1.

a) $[\hat{A}+\hat{B}, \hat{C}] = (\hat{A}+\hat{B})\hat{C} - \hat{C}(\hat{A}+\hat{B})$

$= \hat{A}\cdot\hat{C} + \hat{B}\cdot\hat{C} - \hat{C}\hat{A} - \hat{C}\cdot\hat{B}$

$= (\hat{A}\cdot\hat{C} - \hat{C}\cdot\hat{A}) + (\hat{B}\cdot\hat{C} - \hat{C}\cdot\hat{B})$

$= [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$

$[\hat{A}\hat{B}, \hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B}$

$= \hat{A}\hat{B}\hat{C} + \hat{A}\hat{C}\hat{B} - \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B}$

$= \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) + (\hat{A}\hat{C} - \hat{C}\hat{A})\hat{B}$

$= \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$

테스트 함수

b) $[x^n, \hat{p}]f = (x^n\hat{p} - \hat{p}x^n)f$

$= x^n \frac{\hbar}{i}\frac{\partial f}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}(x^n f)$

$= x^n \frac{\hbar}{i}\frac{\partial f}{\partial x} - \frac{\hbar}{i}\left(n x^{n-1} f + x^n \frac{\partial f}{\partial x}\right)$

$= \cancel{x^n \frac{\hbar}{i}\frac{\partial f}{\partial x}} - \frac{\hbar}{i}\cdot n \cdot x^{n-1} f - \cancel{x^n \frac{\hbar}{i}\frac{\partial f}{\partial x}}$

$= i\hbar \cdot n \cdot x^{n-1} f$

$\therefore [x^n, \hat{p}] = i\hbar\, n x^{n-1}$

c) $[f, \hat{p}]g = (f\hat{p} - \hat{p}f)g$

$= f\cdot \frac{\hbar}{i}\frac{\partial g}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}(fg)$

$= f\cdot \frac{\hbar}{i}\frac{\partial g}{\partial x} - \frac{\hbar}{i}\left(\frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}\right)$

$= \cancel{f\cdot \frac{\hbar}{i}\frac{\partial g}{\partial x}} - \frac{\hbar}{i}\frac{\partial f}{\partial x}\cdot g - \cancel{\frac{\hbar}{i} f\cdot\frac{\partial g}{\partial x}}$

$= i\hbar \frac{\partial f}{\partial x} g$

$\therefore [f, \hat{p}] = i\hbar \frac{\partial f}{\partial x}$

d) $[\hat{H}, \hat{a}_+] = [\hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2}), \hat{a}_+]$

$= \hbar\omega[\hat{a}_+\hat{a}_- + \frac{1}{2}, \hat{a}_+]$

$= \hbar\omega([\hat{a}_+\hat{a}_-, \hat{a}_+] + [\frac{1}{2}, \hat{a}_+])$

$= \hbar\omega(\hat{a}_+[\hat{a}_-, \hat{a}_+] + \cancelto{0}{[\hat{a}_+, \hat{a}_+]}\hat{a}_- + \cancelto{0}{[\frac{1}{2}, \hat{a}_+]})$

$= \hbar\omega\, \hat{a}_+ \quad \left(\because [\hat{a}_-, \hat{a}_+] = \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- \right.$

$\qquad = \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2} - \left(\frac{\hat{H}}{\hbar\omega} - \frac{1}{2}\right)$

$\qquad \left. = 1 \right)$

이어서, $[\hat{H}, \hat{a}_-] = [\hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2}), \hat{a}_-]$

$$= \hbar\omega [\hat{a}_+\hat{a}_- + \tfrac{1}{2}, \hat{a}_-]$$

$$= \hbar\omega([\hat{a}_+\hat{a}_-, \hat{a}_-] + [\tfrac{1}{2}, \hat{a}_-])$$

$$= \hbar\omega(\hat{a}_+\cancel{[\hat{a}_-, \hat{a}_-]}^{0} + [\hat{a}_+, \hat{a}_-]\hat{a}_- + \cancel{[\tfrac{1}{2}, \hat{a}_-]}^{0})$$

$$= \hbar\omega [\hat{a}_+, \hat{a}_-]\hat{a}_-$$

$$= -\hbar\omega\, \hat{a}_- \quad \left(\because [\hat{a}_+, \hat{a}_-] = \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2} - \left(\frac{\hat{H}}{\hbar\omega} + \frac{1}{2}\right) = -1 \right)$$

