

Functions of Continuous R.V.'s: Difficult

If X is a CRV and $Y = g(X)$, then Y is also a R.V.

Example

Let X be a Uniform(0,1) R.V. and let $Y = e^X$

a) Find CDF of Y

$$F_Y(y) = \frac{y-a}{b-a} = \frac{y-0}{1-0} = y, f(x)=1$$

$$R_X = [0; 1], R_Y = [1; e]$$

$$F(Y) = P(Y \leq y)$$

$$= P(e^X \leq y)$$

$$= P(X \leq \ln y)$$

SG

$$F_Y = \begin{cases} 0 & y < 1 \\ \ln y & 1 \leq y < e \\ 1 & y \geq e \end{cases}$$

b) Find pdf of Y :

$$f_Y(y) = F'_Y(y) = \frac{1}{y} \quad \text{for } 1 \leq y \leq e, \text{ else } 0$$

c) Find $E[Y]$:

$$E[Y] = \int_1^e y \cdot \frac{1}{y} dy = y \Big|_1^e = \underline{\underline{e-1}}$$

Using hotus:

$$\begin{aligned} E[Y] &= E[e^X] = \int_0^1 e^x \cdot f_X(x) dx = e^x \Big|_0^1 \\ &= e^1 - e^0 = \underline{\underline{e-1}} \end{aligned}$$

Example

Let $f_X(x) = \begin{cases} 4x^3 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

and let $Y = \frac{1}{X}$. Find PDF.

Method 1: like above we first find CDF

Note $R_Y = [1, \infty]$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P\left(\frac{1}{X} \leq y\right) \\ &= P(X \geq \frac{1}{y}) \\ &= F_X(y) = \int_0^{1/y} 4x^3 dx = x^4 \Big|_0^{1/y} = \frac{1}{y^4} \end{aligned}$$

$$f_Y(y) = F_Y'$$

Method 2: The Method of Transformation

Let X be a CRV and $g: \mathbb{R} \rightarrow \mathbb{R}$ is a strictly monotonic and differentiable function

$$(x^{-1})' = -x^{-2}$$

$$f_Y(y) = \begin{cases} \frac{f_X(x_1)}{|g'(x_1)|} & \text{where } g(x_1) = y \Rightarrow x_1 = g^{-1}(y) = \frac{1}{y} \\ 0 & \text{if } g(x) = y \text{ has no solution} \end{cases}$$

Exponential Distribution:

A CRV X is said to be exponentially distributed with $\lambda > 0$, shown as

$X \sim \text{Exponential}(\lambda)$, if its PDF is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0 \\ 0 & \end{cases}$$

Is often used to model time between events. let's find CDF:

λ is called the rate parameter.

$$E(X) = \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$

Example

Jobs are sent to printer at a rate of 3 jobs per hour (on average)

a) What is expected time between jobs:

b) What is probability that next job is sent within 5 minutes?

Normal Distribution

Characteristics:

Symmetric, bell shaped

Continuous for all intervals

s.t. any $P(X \in [a, b]) \neq 0$

$-\infty \leq X \leq \infty$

Two parameters, μ and σ^2

PDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Notation: $N(\mu, \sigma^2)$ or $X \sim N(\mu, \sigma^2)$

About $2/3$ of all cases lie in $[\mu-\sigma; \mu+\sigma]$

$$P(\mu-\sigma \leq X \leq \mu+\sigma) \approx 0.6826$$

About 95% of all cases lie in $[\mu-2\sigma; \mu+2\sigma]$

$$P(\mu-2\sigma \leq X \leq \mu+2\sigma) \approx 0.9544$$

Standard normal distribution

→ Special case of normal distribution with $\mu=0$ and $\sigma^2=1$, i.e. $N(0, 1)$

