Application of Newton's Method to the Lower Bounds on the Gaussian Q-Function

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I. INTRODUCTION

The Gaussian Q-function defined by

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t \tag{1}$$

appears in most error probability. It is of great importance in performance analysis of communication systems. Due to the infinite range integral form of the definition, we can not apply it to performance analysis. Therefore, a family of the Gaussian Q-function lower bounds were discovered by Wu [2], [3], which are finite range integral. Many approximations and bounds in the literature on the Gaussian Q-function, e.g. [2], [3], are in the form of a product of an exponential function with a rational or irrational function, or a sum of such products. Due to their complexity, lower bounds [2], [3] are obviously not invertible. Because of the irreversibility of the function, we can only approximate the inverse function by approaching the approximate value of x with a fixed y value. We are now going to introduce a Newton's method, for the lower bounds on the Gaussian Q-function that cannot be inverted.

The rest of this paper is organized as follows. The performance of Newton's method on inversion is derived in Section II. The comparison of result in the different way is derived in Section III. Eestimating the inverse value of the Q-function via Newton's method is organized in Section IV. Summaries are made in Section V.

II. THE PERFORMANCE OF NEWTON'S METHOD ON INVERSION

In [2], Wu has already introduced a method for inversion of parametric lower bound [2, eq(5, 6)]

$$\begin{aligned} h\left(x\right) &= \sqrt{\frac{e}{\pi}} \frac{\sqrt{c}}{2c+1} \exp\left(-\frac{2c+1}{4c}x^2\right) \\ c &= \frac{\sqrt{x^4+6x^2+1}+x^2+1}{4}. \end{aligned} \tag{2}$$

However, this approach is limited to the properties of the function. It is merely applicable to parametric functions, and the functions are invertible in parametric form.

The lower bound [3, eq.(9)]

$$g\left(x\right) = \frac{1}{6}\exp\left(-\frac{2\sqrt{3}}{\pi}x^2\right) + \frac{1}{6}\exp\left(-\frac{\sqrt{3}}{\pi}x^2\right) \tag{3}$$

is not invertible in the same way, due to its complexity. The figures of lower bounds, [2, eqs. (5) and (6)] and [3, eq. (9)], on the Gaussian Q-function are shown in Fig. 1 and Fig. 2.

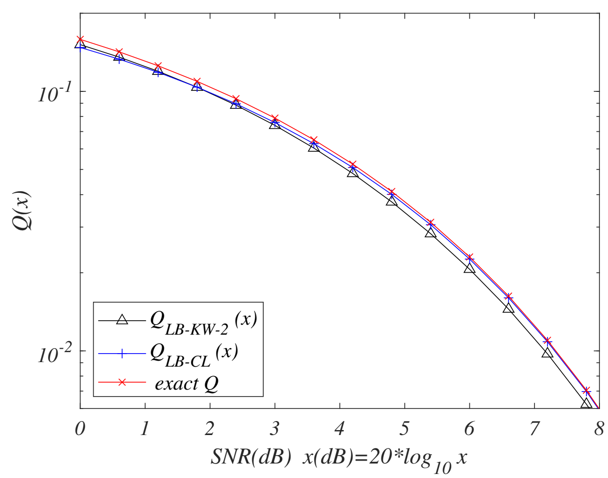

Fig. 1: Lower bounds for small argument values

Newton's method from [1]. In numerical analysis, Newton's method is a root-finding algorithm which produces successively better approximations to the roots (or zeroes) of a real-valued function. Find the root of a continuous, differentiable function $f(x)$, and you know the root you are looking for is near the point $x = x_a(0)$. Then Newton's method tells us that a better approximation for the root is

$$x_a(1) = x_a(0) - \frac{f\left(x_a(0)\right)}{f'\left(x_a(0)\right)} \tag{4}$$

This process may be repeated as many times as necessary to get the desired accuracy. In general, for any x-value $x_a(n)$, the next value is given by

$$x_a(n+1) = x_a(n) - \frac{f\left(x_a(n)\right)}{f'\left(x_a(n)\right)}. \tag{5}$$

The inversion problem can be stated precisely as follows. Given any certain y, we want to find the value x. The idea is to start with an initial x value brought into (6)

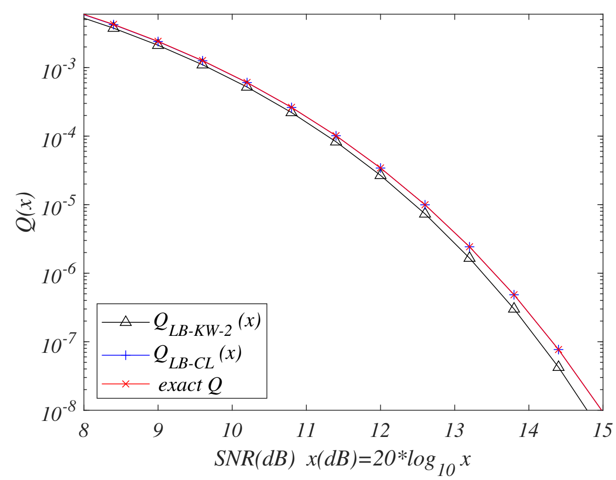

Fig. 2: Lower bounds for large argument values.

$$f(x) = \frac{1}{6}\exp\left(-\frac{2\sqrt{3}}{\pi}x^2\right) + \frac{1}{6}\exp\left(-\frac{\sqrt{3}}{\pi}x^2\right) - \mathrm{Q}_c \quad (6)$$

where Q_c is a constant calculated by (1). From (3), it is clear that for any given y such that $Q(x) = y$, the value of $x = Q^{-1}(y)$ is upper bounded by x obtained by Newton's method. We summarize the iterative algorithm for inverting the lower bound $g(x)$ in Algorithm 1.

