New upper bounds on the Gaussian Q-function via Jensen's inequality and integration by parts, and applications in symbol error probability analysis

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Using Jensen's inequality and integration by parts, some tight upper bounds are derived on the Gaussian Q-function. The tightness of the bounds obtained by Jensen's inequality can be improved by increasing the number of exponential terms, and one of them is invertible. A piece-wise upper bound is obtained and its application in the analysis of the symbol error probability of various modulation schemes in different channel models is shown.

Introduction: The Gaussian Q-function is of great significance in the field of communication. It is defined as

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t. \tag{1}$$

This infinite integral form is not easy to evaluate in practice. Many approximations and bounds have been proposed in the literature to facilitate its easy computation, especially approximations, e.g. [1–3]. However, these might not be helpful because determining the accuracy of each approximation or bound requires in turn accurate knowledge of the Q-function values. In comparison, a pair of tight, upper and lower bounds together can easily provide accurate estimates of the Q-function. Many lower bounds are available [2–5], but only a few upper bounds [2, 3]. Our work here, therefore, focuses on deriving upper bounds.

The Craig's form of Gaussian Q-function is proposed in [6], which stimulated the research on further simplifications and approximations. Using the monotonically increasing property of the integrand of the Craig's form, a set of sums of purely exponential upper and lower bounds is obtained in [2]. Using the Jensen's inequality and the convexity property of the exponential function, we derive a tighter family of purely exponential lower bounds in [4]. The purely exponential form allows easy averaging over the fading distribution using the moment generating function method [2, 4]. The coefficients of lower and upper bounds and approximations in the form of weighted sum of purely exponential functions are numerically optimized in [3]. The approximations and bounds are generalized with coefficients optimized in [7]. Using integration by parts on the canonical form of the Gaussian Q-function in (1), an upper bound is obtained in [1]. Using integration by parts followed by Jensen's inequality on the integral term, tighter upper bounds are derived in [8]. Applying integration by parts twice, a lower bound is obtained in [9] in the form of the product of a 2-term polynomial and exponential function.

We here propose to bound the cumulative distribution function (CDF) of the standard Gaussian distribution, so that the approaches for deriving lower bounds can be used to derive upper bounds. Applying the Jensen's inequality approach on the CDF, a set of exponential upper bounds on the Gaussian Q-function is obtained, one of which is invertible. Our bounds have the similar simple form as the bounds in [2], but are tighter with small argument values. Applying the integration by parts approach on the CDF, the 2-term lower bound in [1, 9] is obtained. We here apply integration by parts again and get a new 3-term upper bound which is tight with large argument values.

Combining the two upper bounds derived using the two approaches, we obtain a tighter upper bound as a piece-wise function. The accuracy of the new upper bound is proved by comparing it with the existing upper bounds in [2, 3, 8].

As for applications, a closed-form upper bound of the inverse Gaussian Q-function is derived and some symbol error probability (SEP) expressions of various digital modulation techniques in different channel models are computed to justify the accuracy of our new upper bounds.

Small-argument bound via Jensen's inequality and its inverse: The CDF of standard normal distribution is defined as

$$F(x) = 1 - Q(x) = \frac{1}{2} + \int_0^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t. \tag{2}$$

First, we split the integration range of $[0, x]$ into n subranges, by arbitrarily choosing $n+1$ values of β_k such that $0 = \beta_0 x < \beta_1 x < \ldots < \beta_k < \ldots < \beta_n x = x$, i.e. $\beta_0 = 0$ and $\beta_n = 1$. Thus, (2) becomes

$$F(x) = \frac{1}{2} + \sum_{k=1}^{n} \int_{\beta_{k-1}x}^{\beta_k x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t. \tag{3}$$

Then, we can apply the Jensen's inequality for each summation term in (3).

Jensen's inequality [10, eq. (12.411)]: Let $f(\theta)$ and $p(\theta)$ be two functions defined for $a \le \theta \le b$ such that $\alpha \le f(\theta) \le \beta$ and $p(\theta) \ge 0$, with $p(\theta) \not\equiv 0$. Let $\phi(u)$ be a convenx function defined on the interval $\alpha \le u \le \beta$, then

$$\phi\left(\frac{\int_a^b f(\theta)p(\theta)\mathrm{d}\theta}{\int_a^b p(\theta)\mathrm{d}\theta}\right) \le \frac{\int_a^b \phi(f)p(\theta)\mathrm{d}\theta}{\int_a^b p(\theta)\mathrm{d}\theta}. \tag{4}$$

Letting

$$\begin{aligned} \phi(u) &= \exp(u), \\ p(\theta) &= \frac{1}{\sqrt{2\pi}}, \\ f(\theta) &= -\frac{t^2}{2}, \\ a &= \beta_{k-1}x, \\ b &= \beta_k x, \end{aligned} \tag{5}$$

we obtain a lower bound on (3) as

$$F_{\mathrm{LB}}(x) = \frac{1}{2} + \sum_{k=1}^{n} a_k x \exp\left(-b_k x^2\right), \tag{6}$$

where

$$a_k = \frac{\beta_k - \beta_{k-1}}{\sqrt{2\pi}}, \, b_k = \frac{\beta_k^2 + \beta_k\beta_{k-1} + \beta_{k-1}^2}{6}, \tag{7}$$

are constant coefficients that are independent of x.

