

Continuous Random Variables

A continuous R.V. has only continuous values i.e. values that are uncountable and are related to real numbers, \mathbb{R} .

- * Time it takes to complete SMP exam

$$R_x = [0; 180]$$

- * Age of fossil

$$R_x = [\text{min age}; \text{max age}]$$

- * Km/h of 1985 BMW 5-Series

$$R_x = [0; \infty]$$

Main difference to discrete:

DRV measured on exact values

CRV measured on intervals

↳ Makes no sense to find probability that exam took exactly 117 min., perhaps it took 117.0132149... min but not exactly 117

$$\hookrightarrow P(X=117) = 0 \rightarrow \text{Note}$$

$$\hookrightarrow P(116 \leq X \leq 118) = 0.71$$

→ It makes perfect sense to find probability of 117 students attending exam: $P(X=117)$

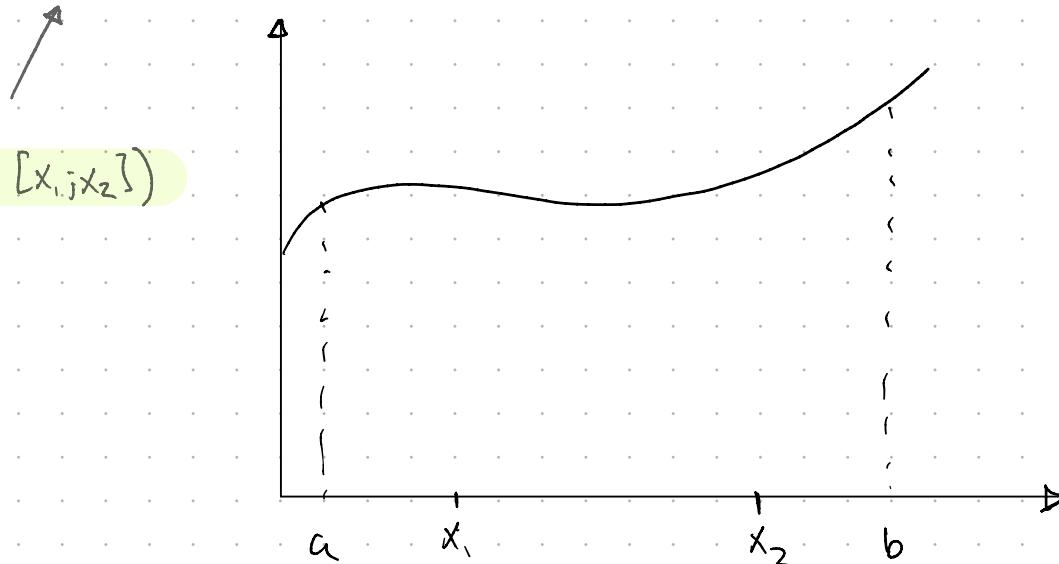
Uniform Distribution

Choose a real number uniformly in $[a; b]$ and denote it X . By uniformly we mean that all intervals of same length in $[a; b]$ have same probability. Find CDF of X :

$$P(X=x) = 0 \quad \text{Means "proportional to"}$$

$$P(x_1 \leq X \leq x_2) \propto (x_2 - x_1), \quad a \leq x_1 \leq x_2 \leq b$$

$P(X \in [x_1, x_2])$



$$P(X \in [a; b]) = 1$$

$$P(X \in [x_1; x_2]) = \frac{x_2 - x_1}{b - a}$$

$$F(x) = P(X \leq x)$$

$$F(x) = 0 \quad \text{for } x < a \quad \text{including or not}$$

$$F(x) = 1 \quad \text{for } x > b \quad \text{including end point}$$

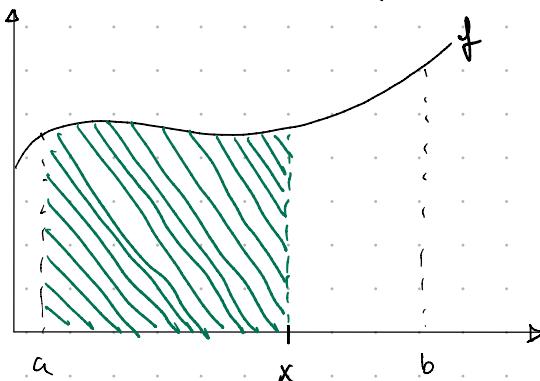
not important.

For $a \leq x \leq b$:

$$F(x) = P(X \leq x)$$

$$= P(X \in [a, x])$$

$$= \frac{x - a}{b - a}$$



To summarise: The CDF of the Uniform CRV

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b \end{cases}$$

More generally:

If X is a continuous R.V., then its CDF is a function s.t.:

$$F(x) = P(X \leq x)$$

1. $0 \leq F(x) \leq 1$

2. $F(-\infty) = 0$, $F(\infty) = 1$ $\xrightarrow{-\infty = \text{lower bound}}$ $\infty = \text{Upper bound}$

3. $F(x)$ is non-decreasing as x increases

4. $P(x_1 \leq X \leq x_2) = F(x_2) - F(x_1)$

5. $P(X \leq x_1) = P(X < x_1)$

6. $P(X > x_1) = 1 - P(X \leq x_1) = 1 - F(x_1)$

Probability Density Function (PDF):

Let X be a continuous R.V. The PDF is a function s.t.

$$f(x) = \lim_{n \rightarrow 0} \frac{F(x+n) - F(x)}{n} = \frac{dF(x)}{dx}$$

1. $f(x) \geq 0$

2. $\int_{-\infty}^{\infty} f(x) dx$

3. $F(x) = \int_{-\infty}^x f(u) du$ \leftarrow Important !!

