Simulation of Many Body System Constrained to Surface Acting Under the Influence of an Inverse Square Law

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1 Abstract

In this project, we create a simulation consisting of 10,000 bodies operating under the influence of an inverse square law force on 3 Dimensional surfaces of varying curvature. The objective of creating these simulations is to see the effect of these surfaces' curvature (Gaussian and Mean curvature in particular) on the formation and stability of orbits.

2 Introduction

We consider a 3-Dimensional surface defined as the set of points $(x, y, f(x, y))$ and an extensive system of particles constrained to move about this surface. Each particle is assigned a mass, and between each set of particles, there is a force:

$$F = \frac{Gm_1m_2}{r^2} \tag{1}$$

Where G is a constant that, for ease, we set equal to 1, and m_1 and m_2 are the individual particles' masses. It is important to note that r is the distance between each particle on the surface. We calculate this distance by drawing a straight line between the two particles in a two-dimensional space. We then map this line to the surface, obtaining a three-dimensional curve. The length of this curve gives us the value r. The distance between the particles in our system depends on the curvature of the surface we are investigating. To find the direction of the forces acting between each of the points, we similarly find the vector in two-dimensional space between the two-dimensional coordinates of the particles under inspection. We then map this vector to the surface to find the unit vector in the force's direction.

3 Methods

To find the particles' trajectories in our system, we integrate the equations of motion in conjunction with a set of simplifying approximations. This section describes essential techniques and measurements used during the simulation.

3.1 Initialization

The first step in our simulation is to initialize the parameters in this system. We use normal distributions to find the starting velocities and positions of the particles in our system. We also use these normal distributions to set the masses of these particles.

3.2 Sorting

After initializing the particles in our system, we employ a sorting algorithm to simplify solving the equations of motion. Our algorithm would be highly inefficient if the program must calculate each of the forces acting on all of the bodies in our system. As a result, we use a sorting algorithm to find the most significant forces acting on our bodies at any particular time. In our case, we find the most influential forces on each body by dividing the entire system into grids. We will then consider the forces acting on the body from particles in its grid. To ensure that the system remains accurate, we will need to frequently resort to our sorting algorithm to ensure that the simulation only accounts for the most significant forces.

3.3 Integrating the Equations of Motion

Once we have initialized our system and have run our sorting algorithm for the first time, we can use these results to find the particles' trajectories in our system. We do so using simple integration using Euler's Method. We calculate the forces acting on the body from the nearest particles found in the sorting algorithm for each body. We then add these forces together and use them to see the acceleration associated with the body. We then use this acceleration to find the velocity. We can use the velocity to find the change in position in the coordinates independent of the surface. Once we have calculated each body's change in position in a small interval of time, we complete a loop. After a few loops, we reemploy the sorting algorithm discussed previously.

3.4 Periodic Boundary Conditions

In simulating our system, we apply rectangular periodic boundary conditions, in which a body is assigned to a random coordinate if the body moves outside of the region.

3.5 Significant Measurements

We allow the bodies to move under the inverse square law force's influence for several integration steps before conducting our measurements. Once this initial period has passed, we assign an interval of integration steps to perform stability measurements. At each stage, we consider a large portion of randomly selected bodies. We consider the particles that have the most significant influence on the body's motion for each body. We then find the center of mass of these particles and then approximate the subsystem as a two-body motion. In such a system, one body is the particle under consideration. The other is a particle located at the center of mass found previously with a total mass equal to the average mass of all the prior bodies. We then measure the particle's energy in the two-body system to see if it will obey the behavior of a closed orbit in the short term. We can then use this two-body approximation to find if this short-term orbit will be stable. We also measure the Gaussian and Mean curvature of the surface at the point at which we measure the orbital behavior in the short term. These measurements will show how these two-dimensional surfaces' curvature impacts short-term orbital behavior and orbital stability.

4 The Results

We have the following data for various curves tested in our simulations.