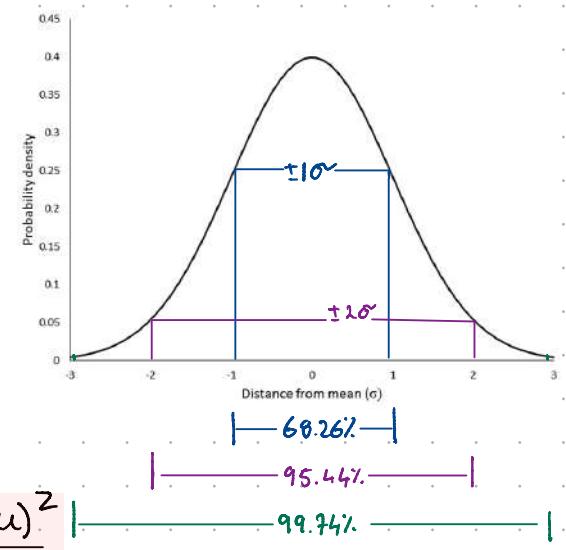
① Convert problem to a standardized normal variable, z-score

$$z = \frac{x - \mu}{\sigma} \sim N(0, 1)$$

② A table of z-scores exists

③ We can convert back:

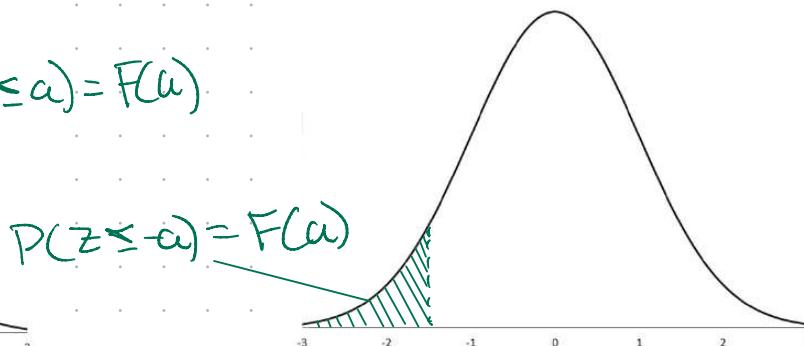
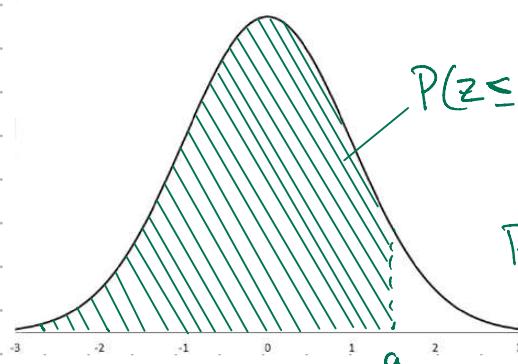
$$x = z \cdot \sigma + \mu$$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003
-3.3	.0005	.0005	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0006	.0005	.0005
-3.1	.0011	.0011	.0011	.0011	.0011	.0011	.0011	.0011	.0011	.0010
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0028	.0025	.0024	.0023	.0023	.0022	.0021	.0020	.0019	.0019
-2.7	.0038	.0032	.0029	.0026	.0024	.0022	.0020	.0018	.0016	.0016
-2.6	.0047	.0044	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0066	.0060	.0057	.0055	.0054	.0052	.0051	.0049	.0048	.0048
-2.4	.0093	.0079	.0067	.0057	.0051	.0047	.0043	.0039	.0035	.0033
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0177	.0174	.0170	.0168	.0162	.0158	.0153	.0150	.0148	.0143
-2.0	.0217	.0212	.0207	.0203	.0202	.0201	.0197	.0194	.0191	.0183
-1.9	.0267	.0281	.0274	.0268	.0262	.0256	.0244	.0239	.0233	.0223
-1.8	.0359	.0351	.0344	.0329	.0322	.0314	.0307	.0301	.0294	.0283
-1.7	.0471	.0459	.0446	.0431	.0421	.0411	.0401	.0391	.0381	.0371
-1.6	.0584	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0686	.0655	.0643	.0618	.0606	.0594	.0582	.0571	.0559	.0559
-1.4	.0803	.0776	.0754	.0749	.0735	.0721	.0708	.0694	.0681	.0676
-1.3	.0934	.0904	.0879	.0854	.0831	.0807	.0783	.0759	.0735	.0723
-1.2	.1115	.1131	.1112	.1093	.1075	.1056	.1036	.1010	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1194	.1170
-1.0	.1641	.1611	.1581	.1551	.1519	.1487	.1455	.1424	.1393	.1362
-0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1633	.1611
-0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.7	.2420	.2389	.2356	.2327	.2296	.2266	.2236	.2206	.2176	.2148
-0.6	.2743	.2707	.2674	.2641	.2608	.2575	.2542	.2509	.2475	.2443
-0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-0.3	.3826	.3787	.3749	.3710	.3671	.3632	.3593	.3553	.3513	.3473
-0.2	.4207	.4169	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
-0.1	.4602	.4562	.4522	.4483	.4443	.4403	.4364	.4325	.4286	.4247
-0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Rules:

