I ♡ SMP

Discret Random Variables

A function that assigns a real number to each outcome:

$$X : S \to \mathbb{R}$$

* Usually an R.V. is denoted by X, Y, Z (Upper Case)
* The outcomes are denoted by x, y, z (lower Case)

e.g. $P(X=x)$ if $x=2$
$P(X=2)$
$P(Y=y)$ if $y=3$
$P(Y=3)$

Random Variables are either Discrete or Continuous:

Discrete

- The range is countable $\{1, 2, 3, \ldots\}$
  * Uniform
  * Binomial
  * Geometric
  * Negative Binomial
  * Hypergeometric
  * Poisson

Continuous

- The range is continuous
  ↳ $\mathbb{R}_+$
  * Uniform
  * Normal Distribution
  * Chi-Squared
  * Log-Normal
  * Exponential

Definition: Discrete Random Variable

Let X be a discrete R.V. with $R_X = \{x_1, x_2 \dots\}$

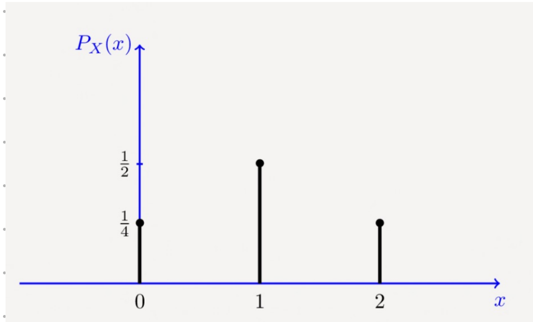
The function

$$P_X(x_k) = P(X = x_k), \quad \text{for } k = 1, 2, 3 \dots$$

is called the probability mass function (PMF) of X.

Properties:

1) $f(x_k) = P_X(x_k)$
2) $0 \leq f(x_k) \leq 1$
3) $\sum_{k=1}^{n} f(x_k) = 1$



Independent Random Variables:

Consider X and Y. We say X and Y are independent if:

$$P((X = x) \cap (Y = y)) = P(X = x) \cdot P(Y = y)$$

It follows:

$$P(Y = y \mid X = x) = P(Y = y)$$

Example:

I toss a coin twice and define X to be the number of heads I observe. Then, I toss the coin two more times and define Y to be the number of heads that I observe this time. Find $P\Big((X < 2) \text{ and } (Y > 1)\Big)$.

$$\begin{aligned}
P(X < 2, Y > 1) &= P(X < 2) \cdot P(Y > 1) \\
&= \big(P(X = 0) + P(X = 1)\big) \cdot P(Y = 2) \\
&= \left(\tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{1}{2}\right) \cdot \left(\tfrac{1}{2} \cdot \tfrac{1}{2}\right) = \left(\tfrac{1}{4} + \tfrac{1}{2}\right) \cdot \tfrac{1}{4} \\
&= \tfrac{3}{16} \approx \underline{\underline{0.1875}}
\end{aligned}$$

Bernoulli Distribution:

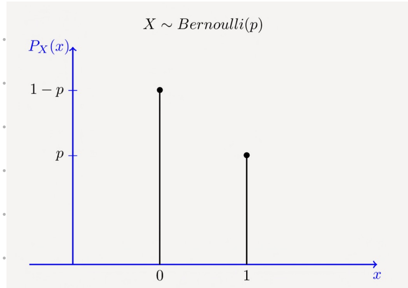
A Bernoulli R.V. can only take two values:

1 : success

0 : failure

Bernoulli PMF:

$$P_X(x) = \begin{cases} p & \text{for } x = 1 \\ 1-p & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$



Binomial Distribution:

Given multiple independent Bernoulli experiment, the resulting R.V. has a Binomial PMF:

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F S F S F

$$P_X(x) = f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

X = number of successes

p = probability of success

n = number of trials/Experiments

$\binom{n}{x}$ = Binomial Coefficient

↳ Combination of subset:

In how many ways can I choose x elements from n when order does not matter:

$$C^n_r = \frac{n!}{r!(n-r)!}$$

We can summarize as

$$\boxed{P(X=x) = \frac{n!}{x!(n-x)!} \cdot p^x (1-p)^{n-x}}$$

* Flip a coin 12 times
  - ↳ each is a Bernoulli trial
  - ↳ together they form $B \sim (n,p)$
  - ↳ $n = 12$, $p = 1/2$ $\quad X \sim Bin(n,p)$

$X \sim Bin(12, 0.5)$

$P(X=2)$ means given 12 coin flips what is the probability of getting exactly 2 heads (if defined as succes) $\quad P(X=2) = \frac{12!}{2! \cdot 10!} \cdot (1/2)^2 \cdot (1/2)^{10} = \underline{\underline{0.016}}$

* A footballer takes 4 penalties. Assume each trial has $p = 0.7$ (scores). Then $X \sim B(4, 0.7)$

Examples:

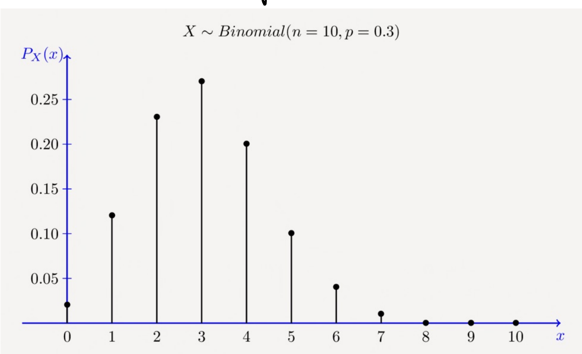
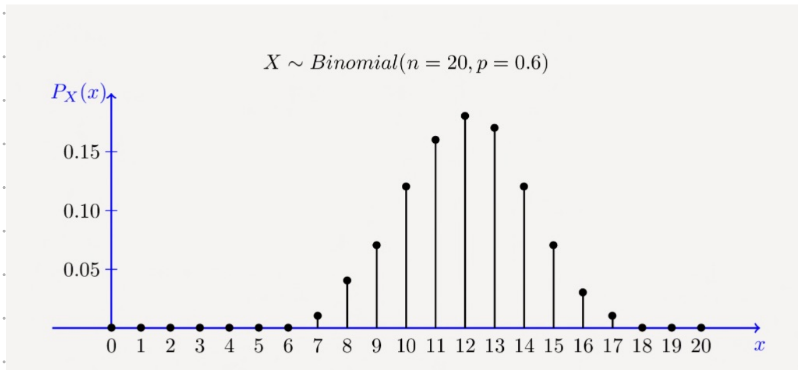


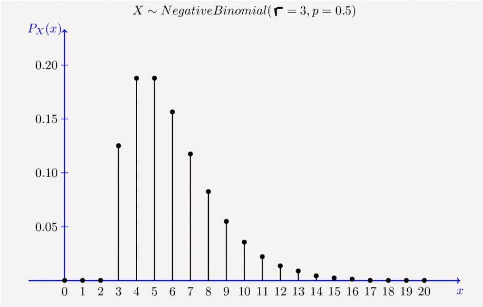
Fig.3.4 - PMF of a $Binomial(10, 0.3)$ random variable.



Negative Binomial Distribution: (Pascal)

Let X denote the number of trials until r successes. Then X has a negative binomial PMF:

$$f(x) = \binom{x-1}{r-1} \cdot (1-p)^{x-r} \cdot p^r$$



Geometric Distribution:

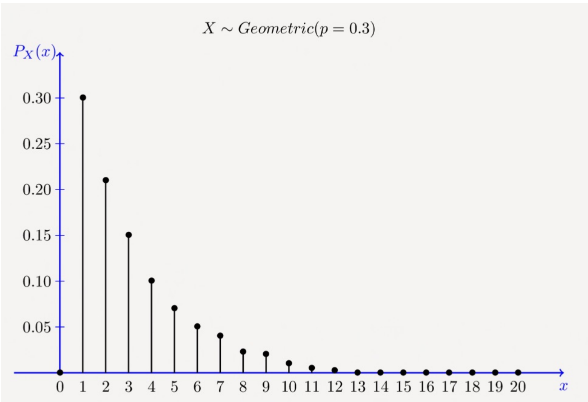
Let X denote the number of trials until the first success. Then X has a geometric PMF:

$$f(x) = (1-p)^{x-1} \cdot p$$

Relation to negative binomial:

Same but $r=1$:

$$f(x) = \binom{x-1}{1-1} \cdot (1-p)^{x-1} \cdot p^1 = (1-p)^{x-1} \cdot p$$



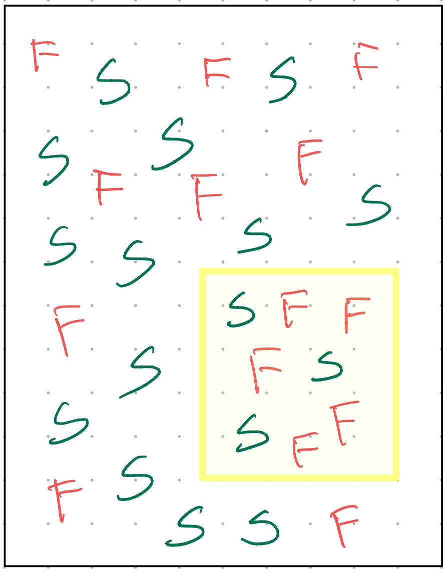
Hypergeometric Distribution

- Given a pool of size N
- With exactly r success
- and a random sample n

Now, let X denote the number of successes in n. Then X is a Hypergeometric R.V. with PMF:

$$f(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

N



$N = 30$

$r = 16$

$n = 8$

$x = 3$

Example: Lottery

$N = 40, \quad r = 7, \quad n = 7$

$$P(X=7) = \frac{\binom{7}{7}\cdot\binom{33}{0}}{\binom{40}{7}} = \frac{1}{\frac{40!}{7!\,33!}} = \frac{7\cdot 6\cdot \ldots \cdot 1}{40\cdot 39\cdot \ldots \cdot 34}$$

$$= \frac{1}{18643560} \approx \underline{\underline{5.36\cdot 10^{-8}}}$$

$$P(X=4) = \frac{\binom{7}{4}\cdot\binom{33}{3}}{18643560} = \frac{\frac{7!}{4!\,3!}\cdot\frac{33!}{3!\,30!}}{18643560} = \frac{190960}{18643560}$$

Poisson Distribution:

Used to model the number of events occurring within a specific time/space interval and has PMF:

$$f(x) = P(X=x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

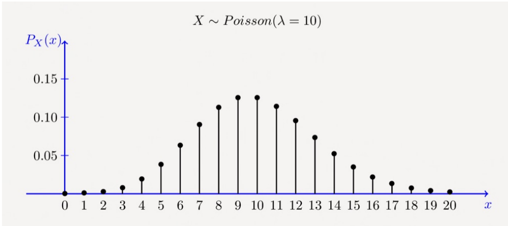


Fig.3.9 - PMF of a $Poisson(10)$ random variable.

λ indicates the average number of events in a $\underbrace{\text{given interval}}_{\text{Important}}$

PMF Summary:

$f(x) = P(X=x)$ Allows us to find probality of exactly x success.

Assume we want to find $P(X \le 22)$ where X is a Poisson R.V.:

$$P(X=0) + P(X=1) + P(X=2) + \ldots\ldots P(X=22)$$

$$P(X \le 22) = \sum_{k=0}^{22} \frac{e^{-\lambda} \cdot \lambda^{k}}{k!}$$

Cumulative Distribution Function:

$$F_X(x) = P(X \leq x) \quad , \ x \in \mathbb{R}$$

Example: Find CDF

Toss a coin twice. Let X denote heads

$$\boxed{X \sim \text{Binomial}(2, 1/2)}$$

$R_X = \{0, 1, 2\}$

$$P(X=0) = \binom{2}{0} \cdot (1/2)^0 \cdot (1-1/2)^2 = 1/4$$
$$P(X=1) = \binom{2}{1} \cdot (1/2)^1 \cdot (1-1/2)^1 = 1/2$$
$$P(X=2) = \binom{2}{2} \cdot (1/2)^2 \cdot (1-1/2)^0 = 1/4$$

$F(x) = P(X \leq x) = 0$ for $x < 0$

$F(x) = P(X \leq x) = 1$ for $x \geq 2$

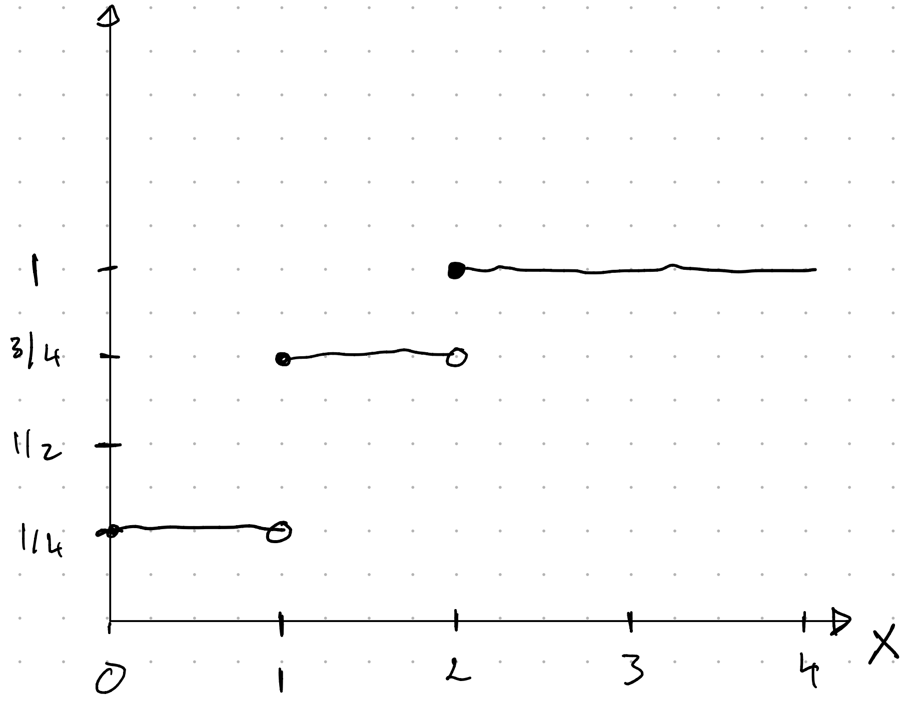
$\underline{0 \leq X < 1}$:

$$P(X \leq x) = 1/4$$

$\underline{1 \leq X < 2}$:

$$P(X \leq x) = P(X=0) + P(X=1) = 1/4 + 1/2 = 3/4$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1/4 & 0 \leq x < 1 \\ 3/4 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$



We get:

$$F_X(x) = \sum_{x_k \le x} P_X(x_k)$$

Intervals:

$$P(3 \le X \le 7) = P(X \le 7) - P(X < 3) = \underbrace{P(X \le 7) - P(X \le 2)}_{\text{use this}}$$

$$P(3 \le X < 7) = P(X < 7) - P(X < 3) = P(X \le 6) - P(X \le 2)$$

$$P(3 < X \le 7) = P(X \le 7) - P(X \le 3) =$$

$$P(3 < X < 7) = P(X \le 6) - P(X \le 3)$$

In general:

$$P(a < X \le b) = F(b) - F(a)$$

$$P(a \le X \le b) = F(b) - F(a-1)$$

Example:

Let X be a discrete R.V with $R_x = \{1, 2, 3 \ldots\}$ and $f(x) = \frac{1}{2^x}$

1. Find CDF

2. $P(2 < X \leq 5) =$

3. $P(X > 4) =$

Expectation:

Expected value:

$$EX = E(X) = \sum_{x_k \in R_X} x_k \cdot P(X = x_k)$$

Recall Flipping two coins:

$P(X=0) = 1/4$, $P(X=1) = 1/2$, $P(X = 1/4)$

$$E(X) = 0 \cdot 1/4 + 1 \cdot 1/2 + 2 \cdot 1/4$$
$$= 1$$

Variance:

$$\mathrm{Var}(X) = E[(X - \mu_X)^2]$$
$$= \sum_{k=1}^{n} x_k^2 \cdot P(X = x_k) - (E(X))^2$$