Table 1: Data for $f(x, y) = x^2y^2$

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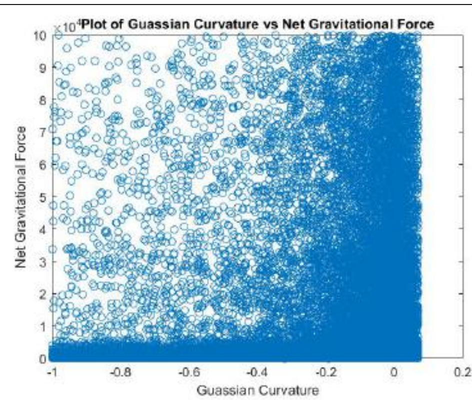
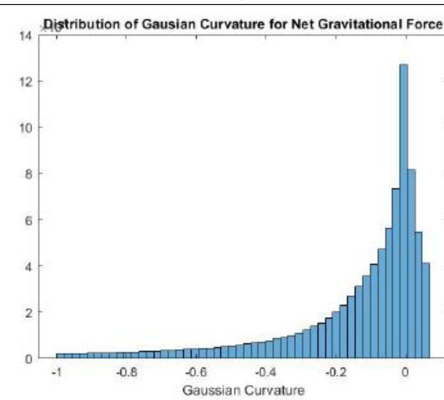
<tr><td>
</td><td>
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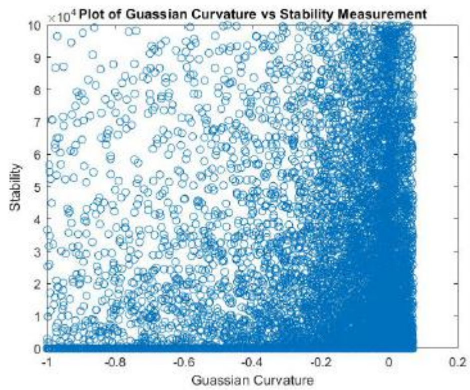
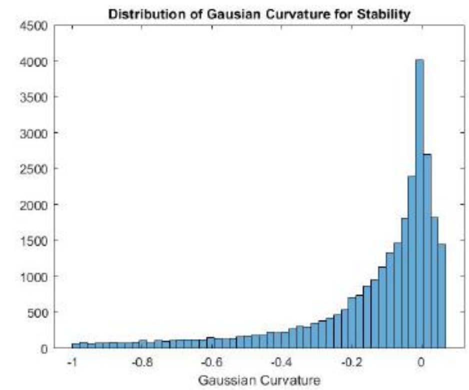
<tr><td>
</td><td>
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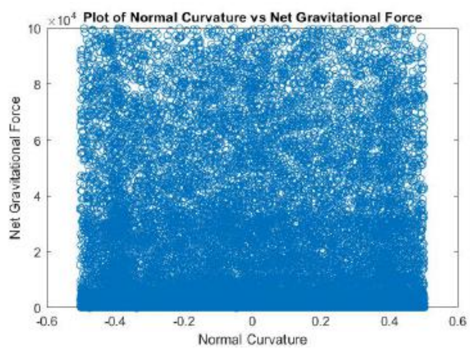
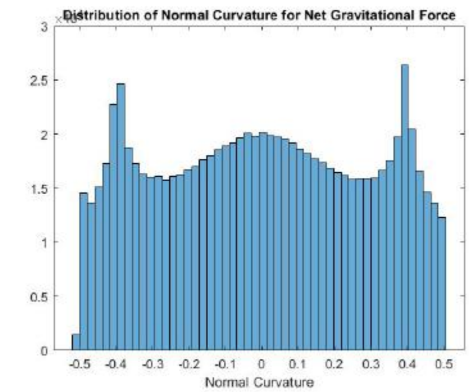
<tr><td>
</td><td>
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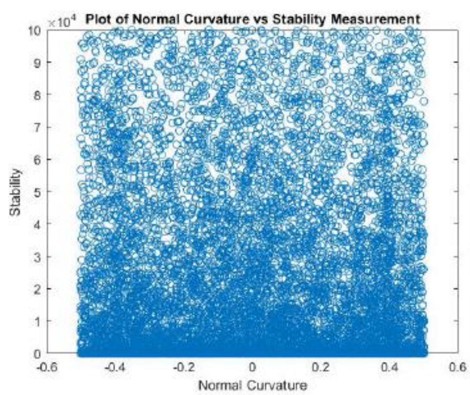
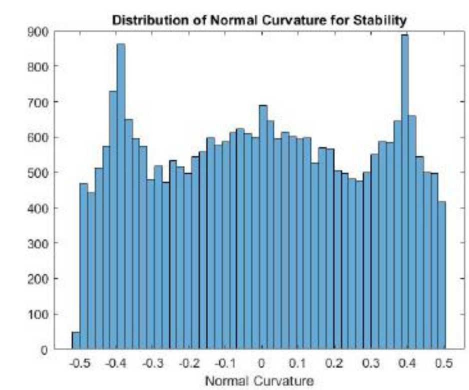
<tr><td>
</td><td>
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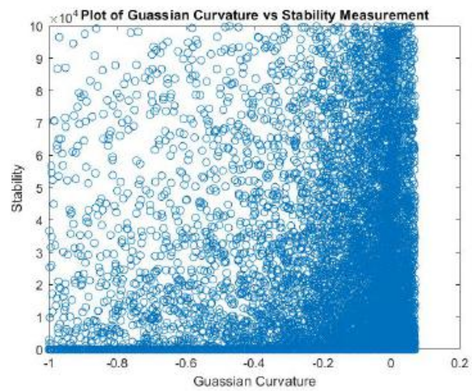

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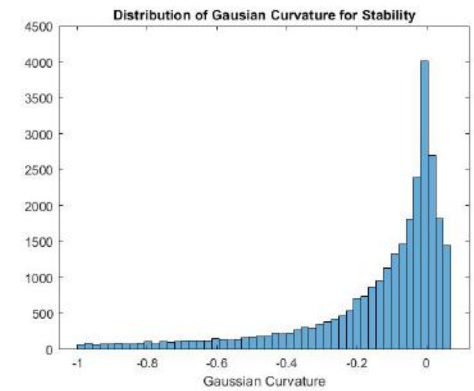

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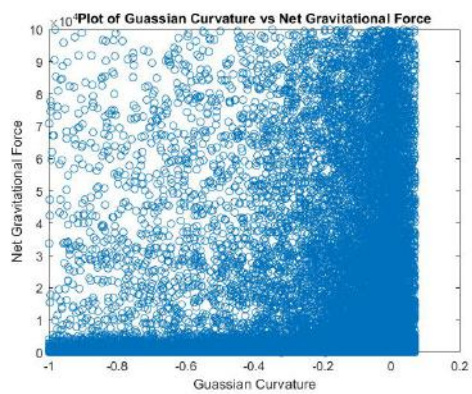

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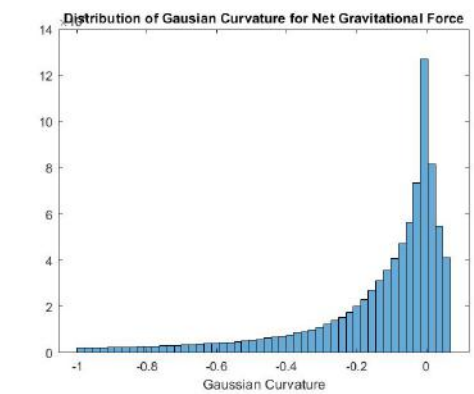

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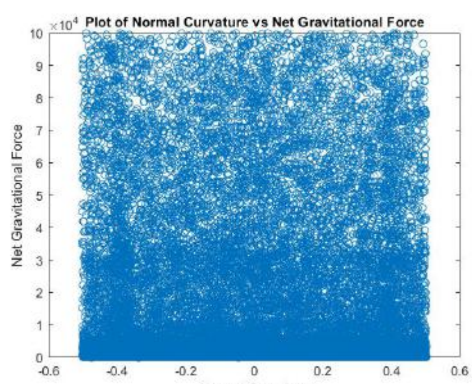

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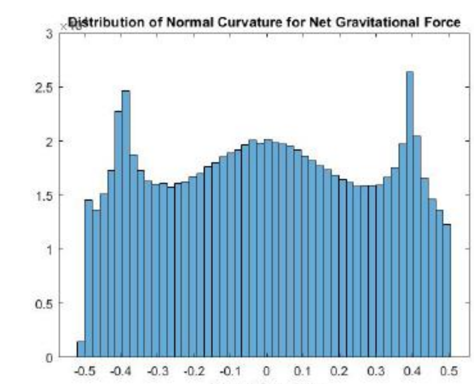

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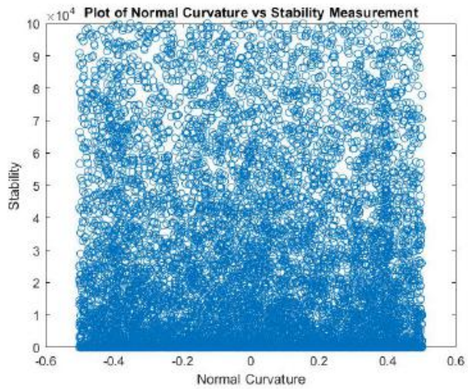
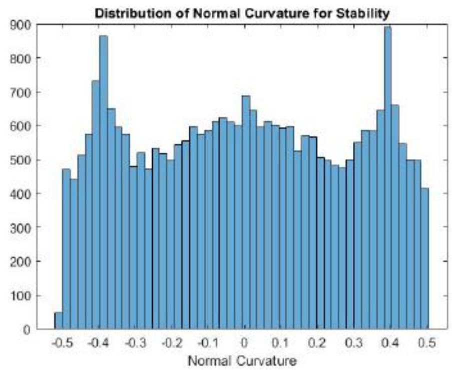
<tr><td>
</td><td>
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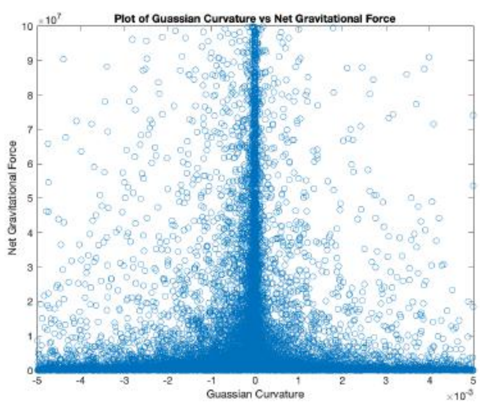
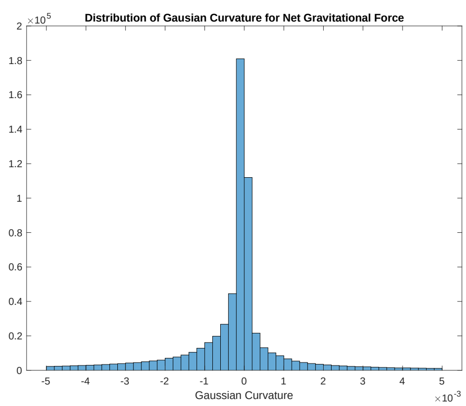
<tr><td>
</td><td>
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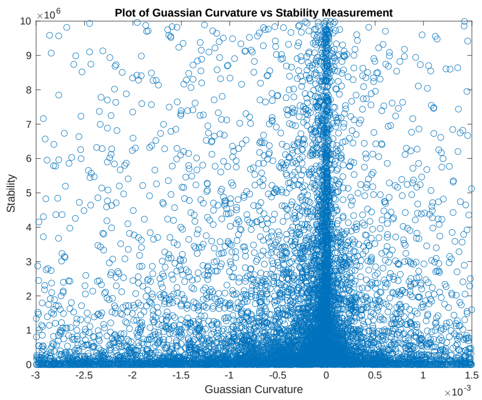
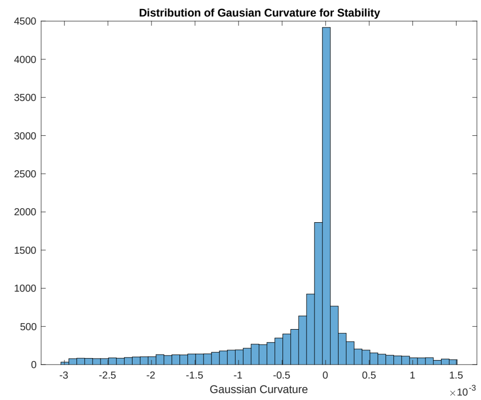
<tr><td>
</td><td>
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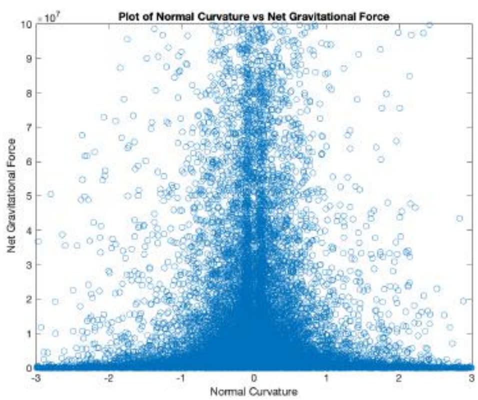
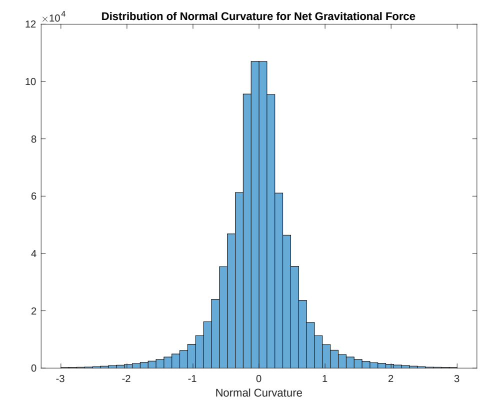
<tr><td>
</td><td>
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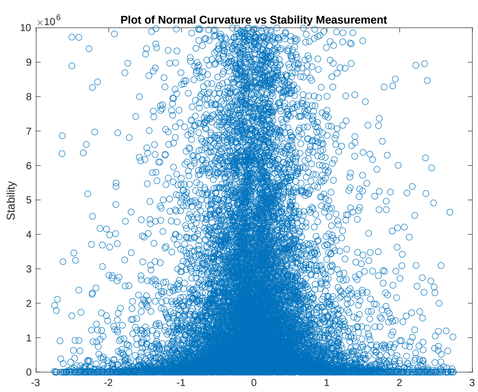
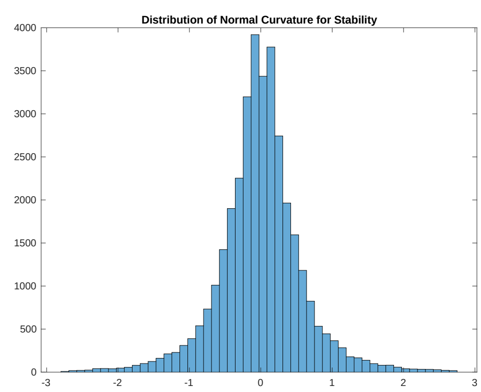
<tr><td>
</td><td>
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Table 4: Data for $f(x,y) = \left(x^2+y^2\right)\sin(x)\cos(y)$

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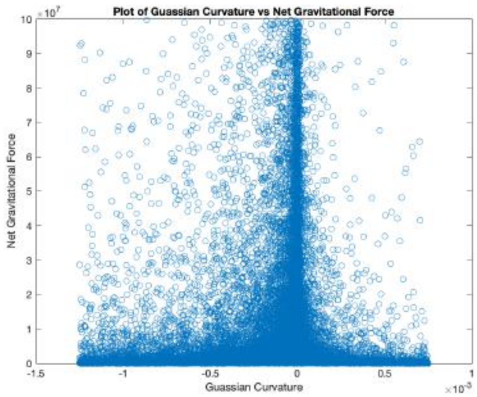

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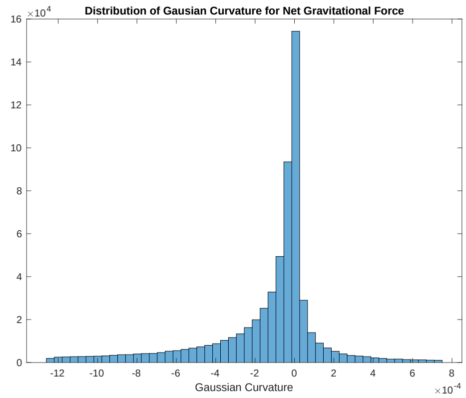

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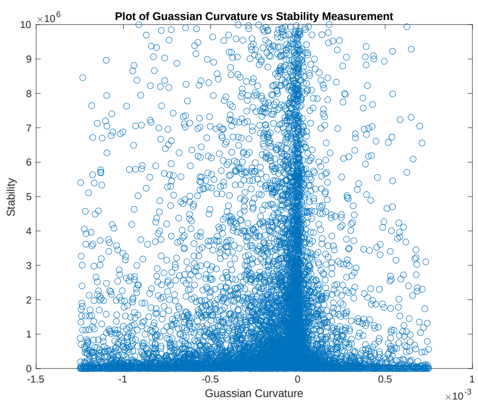

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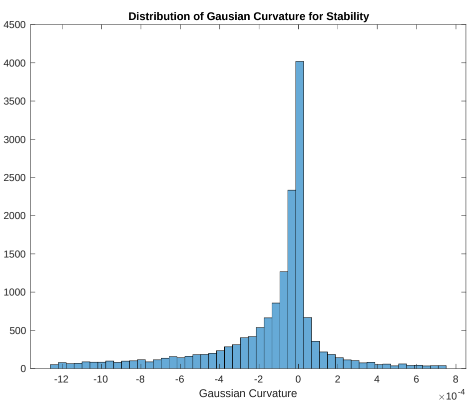

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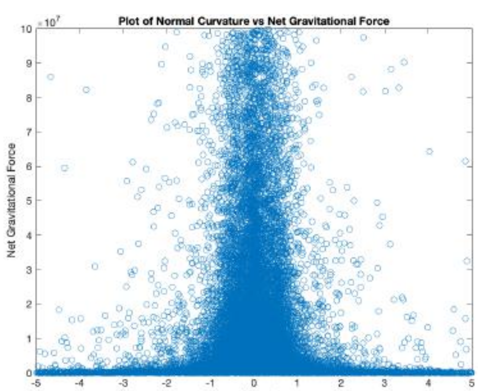

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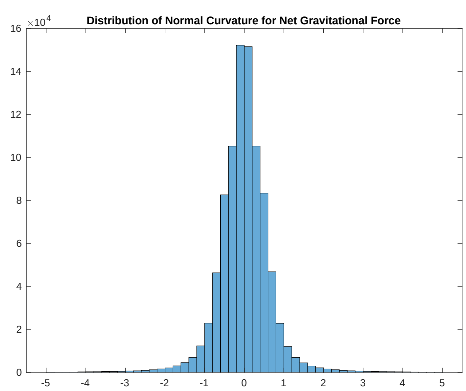

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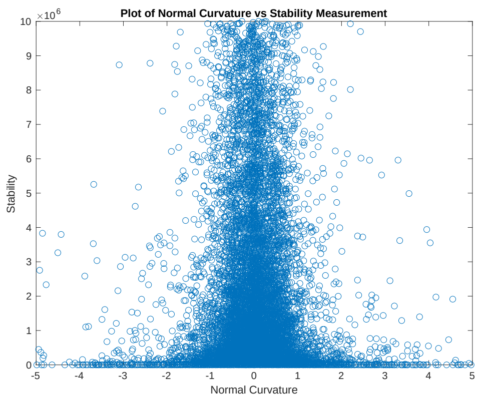
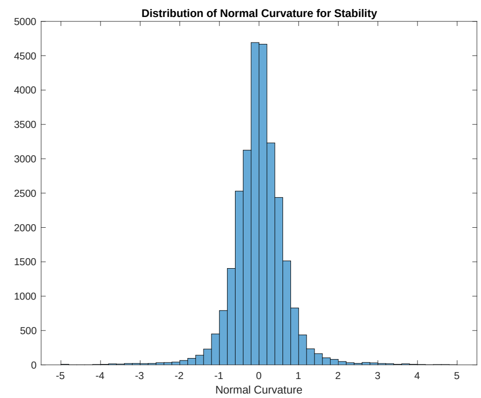
<tr><td>
</td><td>
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Table 5: Data for $f(x, y) = x^2 \sin(y)$

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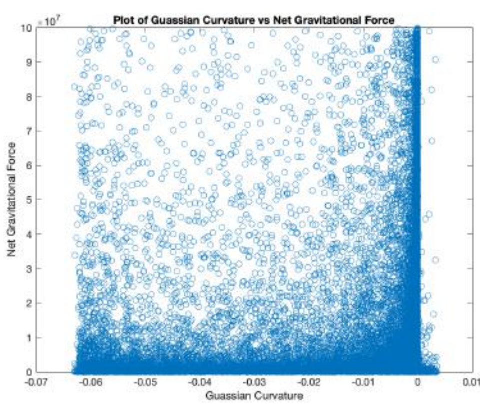
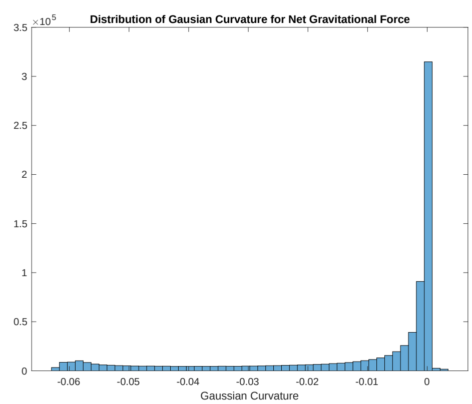
<tr><td>
</td><td>
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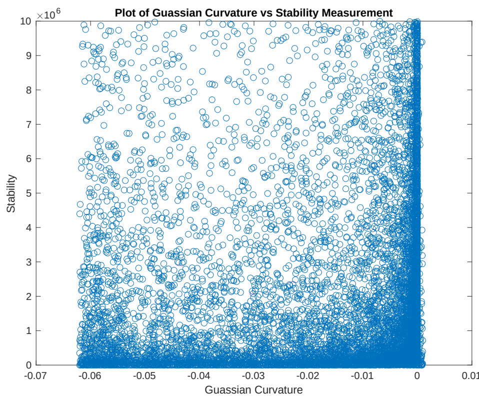
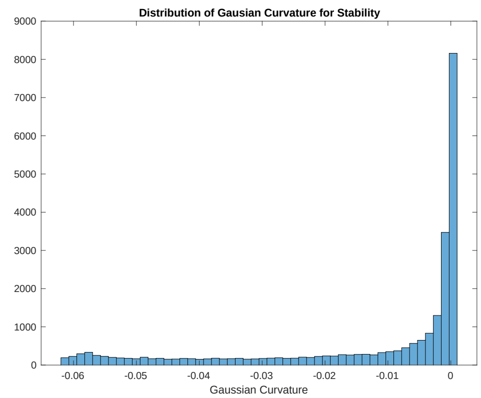
<tr><td>
</td><td>
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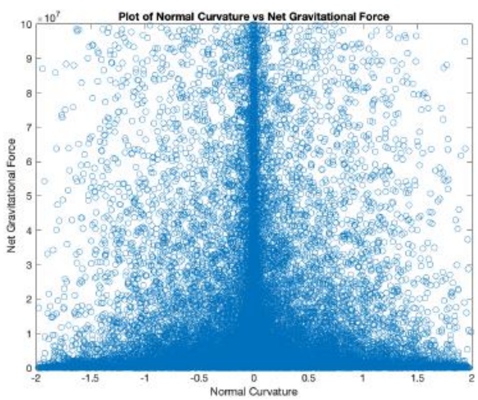
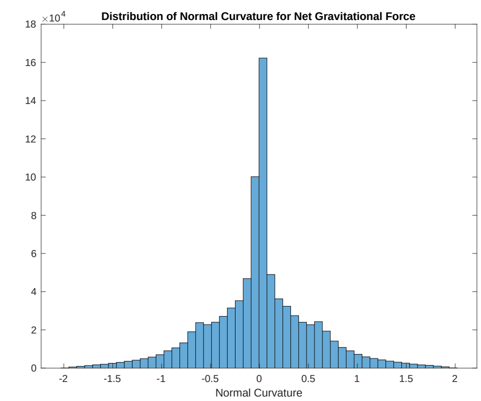
<tr><td>
</td><td>
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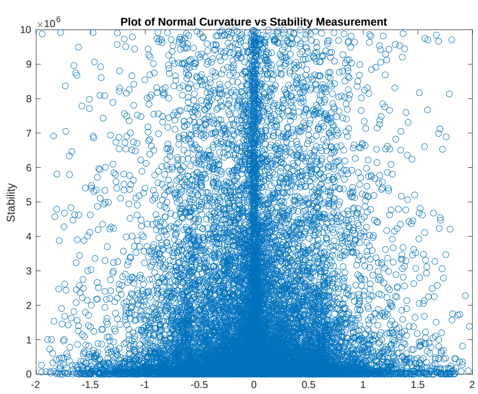
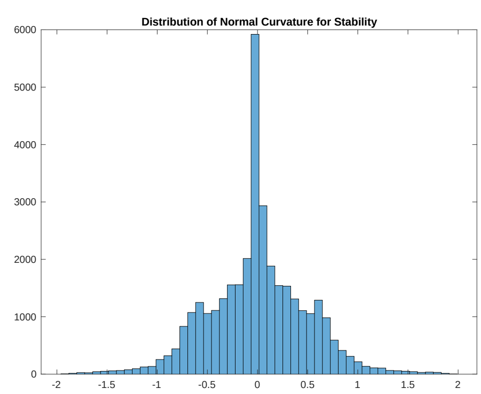
<tr><td>
</td><td>
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Table 6: Data for $f(x,y) = y^2\cos(x)$

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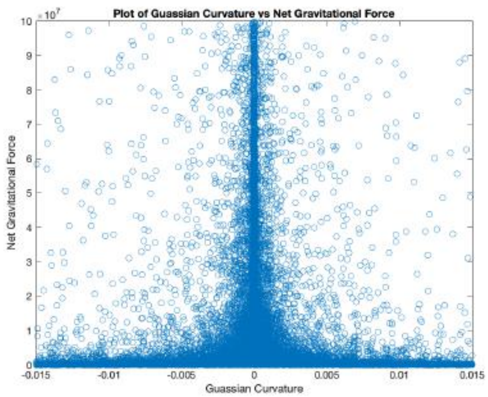
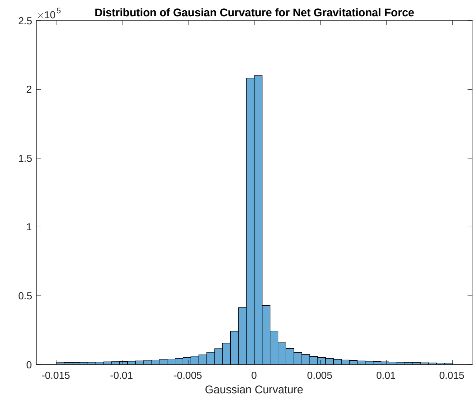
<tr><td>
</td><td>
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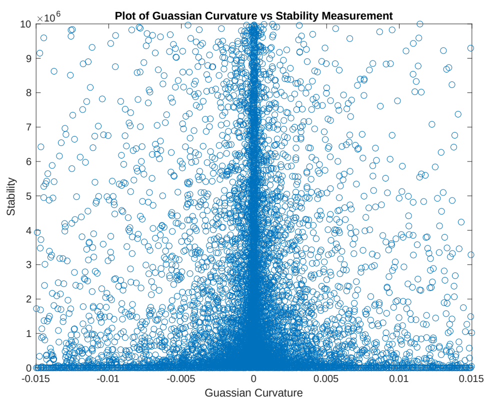
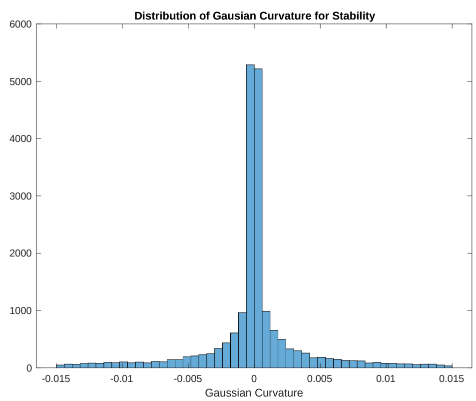
<tr><td>
</td><td>
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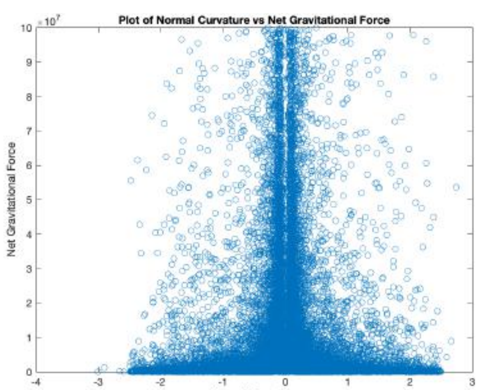
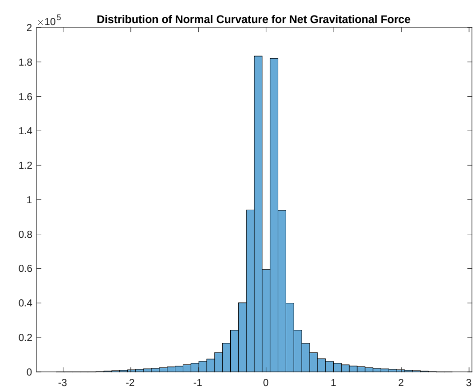
<tr><td>
</td><td>
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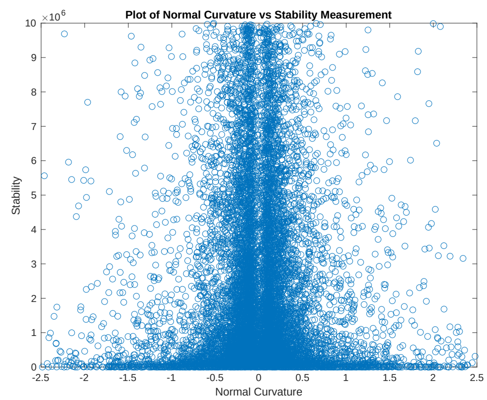
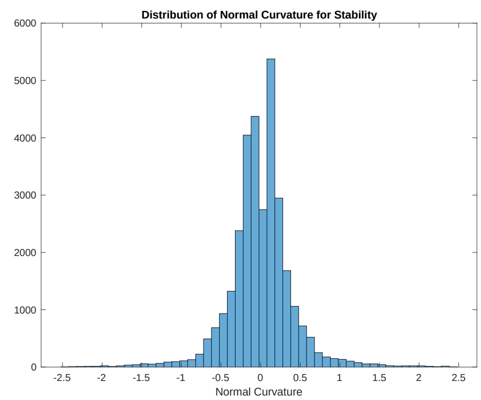
<tr><td>
</td><td>
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Table 7: Data for $f(x, y) = \cos(x)\sin(y)$

