1. $\Psi(x,0) = A[3\psi_1(x) + 4\psi_2(x)]$ ... 무한사각형 우물 내에서 한 입자가 가지는 초기 파동함수는 두개의 정지상태 합으로 주어짐

(a) 파동함수 규격화 조건

$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx$$
$$= \int_{-\infty}^{\infty} \Psi^*(x,0)\cdot\Psi(x,0)\,dx$$
$$= \int_{-\infty}^{\infty} A^*\cdot[3\psi_1^*(x) + 4\psi_2^*(x)]\cdot A[3\psi_1(x) + 4\psi_2(x)]\,dx$$
$$= |A|^2\int_{-\infty}^{\infty}\left(9|\psi_1|^2 + \cancel{12\psi_1^*\psi_2} + \cancel{12\psi_2^*\psi_1} + 16|\psi_2|^2\right)dx$$

$\because$ orthogonality

$$= |A|^2\cdot 25$$

양의 실수인 A를 택하면, $A = \frac{1}{5}$

(b) (a)에 의해 $\Psi(x,0) = \frac{3}{5}\psi_1(x) + \frac{4}{5}\psi_2(x)$

$$\Psi(x,t) = \frac{3}{5}\psi_1(x)e^{-iE_1t/\hbar} + \frac{4}{5}\psi_2(x)e^{-iE_2t/\hbar}$$

by letting $\omega = \frac{\pi^2\hbar}{2ma^2}$

$E_n = \frac{\hbar^2k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2} = n^2\hbar\omega$

$\therefore E_1 = \hbar\omega,\ E_4 = 4\hbar\omega$

$$= \frac{3}{5}\psi_1(x)e^{-i\omega t} + \frac{4}{5}\psi_2(x)e^{-i4\omega t}$$
$$= \frac{3}{5}\sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \frac{4}{5}\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi}{a}x\right)e^{-i4\omega t}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-3\omega ti} = \cos(3\omega t) - i\sin 3\omega t$

$e^{3i\omega t} = \cos 3\omega t + i\sin 3\omega t$

$$|\Psi(x,t)|^2 = \Psi^*(x,t)\cdot\Psi(x,t)$$
$$= \frac{1}{25}\left[9|\psi_1|^2 + 16|\psi_2|^2 + 12\psi_1^*\psi_2 e^{-3i\omega t} + 12\psi_1\psi_2^* e^{3i\omega t}\right]$$
$$= \frac{1}{25}\left[9\cdot\frac{2}{a}\sin^2\left(\frac{\pi}{a}x\right) + 16\cdot\frac{2}{a}\sin^2\left(\frac{2\pi}{a}x\right) + 12\cdot\frac{2}{a}\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\left\{e^{-3i\omega t} + e^{3i\omega t}\right\}\right]$$
$$= \frac{1}{25}\cdot\frac{2}{a}\left[9\sin^2\left(\frac{\pi}{a}x\right) + 16\sin^2\left(\frac{2\pi}{a}x\right) + 12\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cdot 2\cos(3\omega t)\right]$$
$$= \frac{2}{25a}\left[9\sin^2\left(\frac{\pi}{a}x\right) + 16\sin^2\left(\frac{2\pi}{a}x\right) + 24\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos 3\omega t\right]$$

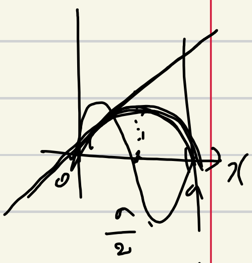
(c) $\langle x\rangle = \int_{-\infty}^{\infty} x\cdot|\Psi(x,t)|^2\,dx$

$$= \int_{-\infty}^{\infty} x\cdot\left[\left(\frac35\psi_1^* e^{iE_1t/\hbar}+\frac45\psi_2^* e^{iE_2t/\hbar}\right)\left(\frac35\psi_1 e^{-iE_1t/\hbar}+\frac45\psi_2 e^{-iE_2t/\hbar}\right)\right]dx$$

$$= \int_{-\infty}^{\infty} x\cdot\left[\left(\frac35\psi_1^* e^{i\omega t}+\frac45\psi_2^* e^{4i\omega t}\right)\left(\frac35\psi_1 e^{-i\omega t}+\frac45\psi_2 e^{-4i\omega t}\right)\right]dx$$