According to the definition of the CDF of the standard Gaussian distribution, a new set of upper bounds are obtained as

$$Q_{\mathrm{UB}}(x) = 1 - F_{\mathrm{LB}}(x) = \frac{1}{2} - \sum_{k=1}^{n} a_k x \exp\left(-b_k x^2\right). \tag{8}$$

We select several groups of β values to determine a_k and b_k, and get a series of new upper bounds:

$$\begin{aligned} Q_{\mathrm{UB-2}}(x) &= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{1}{6}x^2\right), \\ Q_{\mathrm{UB-3}}^{(1)}(x) &= \frac{1}{2} - \frac{1}{3\sqrt{2\pi}} x \exp\left(-\frac{1}{54}x^2\right) - \frac{2}{3\sqrt{2\pi}} x \exp\left(-\frac{13}{54}x^2\right), \\ Q_{\mathrm{UB-3}}^{(2)}(x) &= \frac{1}{2} - \frac{1}{2\sqrt{2\pi}} x \exp\left(-\frac{1}{24}x^2\right) - \frac{1}{2\sqrt{2\pi}} x \exp\left(-\frac{7}{24}x^2\right), \\ Q_{\mathrm{UB-3}}^{(3)}(x) &= \frac{1}{2} - \frac{2}{3\sqrt{2\pi}} x \exp\left(-\frac{2}{27}x^2\right) - \frac{1}{3\sqrt{2\pi}} x \exp\left(-\frac{19}{54}x^2\right), \\ Q_{\mathrm{UB-4}}(x) &= \frac{1}{2} - \frac{1}{3\sqrt{2\pi}} x \exp\left(-\frac{1}{54}x^2\right) - \frac{1}{3\sqrt{2\pi}} x \exp\left(-\frac{7}{54}x^2\right) \\ &\quad - \frac{1}{3\sqrt{2\pi}} x \exp\left(-\frac{19}{54}x^2\right). \end{aligned} \tag{9}$$

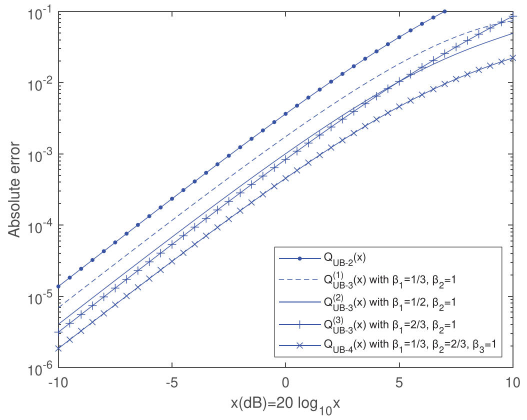

Fig. 1 *Absolute error of the upper bounds on the Gaussian Q-function for small argument values*

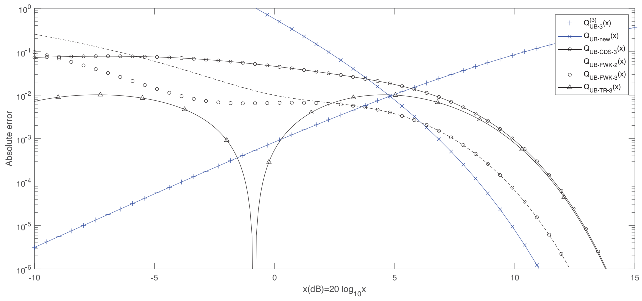

Fig. 2 *Absolute error of the new upper bound on the Gaussian Q-function for small argument values*

The lower bounds on the CDF are tight for small argument values only and they approach the exact value with increasing *n*. Fig. 1 shows that our 2-term, 3-term and 4-term bounds are all tight for small argument values. The 2-term bound $Q_{\mathrm{UB-2}}(x)$ turns out to be invertible. Considering accuracy and conciseness of the expressions, $Q^{(3)}_{\mathrm{UB-3}}(x)$ is chosen for small argument values, as shown in Fig. 2.