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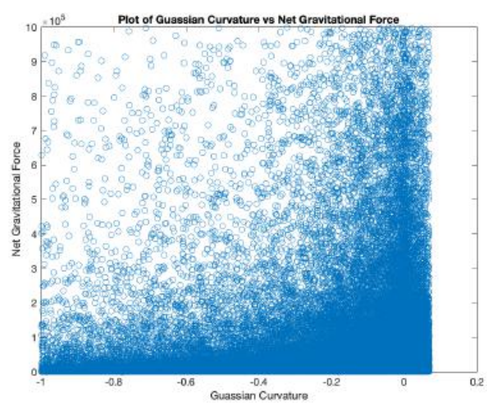
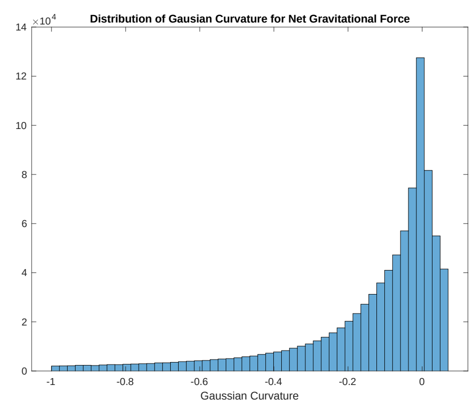
<tr><td>
</td><td>
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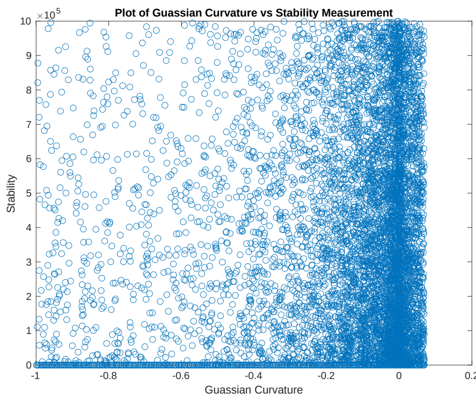
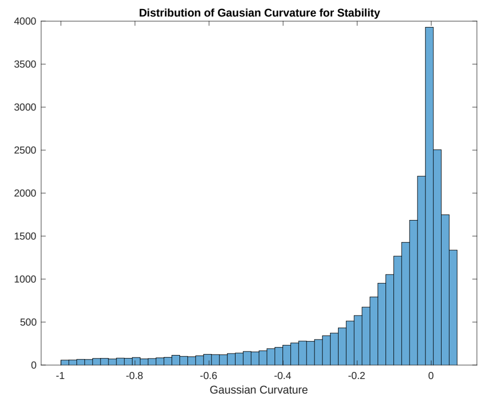
<tr><td>
</td><td>
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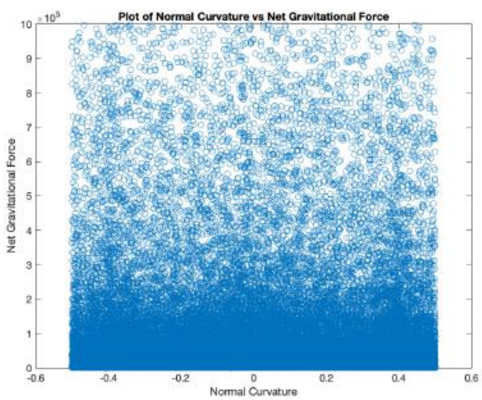
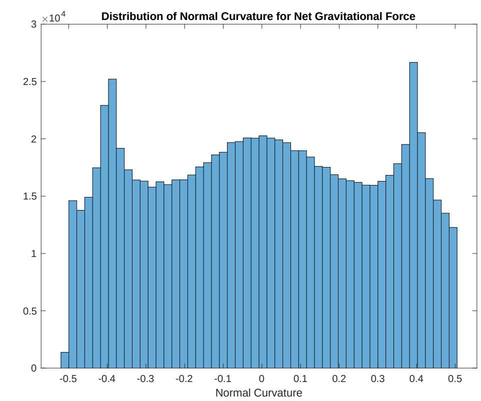
<tr><td>
</td><td>
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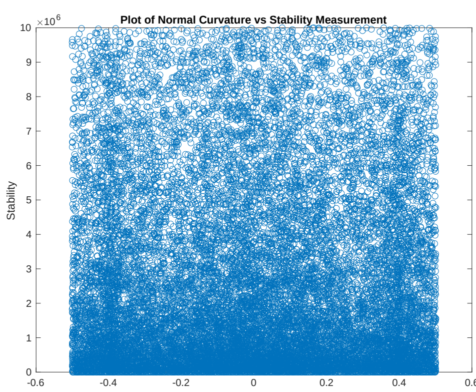
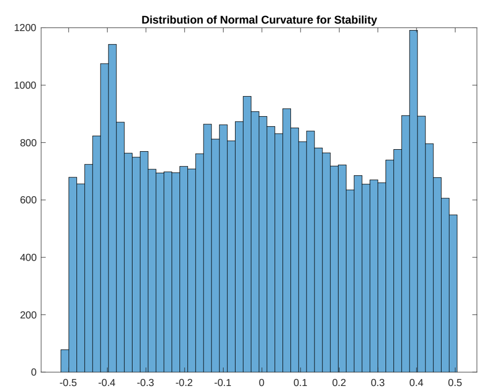
<tr><td>
</td><td>
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Table 8: Data for $f(x,y) = \exp(-\sin(x))\cos(y) + \exp(-\cos(y))\sin(x)$

