Master Thesis - Report

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1 Introduction

1.1 Approach

We replicate Galí and Monacelli (2008) in a two-country setup.
We add features to the model

- We relax the function form assumptions
- We add a country size parameter

Monetary policy is conducted at the Union level while fiscal policy stay at the national level. Though, we will investigate cases where national government are forced to follow Union fiscal policy rules and targets.

1.2 References

Below are the references we used to build the model:

- ENSAE MiE 2 course : AE332, Monetary Economics, Olivier Loisel
- Galí and Monacelli, Optimal monetary and fiscal policy in a currency union, *Journal of International Economics*, 2008
- Marcos Antonio C. da Silveira, Two-country new Keynesian DSGE model : a small open economy as limit case, *Ipea*, 2006
- Cole et al., One EMU fiscal policy for the Euro, *Macroeconomic Dynamics*, 2019
- Forlati, Optimal monetary and fiscal policy in the EMU : does fiscal policy coordination matter?,*Center for Fiscal Policy, EPFL, Chair of International Finance (CFI) Working Paper No. 2009-04*, 2009
- Schäfer, Monetary union with sticky prices and direct spillover channels, *Journal of Macroeconomics*, 2016

2 A currency union model

We model a currency union as a closed system made up of two economies : *Home* and *Foreign*.

The two country form a currency union henceforth call *Union* and abbreviated CU. Variables without asterix (e.g. X) denote *Home* variables and variables with an asterix (e.g. X_t^*) denote *Foreign* variables.

Home is inhabited by a continuum of identical households indexed by j where $j \in [0, h]$ with $0 \leq h \leq 1$. *Foreign* is inhabited by a continuum of identical households indexed by j where $j \in (h, 1]$.

2.1 Households

2.1.1 Objective

Home j-th household seeks to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t^j, N_t^{sj}, \frac{G_t}{h}\right),$$

where U is the instantaneous utility function, N_t^{sj} is the number of work hours supplied by *Home* j-th household, C_t^j is a composite index of *Home* j-th household's consumption, and G_t is an index of *Home*'s government consumption.

2.1.2 Aggregate composite consumption index

More precisely, C_t^j is given by

$$C_t^j \equiv \left[(1-\alpha)^{\frac{1}{\eta}} (C_{H,t}^j)^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} (C_{F,t}^j)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}},$$

where

- $C_{H,t}^j$ is an index of *Home* j-th household's consumption of *Home*-made goods,
- $C_{F,t}^j$ is an index of *Home* j-th household's consumption of *Foreign*-made goods,
- $\alpha \in [0, 1]$ is a measure of *Home*'s **openess** and $1 - \alpha$ is a measure of *Home*'s **home bias**,
- η is *Home*'s elasticity of substitution between *Home*-made goods and *Foreign*-made goods.

2.1.3 Regional consumption indexes

$C_{H,t}^j$ is defined by the CES function

$$C_{H,t}^j \equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}} \int_0^h C_{H,t}^j(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where

- $C^j_{H,t}(i)$ is *Home* j-th household's consumption of *Home*-made good $i \in [0,h]$,
- $\varepsilon > 1$ is the elasticity of substitution between *Home*-made goods.

Similarly, $C^j_{F,t}$ is defined by the CES function

$$C^j_{F,t} \equiv \left[\left(\frac{1}{1-h} \right)^{\frac{1}{\varepsilon}} \int_h^1 C^j_{F,t}(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where

- $C^j_{F,t}(i)$ is *Home* j-th household's consumption of *Foreign*-made good $i \in (h,1]$,
- $\varepsilon > 1$ is the elasticity of substitution between *Foreign*-made goods.