4. $\int_{x_1}^{x_2} f(x) dx = P(x_1 \leq X \leq x_2) = F(x_2) - F(x_1)$

5. $P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2)$
 $= P(x_1 < X < x_2)$

Example:

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is $[0, 20 \text{ mA}]$, and assume that the probability density function of X is $f(x) = 0.05$ for $0 \leq x \leq 20$. What is the probability that a current measurement is less than 10 milliamperes?

$$P(X < 10) = \int_0^{10} 0.05 dx = 0.05x \Big|_0^{10} = \underline{0.5}$$

let's find the CDF:

$$F(x) = \int_0^x 0.05 du = 0.05u \Big|_0^x = 0.05x$$

so we get:

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 0.05x & \text{for } 0 \leq x \leq 20 \\ 1 & \text{for } x > 20 \end{cases}$$

Example

Let the continuous random variable X denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 millimeters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function $f(x) = 20e^{-20(x-12.5)}$, $x \geq 12.5$.

If a part with a diameter larger than 12.60 millimeters is scrapped, what proportion of parts is scrapped?

$$\begin{aligned} P(X > 12.6) &= \int_{12.6}^{\infty} 20e^{-20(x-12.5)} dx, 20e^x ; -20(x-12.5) \\ &= -\frac{1}{20} \cdot 20 e^{-20(x-12.5)} \Big|_{12.6}^{\infty} \\ &= -e^{-20(\infty-12.5)} - (-e^{-20(12.6-12.5)}) \\ &= 0 + e^{-2} \approx \underline{0.135} \end{aligned}$$

OR

$$\begin{aligned} P(X > 12.6) &= 1 - \int_{12.5}^{12.6} 20e^{-20(x-12.5)} dx \\ &= 1 - \left[-e^{-20(x-12.5)} \right]_{12.5}^{12.6} \\ &= 1 + [e^{-2} - e^0] = 1 + e^{-2} - 1 \\ &\approx 0.135 \end{aligned}$$

Let's find CDF:

$$\begin{aligned} F(x) &= \int_{12.5}^x 20e^{-20(u-12.5)} du = -e^{-20(u-12.5)} \Big|_{12.5}^x \\ &= -e^{-20(x-12.5)} - (-e^{-20(12.5-12.5)}) \\ &= 1 - e^{-20(x-12.5)} \end{aligned}$$

So we get

$$f(x) = \begin{cases} 0 & \text{for } x < 12.5 \\ 1 - e^{-20(x-12.5)} & \text{for } 12.5 \leq x \end{cases}$$

Example

The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-0.01x} & 0 \leq x \end{cases}$$

Determine the probability density function of X . What proportion of reactions is complete within 200 milliseconds?

$$f(x) = F'(x) = (1 - e^{-0.01x})' = 0.01e^{-0.01x}$$

So we get

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 0.01e^{-0.01x} & \text{for } x \geq 0 \end{cases}$$

$$\begin{aligned} F(200) &= 1 - e^{-0.01 \cdot 200} \\ &= 1 - e^{-2} = \underline{\underline{0.865}} \end{aligned}$$

Example:

het $f(x) = \begin{cases} 4a \cdot e^{-2x} & , x \geq 0 \\ 0 & \text{else} \end{cases}$

a) Find a:

$$\int_0^{\infty} 4a \cdot e^{-2x} dx = 1 \Rightarrow -\frac{4}{2} a \cdot e^{-2x} \Big|_0^{\infty} = 1 \Leftrightarrow 2ae^0 = 1$$
$$\Rightarrow a = \frac{1}{2}$$

b) Find $F(x)$:

$$\int_0^x 4 \cdot \frac{1}{2} \cdot e^{-2v} dv = -\frac{1}{2} \cdot 2e^{-2v} \Big|_0^x = -e^{-2x} - (-e^{-2 \cdot 0})$$
$$= 1 - e^{-2x}$$

c) Find $P(1 \leq x \leq 3)$

$$P(1 \leq x \leq 3) = F(3) - F(1) = (1 - e^{-2 \cdot 3}) - (1 - e^{-2 \cdot 1})$$
$$= \frac{-e^{-6} + e^{-2}}{2}$$

or $= \int_1^3 2e^{-2x} dx =$

Expected Value:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Variance

$$\text{Var}(X) = E[(X - \mu_x)^2] = E[X^2] - (E[X])^2$$
$$= \int_{-\infty}^{\infty} x^2 \cdot f(x) dx \quad \leftarrow \text{for C.R.V.}$$