$$=\frac{1}{25}\int_{-\infty}^{\infty} x\cdot\left[9|\psi_1|^2+16|\psi_2|^2+12\psi_1^*\psi_2 e^{-3i\omega t}+12\psi_2^*\psi_1 e^{3i\omega t}\right]dx$$

$$=\frac{1}{25}\int_{-\infty}^{\infty} x\cdot\left[9\cdot\frac2a\sin^2\left(\frac{\pi x}{a}\right)+16\cdot\frac2a\sin^2\left(\frac{2\pi x}{a}\right)+12\cdot\frac2a\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)e^{-3i\omega t}\right.$$
$$\left.+12\cdot\frac2a\cdot\sin\left(\frac{2\pi}{a}x\right)\cdot\sin\left(\frac{\pi x}{a}\right)e^{3i\omega t}\right]dx$$



$$=\frac{2}{25a}\int_0^a \underset{\text{odd}}{\left(x-\frac a2\right)}\left[\underset{\text{even}}{9\sin^2\left(\frac{\pi x}{a}\right)}+\underset{\text{even}}{16\sin^2\left(\frac{2\pi x}{a}\right)}+\underset{\text{odd}}{12\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cdot 2\cos(3\omega t)}\right]$$

$(\frac a2, 0)$에 대해

$$+\underset{\text{even}}{\frac a2}\left[9\sin^2\left(\frac{\pi x}{a}\right)+16\sin^2\left(\frac{2\pi x}{a}\right)+12\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cdot 2\cos(3\omega t)\right]$$

$$=\frac{2}{25a}\int_0^a \underbrace{\left(x-\frac a2\right)\left\{12\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cdot 2\cos(3\omega t)\right\}}_{①}+\underbrace{\frac a2\left\{9\sin^2\left(\frac{\pi x}{a}\right)+16\sin^2\left(\frac{2\pi x}{a}\right)\right\}}_{②}dx$$

① $= \int_0^a \left(x-\frac a2\right)\cdot 12\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cdot 2\cos(3\omega t)$

$$= 24\cos(3\omega t)\int_0^a \left(x-\frac a2\right)\sin\left(\frac{\pi}{a}x\right)\cdot\sin\left(\frac{2\pi}{a}x\right)dx$$

$$= 24\cos(3\omega t)\int_0^a \left(x-\frac a2\right)\left[-\frac12\left\{\cos\left(\frac{3\pi}{a}x\right)-\cos\left(\frac{\pi}{a}x\right)\right\}\right]dx$$

$$= 12\cos(3\omega t)\int_0^a\left\{\left(x-\frac a2\right)\cos\left(\frac{\pi}{a}x\right)-\left(x-\frac a2\right)\cos\left(\frac{3\pi}{a}x\right)\right\}dx$$

$$= 12\cos(3\omega t)\left[\cancelto{0}{\left(x-\frac a2\right)\frac{a}{\pi}\sin\left(\frac{\pi}{a}x\right)\Big|_0^a}-\frac a\pi\int_0^a\sin\left(\frac{\pi}{a}x\right)dx-\cancelto{0}{\left(x-\frac a2\right)\cdot\frac{a}{3\pi}\sin\left(\frac{3\pi}{a}x\right)\Big|_0^a}+\frac{a}{3\pi}\int_0^a\sin\left(\frac{3\pi}{a}x\right)dx\right]$$

$$= 12\cos(3\omega t)\left[\frac a\pi\cdot\frac a\pi\cos\left(\frac{\pi}{a}x\right)\Big|_0^a-\frac{a^2}{9\pi^2}\cos\left(\frac{3\pi}{a}x\right)\Big|_0^a\right]$$

$$= 12\cos(3\omega t)\cdot\left[\frac{2a^2}{9\pi^2}-\frac{2a^2}{\pi^2}=\frac{-16a^2}{9\pi^2}\right]$$

$$= -\frac{64a^2}{3\pi^2}\cos(3\omega t)$$

② $= \int_0^a \frac{a}{2}\{9\sin^2(\frac{\pi}{a}x)+16\sin^2(\frac{2\pi}{a}x)\}$

$= \frac{a}{2}\int_0^a\{9\sin^2(\frac{\pi}{a}x)+16\sin^2(\frac{2\pi}{a}x)\}dx$

$= \frac{a}{2}\int_0^a\{9\cdot\frac{(1-\cos\frac{2\pi}{a}x)}{2}+16\cdot\frac{(1-\cos\frac{4\pi}{a}x)}{2}\}dx$