In some applications, the expression for $Q(x)$ needs to be inverted, e.g. to determine the SNR required for a certain error probability. For example invertible *Q*-function can be used for analytical estimation of signal-to-noise ratio penalty and laser linewidth tolerance in coherent optical communications [5]. Therefore, a tight invertible bound for the Gaussian *Q*-function is useful. The 2-term upper bound $Q_{\mathrm{UB-2}}(x)$ in (9) is invertible. Its closed-form inverse function is given by

$$x = Q^{-1}_{\mathrm{UB-2}}(y) = \sqrt{-3W\left(-\frac{\pi}{6}(2y-1)^2\right)}, \tag{10}$$

where the Lambert W function $W(z)$ is defined as the inverse function of [11, eq. (1.5)]

$$W(z)e^{W(z)} = z. \tag{11}$$

Fig. 3 shows the accuracy of $Q_{\mathrm{UB-2}}(x)$, whereas the other bounds shown in comparison are not invertible.

Large-argument bound via integration by parts: Using integration by parts on the CDF in (2), the same expression for the Gaussian *Q*-function as in [9, eqs. (7)(10-11)] is obtained

$$Q(x) = \frac{1}{\sqrt{2\pi}x}\left(1-\frac{1}{x^2}\right)\exp\left(-\frac{x^2}{2}\right) + \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{3}{t^4}\exp\left(-\frac{t^2}{2}\right)\mathrm{d}t. \tag{12}$$

Then, applying integration by parts on the integral part again, (12) becomes

$$Q(x) = \frac{1}{\sqrt{2\pi}x}\left(1-\frac{1}{x^2}+\frac{3}{x^4}\right)\exp\left(-\frac{x^2}{2}\right) - \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{15}{t^6}\exp\left(-\frac{t^2}{2}\right)\mathrm{d}t. \tag{13}$$

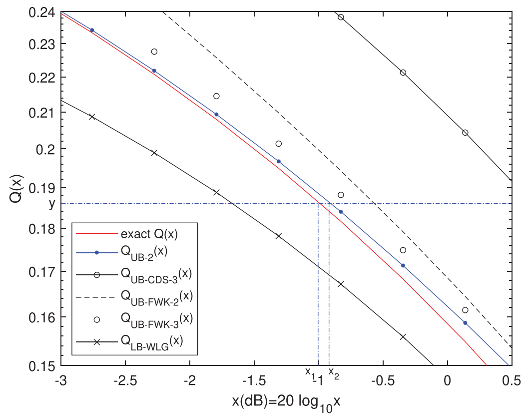

Fig. 3 *The invertible bounds on the Gaussian Q-function*

By omitting the last term in (13), we obtain a new 3-term upper bound on the Gaussian *Q*-function as

$$Q_{\mathrm{UB-new}}(x) = \frac{1}{\sqrt{2\pi}x}\left(1-\frac{1}{x^2}+\frac{3}{x^4}\right)\exp\left(-\frac{x^2}{2}\right). \tag{14}$$

Fig. 2 shows that our upper bound is tighter than the upper bounds in [2, 3, 8] for large argument values. Further applying integration by parts for more times, tighter upper and lower bounds with more terms can be obtained.

A piece-wise upper bound: $Q^{(3)}_{\mathrm{UB-3}}(x)$ is tighter than the upper bounds in [2, 3, 8] for small argument values while $Q_{\mathrm{UB-new}}(x)$ is tighter for large argument values. So we combine the two upper bounds into a piece-wise function, which is written as:

$$Q_{\mathrm{UB}}(x) = \begin{cases} \frac{1}{2} - \frac{2}{3\sqrt{2\pi}}x\exp\left(-\frac{2}{27}x^2\right) - \frac{1}{3\sqrt{2\pi}}x\exp\left(-\frac{19}{54}x^2\right), x \le 4.8 \\ \frac{1}{\sqrt{2\pi}x}\left(1-\frac{1}{x^2}+\frac{3}{x^4}\right)\exp\left(-\frac{x^2}{2}\right), x > 4.8 \end{cases} \tag{15}$$

The intersection of the $Q^{(3)}_{\mathrm{UB-3}}(x)$ and $Q_{\mathrm{UB-new}}(x)$ is at $x = 1.7378 = 4.8$ dB. Fig. 2 shows that the piece-wise upper bound is tighter than the bounds in [2, 3, 8] for both small and large argument values.

Symbol error probability analysis: SEP analysis plays an important role in performance analysis of communication systems. The SEP expressions for various digital modulation techniques, e.g. quadrature phase shift keying (QPSK), differentially encoded QPSK (DE-QPSK), and M-ary pulse amplitude modulation (MPAM), over the additive white Gaussian noise (AWGN) channel are commonly expressed using the Gaussian *Q*-function, and are further extended to derive SEP expressions over fading channels.