General for all R.V.'s

law of the unconscious statistician (LOTUS):

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_x(x) dx$$

PDF

Example

Let

$$f(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

Find $E(X)$:

$$\begin{aligned} E(X) &= \int_0^1 x \cdot 2x \, dx = \int_0^1 2x^2 \, dx = \left. \frac{2}{3}x^3 \right|_0^1 \\ &= \underline{\underline{\frac{2}{3}}} \end{aligned}$$

Example

Let

$$f(x) = \begin{cases} x^2(2x + \frac{3}{2}) & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

If $Y = \frac{2}{X} + 3$, Find $\text{Var}(Y)$

Recall,

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

Note:

$$\text{Var}(Y) = \text{Var}\left(\frac{2}{X} + 3\right) = 4 \text{Var}\left(\frac{1}{X}\right), \text{ so}$$

$$\text{Var}\left(\frac{1}{X}\right) = E\left[\frac{1}{X^2}\right] - E\left[\left(\frac{1}{X}\right)\right]^2$$

$$E\left[\frac{1}{X^2}\right] = \int_0^1 \frac{1}{X^2} \cdot x^2(2x + \frac{3}{2}) \, dx = \int_0^1 (2x + \frac{3}{2}) \, dx$$

$$= x^2 + \frac{3x}{2} \Big|_0^1 = 1 + \frac{3}{2} = \frac{5}{2}$$

$$E\left[\frac{1}{X}\right] = \int_0^1 \frac{1}{X} \cdot x^2(2x + \frac{3}{2}) \, dx = \int_0^1 (2x^2 + \frac{3}{2}) \, dx$$

$$= \left. \frac{2}{3}x^3 + \frac{3}{4}x^2 \right|_0^1 = \frac{2}{3} + \frac{3}{4} = \frac{8+9}{12} = \frac{17}{12}$$

$$\text{Var}(Y) = 4 \left(\frac{5}{2} - \left(\frac{17}{12} \right)^2 \right) = 4 \left(\frac{5}{2} - \frac{289}{144} \right)$$

$$= 4 \left(\frac{360 - 289}{144} \right) = 4 \left(\frac{71}{144} \right) = \underline{\underline{\frac{71}{36}}}$$

Uniform R.V.

Recall $F(x) = \frac{x-a}{b-a}$, so

$$f(x) = F'(x) = \left(\frac{x}{b-a}\right)' - \left(\frac{a}{b-a}\right)' \\ = \frac{1}{b-a}$$

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & x < a \text{ or } x > b \end{cases}$$

Expected Value:

$$E(X) = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{b-a} \cdot \frac{1}{2} x^2 \Big|_a^b \\ = \frac{1}{2} \left(\frac{b^2}{b-a} - \frac{a^2}{b-a} \right) = \frac{1}{2} \frac{b^2 - a^2}{b-a} = \frac{(b-a)(b+a)}{2(b-a)} \\ = \frac{a+b}{2}$$

Variance

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a} dx = \frac{1}{3} \frac{x^3}{b-a} \Big|_a^b \\ = \frac{1}{3} \left(\frac{b^3}{b-a} - \frac{a^3}{b-a} \right) = \frac{1}{3} \left(\frac{b^3 - a^3}{b-a} \right) \\ = \frac{1}{3} \frac{(b-a)(a^2 + ab + b^2)}{b-a} = \frac{a^2 + ab + b^2}{3}$$

So,

$$\text{Var}(X) = \frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2}\right)^2 \\ = \frac{a^2 + ab + b^2}{3} - \frac{a^2 + b^2 + 2ab}{4} \\ = \frac{4a^2 + 4ab + 4b^2 - 3a^2 - 3b^2 - 6ab}{12} \\ = \frac{a^2 - 2ab + b^2}{12} = \frac{(b-a)^2}{12}$$

Functions of Continuous R.V.'s: Difficult

If X is a CRV and $Y = g(X)$, then Y is also a R.V.

Example

Let X be a Uniform(0,1) R.V. and let $Y = e^X$

a) Find CDF of Y

$$F_Y(y) = \frac{y-a}{b-a} = \frac{y-0}{1-0} = y, f(x)=1$$

$$R_X = [0; 1], R_Y = [1; e]$$

$$F(Y) = P(Y \leq y)$$

$$= P(e^X \leq y)$$

$$= P(X \leq \ln y)$$

SG

$$F_Y = \begin{cases} 0 & y < 1 \\ \ln y & 1 \leq y < e \\ 1 & y \geq e \end{cases}$$

b) Find pdf of Y :

$$f_Y(y) = F'_Y(y) = \frac{1}{y} \text{ for } 1 \leq y \leq e, \text{ else } 0$$

c) Find $E[Y]$:

$$E[Y] = \int_1^e y \cdot \frac{1}{y} dy = y \Big|_1^e = \underline{\underline{e-1}}$$

Using hotus:

$$\begin{aligned} E[Y] &= E[e^X] = \int_0^1 e^x \cdot f_X(x) dx = e^x \Big|_0^1 \\ &= e^1 - e^0 = \underline{\underline{e-1}} \end{aligned}$$