2.1.4 Household's budget constraints

Home j-th household faces a sequence of budget constraints

$$\forall t \geq 0, \int_0^h P_{H,t}(i) C^j_{H,t}(i) \mathrm{d}i + \int_h^1 P_{F,t}(i) C^j_{F,t}(i) \mathrm{d}i + \mathbb{E}_t\{Q_{t,t+1} D^j_{t+1}\} \leq D^j_t + W_t N^{sj}_t + \frac{T_t}{h}, \quad (1)$$

where

- $P_{H,t}(i)$ is *Home*'s price of *Home*-made good i,
- $P_{F,t}(i)$ is *Home*'s price of *Foreign*-made good i,
- D^j_{t+1} is the quantity of one-period nominal bonds held by *Home* j-th household,
- W_t is *Home*'s nominal wage,
- T_t denotes *Home*'s lump sum taxes.

2.1.5 Optimal allocation of consumption across goods

Given $C^j_{H,t}$ and $C^j_{F,t}$, a first step is to find the optimal allocations $(C^j_{H,t}(i))_{i\in[0,h]}$ and $(C^j_{F,t}(i))_{i\in(h,1]}$ that minimize the regional expenditures.

Home j-th household's optimal consumption of *Home*-made good $i \in [0,h]$ is given by

$$C^j_{H,t}(i) = \frac{1}{h} \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\varepsilon} C^j_{H,t},$$

where $P_{H,t} \equiv \left[\frac{1}{h} \int_0^h P_{H,t}(i)^{1-\varepsilon} \mathrm{d}i \right]^{\frac{1}{1-\varepsilon}}$ is *Home*'s price index of *Home*-made goods.

Similarly, *Home* j-th household's optimal consumption of *Foreign*-made good $i \in (h,1]$ is given by

$$C^j_{F,t}(i) = \frac{1}{1-h} \left(\frac{P_{F,t}(i)}{P_{F,t}} \right)^{-\varepsilon} C^j_{F,t},$$

where $P_{F,t} \equiv \left[\frac{1}{1-h} \int_h^1 P_{F,t}(i)^{1-\varepsilon} \mathrm{d}i \right]^{\frac{1}{1-\varepsilon}}$ is *Home*'s price index of *Foreign*-made goods.

2.1.6 Optimal allocation of consumption across regions

Given C_t^j, a second step is to find the optimal allocation$(C_{H,t}^j, C_{F,t}^j)$ that minimizes total expenditures.

Home j-th household's optimal consumption of *Home*-made goods is given by

$$C_{H,t}^j = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t^j,$$

and *Home* j-th household's optimal consumption of *Foreign*-made goods is given by

$$C_{F,t}^j = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t^j,$$

where $P_t \equiv \left[(1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}\right]^{\frac{1}{1-\eta}}$ is *Home*'s consumer price index (CPI).

2.1.7 Rewrite household's budget constraints

We show in section B.1, that conditional on an optimal allocation across goods and regions, *Home* j-th household's budget constraints can be rewritten as

$$\forall t \geq 0, P_t C_t^j + \mathbb{E}_t\{Q_{t,t+1} D_{t+1}^j\} \leq D_t^j + W_t N_t^{sj} + \frac{T_t}{h}. \tag{2}$$

2.1.8 Household's intratemporal and intertemporal FOCs

Now, we can derive the first order conditions for *Home* j-th household's optimal consumption level C_t^j as well as for *Home* j-th household's optimal number of hours worked N_t^{sj}.

Home j-th household's **intratemporal** FOC is

$$-\frac{U_{n,t}^j}{U_{c,t}^j} = \frac{W_t}{P_t},$$

and *Home* j-th household's **intertemporal** FOC is

$$\mathbb{E}_t\{Q_{t,t+1}\} = \beta \mathbb{E}_t\left\{\frac{U_{c,t+1}^j}{U_{c,t}^j}\frac{P_t}{P_{t+1}}\right\},$$

where $U_{n,t}^j \equiv \frac{\partial U}{\partial N_t^{sj}}\left(C_t^j, N_t^{sj}, \frac{G_t}{h}\right)$ and $U_{c,t}^j \equiv \frac{\partial U}{\partial C_t^j}\left(C_t^j, N_t^{sj}, \frac{G_t}{h}\right)$.

