

## Problems 1

Richard Brooks

Please contact me if you find any mistakes in the solutions below.

### Exercise 1

Heart failures are due to either natural occurrences (87%) or outside factors (13%). Outside factors are related to induced substances (73%) or foreign objects (27%). Natural occurrences are caused by arterial blockage (56%), disease (27%), and infection (e.g., staph infection) (17%).

- a. Determine the probability that a failure is due to an induced substance.  $0.13 \times 0.73 = \underline{0.0949}$
- b. Determine the probability that a failure is due to disease or infection.  $0.87 \times (0.27 + 0.17) = \underline{0.3828}$

### Exercise 2

Computer keyboard failures are due to faulty electrical connects (12%) or mechanical defects (88%). Mechanical defects are related to loose keys (27%) or improper assembly (73%). Electrical connect defects are caused by defective wires (35%), improper connections (13%), or poorly welded wires (52%).

- a. Find the probability that a failure is due to loose keys.  $0.88 \times 0.27 = \underline{0.2376}$
- b. Find the probability that a failure is due to improperly connected or poorly welded wires.  $0.12 \times (0.13 + 0.52) = \underline{0.078}$

### Exercise 3

Two teams  $A$  and  $B$  play a football match, and we are interested in the winner. The sample space can be defined as:

$$S = \{a, b, d\}$$

where  $a$  shows the outcome that  $A$  wins,  $b$  shows the outcome that  $B$  wins, and  $d$  shows the outcome that they draw. Suppose that we know that (1) the probability that  $A$  wins is  $P(a) = P(\{a\}) = 0.5$  and (2) the probability of a draw is  $P(d) = P(\{d\}) = 0.25$ .

- a. Find the probability that  $B$  wins.  $P(b) = \underline{0.25}$
- b. Find the probability that  $B$  wins or a draw occurs.  $P(\{b, d\}) = \underline{0.50}$

### Exercise 4

Let  $A$  and  $B$  be two events such that:

$$P(A) = 0.4, \quad P(B) = 0.7, \quad P(A \cup B) = 0.9$$

- a. Find  $P(A \cap B) = \underline{0.2}$ .
- b. Find  $P(A^c \cap B) = \underline{0.5}$ .
- c. Find  $P(A - B) = \underline{0.2}$ .

- d. Find  $P(A^c - B) = \underline{\underline{0.1}}$ .
- e. Find  $P(A^c \cup B) = \underline{\underline{0.8}}$ .
- f. Find  $P(A \cap (B \cup A^c)) = \underline{\underline{0.2}}$ .

## Exercise 5

Consider a random experiment with a sample space.

$$S = \{1, 2, 3, \dots\}.$$

Suppose that we know:

$$P(k) = P(\{k\}) = \frac{c}{3^k} \quad \text{for } k = 1, 2, \dots$$

where  $c$  is a constant number.

- a. Find  $c = \underline{\underline{2}}$ .
- b. Find  $P(\{2, 4, 6\}) \approx \underline{\underline{0.25}}$ .
- c. Find  $P(\{3, 4, 5, \dots\}) = \underline{\underline{\frac{1}{9}}}$ .

## Exercise 6

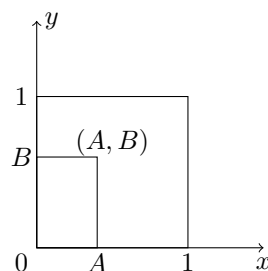
Let  $T$  be the time needed to complete a job at a certain factory. By using the historical data, we know that

$$P(T \leq t) = \begin{cases} \frac{1}{16}t^2 & \text{for } 0 \leq t \leq 4 \\ 1 & \text{for } t > 4 \end{cases}$$

- a. Find the probability that the job is completed in less than one hour, i.e., find  $P(T \leq 1) = \underline{\underline{1/16}}$ .
- b. Find the probability that the job needs more than 2 hours.  $P(T > 2) = 1 - P(T < 2) = \underline{\underline{\frac{3}{4}}}$
- c. Find the probability that  $1 \leq T \leq 3$ .  $P(1 \leq T \leq 3) = P(T \leq 3) - P(T < 1) = \underline{\underline{\frac{9}{16}}}$

## Exercise 7

You choose a point  $(A, B)$  uniformly at random in the unit square  $\{(x, y) : 0 \leq x, y \leq 1\}$ .



What is the probability that the equation

$$AX^2 + X + B = 0$$

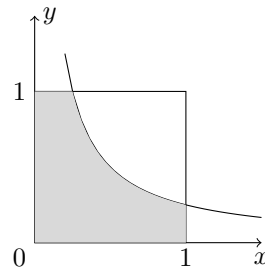
has real solutions?

**Solution:**

The equation has real roots if and only if:

$$1 - 4AB > 0 \quad \text{i.e.} \quad AB < \frac{1}{4}.$$

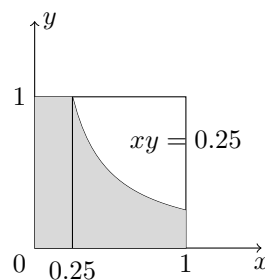
This area is shown here:



Since  $(A, B)$  is uniformly chosen in the square we can say that probability of having real roots is

$$\begin{aligned} P(R) &= \frac{\text{area of the shaded region}}{\text{area of the square}} \\ &= \frac{\text{area of the shaded region}}{1} \end{aligned}$$

To find the area of the shaded region we can set up the following integral:



$$\begin{aligned} \text{Area} &= \frac{1}{4} + \int_{\frac{1}{4}}^1 \frac{1}{4x} dx \\ &= \frac{1}{4} + \frac{1}{4} [\ln(x)]_{\frac{1}{4}}^1 \\ &= \frac{1}{4} + \frac{1}{4} \ln 4 \end{aligned}$$