aligned}\right\} \text{De Morgan's Law}$$

$$= 1 - 0.3 = \underline{\underline{0.7}}$$

b. Rain both days:

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 0.6 + 0.5 - 0.7 \\ &= \underline{\underline{0.4}} \end{aligned}$$

c. Rain today, not tomorrow:

$$\begin{aligned} P(A - B) &= P(A) - P(A \cap B) \\ &= 0.6 - 0.4 \\ &= \underline{\underline{0.2}} \end{aligned}$$

d. Rain today or tomorrow, not both

$$\begin{aligned} P\big((A \cup B) \setminus (A \cap B)\big) &= P(A \cup B) - P(A \cap B) \\ &= 0.7 - 0.4 = \underline{\underline{0.3}} \end{aligned}$$

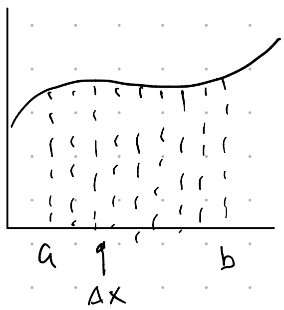
OR

$$\begin{aligned} &= P(A - B) + P(B - A) \\ &= 0.2 + (0.5 - 0.4) = \underline{\underline{0.3}} \end{aligned}$$

Discrete Probability Models

Discrete — **Continuous**

$$\sum$$



$A_1 = \Delta x \cdot f(x_1)$

$A_2 = \Delta x \cdot f(x_2)$

$\vdots$

$A_n = \Delta x \cdot f(x_n)$

$$\int$$

$$\downarrow$$

$$A \approx A_1 + A_2 + A_3 + \dots + A_n \approx \sum_a^b \Delta x \cdot f(x_i) = \int_a^b f(x)\,dx$$

$$S \to \int \to \int \to \int \qquad \Delta x \to 0$$

If a sample space is countable, we use a discrete probability model

If $A \subset S$:

$$P(A) = P\left(\bigcup_{s_j \in A} \{s_j\}\right) = \sum_{s_j \in A} P(s_j)$$

Example:

Assume you win $k-2$ kroner with $p = \frac{1}{2^k}$, $k \in \mathbb{N}$

a. loose 1: $k-2 = -1 \Rightarrow k = 1$ so $P_{-1} = 1/2$

b. loose 0: $k-2 = 0 \Rightarrow k = 2$ so $P_0 = 1/4$

c. Win 1: $k-2 = 1 \Rightarrow k = 3$ so $P_1 = 1/8$

d. win 2: $k-2 = 2 \Rightarrow k = 4$ so $P_2 = 1/16$

e. Win 3: $k-2 = 3 \Rightarrow k = 5$ so $P_3 = 1/32$

f. Win [1;3]: $P_1 + P_2 + P_3 = 7/32$

g. More than 2: $P(3) + P(4) + P(5) + \dots$

$= 1/16 + 1/32 + 1/64 + \dots$

Conditional Probability

$$P(A|B) = \frac{P(A\cap B)}{P(B)}, \quad P(B)>0$$

Some important consequences:

$\rightarrow$ $P(A^c|B) = 1 - P(A|B)$

$\rightarrow$ $P(\emptyset|B) = 0$

$\rightarrow$ If $B\subset A$, $P(A|B) = 1 = \frac{P(A\cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

$\rightarrow$ If $A\subset B$, $P(A|B) = \frac{P(A)}{P(B)}$

Chain Rule

$$P(A_1\cap A_2\cap A_3 \ldots A_k) = P(A_1)\cdot P(A_2|A_1)\cdot P(A_3|A_1,A_2)\cdot \ldots P(A_k|A_{k-1},A_{k-2}\ldots)$$