2.1.9 Functional form of the instantaneous utility function

We assume that the instantaneous utility takes the specific form

$$U(C_t^j, N_t^{sj}, G_t/h) = \chi_C \frac{(C_t^j)^{1-\sigma} - 1}{1-\sigma} + \chi_G \frac{(G_t/h)^{1-\gamma} - 1}{1-\gamma} - \frac{(N_t^{sj})^{1+\varphi}}{1+\varphi}$$

where $\varphi > 0$ while χ_G and χ_C are used to calibrate the steady state of the economy.

2.1.10 Rewrite household's intratemporal and intertemporal FOCs under the functional form assumptions

Under the functional form assumptions, *Home* j-th household **intratemporal** FOC becomes

$$(N_t^{sj})^{\varphi}\frac{(C_t^j)^{\sigma}}{\chi_C} = \frac{W_t}{P_t},$$

and *Home* j-th household's **intertemporal** FOC becomes

$$\mathbb{E}_t\{Q_{t,t+1}\} = \beta\mathbb{E}_t\left\{\left(\frac{C_{t+1}^j}{C_t^j}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right\}.$$

2.2 Aggregating optimal allocation

Home's optimal consumption of *Home*-made good $i \in [0,h]$ and of *Foreign*-made good $i \in (h,1]$ are given by

$$C_{H,t}(i) \equiv \int_0^h C_{H,t}^j(i)\mathrm{d}j = \frac{1}{h}\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon}C_{H,t},$$

$$C_{F,t}(i) \equiv \int_0^h C_{F,t}^j(i)\mathrm{d}j = \frac{1}{1-h}\left(\frac{P_{F,t}(i)}{P_{F,t}}\right)^{-\varepsilon}C_{F,t},$$

where *Home*'s optimal consumption of *Home*-made goods and of *Foreign*-made goods are given by

$$C_{H,t} \equiv \int_0^h C_{H,t}^j\mathrm{d}j = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t,$$

$$C_{F,t} \equiv \int_0^h C_{F,t}^j\mathrm{d}j = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta}C_t,$$

while the composite index of *Home*'s consumption is given by

$$C_t \equiv \int_0^h C_t^j\mathrm{d}j = hC_t^j,$$

since all *Home* households are identical.

Similarly, we define the number of work hours supplied by *Home* households by

$$N_t^s \equiv \int_0^h N_t^{sj}\mathrm{d}j = hN_t^{sj}.$$

2.3 Aggregating optimal intratemporal and intertemporal FOCs

Using the previous results, we can write the intratemporal and intertemporal choices at the aggregate level.

At the aggregate level, **intratemporal** FOC becomes

$$\frac{1}{h^{\varphi+\sigma}}(N_t^s)^{\varphi}\frac{C_t^{\sigma}}{\chi_C} = \frac{W_t}{P_t},$$

and **intertemporal** FOC becomes

$$\mathbb{E}_t\{Q_{t,t+1}\} = \beta\mathbb{E}_t\left\{\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right\}.$$

2.3.1 Aggregate FOCs in log-linearized form

Home RH's **intratemporal** FOC in log form is

$$w_t - p_t = -(\varphi + \sigma)\log(h) + \sigma c_t + \varphi n_t^s - \log(\chi_C),$$

and *Home* RH's **intertemporal** FOC in log form is

$$c_t = \mathbb{E}_t\{c_{t+1}\} - \frac{1}{\sigma}(i_t^{CU} - \mathbb{E}_t\{\pi_{t+1}\} - \bar{i}),$$

where $i_t^{CU} \equiv log\left(\frac{1}{\mathbb{E}_t\{Q_{t,t+1}\}}\right)$ is referred to as the **Currency Union short-term nominal interest rate**, $\pi_t \equiv p_t - p_{t-1}$ is *Home*'s **CPI inflation**, and $\bar{i} \equiv -\log(\beta)$.

2.3.2 Summary of household's optimal allocation

Analogous results hold for the *Foreign* and are given in section A.1.1 and A.1.2.

2.4 Definitions, identities and international risk sharing

2.4.1 The law of one price

Since we are in a currency union, the exchange rate is equal to 1. Therefore, the law of one price (LOP) states that $P_{H,t}(i) = P^*_{H,t}(i)$ and $P_{F,t}(i) = P^*_{F,t}(i)$ which imply $P_{H,t} = P^*_{H,t}$ and $P_{F,t} = P^*_{F,t}$.

2.4.2 Terms of trade

We derive the relationship between inflation, terms of trade and real exchange rate. *Home*'s terms of trade is defined as

$$S_t \equiv \frac{P_{F,t}}{P_{H,t}},$$

and *Foreign*'s terms of trade is defined as

$$S_t^* \equiv \frac{P^*_{H,t}}{P^*_{F,t}}.$$

The terms of trade is simply the relative price of imported goods in terms of domestic goods.

Using the LOP, we have

$$S_t^* = \frac{1}{S_t}.$$

2.4.3 Home bias (not detailed)

It is crucial to understand the role of the parameter α which is *Home*'s degree of openess to *Foreign*. We follow Da Silveira (2006) and assume that α and α^* are linked to h by

$$\alpha = \bar{\alpha}(1-h)$$
$$\alpha^* = \bar{\alpha}h$$

where $\bar{\alpha}$ is the degree of openess of a small open economy. When *Home* is a big economy (i.e. h is big), the degree of openess α is moved down from $\bar{\alpha}$ so that *Home*'s home bias increases. See Da Silveira page 16.

2.4.4 Price level and inflation identities

Using the definitions of P_t, P_t^*, S_t, and S_t^*, we get

$$\begin{aligned}
\frac{P_t}{P_{H,t}} &= \left[(1-\alpha)+\alpha(S_t)^{1-\eta}\right]^{\frac{1}{1-\eta}} \equiv g(S_t) \\
\frac{P_t}{P_{F,t}} &= \frac{P_t}{P_{H,t}}\frac{P_{H,t}}{P_{F,t}} = \frac{g(S_t)}{S_t} \equiv h(S_t) \\
\frac{P_t^*}{P_{H,t}^*} &= \left[\alpha^*+(1-\alpha^*)(S_t)^{1-\eta}\right]^{\frac{1}{1-\eta}} \equiv g^*(S_t) \\
\frac{P_t^*}{P_{F,t}^*} &= \frac{P_t^*}{P_{H,t}^*}\frac{P_{H,t}^*}{P_{F,t}^*} = \frac{g^*(S_t)}{S_t} \equiv h^*(S_t).
\end{aligned}$$

Log-linearizing $g(S_t)$,$h(S_t)$,$g^*(S_t)$ and $h^*(S_t)$ around $S_t = 1$, we get

$$\begin{aligned}
p_t - p_{H,t} &= \alpha s_t \\
p_t - p_{F,t} &= -(1-\alpha)s_t \\
p_t^* - p_{H,t}^* &= (1-\alpha^*)s_t \\
p_t^* - p_{F,t}^* &= -\alpha^* s_t.
\end{aligned}$$

Using the expression of home bias as a function of $\bar{\alpha}$ and h, we get

$$\begin{aligned}
\pi_t &= \pi_{H,t} + \bar{\alpha}(1-h)\Delta s_t \\
\pi_t^* &= \pi_{F,t}^* - \bar{\alpha}h\Delta s_t,
\end{aligned}$$

where *Home* and *Foreign* inflation of domestic price indexes are respectively given by $\pi_{H,t} = p_{H,t} - p_{H,t-1}$ and $\pi_{F,t}^* = p_{F,t}^* - p_{F,t-1}^*$.

2.4.5 Real exchange rate

Using the LOP, *Home*'s real exchange rate denoted $\mathcal{Q}_t$ is given by

$$\mathcal{Q}_t \equiv \frac{P_t^*}{P_t} = \frac{g^*(S_t)}{g(S_t)}.$$

A first order approximation around $S_t = 1$ gives

$$\mathcal{Q}_t \simeq 1 + (1-\alpha^*-\alpha)(S_t - 1).$$

Therefore, around $S_t = 1$ (which implies $\mathcal{Q}_t = 1$), we have

$$q_t = (1-\bar{\alpha})s_t.$$

since $\alpha^* + \alpha = \bar{\alpha}$.

2.4.6 International risk sharing (not detailed)

The international risk sharing (IRS) condition implies that

$$C_t = \frac{h}{1-h}\vartheta \mathcal{Q}_t^{\frac{1}{\sigma}} C_t^*.$$

We assume the same initial conditions for *Home* and *Foreign* households, so that $\vartheta = 1$. In log form, the IRS condition writes

$$c_t = \log(\frac{h}{1-h}) + \frac{1}{\sigma}q_t + c_t^*.$$

2.5 Government

2.5.1 Government consumption index

Home's public consumption index is given by the CES function

$$G_t \equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}} \int_0^h G_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where $G_t(i)$ is the quantity of *Home*-made good i purchased *Home*'s government.

2.5.2 Government demand schedules

For any level of public consumption G_t, the government demand schedules are analogous to those obtain for private consumption, namely

$$G_t(i) = \frac{1}{h}\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} G_t.$$

Similar results hold for *Foreign*'s government consumption and are summarized in section A.1.3.

2.6 Firms

Each country has a continuum of firms represented by the interval $[0, h]$ for *Home* and by the interval $(h, 1]$ for *Foreign*. Each firm produces a differentiated good.

2.6.1 Technology

All *Home* firms use the same technology, represented by the production function

$$Y_t(i) = A_t N_t(i),$$

where A_t is *Home*'s productivity.

2.6.2 Labor demand

The technology constraint implies that *Home* i-th firm's labor demand is given by

$$N_t(i) = \frac{Y_t(i)}{A_t}.$$

2.6.3 Aggregate labor demand

Home's aggregate labor demand is defined as

$$N_t \equiv \int_0^h N_t(i)\mathrm{d}i = \frac{Y_t Z_t}{A_t},$$

where

$$Y_t \equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}} \int_0^h Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

is the aggregate production index while $Z_t \equiv \int_0^h \frac{Y_t(i)}{Y_t}\mathrm{d}i$ is a measure of the dispersion of *Home* firms' output.

2.6.4 Aggregate production function

In log form, we have a relationship between *Home*'s aggregate employment and *Home*'s output

$$y_t = a_t + n_t,$$

because the variation of $z_t \equiv \log(Z_t)$ around the steady state are of second order. (Admitted for now)

2.6.5 Marginal cost

Home's nominal marginal cost is given by

$$MC_t^n = \frac{(1-\tau)W_t}{MPN_t},$$

where MPN_t is *Home*'s average marginal product of labor at t defined as

$$MPN_t \equiv \frac{1}{h}\int_0^h \frac{\partial Y_t(i)}{\partial N_t(i)}\mathrm{d}i = A_t,$$

and where τ is *Home*'s (constant) employment subsidy. This subsidy will be used latter to offset the monopolistic distortion at steady state.

The real marginal cost (express in terms of domestic goods) is the same across firms in any given country.
Home firms' real marginal cost is given by

$$MC_t \equiv \frac{MC_t^n}{P_{H,t}} = \frac{(1-\tau)W_t}{A_t P_{H,t}}.$$

In log form, we get

$$mc_t = \log(1-\tau) + w_t - p_{H,t} - a_t.$$

2.6.6 Firm's problem : price setting

We assume a price setting *à la Calvo*. At each date t, all *Home* firms resetting their prices will choose the same price denoted $\bar{P}_{H,t}$ because they face the same problem.

