

# Recap - CRV Part 1

## Probability Density Function

For a continuous random variable  $X$ , a **probability density function** is a function such that

- (1)  $f(x) \geq 0$
- (2)  $\int_{-\infty}^{\infty} f(x) dx = 1$
- (3)  $P(a \leq X \leq b) = \int_a^b f(x) dx = \text{area under } f(x) \text{ from } a \text{ to } b \text{ for any } a \text{ and } b$  (4.1)

If  $X$  is a continuous random variable, for any  $x_1$  and  $x_2$ ,

$$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2) \quad (4.2)$$

## Cumulative Distribution Function

The **cumulative distribution function** of a continuous random variable  $X$  is

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(u) du \quad (4.3)$$

for  $-\infty < x < \infty$ .

## Probability Density Function from the Cumulative Distribution Function

Given  $F(x)$ ,

$$f(x) = \frac{dF(x)}{dx}$$

as long as the derivative exists.

## Mean and Variance

Suppose that  $X$  is a continuous random variable with probability density function  $f(x)$ . The **mean** or **expected value** of  $X$ , denoted as  $\mu$  or  $E(X)$ , is

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x) dx \quad (4.4)$$

The **variance** of  $X$ , denoted as  $V(X)$  or  $\sigma^2$ , is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$$

The **standard deviation** of  $X$  is  $\sigma = \sqrt{\sigma^2}$ .

## Expected Value of a Function of a Continuous Random Variable

If  $X$  is a continuous random variable with probability density function  $f(x)$ ,

$$E[h(X)] = \int_{-\infty}^{\infty} h(x)f(x) dx \quad (4.5)$$

LOTUS

# Assignment 1 (20%)

Let  $X$  denote a continuous stochastic variable with the following probability density function

$$f(x) = \begin{cases} cx^4 & \text{for } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}, \text{ where } c \text{ is a constant}$$

a) Show that the cumulative probability function of  $X$  is

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{1}{5}c(x^5 + 1) & \text{for } -1 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases}$$

$$\begin{aligned} \int_{-1}^x c \cdot v^4 dv &= \left. \frac{1}{5} c v^5 \right|_{-1}^x = \frac{1}{5} \cdot c \cdot x^5 - \left( \frac{1}{5} \cdot c \cdot (-1)^5 \right) \\ &= \frac{1}{5} c x^5 + \frac{1}{5} c = \frac{1}{5} c (x^5 + 1) \end{aligned}$$

b) Determine the constant  $c$  and restate both the probability density function and the cumulative probability function using the actual value of  $c$

$$\begin{aligned} \int_{-1}^1 f(x) dx &= 1 \Leftrightarrow F(1) - F(-1) = 1 \\ &= \frac{2}{5} c = 1 \Rightarrow c = \frac{5}{2} \end{aligned}$$

$$f(x) = \begin{cases} \frac{5}{2} x^4 & x \in [-1; 1] \\ 0 & \text{else} \end{cases}$$

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}(x^5 + 1) & x \in [-1; 1] \\ 1 & x > 1 \end{cases}$$

c) Compute  $P\left(-\frac{1}{2} < X < \frac{1}{2}\right)$  and  $P(X > 0)$   $\rightarrow F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) = \frac{1}{2} \left( \frac{1}{32} + 1 \right) - \frac{1}{2} \left( 1 - \frac{1}{32} \right) = \frac{1}{32}$

d) Find the expected value and variance of  $X$

$$E[X] = \int_{-1}^1 x \cdot \frac{5}{2} x^4 dx = \left. \frac{5}{12} x^6 \right|_{-1}^1 = 0 \quad 1 - F(0) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$V[X] = \text{Var}(X) = \int_{-1}^1 x^2 \cdot \frac{5}{2} x^4 dx - 0^2 = \left. \frac{5}{14} x^7 \right|_{-1}^1 = 2 \cdot \frac{5}{14} = \frac{10}{14} = \frac{5}{7}$$

# Assignment 1 (15%)

Compute the expected value,  $E(X)$ , if  $X$  has a density function as follows:

$$\begin{aligned} \text{a. } f(x) &= \begin{cases} \frac{1}{4} x e^{-\frac{x}{2}} & x > 0 \\ 0 & \text{otherwise} \end{cases} & \int_0^{\infty} x \cdot \frac{1}{4} x e^{-\frac{x}{2}} = \frac{-(x^2 + 4x + 8) \cdot e^{-\frac{x}{2}}}{2} \Big|_0^{\infty} \\ & & = 0 - (-4) = \underline{4} \\ \text{b. } f(x) &= \begin{cases} 5x^{-2} & x > 5 \\ 0 & \text{otherwise} \end{cases} & \int_5^{\infty} x \cdot 5x^{-2} dx = \int_5^{\infty} 5x^{-1} dx \\ & & = 5 \cdot \log x \Big|_5^{\infty} = \infty \end{aligned}$$

The density function of  $X$  is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

c. If  $E(X) = \frac{3}{5}$ , find  $a$  and  $b$ .

$$\textcircled{1} \int_0^1 (a + bx^2) dx = 1 \Rightarrow ax + \frac{1}{3}bx^3 \Big|_0^1 = 1$$

$$\underline{a + \frac{1}{3}b = 1} \Rightarrow a = -\frac{1}{3}b + 1$$

$$\textcircled{2} \int_0^1 x(a + bx^2) dx = \frac{3}{5} \Rightarrow \int_0^1 (ax + bx^3) dx = \frac{3}{5}$$

$$= \frac{1}{2}ax^2 + \frac{1}{4}bx^4 \Big|_0^1 = \frac{3}{5}$$

$$= \underline{\frac{1}{2}a + \frac{1}{4}b = \frac{3}{5}}$$

$$\frac{1}{2}\left(-\frac{1}{3}b + 1\right) + \frac{1}{4}b = \frac{3}{5} \Rightarrow \underline{\underline{b = \frac{6}{5}}}$$

$$a + \frac{1}{3} - \frac{6}{5} = 1 \Rightarrow \underline{\underline{a = \frac{3}{5}}}$$