2. $\hat{H}|\psi\rangle = \epsilon(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|)(c_1|1\rangle + c_2|2\rangle)$

$$= \epsilon(|1\rangle\langle 1|c_1|1\rangle + |1\rangle\langle 1|c_2|2\rangle - |2\rangle\langle 2|c_1|1\rangle - |2\rangle\langle 2|c_2|2\rangle + |1\rangle\langle 2|c_1|1\rangle + |1\rangle\langle 2|c_2|2\rangle + |2\rangle\langle 1|c_1|1\rangle + |2\rangle\langle 1|c_2|2\rangle)$$

$$= \epsilon(\underline{c_1|1\rangle\langle 1|1\rangle} + c_2|1\rangle\langle 1|2\rangle - c_1|2\rangle\langle 2|1\rangle - \underline{c_2|2\rangle\langle 2|2\rangle} + c_1|1\rangle\langle 2|1\rangle + \underline{c_2|1\rangle\langle 2|2\rangle} + \underline{c_1|2\rangle\langle 1|1\rangle} + c_2|2\rangle\langle 1|2\rangle)$$

$$= \epsilon(c_1|1\rangle - c_2|2\rangle + c_2|1\rangle + c_1|2\rangle)$$

$$= \epsilon\{(c_1+c_2)|1\rangle + (c_1-c_2)|2\rangle\} \quad \cdots ①$$

한편, $\hat{H}|\psi\rangle = E|\psi\rangle$

$$= E(c_1|1\rangle + c_2|2\rangle) \quad \cdots ②$$ 이므로,

$① = ②$; $\left.\begin{array}{l} \epsilon(c_1+c_2) = Ec_1 \\ \epsilon(c_1-c_2) = Ec_2 \end{array}\right\}$ 연립

$$c_2 = \frac{1}{\epsilon}(E-\epsilon)c_1 \quad \cdots ③$$

$$\epsilon\{c_1 - \tfrac{1}{\epsilon}(E-\epsilon)c_1\} = E\cdot\tfrac{1}{\epsilon}(E-\epsilon)c_1$$

$$\epsilon\cancel{c_1} - (E-\epsilon)\cancel{c_1} = \frac{E}{\epsilon}(E-\epsilon)\cancel{c_1}$$

$$\epsilon - E + \epsilon = \frac{E^2}{\epsilon} - E$$

$$2\epsilon = \frac{E^2}{\epsilon}$$

$$\therefore E = \pm\sqrt{2}\,\epsilon$$

$E = \pm\sqrt{2}\epsilon$, 이므로, ⑦ ; $c_2 = \frac{1}{\epsilon}(\pm\sqrt{2}\epsilon - \epsilon)\, c_1$

$$= (\pm\sqrt{2}-1)\, c_1$$

$$\therefore |\psi_\pm\rangle = c_1|1\rangle \pm c_2|2\rangle$$

$$= c_1|1\rangle + (\pm\sqrt{2}-1)|2\rangle$$

$$= c_1\left[\,|1\rangle + (\pm\sqrt{2}-1)|2\rangle\,\right]$$

Since eigenvalues of $\hat{H}$ corresponding to its eigenvectors are $\pm\sqrt{2}\epsilon$, matrix representation of $\hat{H}$ with respect to eigen basis is orthogonal 2x2 matrix $\begin{pmatrix} +\sqrt{2}\epsilon & 0 \\ 0 & -\sqrt{2}\epsilon \end{pmatrix}$

3.

a) $(\sin\hat{D})x^5 = \sin\left(\frac{d}{dx}\right)x^5$

$$= \left\{\frac{d}{dx} - \frac{1}{3!}\left(\frac{d}{dx}\right)^3 + \frac{1}{5!}\left(\frac{d}{dx}\right)^5 - \frac{1}{7!}\left(\frac{d}{dx}\right)^7 + \cdots\right\} x^5$$

$$= \frac{dx^5}{dx} - \frac{1}{3!}\frac{d^3x^5}{dx^3} + \frac{1}{5!}\frac{d^5x^5}{dx^5} - \cancel{\frac{1}{7!}\frac{d^7x^5}{dx^7}}^{\,0} + 0$$

$$= 5x^4 - \frac{1}{3!}\cdot 5\cdot4\cdot3\cdot x^2 + \frac{1}{5!}\cdot 5\cdot4\cdot3\cdot2\cdot1\cdot x^0$$

$$= 5x^4 - 10x^2 + 1$$

4.

시간 t에서의 State $|S(t)\rangle$에 대하여 Schrödinger eqn. $i\hbar\frac{\partial}{\partial t}|S(t)\rangle = H|S(t)\rangle$ 성립

이를 위해 먼저 H의 eigen states를 구하면, 이 $|S\rangle$들은 3x1 column matrix로 표현된다.

$$H|S\rangle = E|S\rangle$$

$$(H - E\mathbb{1}_{3\times3})|S\rangle = 0 \ , \ \mathbb{1}_{3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det|H - E\mathbb{1}| = \begin{pmatrix} a-E & 0 & b \\ 0 & c-E & 0 \\ b & 0 & a-E \end{pmatrix} = (a-E)(c-E)(a-E) + b\{-b(c-E)\}$$

$$= (c-E)\{(a-E)^2 - b^2\}$$

$$= (c-E)(a-E+b)(a-E-b) = 0$$

i) $E = c$,

$$\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax+bz \\ cy \\ bx+az \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix} \quad \therefore \ (a-c)x + bz = 0, \quad bx + (a-c)z = 0$$

normalized eigen state $|c\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$x = \frac{-b}{a-c}z = \frac{b^2}{(a-c)^2}x$

for general a, b, c, $x = z = 0$

ii) $E = a+b$

$$\begin{pmatrix} ax+bz \\ cy \\ bx+az \end{pmatrix} = \begin{pmatrix} ax+bx \\ ay+by \\ az+bz \end{pmatrix}$$

$x = z$

$(a+b-c)y = 0 \rightarrow y = 0$

normalized eigen state $|a+b\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

iii) $E = a-b$

$$\begin{pmatrix} ax+bz \\ cy \\ bx+az \end{pmatrix} = \begin{pmatrix} ax-bx \\ ay-by \\ az-bz \end{pmatrix}$$

$x = -z$

$(a-b-c)y = 0 \rightarrow y = 0$

normalized eigen state $|a-b\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

만약, initial state $|S(0)\rangle = \alpha|c\rangle + \beta|a+b\rangle + \gamma|a-b\rangle \quad (\alpha^2+\beta^2+\gamma^2=1)$

$$\longrightarrow |S(t)\rangle = \alpha|c\rangle e^{-ict/\hbar} + \beta|a+b\rangle e^{-i(a+b)t/\hbar} + \gamma|a-b\rangle e^{-i(a-b)t/\hbar}$$

(time dep. schrödinger eqn.으로부터 구한 time dep. quantum state)

(1) $|S(0)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |c\rangle$

$$|S(t)\rangle = e^{-ict/\hbar}|c\rangle$$

(2) $|S(0)\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}|a+b\rangle + \frac{1}{\sqrt{2}}|a-b\rangle$

$$|S(t)\rangle = \frac{1}{\sqrt{2}}e^{-i(a+b)t/\hbar}|a+b\rangle + \frac{1}{\sqrt{2}}e^{-i(a-b)t/\hbar}|a-b\rangle$$

5.

$$Y_0^0 = \sqrt{\frac{(2\cdot 0+1)}{4\pi}\cdot\frac{(0-|0|)!}{(0+|0|)!}}\, e^{i0\phi} P_0^0(\cos\theta) = \frac{1}{\sqrt{4\pi}} P_0^0(\cos\theta)$$

$$= \frac{1}{\sqrt{4\pi}} (1-\cos^2\theta)^{|0|/2} \left(\frac{d}{dx}\right)^{|0|} P_0(\cos\theta) = \frac{1}{\sqrt{4\pi}} P_0(\cos\theta)$$

$$= \frac{1}{\sqrt{4\pi}} \cdot \frac{1}{2^0 \cdot 0!} \left(\frac{d}{d\cos\theta}\right)^0 (\cos^2\theta - 1)^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_2^1 = -\sqrt{\frac{(2\cdot 2+1)}{4\pi}\cdot\frac{(2-|1|)!}{(2+|1|)!}}\, e^{i\cdot 1\cdot\phi} P_2^1(\cos\theta) = -\sqrt{\frac{5}{24\pi}}\, e^{i\phi} P_2^1(\cos\theta)$$

$$= -\sqrt{\frac{5}{24\pi}}\, e^{i\phi} (1-\cos^2\theta)^{|1|/2} \cdot \left(\frac{d}{d\cos\theta}\right)^{|1|} \cdot P_2(\cos\theta)$$

$$= -\sqrt{\frac{5}{24\pi}}\, e^{i\phi} (1-\cos^2\theta)^{\frac{1}{2}} \cdot \left(\frac{d}{d\cos\theta}\right) \left[\frac{1}{2^2 \cdot 2!} \left(\frac{d}{d\cos\theta}\right)^2 \cdot (\cos^2\theta - 1)^2 \right]$$

$$= -\sqrt{\frac{5}{24\pi}}\, e^{i\phi} (1-\cos^2\theta)^{\frac{1}{2}} \cdot \left(\frac{d}{d\cos\theta}\right) \underbrace{\left[\frac{1}{8} \underbrace{\left(\frac{d}{d\cos\theta}\right)^2 \cdot (\cos^4\theta - 2\cos^2\theta + 1)}_{\frac{d}{d\cos\theta}(4\cos^3\theta - 4\cos\theta) \; = 12\cos^2\theta - 4} \right]}_{\frac{1}{2}(3\cos^2\theta - 1)}$$

$$= -\sqrt{\frac{5}{24\pi}}\, e^{i\phi} (1-\cos^2\theta)^{\frac{1}{2}} (3\cos\theta)$$

$$= -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta\, e^{i\phi}$$

Normalization;

$$\int_0^{2\pi}\int_0^{\pi} Y_\ell^{n\,*}\, Y_\ell^{n'} \sin\theta \, d\theta\, d\phi$$

$$\int_0^{2\pi}\int_0^{\pi} (Y_0^0)^* \cdot Y_0^0 \sin\theta\, d\theta\, d\phi$$

$$\frac{1}{4\pi} \underbrace{\int_0^{2\pi} \underbrace{\int_0^{\pi} \sin\theta\, d\theta}_{2}\, d\phi}_{4\pi} = 1$$

$$\int_0^{2\pi}\int_0^{\pi} (Y_2^1)^* Y_2^1 \sin\theta\, d\theta\, d\phi = \frac{15}{8\pi} \int_0^{2\pi}\int_0^{\pi} \sin\theta\cos\theta\, e^{-i\phi} \sin\theta\cos\theta\, e^{i\phi} \sin\theta\, d\theta\, d\phi$$

$$= \frac{15}{8\pi} \int_0^{2\pi}\int_0^{\pi} \cos^2\theta \sin^3\theta\, d\theta\, d\phi$$

$$= \frac{15}{8\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta (1-\cos^2\theta) \sin\theta\, d\theta$$

$$= \frac{15}{8\pi} \int_0^{2\pi} d\phi \int_1^{-1} (\cos^2\theta - \cos^4\theta)(-d\cos\theta)$$

$$= \frac{15}{8\pi} \int_0^{2\pi} d\phi \int_{-1}^{1} (\cos^2\theta - \cos^4\theta)\, d\cos\theta$$

$$= \frac{15}{8\pi} \cdot 2\pi \cdot \frac{4}{15} = 1$$