Home firms' resetting price problem is

$$\max_{\bar{P}_{H,t}} \sum_{k=0}^{+\infty} \theta^k \mathbb{E}_t \left\{ Q_{t,t+k} \left[\bar{P}_{H,t} Y_{t+k|t} - \Psi_{t+k}(Y_{t+k|t}) \right] \right\},$$

where

- $Q_{t,t+k} \equiv \beta^k \frac{C_t}{C_{t+k}} \frac{P_t}{P_{t+k}}$ is *Home* firms' stochastic discount factor for nominal payoffs between t and $t+k$,
- $Y_{t+k|t}$ is output at $t+k$ for a firm that last resetted its price at t,
- $\Psi_t(\cdot)$ is *Home*'s nominal cost function at t,

subject to $Y_{t+k|t} = \left(\frac{\bar{P}_{H,t}}{P_{H,t+k}} \right)^{-\varepsilon} (C_{H,t+k} + C^*_{H,t+k} + G_{t+k})$ for $k \in \mathbb{N}$, taking $(C_{t+k})_{k\in\mathbb{N}}$ and $(P_{t+k})_{k\in\mathbb{N}}$ as given.

2.6.7 Firm's FOC

Noticing that $\frac{\partial Y_{t+k|t}}{\partial \bar{P}_{H,t}} = -\varepsilon \frac{Y_{t+k|t}}{\bar{P}_{H,t}}$, *Home* firms' FOC is

$$\max_{\bar{P}_{H,t}} \sum_{k=0}^{+\infty} \theta^k \mathbb{E}_t \left\{ Q_{t,t+k} Y_{t+k|t} \left[\bar{P}_{H,t} - \mathcal{M}\psi_{t+k|t} \right] \right\} = 0, \tag{3}$$

where $\psi_{t+k|t} \equiv \Psi'_{t+k}(Y_{t+k|t})$ denotes the nominal marginal cost at $t+k$ for a firm that last reset its price at t, and $\mathcal{M} \equiv \frac{\varepsilon}{\varepsilon-1}$. Under flexible prices ($\theta = 0$), *Home* firms' FOC collapses to $\bar{P}_{H,t} = \mathcal{M}\psi_{t|t}$, so that $\mathcal{M}$ is the "desired" (or frictionless) markup.

Following the definition of the Zero Inflation Steady State (ZIRSS) given in section B.2.1, a log-linearization of *Home* firms' FOC around the ZIRSS yields

$$\bar{p}_{H,t} = (1-\beta\theta) \sum_{k=0}^{+\infty} (\beta\theta)^k \mathbb{E}_t \{ \mu + mc_{t+k|t} + p_{H,t+k} \},$$

where $\bar{p}_{H,t}$ denotes the (log) of newly set prices in *Home* (same for all firms reoptimizing), and $\mu \equiv \log(\frac{\varepsilon}{\varepsilon-1})$.

2.6.8 Aggregate price level dynamics

As only a fraction $1-\theta$ of firms adjusts price each period, we have

$$P_{H,t} = \left[\theta (P_{H,t-1})^{1-\varepsilon} + (1-\theta)(\bar{P}_{H,t})^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}.$$

Log-linearizing around the ZIRSS, we get

$$\pi_{H,t} = (1-\theta)(\bar{p}_{H,t} - p_{H,t}).$$

2.6.9 Rewrite log-linearized firms' FOC

Combining the results of section B.2.2 with the aggregate price level dynamics equation, we get

$$\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \lambda(\mu + mc_t)$$

where $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$.

2.6.10 Summary firm results

Analogous results hold for *Foreign* firms and are reported in section A.1.4.

3 Equilibrium dynamics

3.1 Aggregate demand and output determination

3.1.1 Labor market

At equilibrium, labor supply equals labor demand

$$N_t^s = N_t \Rightarrow n_t^s = n_t.$$

3.1.2 Good markets

The world demand of *Home*-made good i is given by

$$\begin{aligned} Y_t^d(i) &\equiv C_{H,t}(i) + C_{H,t}^*(i) + G_t(i) \\ &= \frac{1}{h}\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon}(C_{H,t} + C_{H,t}^* + G_t). \end{aligned}$$

The market of all *Home* and *Foreign* goods clear in equilibrium so that

$$Y_t(i) = Y_t^d(i), \forall i \in [0,1].$$

Following section B.3, the good-market clearing condition at the aggregate level writes

$$Y_t = \left(\frac{P_{H,t}}{P_t}\right)^{-\eta}\left[(1-\alpha) + \alpha^*\frac{1-h}{h}\mathcal{Q}_t^{\eta-\frac{1}{\sigma}}\right]C_t + G_t.$$

Using $\alpha^* = \frac{h}{1-h}\alpha$, we get

$$Y_t = \left(\frac{P_{H,t}}{P_t}\right)^{-\eta}\left[(1-\alpha) + \alpha\mathcal{Q}_t^{\eta-\frac{1}{\sigma}}\right]C_t + G_t. \tag{4}$$

3.1.3 Log-linearization of the good-market clearing conditions

We define $\hat{x}_t \equiv x_t - x$ the log-deviation of the variable x_t from its steady state value. Also, $\delta \equiv \frac{G}{Y}$ be the steady state share of government spending.

Log-linearizing (4) around $S_t = 1$ (or $\mathcal{Q}_t = 1$), we get

$$\begin{aligned} \frac{1}{1-\delta}(\hat{y}_t - \delta \hat{g}_t) &= \hat{c}_t + \frac{\bar{\alpha}(1-h)w_{\bar{\alpha}}}{\sigma} s_t, \\ \frac{1}{1-\delta}(\hat{y}^*_t - \delta \hat{g}^*_t) &= \hat{c}^*_t - \frac{\bar{\alpha} h w_{\bar{\alpha}}}{\sigma} s_t, \end{aligned} \tag{5}$$

where

$$w_{\bar{\alpha}} = 1 + (2-\bar{\alpha})(\sigma\eta - 1) > 0.$$

Equivalently (5) writes

$$\begin{aligned} \tilde{\sigma}(\hat{y}_t - \delta \hat{g}_t) &= \sigma \hat{c}_t + \bar{\alpha}(1-h)w_{\bar{\alpha}} s_t, \\ \tilde{\sigma}(\hat{y}^*_t - \delta \hat{g}^*_t) &= \sigma \hat{c}^*_t - \bar{\alpha} h w_{\bar{\alpha}} s_t, \end{aligned} \tag{6}$$

where $\tilde{\sigma} \equiv \frac{\sigma}{1-\delta}$.

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3.1.4 IRS condition at equilibrium

As shown in section B.4, we can re-write the IRS condition as

$$s_t = \tilde{\sigma}_{\bar{\alpha}}[\hat{y}_t - \hat{y}^*_t - \delta(\hat{g}_t - \hat{g}^*_t)], \tag{7}$$

where $\tilde{\sigma}_{\bar{\alpha}} \equiv \frac{\tilde{\sigma}}{1+\bar{\alpha}\Theta_{\bar{\alpha}}}$ and $\Theta_{\bar{\alpha}} \equiv w_{\bar{\alpha}} - 1$.

3.1.5 IS equations in log-deviation form

Following section B.5, we obtain a version of the IS equation in log-deviation form

$$\hat{y}_t = \mathbb{E}_t\{\hat{y}_{t+1}\} - \frac{1}{\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}}(\hat{i}^{CU}_t - \mathbb{E}_t\{\pi_{H,t+1}\}) - \delta\mathbb{E}_t\{\Delta\hat{g}_{t+1}\} + \frac{\bar{\alpha}(1-h)\Theta_{\bar{\alpha}}}{\Omega_{\bar{\alpha},h}}[\mathbb{E}_t\{\Delta\hat{y}^*_{t+1}\} - \delta\mathbb{E}_t\{\Delta\hat{g}^*_{t+1}\}], \tag{8}$$