$$1. P(Z \leq a) = F(a)$$



Example:

$$\text{Find } P(Z \leq a) \text{ for } a = \{-1.65, -1.00, 1.00, 1.65\}$$

$$P(Z \leq -1.65) =$$

$$P(Z \leq -1.00) =$$

$$P(Z \leq 1.00) =$$

$$P(Z \leq 1.65) =$$

$$\text{Find } a \text{ for } \{0.6026, 0.9750, 0.3446\}$$

$$P(Z \leq a) =$$

$$a =$$

$$P(Z \leq a) =$$

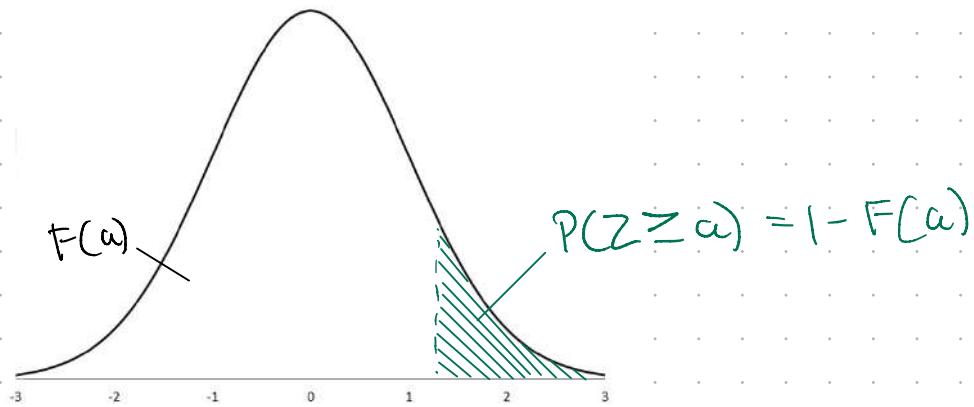
$$a =$$

$$P(Z \leq a) =$$

$$a =$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
-3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0179	.0174	.0170	.0165	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1036	.1020	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-0.9	.1841	.1814	.1784	.1752	.1738	.1711	.1685	.1660	.1635	.1611
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-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
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-0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5949	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8103	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9812	.9817	
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9980	.9981	
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9993	.9993	
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	

$$2. P(Z \geq a) = 1 - F(a) = F(-a) = P(Z \leq -a)$$



Example:

$$P(Z \geq 1.5) =$$

$$P(Z \geq 2) =$$

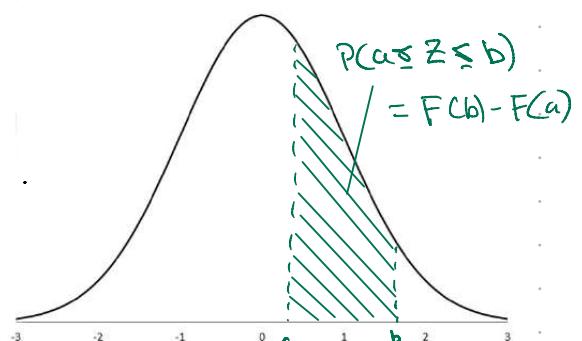
$$3. P(a \leq Z \leq b) = F(b) - F(a), \quad b \geq a$$

Example:

$$P(-1 \leq Z \leq 1.5) =$$

$$=$$

$$=$$



Example:

Family income $\sim N(\$25000, \$10.000^2)$

If poverty level is \$10,000, what percentage of population live in Poverty:

$X = \text{Family income}$: