1. $\Psi(x,t) = Ae^{-\lambda x} \cdot e^{-i\omega t}$ $(x>0)$ 단, A, λ, ω는 양의 상수

(a) Ψ 규격화

$$1 = \int_0^\infty \Psi^*(x,t)\Psi(x,t)\,dx$$
$$= \int_0^\infty A^* e^{-\lambda x} \cdot e^{i\omega t} \cdot Ae^{-\lambda x} \cdot e^{-i\omega t}\,dx$$
$$= \int_0^\infty |A|^2 e^{-2\lambda x}\,dx$$
$$= -\frac{|A|^2}{2\lambda} e^{-2\lambda x}\Big|_0^\infty$$
$$= 0 + \frac{|A|^2}{2\lambda} = 1$$
$$\therefore A = \sqrt{2\lambda} \quad (A>0)$$

(b) $$\langle x \rangle = \int_0^\infty \Psi^* \cdot x\,\Psi\,dx$$
$$= \int_0^\infty A^* e^{-\lambda x} e^{i\omega t} x \cdot Ae^{-\lambda x} e^{-i\omega t}\,dx$$
$$= A^* \cdot A \cdot \int_0^\infty e^{-2\lambda x} \cdot x\,dx$$
$$= |A|^2 \cdot \left[\frac{-1}{2\lambda} e^{-2\lambda x} \cdot x \Big|_0^\infty + \frac{1}{2\lambda}\int_0^\infty e^{-2\lambda x}\,dx \right]$$
$$= \int_0^\infty e^{-2\lambda x}\,dx$$
$$= \frac{-1}{2\lambda} e^{-2\lambda x}\Big|_0^\infty = \frac{1}{2\lambda}$$

$$\langle x^2 \rangle = \int_0^\infty \Psi^* \cdot x^2 \cdot \Psi\,dx$$
$$= \int_0^\infty A^* e^{-\lambda x} e^{i\omega t} \cdot x^2 \cdot Ae^{-\lambda x} e^{-i\omega t}\,dx$$
$$= |A|^2 \int_0^\infty e^{-2\lambda x} x^2\,dx$$
$$= 2\lambda \cdot \left[\frac{-1}{2\lambda} e^{-2\lambda x} \cdot x^2 \Big|_0^\infty + \int_0^\infty \frac{1}{2\lambda} e^{-2\lambda x} \cdot 2x\,dx \right.$$
$$= 2\int_0^\infty e^{-2\lambda x} \cdot x\,dx$$
$$= 2\left[\frac{-1}{2\lambda} e^{-2\lambda x} \cdot x \Big|_0^\infty + \frac{1}{2\lambda}\int_0^\infty e^{-2\lambda x}\,dx \right]$$
$$= \frac{1}{\lambda} \cdot \left(-\frac{1}{2\lambda}\right) e^{-2\lambda x}\Big|_0^\infty = +\frac{1}{2\lambda^2}$$

$$\therefore\ \sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$
$$= \sqrt{\frac{1}{2\lambda^2} - \left(\frac{1}{2\lambda}\right)^2}$$
$$= \sqrt{\frac{1}{4\lambda^2}} = \frac{1}{2\lambda} \quad (\because \lambda > 0)$$

2. $$\langle p\rangle = m\cdot\frac{d\langle x\rangle}{dt} = -i\hbar\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}dx$$

$$\Bigg(\ \because\ \langle x\rangle = \int_{-\infty}^{\infty}\Psi^* x\,\Psi\,dx$$
$$\frac{d\langle x\rangle}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty}\Psi^* x\,\Psi\,dx$$
$$= \int_{-\infty}^{\infty} x\,\frac{\partial}{\partial t}(\Psi^*\Psi)\,dx$$
$$= \int_{-\infty}^{\infty} x\cdot\left(\Psi^*\frac{\partial\Psi}{\partial t} + \Psi\frac{\partial\Psi^*}{\partial t}\right)dx \quad -①$$

Since $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$, $\quad -i\hbar\frac{\partial\Psi^*}{\partial t} = -\frac{\hbar^2}{2m}\cdot\frac{\partial^2\Psi^*}{\partial x^2} + V\Psi^*$

$$\frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\cdot\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi, \quad \frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\cdot\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*$$

$$\frac{\partial}{\partial t}(\Psi^*\Psi) = \Psi^*\frac{\partial\Psi}{\partial t} + \Psi\frac{\partial\Psi^*}{\partial t}$$

적분시 상쇄

$$= \frac{i\hbar}{2m}\Psi^*\cdot\frac{\partial^2\Psi}{\partial x^2} - \cancel{\frac{i}{\hbar}\Psi^* V\Psi} - \frac{i\hbar}{2m}\Psi\cdot\frac{\partial^2\Psi^*}{\partial x^2} + \cancel{\frac{i}{\hbar}\Psi\cdot V\Psi^*}$$
$$= \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \Psi\cdot\frac{\partial^2\Psi^*}{\partial x^2}\right)$$
$$= \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right)$$

$$\therefore\ ① = \frac{i\hbar}{2m}\int_{-\infty}^{\infty} x\,\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right)$$

Integ. by parts

$$= \frac{i\hbar}{2m}\left[x\cdot\left(\cancel{\Psi^*}^{\,0}\frac{\partial\Psi}{\partial x} - \cancel{\Psi}^{\,0}\frac{\partial\Psi^*}{\partial x}\right)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right)dx$$
$$= -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right)dx$$
$$= -\frac{i\hbar}{m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}dx \quad ②$$

(정규화조건)

$$② = \int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}dx - \cancel{\Psi\Psi^*}\Big|_{-\infty}^{\infty}{}^{\,0} + \int_{-\infty}^{\infty}\frac{\partial\Psi}{\partial x}\cdot\Psi^* dx$$
$$= 2\cdot\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}dx \ \Bigg)$$

($\because$ Ψ^*와 $\frac{\partial\Psi}{\partial x}$는 x에 대한 함수이므로 commute)

$$\therefore \frac{d\langle p\rangle}{dt} = -i\hbar \int \frac{\partial}{\partial t}\left(\Psi^* \frac{\partial \Psi}{\partial x}\right) dx$$

$$= -i\hbar \int \left\{ \frac{\partial \Psi^*}{\partial t}\cdot\frac{\partial \Psi}{\partial x} + \Psi^* \cdot \frac{\partial}{\partial t}\left(\frac{\partial \Psi}{\partial x}\right)\right\} dx$$

Schrödinger eqn.

$$= -i\hbar \int \left[\left(-\frac{i\hbar}{2m}\cdot\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V \Psi^*\right)\frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial}{\partial x}\left(\frac{i\hbar}{2m}\cdot\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V\Psi\right)\right] dx$$

$$= -i\hbar \int \left[\frac{-i\hbar}{2m}\left(\frac{\partial^2 \Psi^*}{\partial x^2}\frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial^3 \Psi}{\partial x^3}\right) + \frac{i}{\hbar}\left(V\Psi^* \frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial}{\partial x}(V\Psi)\right)\right] dx$$

$$= -i\hbar \left[\underbrace{\int \frac{-i\hbar}{2m}\left(\frac{\partial^2 \Psi^*}{\partial x^2}\cdot\frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial^3 \Psi}{\partial x^3}\right) dx}_{①} + \underbrace{\int \frac{i}{\hbar}\left\{V\Psi^* \frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial}{\partial x}(V\Psi)\right\} dx}_{②}\right]$$

$$① = \int_{-\infty}^{\infty} -\frac{i\hbar}{2m}\left(\frac{\partial^2 \Psi^*}{\partial x^2}\cdot\frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial^3 \Psi}{\partial x^3}\right) dx$$

$$= -\frac{i\hbar}{2m}\left[\cancel{\frac{\partial \Psi^*}{\partial x}\cdot\frac{\partial \Psi}{\partial x}\Big|_{-\infty}^{\infty}} - \cancel{\int_{-\infty}^{\infty}\frac{\partial \Psi^*}{\partial x}\cdot\frac{\partial^2 \Psi}{\partial x^2} dx} - \cancel{\Psi^* \frac{\partial^2 \Psi}{\partial x^2}\Big|_{-\infty}^{\infty}} + \cancel{\int_{-\infty}^{\infty}\frac{\partial \Psi^*}{\partial x}\cdot\frac{\partial^2 \Psi}{\partial x^2} dx}\right]$$

0 $\left(\because \frac{\partial \Psi}{\partial x} \to 0,\ \text{as } x \to \pm\infty\right)$ 0 $\left(\because \Psi^* \to 0,\ \text{as } x \to \pm\infty\right)$

정규화 조건

$$= 0$$

$$② = \int_{-\infty}^{\infty} \frac{i}{\hbar}\left\{V\Psi^* \frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial}{\partial x}(V\Psi)\right\} dx$$

$$= \frac{i}{\hbar}\int_{-\infty}^{\infty}\left(V\Psi^* \frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial V}{\partial x}\Psi - \Psi^* V \frac{\partial \Psi}{\partial x}\right) dx$$

$$= \frac{i}{\hbar}\int_{-\infty}^{\infty} \Psi^*\left(-\frac{\partial V}{\partial x}\right)\Psi\, dx$$

$$= \frac{i}{\hbar}\left\langle -\frac{\partial V}{\partial x}\right\rangle$$

$$\therefore \frac{d\langle p\rangle}{dt} = -i\hbar\cdot\left(0 + \frac{i}{\hbar}\left\langle -\frac{\partial V}{\partial x}\right\rangle\right)$$

$$= \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

/* 뉴턴 제2법칙 $\frac{dp}{dt} = -\frac{dV}{dx}$ 에 대한, 양자역학적 analogue 이다. */

3. $\Psi(x) = \begin{cases} A(a^2-x^2), & |x| \le a \\ 0, & |x| > a \end{cases}$

(a) $1 = \int_{-\infty}^{\infty} \Psi^*(x) \cdot \Psi(x)\, dx$

$= \int_{-a}^{a} A^* (a^2-x^2)^* \cdot A \cdot (a^2-x^2)\, dx$

$= A \cdot A^* \cdot \int_{-a}^{a} (a^2-x^2)^* \cdot (a^2-x^2)\, dx$

$= |A|^2 \cdot \int_{-a}^{a} |a^2-x^2|^2\, dx$

$= 2|A|^2 \cdot \int_0^a (a^4 - 2a^2x^2 + x^4)\, dx$

$= 2|A|^2 \cdot \left(a^4 x - \frac{2a^2}{3} x^3 + \frac{1}{5} x^5 \Big|_0^a \right)$

$= 2|A|^2 \cdot \left(a^5 - \frac{2a^5}{3} + \frac{1}{5} a^5 \right)$

$= 2|A|^2 \cdot \frac{15-10+3}{15} a^5 = 2|A|^2 \cdot \frac{8}{15} a^5$

$\therefore |A| = \sqrt{\frac{15}{16a^5}}$

(b) $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$

$= \int_{-a}^{a} \Psi^* x^2 \Psi\, dx - \left(\int_{-a}^{a} \Psi^* x \Psi\, dx \right)^2$

$= \int_{-a}^{a} x^2 \cdot |A|^2 (a^2-x^2)^2\, dx - \left(\int_{-a}^{a} x \cdot |A|^2 (a^2-x^2)\, dx \right)^2$ Considering even & oddness

$= 2|A|^2 \cdot \int_0^a x^2 (a^2-x^2)^2\, dx - 0$

$= 2|A|^2 \cdot \int_0^a x^2 \cdot (a^4 - 2a^2x^2 + x^4)\, dx$

$= 2|A|^2 \int_0^a (a^4 x^2 - 2a^2 x^4 + x^6)\, dx$

$= 2|A|^2 \cdot \left(\frac{a^4}{3} x^3 - \frac{2a^2}{5} x^5 + \frac{1}{7} x^7 \Big|_0^a \right)$ (a) 결과이용