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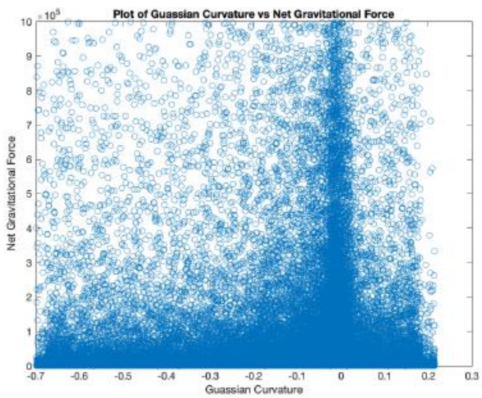
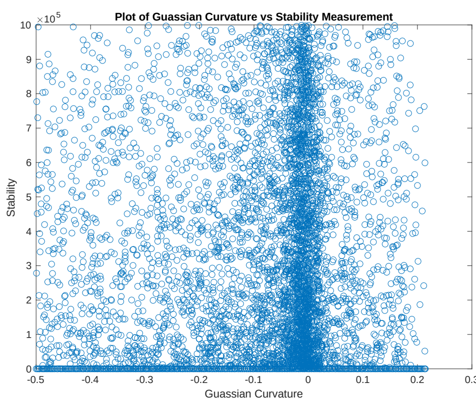
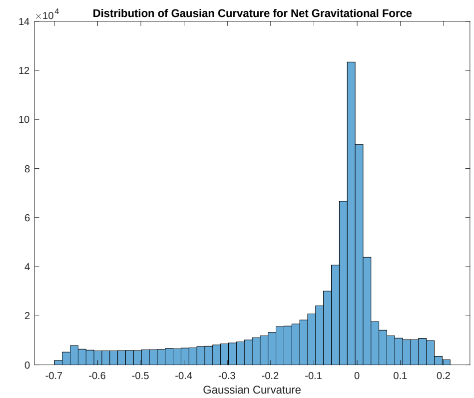
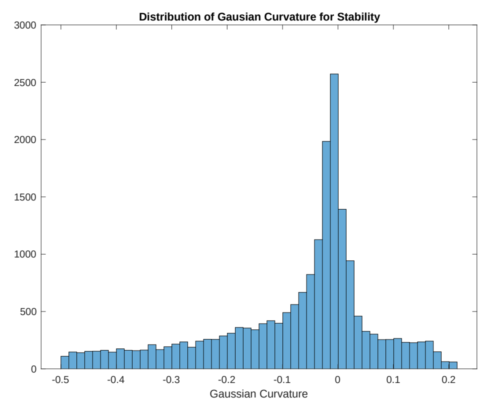
<tr><td>


</td><td>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<table>
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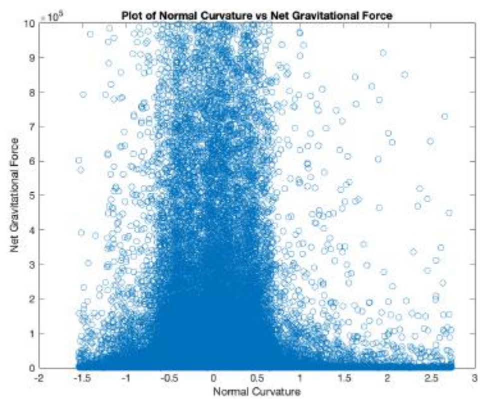
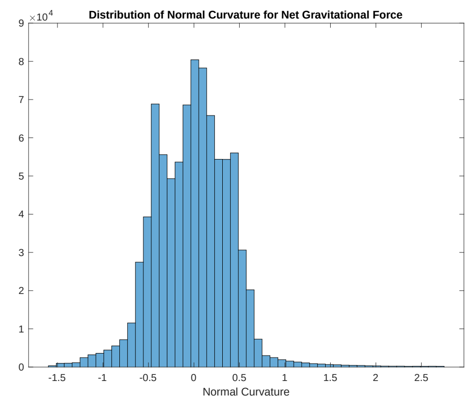
<tr><td>
</td><td>
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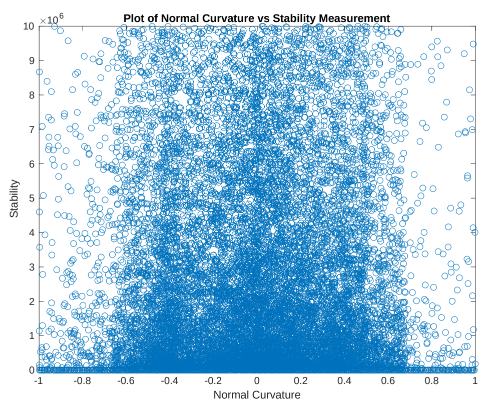
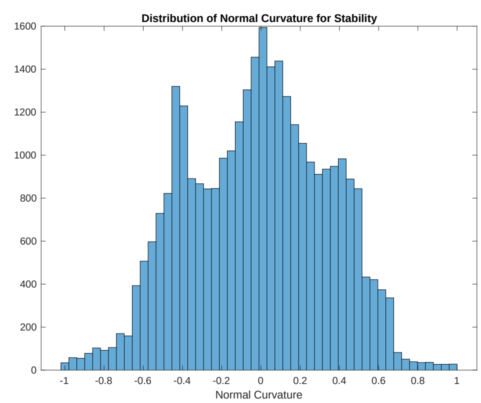
<tr><td>
</td><td>
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5 Conclusions

The results we gathered suggest a strong pattern concerning Gaussian curvature and the formation and stability of orbits. The histograms appear skewed because the surfaces have all positive or negative Gaussian curvature. The distribution of Gaussian curvature for bodies that enter orbits tends to center at a value of 0, with only minor variance. The distribution of Gaussian curvature for bodies that enter stable orbits also seems to be sharply centered at a value of 0. Additionally, the behavior of the normal curvature for bodies in rotation and bodies in stable orbits exhibit similar behavior. In other words, the distribution of normal curvature for bodies that form orbits and for bodies that form stable orbits center at a value of 0 with minor variance. These results indicate that flat geometries are the most conducive to the formation of orbits and the formation of stable orbits.

There are numerous additional questions to consider for future simulations. Firstly, will we see similar results in higher dimensions? In other words, if we were to repeat this simulation on 3-Dimensional or 4-Dimensional manifolds of varying curvature, would we expect to see similar results. If we were to consider the stability of orbits over extended periods or incorporate relativistic effects into our simulation, would we see significant differences? Additionally, the distribution of mass that we used in this simulation may have affected the results. Would a different distribution yield different effects from the curvature of our geometry?

References

[1] Kristopher Tapp. *Differential Geometry of Curves and Surfaces*

[2] Stephen T. Thornton, Jerry B. Marion. *Classical Dynamics of Particles and Systems*

[3] Sverre J. Aarseth. *Gravitational N-Body Simulations: Tools and Algorithms*