Algorithm 1 Iterative Algorithm For Inverse Q-Function

Initialize with $x_a(0) = \sqrt{-\frac{\pi}{2}\ln 4y}$,
Find $x_a(1) = x_a(0) - \frac{f(x_a(0))}{f'(x_a(0))}$
do
then $x_a(n+1) = x_a(n) - \frac{f(x_a(n))}{f'(x_a(n))}$
while $x_a(n+1) - x_a(n) < \epsilon$
return $x_f = x_a(n+1)$

Firstly, we choose the initial guess $x_a(0)$ using the bound (7)[3, eqs. (5) and (6)] with $\theta = \pi/4$ for convenience.

$$Q_{LB-KW-1} = \left(\frac{1}{2} - \frac{\theta}{\pi}\right)\exp\left(-\frac{\cot\theta}{\pi - 2\theta}x^2\right) \quad (7)$$

Next, for each $i = 0, 1, 2, ...$, a better approximation for the root $x_a(n+1)$ is computed with (5) ,where $f'(x_a(n))$ is (8)

$$f'(x) = -\frac{2\sqrt{3}}{3\pi}x\exp\left(-\frac{2\sqrt{3}}{\pi}x^2\right) - \frac{\sqrt{3}}{3\pi}x\exp\left(-\frac{\sqrt{3}}{\pi}x^2\right). \quad (8)$$

The iteration terminates once the change in $x_a(i)$ falls below a specified threshold ϵ and gives the final value x_f. TABLE I shows the accuracy of the proposed algorithm with $\epsilon = 10^{-3}$ and $\epsilon = 10^{-16}$ for the values of $x = 1, 2, 3, ..., 8$ that we feed in. It can be seen from the above results that Newton's Method has a high precision in changing $x_a(i)$ and less iterations.

TABLE I: INVERSE Q-FUNCTION VALUES.

x	1	2	3	4
y=Q(x)	0.1587	2.2750E-02	1.3499E-03	3.1671E-05
$x_f(\epsilon = 10^{-3})$	0.9683	1.9546	2.9580	3.9423
iterations	3	2	3	4
$x_f(\epsilon = 10^{-16})$	0.9683	1.9546	2.9580	3.9423
iterations	5	5	6	7
x	5	6	7	8
y=Q(x)	2.8665E-07	9.8659E-10	1.2798E-12	6.2210E-16
$x_f(\epsilon = 10^{-3})$	4.9066	5.8619	6.8132	7.7626
iterations	5	6	6	7
$x_f(\epsilon = 10^{-16})$	4.9066	5.8619	6.8132	7.7626
iterations	7	8	10	10

III. COMPARISON OF DIFFERENT METHOD WITH A LOWER BOUND

We use different methods for the same function (2) to compare the superiority of the two methods. The comparison is based on comparing the number of iterations when the $x_a(i)$ in the same accuracy. We set the accuracy of different algorithm with $\epsilon = 10^{-12}$. The results are shown in Fig.3.

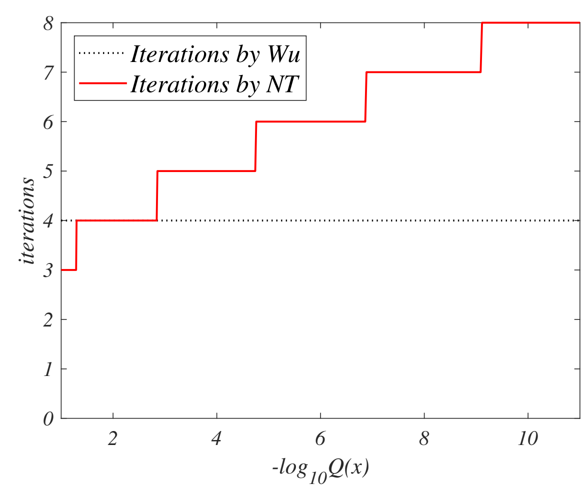

Fig. 3: Number of iterations for different method.

IV. ESTIMATE THE INVERSE VALUE OF THE Q-FUNCTION

In summary, Newtonian iterations have universality for continuous derivative functions. If the objective function is (1), use the above method, then we get (9) and (10).

$$f(x) = Q(x) - \mathrm{Q}_c \quad (9)$$

$$f'(x) = -\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \quad (10)$$

We can directly make an approximate estimate of the inverse function of the Q-function. The result is shown in Fig.4

V. CONCLUSION

To sum up, the basic requirement of this method for the lower bound function is derivable. The limitation is that the expression of x with respect to y can not be written.

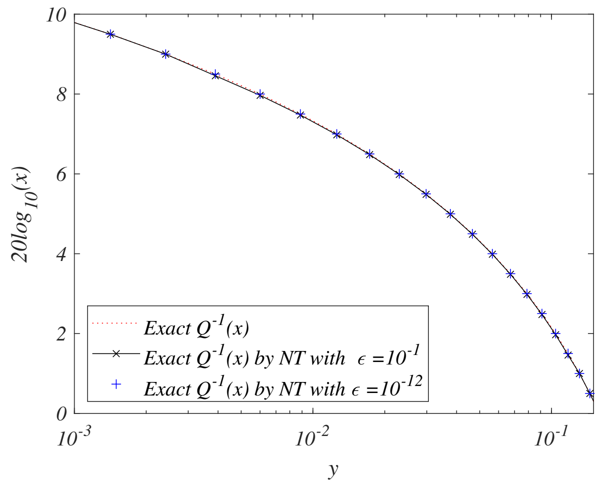

Fig. 4: Application of Newton's Method to the Gaussian Q-Function.

References

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