$= \frac{a}{4}\int_0^a(25-9\cos\frac{2\pi}{a}x-16\cos\frac{4\pi}{a}x)dx$

$= \frac{a}{4}\left[25x-9\cdot\frac{a}{2\pi}\sin\frac{2\pi}{a}x-16\cdot\frac{a}{4\pi}\cdot\sin\frac{4\pi}{a}x\right]_0^a dx$

$= \frac{a}{4}\cdot 25a = \frac{25}{4}a^2$

① + ② $= -\frac{64a^2}{3\pi^2}\cos(3wt)+\frac{25}{4}a^2$

$\therefore \langle x\rangle = \frac{2}{25a}\cdot\left(-\frac{64a^2}{3\pi^2}\cos 3wt+\frac{25}{4}a^2\right)$

$= \frac{a}{2}-\frac{128a}{75\pi^2}\cos 3wt$

$\langle p\rangle = m\cdot\frac{d\langle x\rangle}{dt}$

$= m\cdot\left\{\frac{-128a}{75\pi^2}\cdot(3w)\cdot(-\sin(3wt))\right\}$

$= \frac{128ma}{25\pi^2}w\cdot\sin(3wt)$

$= \frac{128ma}{25\pi^2}\cdot\frac{\pi^2\hbar}{2ma^2}\sin(3wt)$

$= \frac{64\hbar}{25a}\cdot\sin(3wt)$

또는, $\langle p \rangle = \int_0^a \Psi^*(x,t)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi(x,t)\,dx$

$$= \frac{\hbar}{i}\int_0^a \Psi^*(x,t)\frac{\partial}{\partial x}\Psi(x,t)\,dx$$

$$= \frac{\hbar}{i}\int_0^a \left(\frac{3}{5}\psi_1^*(x)e^{i\omega t} + \frac{4}{5}\psi_2^*(x)e^{4i\omega t}\right)\frac{\partial}{\partial x}\left(\frac{3}{5}\psi_1(x)e^{-i\omega t} + \frac{4}{5}\psi_2(x)e^{-4i\omega t}\right)dx$$

$$= \frac{\hbar}{i}\cdot\int_0^a \left(\frac{3}{5}\psi_1^*(x)e^{i\omega t} + \frac{4}{5}\psi_2^*(x)e^{4i\omega t}\right)\cdot\left(\frac{3}{5}\frac{\partial\psi_1(x)}{\partial x}\cdot e^{-i\omega t} + \frac{4}{5}\frac{\partial\psi_2(x)}{\partial x}\cdot e^{-4i\omega t}\right)dx$$

$$= \frac{\hbar}{25i}\int_0^a \left(3\psi_1^* e^{i\omega t} + 4\psi_2^* e^{4i\omega t}\right)\left[3\sqrt{\frac{2}{a}}\cdot\frac{\pi}{a}\cdot\cos\left(\frac{\pi x}{a}\right)e^{-i\omega t} + 4\cdot\sqrt{\frac{2}{a}}\cdot\frac{2\pi}{a}\cdot\cos\left(\frac{2\pi x}{a}\right)e^{-4i\omega t}\right]dx$$

$$= \frac{\hbar}{25i}\cdot\sqrt{\frac{2}{a}}\cdot\frac{\pi}{a}\cdot\int_0^a \left(3\psi_1^* e^{i\omega t} + 4\psi_2^* e^{4i\omega t}\right)\left(3\cos\left(\frac{\pi x}{a}\right)e^{-i\omega t} + 8\cos\left(\frac{2\pi x}{a}\right)\cdot e^{-4i\omega t}\right)dx$$

$$= \frac{\hbar}{25i}\sqrt{\frac{2}{a}}\cdot\frac{\pi}{a}\int_0^a \left(3\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) + 4\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi}{a}x\right)e^{3i\omega t}\right)\left(3\cos\left(\frac{\pi}{a}x\right) + 8\cos\left(\frac{2\pi}{a}x\right)e^{-3i\omega t}\right)dx$$

$$= \frac{\hbar}{25i}\,\frac{2}{a}\cdot\frac{\pi}{a}\cdot\int_0^a \Big[\cancel{9\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)} + 24\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{2\pi}{a}x\right)e^{-3i\omega t}$$
$$+ 12\sin\left(\frac{2\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)e^{3i\omega t} + \cancel{32\sin\left(\frac{2\pi}{a}x\right)\cos\left(\frac{2\pi}{a}x\right)}\Big]dx$$

