$Beckmann$ 课题结课报告

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1 Beckmann 随机变量定义，PDF、CDF 推导

1.1 笛卡尔坐标

Beckmann 随机变量是独立的二元高斯随机向量的范数 [1, eq.8]

$$R=\sqrt{X^2+Y^2}, \tag{A}$$

其中 X 和 Y 是独立的高斯过程，且 $X\sim\left(\mu_X,\sigma_X^2\right),Y\sim\left(\mu_Y,\sigma_Y^2\right)$。由此可以直接写出 Beckmann 随机变量的联合概率密度函数 (PDF)(B-1).

$$p(x,y)=\frac{1}{2\pi\sigma_X\sigma_Y}\exp\left[-\frac{\left(x-\mu_X\right)^2}{2\sigma_X^2}-\frac{\left(y-\mu_Y\right)^2}{2\sigma_Y^2}\right] \tag{B-1}$$

对 (B-1) 进行二重积分，可以得到 Beckmann 随机变量的积累分布函数 (CDF)

$$\begin{aligned}F_R&=\Pr\left(\sqrt{X^2+Y^2}\le r\right)\\&=\iint\limits_{\sqrt{x^2+y^2}\le r} f_X\left(x\right)f_Y\left(y\right)dxdy\\&=\frac{1}{2\pi\sigma_X\sigma_Y}\iint\limits_{\sqrt{x^2+y^2}\le r}\exp\left[-\frac{\left(x-\mu_X\right)^2}{2\sigma_X^2}-\frac{\left(y-\mu_Y\right)^2}{2\sigma_Y^2}\right]dxdy\end{aligned} \tag{D-1}$$

1.2 极坐标

对 (B1) 进行换元。二元随机变量的变换有以下规律 [2, eq.6-104]，$p(x,y)=\left|J\left(z,w\right)\right|p(z,w)$。进行以下换元，$X=r\cos\theta,Y=r\sin\theta$，则其中的 $\left|J\left(r,\theta\right)\right|=\begin{vmatrix}\frac{\partial g}{\partial z}&\frac{\partial g}{\partial w}\\\frac{\partial h}{\partial z}&\frac{\partial h}{\partial w}\end{vmatrix}$，[2, eq.6-114]，$x=g\left(z,w\right)$，$y=h\left(z,w\right)$。可以得到

$$\begin{aligned}p(r,\theta)&=\frac{\left|J\left(r,\theta\right)\right|}{2\pi\sigma_X\sigma_Y}\exp\left[-\frac{\left(r\cos\theta-\mu_X\right)^2}{2\sigma_X^2}-\frac{\left(r\sin\theta-\mu_Y\right)^2}{2\sigma_Y^2}\right]\\&=\frac{r}{2\pi\sigma_X\sigma_Y}\exp\left[-\frac{\left(r\cos\theta-\mu_X\right)^2}{2\sigma_X^2}-\frac{\left(r\sin\theta-\mu_Y\right)^2}{2\sigma_Y^2}\right]\end{aligned} \tag{B-2}$$

若要得到只与 r 相关的表达式，则对 θ 进行积分。

$$p\left(r\right)=\frac{r}{2\pi\sigma_X\sigma_Y}\times\int_0^{2\pi}\exp\left[-\frac{\left(r\cos\left(\theta\right)-\mu_X\right)^2}{2\sigma_X^2}-\frac{\left(r\sin\left(\theta\right)-\mu_Y\right)^2}{2\sigma_Y^2}\right]d\theta \tag{C-1}$$

上述表达式 (C-1) 与 [3, eq.2] 一致。(C-1) 经过化简，可以写作

$$
\begin{aligned}
p\left(r\right) &= \frac{r\exp\left(-\frac{{\mu_X}^2}{2\sigma_X^2}-\frac{{\mu_Y}^2}{2\sigma_Y^2}\right)}{2\pi\sigma_X\sigma_Y}\int_0^{2\pi}\exp\left[-\frac{\left(r\cos\left(\theta\right)\right)^2-2r\mu_X\cos\theta}{2\sigma_X^2}-\frac{\left(r\sin\left(\theta\right)\right)^2-2r\mu_y\sin\theta}{2\sigma_Y^2}\right]d\theta \\
&= \frac{r\exp\left(-\frac{{\mu_X}^2}{2\sigma_X^2}-\frac{{\mu_Y}^2}{2\sigma_Y^2}\right)}{2\pi\sigma_X\sigma_Y}\int_0^{2\pi}\exp\left[-r^2\left(\frac{\left(\cos\left(\theta\right)\right)^2}{2\sigma_X^2}+\frac{\left(\sin\left(\theta\right)\right)^2}{2\sigma_Y^2}\right)+r\left(\frac{\mu_X\cos\theta}{\sigma_X^2}+\frac{\mu_y\sin\theta}{\sigma_Y^2}\right)\right]d\theta
\end{aligned}
\tag{1}
$$

由于 (1) 较为复杂，进行以下化简。令 $A=\sqrt{{\mu_X}^2+{\mu_Y}^2}$，$\tan\theta_0=\frac{\mu_Y}{\mu_X}$ 则可以得到 $\cos\theta_0=\frac{\mu_X}{\sqrt{{\mu_X}^2+{\mu_Y}^2}}$，$\sin\theta_0=\frac{\mu_Y}{\sqrt{{\mu_X}^2+{\mu_Y}^2}}$，$\theta_0=\arctan\left(\frac{\mu_Y}{\mu_X}\right)$ 可得到

$$
\begin{aligned}
p\left(r\right) = &\frac{r\exp\left(-A^2\frac{\left(\cos\left(\theta_0\right)\right)^2}{2\sigma_X^2}-A^2\frac{\left(\sin\left(\theta_0\right)\right)^2}{2\sigma_Y^2}\right)}{2\pi\sigma_X\sigma_Y} \\
&\times\int_0^{2\pi}\exp\left[-r^2\left(\frac{\left(\cos\left(\theta\right)\right)^2}{2\sigma_X^2}+\frac{\left(\sin\left(\theta\right)\right)^2}{2\sigma_Y^2}\right)+A\frac{\cos\theta_0\cos\theta}{\sigma_X^2}r+A\frac{\sin\theta_0\sin\theta}{\sigma_Y^2}r\right]d\theta
\end{aligned}
\tag{2}
$$

再令 $\gamma\left(\theta\right)=\frac{\cos^2\theta}{2\sigma_X^2}+\frac{\sin^2\theta}{2\sigma_Y^2}$，$\rho\left(\theta\right)=A\left(\frac{\cos\theta\cos\theta_0}{\sigma_1^2}+\frac{\sin\theta\sin\theta_0}{\sigma_2^2}\right)$

$$
p\left(r\right)=\frac{r\exp\left(-A^2\gamma\left(\theta_0\right)\right)}{2\pi\sigma_X\sigma_Y}\times\int_0^{2\pi}\exp\left[-r^2\left(\gamma\left(\theta\right)\right)+r\rho\left(\theta\right)\right]d\theta \tag{C-2}
$$

上述表达式和论文 [4, eq.4] 一致。由可以得到 CDF 的表达式

$$
\begin{aligned}
F\left(r\right) &= \int_0^r p\left(z\right)dz \\
&= \frac{\exp\left(-A^2\gamma\left(\theta_0\right)\right)}{2\pi\sigma_X\sigma_Y}\times\int_0^{2\pi}\int_0^r z\exp\left[-z^2\left(\gamma\left(\theta\right)\right)+z\rho\left(\theta\right)\right]dzd\theta
\end{aligned}
\tag{3}
$$

根据已有表达式 [5, eq.2.33.1]

$$
\int\exp\left[-\left(ax^2+bx+c\right)\right]\mathrm{dx}=\frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2-ac}{a}\right)\mathrm{erf}\left(\sqrt{a}x+\frac{b}{\sqrt{a}}\right) \tag{4}
$$

我们根据 (3)，(4) 很容易联想到分部积分法 (5)。

$$
\int u\left(x\right)v'\left(x\right)dx=u\left(x\right)v\left(x\right)-\int u'\left(x\right)v\left(x\right)dx \tag{5}
$$

设 $v(z)=\exp\left[-z^2(\gamma(\theta))+\rho(\theta)\right]$ ，$g(r)=\int_0^r z\exp\left[-z^2(\gamma(\theta))+\rho(\theta)\right]dz$。则可以得到

$$\begin{aligned}
g(r) &= \int_0^r z\exp\left[-z^2(\gamma(\theta))+\rho(\theta)z\right]dz \\
&= \int_0^r z\left[\frac{1}{2}\sqrt{\frac{\pi}{(\gamma(\theta))}}\exp\left(\frac{\rho^2(\theta)}{\gamma(\theta)}\right)\operatorname{erf}\left(\sqrt{\gamma(\theta)}z-\frac{\rho(\theta)}{\sqrt{\gamma(\theta)}}\right)\right]'dz \\
&= \left[\frac{1}{2}z\sqrt{\frac{\pi}{(\gamma(\theta))}}\exp\left(\frac{\rho^2(\theta)}{\gamma(\theta)}\right)\operatorname{erf}\left(\sqrt{\gamma(\theta)}z-\frac{\rho(\theta)}{\sqrt{\gamma(\theta)}}\right)\right]\Bigg|_0^r \\
&-\frac{1}{2}\sqrt{\frac{\pi}{(\gamma(\theta))}}\exp\left(\frac{\rho^2(\theta)}{\gamma(\theta)}\right)\int_0^r\operatorname{erf}\left(\sqrt{\gamma(\theta)}z-\frac{\rho(\theta)}{\sqrt{\gamma(\theta)}}\right)dz \\
&= \left[\frac{1}{2}\mathrm{r}\sqrt{\frac{\pi}{(\gamma(\theta))}}\exp\left(\frac{\rho^2(\theta)}{\gamma(\theta)}\right)\operatorname{erf}\left(\sqrt{\gamma(\theta)}\mathrm{r}-\frac{\rho(\theta)}{\sqrt{\gamma(\theta)}}\right)\right] \\
&-\frac{1}{2}\sqrt{\frac{\pi}{(\gamma(\theta))}}\exp\left(\frac{\rho^2(\theta)}{\gamma(\theta)}\right)\int_0^r\operatorname{erf}\left(\sqrt{\gamma(\theta)}z-\frac{\rho(\theta)}{\sqrt{\gamma(\theta)}}\right)dz
\end{aligned} \tag{6}$$

根据 Maple 求导，我们可以的得到以下结论

$$\begin{aligned}
F(r) = \frac{1}{2\pi\sigma_X\sigma_Y}\exp\left(-A^2\gamma(\theta_0)\right)\times\int_0^{2\pi}\frac{1}{2\gamma(\theta)}\left(1-\exp\left(-\gamma(\theta)r^2+\rho(\theta)r\right)\right) \\
+\frac{\rho(\theta)\sqrt{\pi}}{4\gamma^{\frac{3}{2}}(\theta)}\exp\left(\frac{\rho^2(\theta)}{4\gamma(\theta)}\right)\times\left(\operatorname{erf}\left[\frac{\rho(\theta)}{2\sqrt{\gamma(\theta)}}\right]+\operatorname{erf}\left[\frac{2\gamma(\theta)r-\rho(\theta)}{2\sqrt{\gamma(\theta)}}\right]\right)d\theta
\end{aligned} \tag{D-2}$$

与 [4, eq.8] 一致。

2 Beckmann 随机变量上下界推导

2.1 笛卡尔坐标中的下界

Beckmann 随机变量的积累分布函数 (CDF)(D-1) 如下

$$\begin{aligned} F_R(r) &= \Pr\left(\sqrt{X^2+Y^2} \le r\right) \\ &= \iint\limits_{\sqrt{x^2+y^2}\le r} f_X(x) f_Y(y)\,dxdy \\ &= \frac{1}{2\pi\sigma_X\sigma_Y} \iint\limits_{\sqrt{x^2+y^2}\le r} \exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2} - \frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right] dxdy \end{aligned} \tag{D-1}$$

2.1.1 下界 1

记 (D-1) 中的第一次积分为 $i(r,x)$

$$\begin{aligned} i(r,x) &= \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2} - \frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right] dy \\ &= \int_{-\sqrt{r^2-x^2}}^{0} \exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2} - \frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right] dy \\ &+ \int_{0}^{\sqrt{r^2-x^2}} \exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2} - \frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right] dy \end{aligned} \tag{7}$$

Jensen's Inequality：$f(\theta)$ 和 $g(\theta)$，θ 的范围 $a \le \theta \le b$，得 $\alpha \le f(\theta) \le \beta$ 和 $g(\theta) \ge 0$，且 $g(\theta) \ne 0$。设 $\phi(\mu)$ 是区间在 $\alpha \le \mu \le \beta$ 的凸函数，据此可得到

$$\phi\left(\frac{\int_a^b f(\theta)\,g(\theta)\,d\theta}{\int_a^b g(\theta)\,d\theta}\right) \le \frac{\int_a^b \phi[f(\theta)]g(\theta)\,d\theta}{\int_a^b g(\theta)\,d\theta}. \tag{8}$$

令

$$\begin{aligned} \phi(u) &= \exp(u), \\ f(\theta) &= -\frac{(x-\mu_X)^2}{2\sigma_X^2} - \frac{(y-\mu_Y)^2}{2\sigma_Y^2}, \\ g(\theta) &= 1, \\ a &= 0, \\ b &= 2\pi. \end{aligned} \tag{9}$$

可以得到如下关系

$$i(r,x) \ge i_1(r,x) = 2\sqrt{r^2-x^2}\exp\left(\frac{\int\limits_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} -\frac{(x-\mu_X)^2}{2\sigma_X^2} - \frac{(y-\mu_Y)^2}{2\sigma_Y^2} dy}{2\sqrt{r^2-x^2}}\right) \tag{10}$$

将 $i_1(r,x)$ 替代 $i(r,x)$，可得到如下关系

$$F_R(r) \ge F_{LB1-1}(r) = \int_{-r}^{r} \frac{\sqrt{r^2-x^2}}{\pi\sigma_X\sigma_Y} \exp\left(\frac{h_1(x)}{2\sqrt{r^2-x^2}}\right) dx \tag{11}$$

其中

$$h_1(x,r) = -\frac{\left(r^2-x^2\right)^{3/2}}{3{\sigma_y}^2} - \frac{{\mu_y}^2\sqrt{r^2-x^2}}{{\sigma_y}^2} - \frac{(x-\mu_x)^2\sqrt{r^2-x^2}}{{\sigma_x}^2} \tag{12}$$

同理 $i(r,x)$ 也可以写做

$$\begin{aligned} i(r,x) = &\int_{-\sqrt{r^2-x^2}}^{0} \exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2} - \frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right] dy \\ &+ \int_0^{\sqrt{r^2-x^2}} \exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2} - \frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right] dy \end{aligned} \tag{13}$$

可以得到

$$F_R(r) \geq F_{LB1-2}(r) = \int_{-r}^{r} \frac{\sqrt{r^2-x^2}}{2\pi\sigma_X\sigma_Y} \exp\left(\frac{h_2(x)}{\sqrt{r^2-x^2}}\right) dx + \int_{-r}^{r} \frac{\sqrt{r^2-x^2}}{2\pi\sigma_X\sigma_Y} \exp\left(\frac{h_3(x)}{\sqrt{r^2-x^2}}\right) dx \tag{14}$$

其中

$$\begin{aligned} h_2(x) &= -\frac{\left(r^2-x^2\right)^{3/2}}{6{\sigma_y}^2} - \frac{{\mu_y}^2\sqrt{r^2-x^2}}{2{\sigma_y}^2} - \frac{(x-\mu_x)^2\sqrt{r^2-x^2}}{2{\sigma_x}^2} + \frac{\mu_y\left(-r^2+x^2\right)}{2{\sigma_{\mathrm{y}}}^2} \\ h_3(x) &= -\frac{\left(r^2-x^2\right)^{3/2}}{6{\sigma_y}^2} - \frac{{\mu_y}^2\sqrt{r^2-x^2}}{2{\sigma_y}^2} - \frac{(x-\mu_x)^2\sqrt{r^2-x^2}}{2{\sigma_x}^2} - \frac{\mu_y\left(-r^2+x^2\right)}{2{\sigma_{\mathrm{y}}}^2} \end{aligned} \tag{15}$$

根据不同不同的均值和方差，我们可以画出不同程度贴合原函数的下界

2.1.2 下界 2

切线不等式可以写做 (16)

$$v \geq c\left(\ln v - \ln c + 1\right), c > 0 \tag{16}$$

令 $v = \exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2} - \frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right]$，将 (16) 带入 (D-1)，可以得到

$$\begin{aligned} F_R(r) \geq F_{LB2}(x) &= \frac{c_1}{2\pi\sigma_X\sigma_Y} dydx \\ &= \frac{c_1}{2\pi\sigma_X\sigma_Y}\int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \left[-\frac{(x-\mu_X)^2}{2\sigma_X^2} - \frac{(y-\mu_Y)^2}{2\sigma_Y^2} - \ln c_1 + 1\right] dydx \\ &= \frac{c_1}{2\pi\sigma_X\sigma_Y}\int_{-r}^{r} -\frac{\left(r^2-x^2\right)^{3/2}}{3{\sigma_Y}^2} - \frac{{\mu_Y}^2\sqrt{r^2-x^2}}{{\sigma_Y}^2} \\ &- \frac{(x-\mu_X)^2\sqrt{r^2-x^2}}{{\sigma_X}^2} - 2\ln c_1\sqrt{r^2-x^2} + 2\sqrt{r^2-x^2}dx \end{aligned} \tag{17}$$

2.2 极坐标中的下界

Beckmann 随机变量的 PDF(C-1) 的如下

$$p(r) = \frac{r}{2\pi\sigma_X\sigma_Y} \times \int_0^{2\pi} \exp\left[-\frac{(r\cos(\theta)-\mu_X)^2}{2\sigma_X^2} - \frac{(r\sin(\theta)-\mu_Y)^2}{2\sigma_Y^2}\right] d\theta \tag{C-1}$$

Beckmann 随机变量的 CDF(D-2) 的如下

$$F_R(r)=\frac{1}{2\pi\sigma_X\sigma_Y}\exp\left(-A^2\gamma\left(\theta_0\right)\right)\times\int\limits_0^{2\pi}\frac{1}{2\gamma\left(\theta\right)}\left(1-\exp\left(-\gamma\left(\theta\right)r^2+\rho\left(\theta\right)r\right)\right)$$
$$+\frac{\rho\left(\theta\right)\sqrt{\pi}}{4\gamma^{\frac{3}{2}}\left(\theta\right)}\exp\left(\frac{\rho^2\left(\theta\right)}{4\gamma\left(\theta\right)}\right)\times\left(\operatorname{erf}\left[\frac{\rho\left(\theta\right)}{2\sqrt{\gamma\left(\theta\right)}}\right]+\operatorname{erf}\left[\frac{2\gamma\left(\theta\right)r-\rho\left(\theta\right)}{2\sqrt{\gamma\left(\theta\right)}}\right]\right)d\theta \tag{D-2}$$

2.2.1 下界 3

对 (C-1) 使用 Jensen 不等式，令

$$\begin{aligned}\phi\left(u\right)&=\exp\left(u\right),\\ f\left(\theta\right)&=-\frac{\left(r\cos\left(\theta\right)-\mu_X\right)^2}{2\sigma_X^2}-\frac{\left(r\sin\left(\theta\right)-\mu_Y\right)^2}{2\sigma_Y^2},\\ g\left(\theta\right)&=1,\\ a&=0,\\ b&=2\pi.\end{aligned} \tag{18}$$

通过 $Jensen$ 不等式可以得到以下不等式.

$$\begin{aligned}p_R\left(r\right)\geq p_{LB-1}\left(r\right)&=\frac{r}{\sigma_X\sigma_Y}\exp\left(\frac{1}{2\pi}\int_0^{2\pi}-\frac{\left(r\cos\left(\theta\right)-\mu_X\right)^2}{2\sigma_X^2}-\frac{\left(r\sin\left(\theta\right)-\mu_Y\right)^2}{2\sigma_Y^2}d\theta\right)\\&=\frac{r}{\sigma_X\sigma_Y}\exp\left(-\frac{r^2{\sigma_X}^2+r^2{\sigma_Y}^2+2{\mu_Y}^2{\sigma_X}^2+2{\mu_X}^2{\sigma_Y}^2}{4{\sigma_X}^2{\sigma_Y}^2}\right)\end{aligned} \tag{19}$$

若对 (19) 进行积分，可以得到新的下界函数

$$\begin{aligned}F_R\left(r\right)\geq F_{LB3-1}\left(r\right)=&\frac{2\sigma_X\sigma_Y}{{\sigma_X}^2+{\sigma_Y}^2}\exp\left(-\frac{2{\mu_Y}^2{\sigma_X}^2+2{\mu_X}^2{\sigma_Y}^2}{4{\sigma_X}^2{\sigma_Y}^2}\right)\\&-\frac{2\sigma_X\sigma_Y}{{\sigma_X}^2+{\sigma_Y}^2}\exp\left(-\frac{r^2{\sigma_X}^2+r^2{\sigma_Y}^2+2{\mu_Y}^2{\sigma_X}^2+2{\mu_X}^2{\sigma_Y}^2}{4{\sigma_X}^2{\sigma_Y}^2}\right)\end{aligned} \tag{20}$$

PDF 同样也可以写做

$$\begin{aligned}p_R\left(r\right)=&\frac{r}{2\pi\sigma_X\sigma_Y}\times\int_0^{\pi}\exp\left[-\frac{\left(r\cos\left(\theta\right)-\mu_X\right)^2}{2\sigma_X^2}-\frac{\left(r\sin\left(\theta\right)-\mu_Y\right)^2}{2\sigma_Y^2}\right]d\theta\\&+\frac{r}{2\pi\sigma_X\sigma_Y}\times\int_{\pi}^{2\pi}\exp\left[-\frac{\left(r\cos\left(\theta\right)-\mu_X\right)^2}{2\sigma_X^2}-\frac{\left(r\sin\left(\theta\right)-\mu_Y\right)^2}{2\sigma_Y^2}\right]d\theta\end{aligned} \tag{21}$$

用 $Jensen$ 可以得到

$$\begin{aligned}p_R\left(r\right)\geq p_{LB-2}\left(r\right)=&\frac{r}{\sigma_X\sigma_Y}\exp\left(-\frac{\pi r^2{\sigma_X}^2+\pi r^2{\sigma_Y}^2+2\pi{\mu_Y}^2{\sigma_X}^2+2\pi{\mu_X}^2{\sigma_Y}^2-8r\mu_Y{\sigma_X}^2}{8\pi{\sigma_X}^2{\sigma_Y}^2}\right)\\&+\frac{r}{\sigma_X\sigma_Y}\exp\left(-\frac{\pi r^2{\sigma_X}^2+\pi r^2{\sigma_Y}^2+2\pi{\mu_Y}^2{\sigma_X}^2+2\pi{\mu_X}^2{\sigma_Y}^2+8r\mu_Y{\sigma_X}^2}{8\pi{\sigma_X}^2{\sigma_Y}^2}\right)\end{aligned}t \tag{22}$$

同理可以推出 p_{LB-2} 的 CDF 表达式

$$F_{LB3-2}\left(r\right)=\int_0^r p_{LB-2}\left(x\right)dx$$

2.2.2 上界 4

在 (D-2) 中，若对误差函数使用 Jensen 不等式，则可以得到误差函数的下界

$$\mathrm{erf}_{\mathrm{LB}}=\frac{2}{\sqrt{\pi}}x\exp\left(-\frac{x^2}{3}\right) \tag{23}$$

若将此函数带到原函数 (D-2) 上，我们在图像上得到一个新的上界

$$F_{UB}(r)=\frac{1}{2\pi\sigma_X\sigma_Y}\exp\left(-A^2\gamma\left(\theta_0\right)\right)\times\int_0^{2\pi}\frac{1}{2\gamma\left(\theta\right)}\left(1-\exp\left(-\gamma\left(\theta\right)r^2+\rho\left(\theta\right)r\right)\right)$$
$$+\frac{\rho\left(\theta\right)\sqrt{\pi}}{4\gamma^{\frac{3}{2}}\left(\theta\right)}\exp\left(\frac{\rho^2\left(\theta\right)}{4\gamma\left(\theta\right)}\right)\times\left(\mathrm{erf}_{\mathrm{LB}}\left[\frac{\rho\left(\theta\right)}{2\sqrt{\gamma\left(\theta\right)}}\right]+\mathrm{erf}_{\mathrm{LB}}\left[\frac{2\gamma\left(\theta\right)r-\rho\left(\theta\right)}{2\sqrt{\gamma\left(\theta\right)}}\right]\right)d\theta$$

待核对！

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A 下界 1

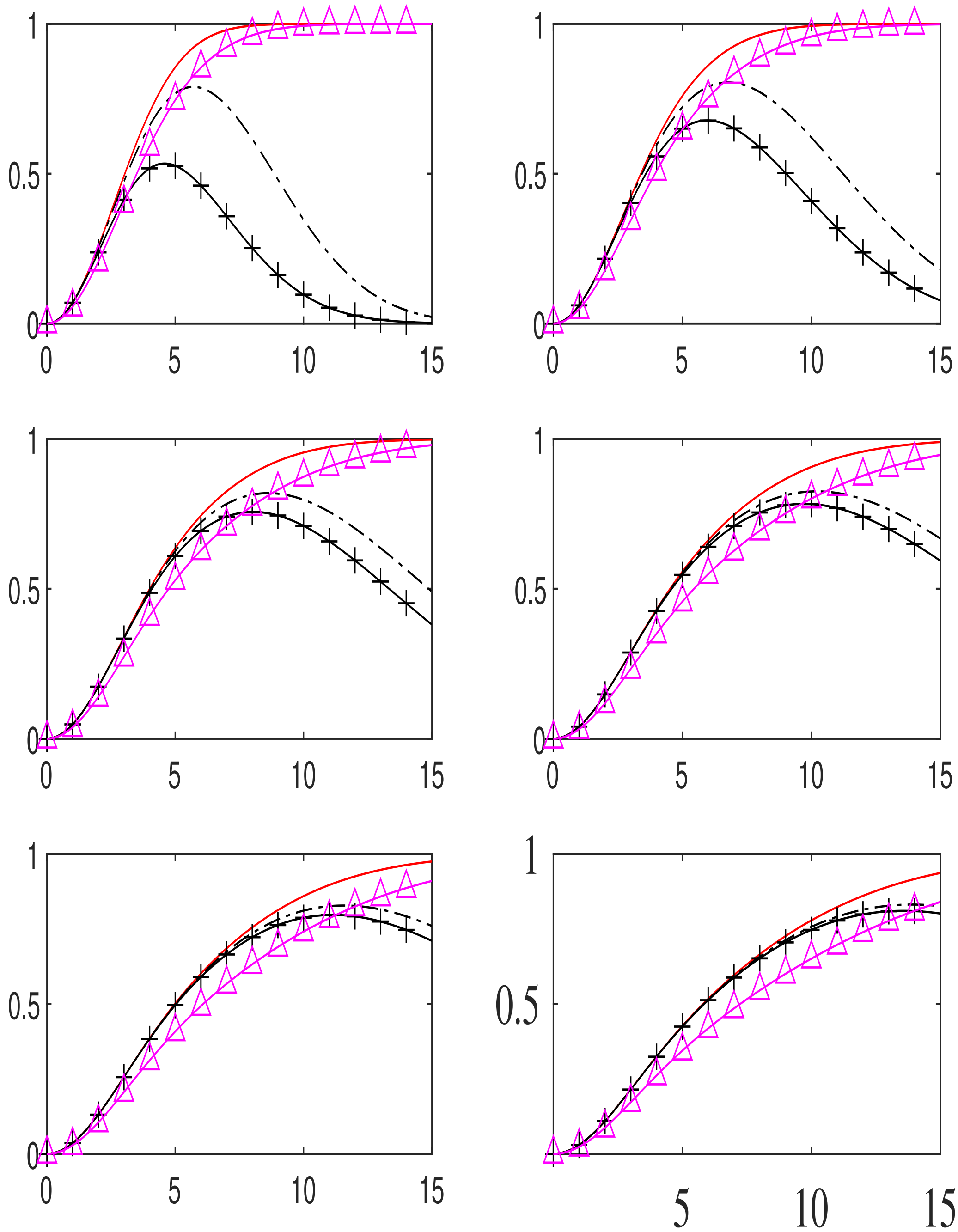
下界 1 在给定参数 σ_x 和 σ_y 的情况下，μ_x、μ_y 的值越大与原函数越不贴合。在多次循环尝试过程中，发现下界 1 的贴合程度主要由参数 σ_y 决定，σ_y 越大，下界离原函数越紧。其中参数 $\mu_x=1$、$\mu_y=2$、$\sigma_x=\sqrt{3}$、σ_y 为 $\sqrt{5}$、$\sqrt{10}$、$\sqrt{20}$、$\sqrt{30}$、$\sqrt{40}$、$\sqrt{60}$。参考图像.1

B 下界 2

在给定参数 σ_x 和 σ_y 的情况下，μ_x、μ_y 的值越大与原函数越不贴合。多次循环尝试过程中，发现下界 2 的贴合程度主要由参数 σ_y 和 σ_x 共同决定，$\sigma_y^2+\sigma_x^2$ 的数值越大，下界 2 越紧. 设 $\sigma_{xy}^2=\sigma_y^2+\sigma_x^2$，当 $\mu_x=0.2$、$\mu_y=0.1$、σ_{xy}^2 分别为 $\sqrt{5}$、$\sqrt{10}$、$\sqrt{20}$、$\sqrt{30}$、$\sqrt{40}$、$\sqrt{60}$。参考图像.2

C 下界 3

在给定参数 σ_x 和 σ_y 的情况下，μ_x、μ_y 的值越大与原函数越不贴合。在多次尝试下，发现 σ_x 和 σ_y 的值不是很大的情况下（其中一个小于 $\sqrt{15}$ 左右），两者数值越接近，就越紧。当 $\mu_x=0.2$、$\mu_y=0.1$、σ_x 分别为 $\sqrt{2}$、$\sqrt{2}$、$\sqrt{6}$、$\sqrt{6}$、$\sqrt{10}$、$\sqrt{10}$，分别为 $\sqrt{2}$、$\sqrt{3}$、$\sqrt{4}$、$\sqrt{5}$、$\sqrt{6}$、$\sqrt{7}$。



1
0.5
0
0
5
10
15
1
0.5
0
0
5
10
15
1
0.5
0
0
5
10
15
1
0.5
0
0
5
10
15
1
0.5
0
0
5
10
15
1
0.5
5
10
15

Figure 1: 下界 1

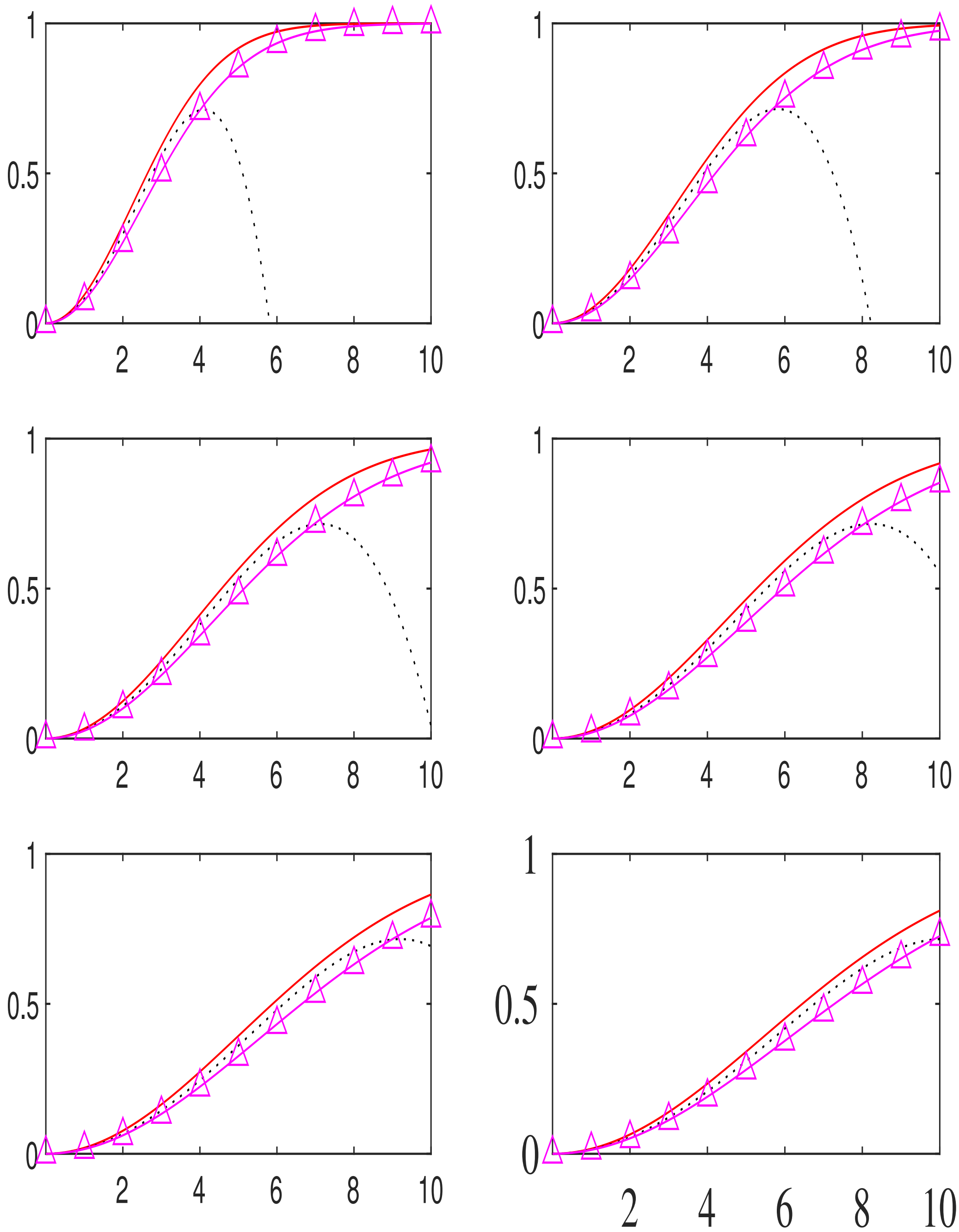


Figure 2: 下界 2

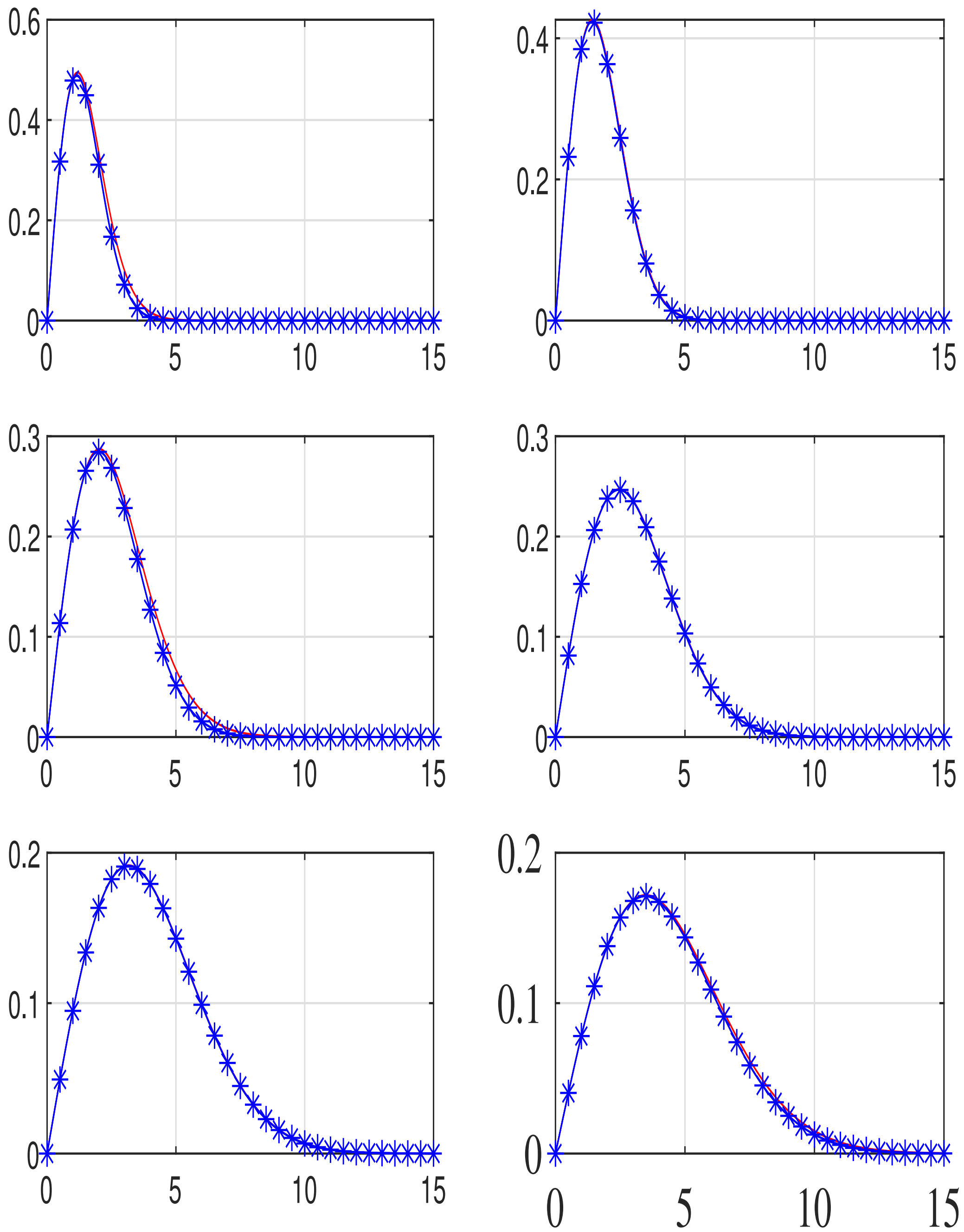

0.6
0.4
0.2
0
0
5
10
15
0.4
0.2
0
0
5
10
15
0.3
0.2
0.1
0
0
5
10
15
0.3
0.2
0.1
0
0
5
10
15
0.2
0.1
0
0
5
10
15
0.2
0.1
0
0
5
10
15

Figure 3: 下界 3