$= 2|A|^2 \left(\frac{a^7}{3} - \frac{2a^7}{5} + \frac{1}{7} a^7 \right)$

$= 2|A|^2 \cdot \frac{(35-42+15)a^7}{105} = 2|A|^2 \cdot \frac{8}{105} a^7 = 2 \cdot \frac{15}{16a^5} \cdot \frac{8}{105} a^7 = \frac{1}{7} a^2$

$$\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2$$

$$= \int \Psi^* \left(-i\hbar \frac{\partial}{\partial x}\right)^2 \Psi \, dx - \left(\int \Psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \Psi \, dx \right)^2$$

$$= -\hbar^2 \int \Psi^* \frac{\partial^2 \Psi}{\partial x^2} dx - \left(-i\hbar \int \Psi^* \frac{\partial \Psi}{\partial x} dx \right)^2$$

$$= -\hbar^2 \cdot \int A^* (a^2 - x^2)^* \cdot (-2A) - \left(-i\hbar \int A^* (a^2 - x^2)^* \cdot (-2Ax) dx \right)^2$$

$$= 2|A|^2 \cdot \hbar^2 \cdot \int (a^2 - x^2) dx - \left(2|A|^2 i\hbar \int (a^2 - x^2) x \, dx \right)^2$$

$$= 2|A|^2 \cdot \hbar^2 \left(a^2 x - \frac{1}{3} x^3 \Big|_{-a}^{a} \right) - 0$$

$$= 2|A|^2 \cdot \hbar^2 \cdot \left(a^3 - \frac{1}{3} a^3 + a^3 - \frac{1}{3} a^3 \right)$$

$$= 2 \cdot \frac{15}{16a^5} \cdot \hbar^2 \cdot \frac{4}{3} a^3$$

$$= \frac{5\hbar^2}{2a^2}$$

$$\therefore \ \sigma_x = \frac{a}{\sqrt{7}}, \quad \sigma_p = \sqrt{\frac{5}{2}} \cdot \frac{\hbar}{a}$$

(C) $$\sigma_x \cdot \sigma_p = \frac{a}{\sqrt{7}} \cdot \frac{\sqrt{5}}{\sqrt{2}} \cdot \frac{\hbar}{a}$$

$$= \frac{\sqrt{5}}{\sqrt{14}} \hbar \fallingdotseq 0.5976\hbar > \frac{\hbar}{2}$$

∴ 주어진 파동함수는 불확정성 원리를 만족

4. (a) $E = \frac{1}{2}mv^2(x) + V(x)$, Conservation of Energy

$$\therefore v(x) = \sqrt{\frac{2}{m}\{E - V(x)\}}$$

(b) $$\rho(x) = \frac{1}{v(x)T}$$

$$= \frac{1}{T} \cdot \sqrt{\frac{m}{2\{E - V(x)\}}}$$

$V(x) = \frac{1}{2}kx^2$

$$= \frac{1}{T} \cdot \sqrt{\frac{m}{2E - kx^2}}$$

since $T = \int_a^b \frac{1}{v(x)}\,dx$

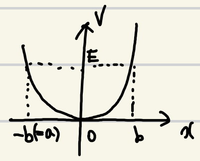
(a) 결과에 의해

$$= \int_a^b \sqrt{\frac{m}{2\{E - V(x)\}}}\,dx$$

$$= \int_a^b \sqrt{\frac{m}{2\{E - \frac{1}{2}kx^2\}}}\,dx$$

$V(a) = \frac{1}{2}ka^2 = \frac{1}{2}kb^2 = V(b) = E$

$a = -b$ with $a < 0 < b$



$$= 2 \cdot \int_0^b \sqrt{\frac{m}{2E - kx^2}}\,dx$$

$$= 2 \cdot \int_0^b \sqrt{\frac{m}{2E(1 - \frac{k}{2E}x^2)}}\,dx$$

Let, $x = \sqrt{\frac{2E}{k}}\cos\theta$

$\therefore \theta \to \frac{\pi}{2}$ $(x \to 0)$

$\theta \to 0$ $(x \to b)$

$(\because \frac{1}{2}kb^2 = E$

$b = \sqrt{\frac{2E}{k}})$

$dx = -\sqrt{\frac{2E}{k}}\sin\theta\,d\theta$

$$= 2 \cdot \sqrt{\frac{m}{2E}} \int_0^b \frac{1}{\sqrt{1 - \frac{k}{2E}x^2}}\,dx$$