$$= \frac{2\pi\hbar}{25ia^2}\int_0^a \left[24\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{2\pi}{a}x\right)e^{-3i\omega t} + 12\sin\left(\frac{2\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)e^{3i\omega t}\right]dx$$

$$= \frac{2\pi\hbar}{25ia^2}\int_0^a 12\left[\sin\frac{3\pi}{a}x - \sin\frac{\pi}{a}x\right]e^{-3i\omega t} + 6\left[\sin\frac{3\pi}{a}x + \sin\frac{\pi}{a}x\right]e^{3i\omega t}\,dx$$

$$= \frac{12\pi\hbar}{25ia^2}\left[2\left\{\frac{-a}{3\pi}\cos\frac{3\pi}{a}x + \frac{a}{\pi}\cos\frac{\pi}{a}x\right\}e^{-3i\omega t} + \left\{\frac{-a}{3\pi}\cos\frac{3\pi}{a}x - \frac{a}{\pi}\cos\frac{\pi}{a}x\right\}\right]_0^a$$

$$= \frac{12\pi\hbar}{25ia^2}\left[2\left\{+\frac{a}{3\pi} - \frac{a}{\pi} + \frac{a}{3\pi} - \frac{a}{\pi}\right\}e^{-3i\omega t} + \left(\frac{a}{3\pi} + \frac{a}{\pi} + \frac{a}{3\pi} + \frac{a}{\pi}\right)e^{3i\omega t}\right]$$

$$= \frac{12\pi\hbar}{25ia^2}\left[\frac{2a}{\pi}\cdot\left(-\frac{4}{3}\right)e^{-3i\omega t} + \frac{a}{\pi}\cdot\left(\frac{8}{3}\right)e^{3i\omega t}\right]$$

$$= \frac{12\hbar}{25ia}\left[\frac{8}{3}e^{3i\omega t} - \frac{8}{3}e^{-3i\omega t}\right]$$

$$= \frac{32}{25}\cdot\frac{\hbar}{ia}\left[e^{3i\omega t} - e^{-3i\omega t}\right]$$

$$= \frac{32\hbar}{25ia}\cdot 2i\sin 3\omega t = \frac{64\hbar}{25a}\sin 3\omega t$$

2. $\Psi(x,0) = \begin{cases} A, & 0 \le x \le \frac{a}{2} \\ 0, & \frac{a}{2} \le x \le a \end{cases}$

(a) normalization

$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx$$

$$= \int_0^{\frac{a}{2}} |A|^2 dx + \int_{\frac{a}{2}}^{a} 0 \, dx$$

$$= |A|^2 \cdot \left(\frac{a}{2}\right) + 0$$

양수인 A를 택하면 $A = \sqrt{\frac{2}{a}}$

(b) $\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_n t/\hbar}$

$\Psi(x,0) = \sum_n c_n \psi_n(x)$

$= c_1\psi_1 + c_2\psi_2 + c_3\psi_3 + \cdots + c_n\psi_n + \cdots$

$\int_{-\infty}^{\infty} \psi_n^* \cdot \Psi(x,0) dx = \int_{-\infty}^{\infty} (\cancel{c_1\psi_1\psi_n^*} + \cancel{c_2\psi_2\psi_n^*} + \cdots + c_n\psi_n\psi_n^* + \cdots) dx$

$\because$ orthonormality of $\psi_n$

$\int_0^{\frac{a}{2}} \psi_n^* \cdot A dx = \int_{-\infty}^{\infty} c_n |\psi_n|^2 dx$

$\Leftrightarrow A \cdot \int_0^{\frac{a}{2}} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \cdot dx = c_n$

$\Leftrightarrow A \cdot \sqrt{\frac{2}{a}} \cdot \left(\frac{-a}{n\pi}\right) \cdot \cos\left(\frac{n\pi}{a}x\right)\Big|_0^{\frac{a}{2}} = c_n$

$\Leftrightarrow \frac{-A\sqrt{2a}}{n\pi}\left(\cos\frac{n\pi}{2} - 1\right) = c_n = \frac{-2}{n\pi}\left(\cos\frac{n\pi}{2} - 1\right)$