Independence

$P(A|B) = P(A)$ iff. A and B are independent

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$\Downarrow$$

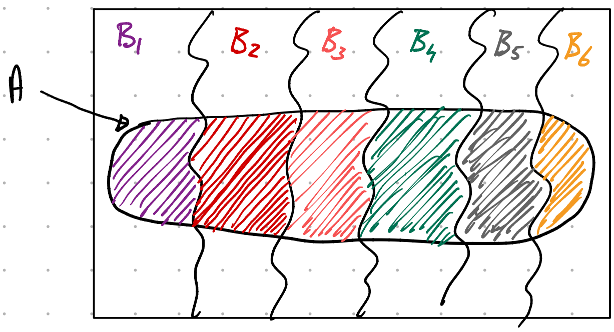
$$P(A\cap B) = P(A|B)\cdot P(B) = P(A)\cdot P(B)$$

Law of total probability

$$P(A) = \sum P(A\cap B_i)$$

$$P(A) = P(A\cap B_1) + P(A\cap B_2) + \ldots$$

$$P(A) = \sum P(A|B_i)\cdot P(B_i)$$



Bayes' Rule

$$P(A|B) \cdot P(B) = P(A \cap B) = P(B|A) \cdot P(A)$$

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

$$P(B_j|A) = \frac{P(A|B_j) \cdot P(B_j)}{\sum_i^k P(A|B_i) \cdot P(B_i)}$$

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A|B) \cdot P(B) + P(A|\bar{B}) \cdot P(\bar{B})}$$

Conditional Independence (not recap)

Two events A and B are said to be conditionally independent given an event C if

$$P(A \cap B|C) = P(A|C) \cdot P(B|C) \quad , P(C) > 0$$

Understanding Conditional independence:

$P(A|B,C) =$ A given B and C

If A and B are conditionally independent, then $P(A|B,C)$ does not depend on B and we get $P(A|B,C) = P(A|C)$ and also $\boxed{P(A,B|C) = P(B|C)}$, i.e. A and B are independent given C.

Example

A box contains two coins: a regular coin and one fake two-headed coin ($P(H) = 1$). I choose a coin at random and toss it twice. Define the following events.

- A= First coin toss results in an H.
- B= Second coin toss results in an H.
- C= Coin 1 (regular) has been selected.

Find $P(A|C), P(B|C), P(A \cap B|C), P(A), P(B)$, and $P(A \cap B)$. Note that A and B are NOT independent, but they are *conditionally* independent given C.

Regular Coin: $P(H) = 1/2$, $P(T) = 1/2$

Fake Coin : $P(H) = 1$, $P(T) = 0$

$$P(A|C) = 1/2$$

$$P(B|C) = 1/2$$

$$P(A \cap B|C) = 1/2 \cdot 1/2 = 1/4$$

$$\begin{aligned} P(A) &= P(A|C) \cdot P(C) + P(A|\bar{C}) \cdot P(\bar{C}) \\ &= 1/2 \cdot 1/2 + 1 \cdot 1/2 \\ &= 3/4 \end{aligned}$$

$$\begin{aligned} P(B) &= \text{Same as } P(A) \\ &= 3/4 \end{aligned}$$

$$\begin{aligned} P(A \cap B) &= P(A \cap B|C) \cdot P(C) + P(A \cap B|\bar{C}) \cdot P(\bar{C}) \\ &= P(A|C) \cdot P(B|C) \cdot P(C) + P(A|\bar{C}) \cdot P(B|\bar{C}) \cdot P(\bar{C}) \\ &= 1/2 \cdot 1/2 \cdot 1/2 + 1 \cdot 1 \cdot 1/2 \\ &= \frac{1}{8} + \frac{1}{2} = \underline{\underline{\frac{5}{8}}} \end{aligned}$$

Random Variables

A random variable X is a function from the sample space to $\mathbb{R}$:

$$X: S \to \mathbb{R}$$

The range R_X is the set of all possible values of X

Ex:

- Toss a coin 100 times. Let X be the number of heads: $R_X = \{0, 1 \ldots 100\}$
- Toss a coin until the first head. Let Y be the number of tosses until first head:

$$R_Y = \mathbb{N}_+$$

- Let Z denote the number of times you have to stop at a red light on your way to school:

$$P(Z=0)$$

$$P(Z=1)$$

$$P(Z \leq 2)$$

$\vdots$

etc.