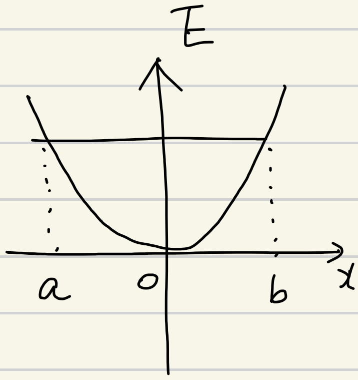
$$= \sqrt{\frac{2m}{E}} \int_{\frac{\pi}{2}}^0 \frac{1}{\sqrt{1 - \cos^2\theta}} \left(-\sqrt{\frac{2E}{k}}\sin\theta\right) d\theta$$

$$= 2\sqrt{\frac{m}{k}} \int_0^{\frac{\pi}{2}} d\theta = \pi\sqrt{\frac{m}{k}}$$

(용수철 단조화 진동자의 $\frac{1}{2}$주기 $= \pi\sqrt{\frac{m}{k}}$ 과 일치)

$$\therefore \ \rho(x) = \frac{1}{\pi}\sqrt{\frac{k}{m}} \cdot \sqrt{\frac{m}{2E-kx^2}}$$

$$= \frac{1}{\pi} \cdot \sqrt{\frac{k}{2E-kx^2}}$$



(c) $\langle x \rangle = \int_a^b x\,\rho(x)\,dx$

$$= \int_a^b \frac{x}{\pi}\sqrt{\frac{k}{2E-kx^2}}\,dx$$

$= 0$ ($\because$ $a=-b$, 피적분함수는 odd func.)

$$\langle x^2 \rangle = \int_a^b x^2 \cdot \rho(x)\,dx$$

$$= \int_a^b x^2 \cdot \frac{1}{\pi}\sqrt{\frac{k}{2E-kx^2}}\,dx$$

$$= \frac{2}{\pi}\int_0^b x^2\sqrt{\frac{k}{2E-kx^2}}\,dx$$

Let, $x = \sqrt{\frac{2E}{k}}\cos\theta$

$dx = -\sqrt{\frac{2E}{k}}\sin\theta\,d\theta$

$\theta \to \frac{\pi}{2}$

$\theta \to \cos^{-1}\left(b\cdot\sqrt{\frac{k}{2E}}\right)$

$= \cos^{-1}\left(\sqrt{\frac{2E}{k}}\cdot\sqrt{\frac{k}{2E}}\right)$

$= 0$

$$= \frac{2}{\pi}\int_0^b x^2\sqrt{\frac{k}{2E\left(1-\frac{k}{2E}x^2\right)}}\,dx$$

$$= \frac{2}{\pi}\cdot\sqrt{\frac{k}{2E}}\int_{\frac{\pi}{2}}^{0} \frac{2E}{k}\cos^2\theta\,\frac{1}{\sqrt{1-\cos^2\theta}}\cdot\left(-\sqrt{\frac{2E}{k}}\sin\theta\right)d\theta$$

$$= \frac{2}{\pi}\cdot\int_{\frac{\pi}{2}}^{0}\left(-\frac{2E}{k}\cos^2\theta\right)d\theta$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{2E}{k}\cdot\frac{(1+\cos 2\theta)}{2}\,d\theta$$

$$= \frac{2}{\pi}\cdot\frac{E}{k}\cdot\left(\theta + \frac{1}{2}\sin 2\theta \Big|_0^{\frac{\pi}{2}}\right)$$

$$= \frac{2E}{\pi k}\cdot\left(\frac{\pi}{2}\right) = \frac{E}{k}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= \sqrt{\frac{E}{k}}$$

$$E = \frac{1}{2} k b^2$$

$$= \sqrt{\frac{1}{2} b^2}$$

$$= \frac{b}{\sqrt{2}}$$

$\therefore$ 용수철 진동자에서 위치 x의 표준편차 σ_x는
용수철 상수 k에 의존하지 않고, 초기에 당긴 위치 b에만 의존