이때, $\cos\frac{n\pi}{2} - 1 = \begin{cases} \cos\frac{\pi}{2} - 1 = -1 & (n=1) \\ \cos\pi - 1 = -2 & (n=2) \\ \cos\frac{3\pi}{2} - 1 = -1 & (n=3) \\ \cos 2\pi - 1 = 0 & (n=4) \\ \vdots & \vdots \end{cases}$

$\therefore c_n = \begin{cases} \frac{2}{n\pi} & (n = 4k-3) \\ \frac{4}{n\pi} & (n = 4k-2) \\ \frac{2}{n\pi} & (n = 4k-1) \\ 0 & (n = 4k) \end{cases}$ ① ⋯ $k$는 자연수

$\therefore \Psi(x,t) = \sum c_n \psi_n e^{-iE_n t/\hbar}$

$= \sum c_n \cdot \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-iE_n t/\hbar}$

$= \sum c_n \cdot \sqrt{\frac{2}{a}} \cdot \sin\left(\frac{n\pi}{a}x\right) e^{-in^2\omega t}$

(단, $c_n$은 ①의 결과를 따른다.)

(C) $E_1$ 측정 확률 $= |C_1|^2$

$$\frac{-A\sqrt{2a}}{n\pi}\left(\cos\frac{n\pi}{2}-1\right) = C_n$$ 에서

$$C_1 = \frac{-A\sqrt{2a}}{\pi}(-1)$$

$$= \frac{A\sqrt{2a}}{\pi}$$

$$\therefore |C_1|^2 = \frac{|A|^2 \cdot 2a}{\pi} = \frac{\frac{2}{a}\cdot 2a}{\pi} = \frac{4}{\pi}$$

3. $\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x - i\hat{p})$

+) $\hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x + i\hat{p})$

$$\hat{a}_+ + \hat{a}_- = \frac{2}{\sqrt{2\hbar m\omega}}(m\omega x)$$

$$\therefore x = \frac{\sqrt{2\hbar m\omega}}{2m\omega}(\hat{a}_+ + \hat{a}_-)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$$

$$\therefore \langle x^4\rangle = \left\langle \frac{\hbar^2}{(2m\omega)^2}(\hat{a}_+ + \hat{a}_-)^4 \right\rangle$$

$$= \frac{\hbar^2}{4m^2\omega^2}\int_{-\infty}^{\infty} \psi_n^*(\hat{a}_+ + \hat{a}_-)^4 \psi_n \, dx$$

$$= \frac{\hbar^2}{4m^2\omega^2}\int_{-\infty}^{\infty} \psi_n^*(\hat{a}_+ + \hat{a}_-)(\hat{a}_+ + \hat{a}_-)(\hat{a}_+ + \hat{a}_-)(\hat{a}_+ + \hat{a}_-)\psi_n \, dx$$

총 4번의 시행 중, $\hat{a}_+$ 2회, $\hat{a}_-$ 2회의 시행으로 $\psi_n$이 그대로 나오는 항만이 orthogonality에 의해 0이 되지 않고, 살아남는다. 단, 내림연산자에 의해 $\psi_n$이 사라지는 경우를 피하기 위해 여기서는 $n \geq 2$ 라 가정.

$$= \frac{\hbar^2}{4m^2\omega^2}\int_{-\infty}^{\infty} \psi_n^*(\hat{a}_+\hat{a}_+\hat{a}_-\hat{a}_- + \hat{a}_+\hat{a}_-\hat{a}_+\hat{a}_- + \hat{a}_+\hat{a}_-\hat{a}_-\hat{a}_+ + \hat{a}_-\hat{a}_+\hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+\hat{a}_-\hat{a}_+ + \hat{a}_-\hat{a}_-\hat{a}_+\hat{a}_+)\psi_n \, dx$$

$$= \frac{\hbar^2}{4m^2\omega^2}\int_{-\infty}^{\infty} \psi_n^*\{(\hat{a}_+\hat{a}_+\hat{a}_-\sqrt{n}\,\psi_{n-1}) + (\hat{a}_+\hat{a}_-\hat{a}_+\sqrt{n}\,\psi_{n-1}) + (\hat{a}_+\hat{a}_-\hat{a}_-\sqrt{n+1}\,\psi_{n+1}) + (\hat{a}_-\hat{a}_+\hat{a}_+\sqrt{n}\,\psi_{n-1}) + \hat{a}_-\hat{a}_+\hat{a}_-(\sqrt{n+1}\,\psi_{n+1}) + \hat{a}_-\hat{a}_-\hat{a}_+(\sqrt{n+1}\,\psi_{n+1})\}\,dx$$

$$= \frac{\hbar^2}{4m^2\omega^2}\int_{-\infty}^{\infty} \psi_n^*\cdot\{(\hat{a}_+\hat{a}_+\sqrt{n(n-1)}\,\psi_{n-2}) + (\hat{a}_+\hat{a}_-\sqrt{n\cdot n}\,\psi_n) + (\hat{a}_+\hat{a}_-\sqrt{(n+1)^2}\,\psi_n) + (\hat{a}_-\hat{a}_+\sqrt{n^2}\,\psi_n) + (\hat{a}_-\hat{a}_+\sqrt{(n+1)^2}\,\psi_n) + (\hat{a}_-\hat{a}_-\sqrt{(n+1)(n+2)}\,\psi_{n+2})\}\,dx$$

$$= \frac{\hbar^2}{4m^2\omega^2}\int_{-\infty}^{\infty} \psi_n^*\cdot\{(\hat{a}_+\sqrt{n(n-1)^2}\,\psi_{n-1}) + (\hat{a}_+\sqrt{n^3}\,\psi_{n-1}) + (\hat{a}_+\sqrt{n(n+1)^2}\,\psi_{n-1}) + (\hat{a}_-\sqrt{n^2(n+1)}\,\psi_{n+1}) + (\hat{a}_-\sqrt{(n+1)^3}\,\psi_{n+1}) + (\hat{a}_-\sqrt{(n+1)(n+2)^2}\,\psi_{n+1})\}\,dx$$

$$= \frac{\hbar^2}{4m^2\omega^2}\int_{-\infty}^{\infty} \psi_n^*\cdot\{\sqrt{n^2(n-1)^2}\,\psi_n + \sqrt{n^4}\,\psi_n + \sqrt{n^2(n+1)^2}\,\psi_n + \sqrt{n^2(n+1)^2}\,\psi_n + \sqrt{(n+1)^4}\,\psi_n + \sqrt{(n+1)^2(n+2)^2}\,\psi_n\}\,dx$$

$$= \frac{\hbar^2}{4m^2\omega^2}\int_{-\infty}^{\infty} \psi_n^*\{n(n-1) + n^2 + n(n+1) + n(n+1) + (n+1)^2 + (n+1)(n+2)\}\psi_n \, dx$$

$$= \frac{\hbar^2}{4m^2\omega^2}(n^2 - n + n^2 + n^2 + n + n^2 + n + n^2 + 2n + 1 + n^2 + 3n + 2)$$

$$= \frac{\hbar^2}{4m^2\omega^2}(6n^2 + 6n + 3)$$

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x - ip)$$

$$-)\ \hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x + ip)$$

$$\hat{a}_+ - \hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}(-2ip)$$

$$\therefore p = -\frac{\sqrt{2\hbar m\omega}}{2i}(\hat{a}_+ - \hat{a}_-)$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)$$

$$\therefore p^4 = \frac{\hbar^2 m^2\omega^2}{4}(\hat{a}_+ - \hat{a}_-)^4$$

$$\langle p^4\rangle = \frac{\hbar^2 m^2\omega^2}{4}\int \Psi_n^*(\hat{a}_+ - \hat{a}_-)^4\Psi_n\,dx$$

$$= \frac{\hbar^2 m^2\omega^2}{4}\cdot(6n^2+6n+3)$$

$(-\hat{a}_-)$는 두 번 곱해지므로 부호는 $+$인 경우와 마찬가지로 계산함.