The SEP of QPSK is given as [9, eq. 58]:

$$P^{\mathrm{AWGN}}_{\mathrm{QPSK}} = 2Q\left(\sqrt{\gamma}\right) - Q^2\left(\sqrt{\gamma}\right). \tag{16}$$

The SEP of DE-QPSK (coherent detection) is given as [9, eq. (59)] and [12, eq. (11)]:

$$P^{\mathrm{AWGN}}_{\mathrm{DE-QPSK}} = 4Q\left(\sqrt{\gamma}\right) - 8Q^2\left(\sqrt{\gamma}\right) + 8Q^3\left(\sqrt{\gamma}\right) - 4Q^4\left(\sqrt{\gamma}\right). \tag{17}$$

Fig. 4 and Fig. 5 show that the SEP performance of QPSK and DE-QPSK over AWGN channel using the new piece-wise upper bound is tighter than the upper bounds in [2, 3, 8], except for the value near the intersection of the piece-wise function. Therefore, our upper bound is very suitable for low or high SNR, for example, when the characteristic signal itself is really weak or when the SNR is reduced due to the strong noise interference. It shows that our bound also works well with powers of the *Q*-function when high order modulations are studied.

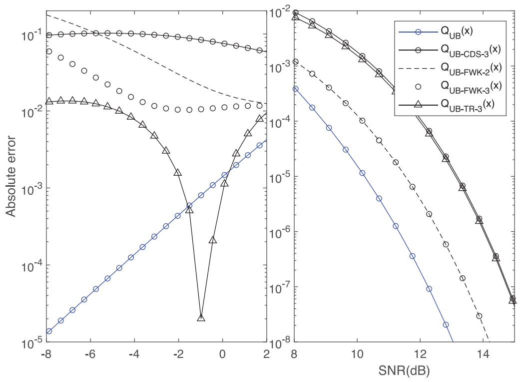

Fig. 4 *Absolute error of upper bounds on the SEP of QPSK over AWGN*

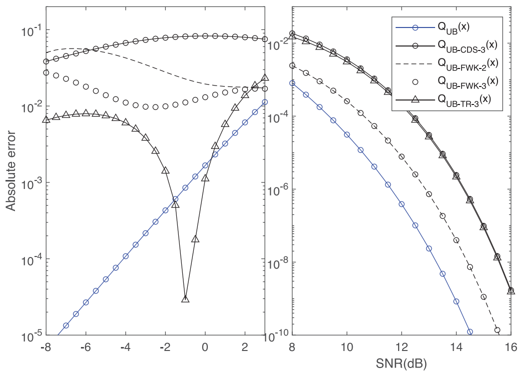

Fig. 5 *Absolute error of upper bounds on the SEP of DE-QPSK over AWGN*

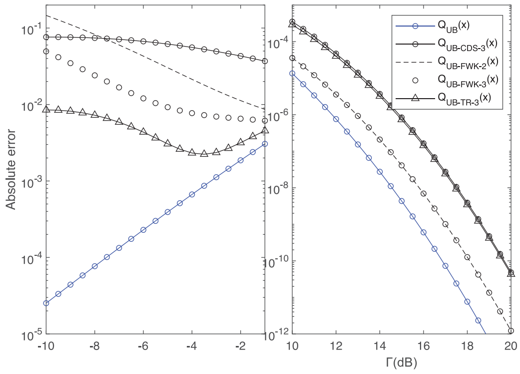

Fig. 6 *Absolute error of upper bounds on the SEP of MPAM over Nakagami-m fading ($M = 2, \sigma = 2$)*

The average SEP of MPAM signals over Nakagami-*m* fading can be derived as [13, eq. (12)]:

$$P_{\mathrm{e}} = \frac{40(M-1)/\ln 10}{M\sqrt{2\pi\sigma^2}} \int_0^\infty \frac{Q(t)}{t} e^{-\frac{\left(20\lg t + 10\lg\frac{M^2-1}{6} - \mu\right)^2}{2\sigma^2}} \mathrm{d}t \tag{18}$$

with μ and σ being the logarithmic mean and logarithmic standard deviation, respectively. In the comparison, $\Gamma = E\{\gamma\} = e^{\frac{\ln 10}{10}\mu + \frac{\ln^2 10}{200}\sigma^2}$. Fig. 6 shows that the average SEP performance of MPAM over Nakagami-*m* fading used by our new piece-wise upper bound is also tighter than the upper bounds in [2, 3, 8] except for the value near the intersection of the piece-wise function. In this case, the new upper bound fits a large or small Γ value.

Conclusions: We obtain a set of new exponential upper bounds on the Gaussian *Q*-function by the Jensen's inequality and integration by parts. A piece-wise upper bound is obtained combining the two. Besides, we also obtain an invertible upper bound for small argument values. Applications of the piece-wise upper bound to the SEP of various digital modulation techniques over AWGN and Nakagami-*m* fading show the tightness of the bounds. The upper bounds together with lower bounds, e.g. those in [4, 5], alone can show the tightness of the bounds, without having to compare with exact Gaussian *Q*-function values.

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