4. $\Psi(x,0) = A[\psi_0(x) - \psi_1(x)]$

(a) normalization;

$$1 = \int_{-\infty}^{\infty} |A|^2 (|\psi_0|^2 + |\psi_1|^2 - \psi_0^*\psi_1 - \psi_1^*\psi_0)\,dx$$

$$= |A|^2 \cdot 2$$

$$\therefore A = \sqrt{\frac{1}{2}}$$

(b) $\Psi(x,t) = \sum_n c_n \psi_n e^{-iE_nt/\hbar}$ $\quad E_n = (n+\frac{1}{2})\hbar\omega$

$$= \sum_n c_n \psi_n e^{-i(n+\frac{1}{2})\omega t}$$

$$= \frac{1}{\sqrt{2}}\psi_0 e^{-i\omega t/2} - \frac{1}{\sqrt{2}}\psi_1 e^{-i3\omega t/2}$$

$$\langle x\rangle = \int \Psi^* x \Psi\,dx$$

$$= \int \Psi^* \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)\Psi\,dx$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\int \Psi^*(\hat{a}_+ + \hat{a}_-)\Psi\,dx$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cdot\sqrt{\frac{1}{2}}\cdot\int\left(\psi_0^* e^{i\omega t/2} - \psi_1^* e^{3i\omega t/2}\right)(\hat{a}_+ + \hat{a}_-)\left(\psi_0 e^{-i\omega t/2} - \psi_1 e^{-3i\omega t/2}\right)dx$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\int\left(\psi_0^* e^{i\omega t/2} - \psi_1^* e^{3i\omega t/2}\right)\left(\sqrt{1}\cdot\psi_1 e^{-i\omega t/2} - \sqrt{2}\psi_2 e^{-3i\omega t/2} - \psi_0 e^{-3i\omega t/2}\right)dx$$

$$= \frac{\sqrt{\hbar}}{2\sqrt{m\omega}}(-e^{-i\omega t} - e^{i\omega t}) = -\frac{\sqrt{\hbar}}{2\sqrt{m\omega}}(+2\cos\omega t) = -\frac{\sqrt{\hbar}}{\sqrt{m\omega}}\cos\omega t$$

$$\langle p \rangle = \int \Psi^* \, i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)\, \Psi \, dx$$

$$= i\cdot\sqrt{\frac{\hbar m\omega}{2}} \int \Psi^* (\hat{a}_+ - \hat{a}_-) \Psi \, dx$$

$$= i\cdot\sqrt{\frac{\hbar m\omega}{2}} \cdot \int \left(\psi_0^* e^{i\omega t/2} - \psi_1^* e^{3i\omega t/2}\right)(\hat{a}_+ - \hat{a}_-)\left(\psi_0 e^{-i\omega t/2} - \psi_1 e^{-3i\omega t/2}\right) dx$$

$$= i\cdot\frac{\sqrt{\hbar m\omega}}{2} \cdot \int \left(\psi_0^* e^{i\omega t/2} - \psi_1^* e^{3i\omega t/2}\right)\left(\psi_1 e^{-i\omega t/2} - \sqrt{2}\,\psi_2 e^{-3i\omega t/2} + \psi_0 e^{-3i\omega t/2}\right) dx$$

$$= i\cdot\frac{\sqrt{\hbar m\omega}}{2}\cdot\left(e^{-i\omega t} - e^{i\omega t}\right)$$

$$= i\,\frac{\sqrt{\hbar m\omega}}{2}\cdot(-2i)\sin(\omega t)$$

$$= \sqrt{\hbar m\omega}\cdot\sin(\omega t)$$

$$m\cdot\frac{d\langle x\rangle}{dt} = m\cdot\frac{d}{dt}\left(-\frac{\sqrt{\hbar}}{\sqrt{m\omega}}\cos\omega t\right)$$

$$= m\cdot\left\{-\sqrt{\frac{\hbar}{m\omega}}\,(-\omega)\sin(\omega t)\right\}$$

$$= \sqrt{\hbar m\omega}\,\sin(\omega t) = \langle p \rangle$$ 임을 확인할수있다.

$$\therefore \frac{d\langle p\rangle}{dt} = \sqrt{\hbar m}\,\omega^{\frac{3}{2}}\cdot\cos(\omega t)$$

한편, $\left\langle -\frac{\partial V}{\partial x}\right\rangle = \langle -m\omega^2 x \rangle$

$$= -m\omega^2\langle x\rangle$$

$$= -m\omega^2\cdot\left(-\sqrt{\frac{\hbar}{m\omega}}\cos\omega t\right)$$

$$= \sqrt{\hbar m}\,\omega^{\frac{3}{2}}\cdot\cos(\omega t)$$

$$\therefore \frac{d\langle p\rangle}{dt} = \sqrt{\hbar m}\,\omega^{\frac{3}{2}}\cdot\cos(\omega t) = \left\langle -\frac{\partial V}{\partial x}\right\rangle$$