$$\hat{y}^*_t = \mathbb{E}_t\{\hat{y}^*_{t+1}\} - \frac{1}{\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h}}(\hat{i}^{CU}_t - \mathbb{E}_t\{\pi^*_{F,t+1}\}) - \delta\mathbb{E}_t\{\Delta\hat{g}^*_{t+1}\} + \frac{\bar{\alpha}h\Theta_{\bar{\alpha}}}{\Omega_{\bar{\alpha},1-h}}[\mathbb{E}_t\{\Delta\hat{y}_{t+1}\} - \delta\mathbb{E}_t\{\Delta\hat{g}_{t+1}\}], \tag{9}$$

where $\Omega_{\bar{\alpha},h} \equiv 1 + \bar{\alpha}h\Theta_{\bar{\alpha}}$.

To interpret (8-9), it is convenient to note that $\frac{\bar{\alpha}(1-h)\Theta_{\bar{\alpha}}}{\Omega_{\bar{\alpha},h}} = \frac{1+\bar{\alpha}\Theta_{\bar{\alpha}}}{\Omega_{\bar{\alpha},h}} - 1$.[1]
INTERPRET.

3.1.6 IS equation when *Foreign* is a small open economy

In the limit case where *Foreign* is a small open economy (i.e. $1 - h = 0$), we have $\Omega_{\bar{\alpha},1-h} = 1$ and *Foreign*'s IS equation becomes

$$\hat{y}^*_t = \mathbb{E}_t\{\hat{y}^*_{t+1}\} - \frac{1}{\tilde{\sigma}_{\bar{\alpha}}}(\hat{i}^{CU}_t - \mathbb{E}_t\{\pi^*_{F,t+1}\}) - \delta\mathbb{E}_t\{\Delta\hat{g}^*_{t+1}\} + \bar{\alpha}\Theta_{\bar{\alpha}}[\mathbb{E}_t\{\Delta\hat{y}_{t+1}\} - \delta\mathbb{E}_t\{\Delta\hat{g}_{t+1}\}].$$

[1] By definition, we also have $\frac{\bar{\alpha}h\Theta_{\bar{\alpha}}}{\Omega_{\bar{\alpha},1-h}} = \frac{1+\bar{\alpha}\Theta_{\bar{\alpha}}}{\Omega_{\bar{\alpha},1-h}} - 1$.

In addition, when $\delta = 0$ we recover the equation of a small open economy without government spending.

INTERPRET LIMIT CASE.

3.2 The supply side: marginal cost and inflation dynamics

3.2.1 Marginal cost

As show in section B.6, real marginal cost at equilibrium writes

$$\hat{mc}_t = (\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h} + \varphi)\hat{y}_t - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}\delta\hat{g}_t + (\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h})(\hat{y}^*_t - \delta\hat{g}^*_t) - (1+\varphi)a_t, \tag{10}$$
$$\hat{mc}^*_t = (\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h} + \varphi)\hat{y}^*_t - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h}\delta\hat{g}^*_t + (\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h})(\hat{y}_t - \delta\hat{g}_t) - (1+\varphi)a^*_t. \tag{11}$$

3.2.2 NKPCs in log-deviation form

Combining the previous expressions with *Home* and *Foreign* firms' FOCs, we obtain the New Keynesian Phillips Curves in log-deviation form

$$\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \lambda[(\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h} + \varphi)\hat{y}_t - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}\delta\hat{g}_t + (\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h})(\hat{y}^*_t - \delta\hat{g}^*_t) - (1+\varphi)a_t], \tag{12}$$
$$\pi^*_{F,t} = \beta\mathbb{E}_t\{\pi^*_{F,t+1}\} + \lambda^*[(\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h} + \varphi)\hat{y}^*_t - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h}\delta\hat{g}^*_t + (\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h})(\hat{y}_t - \delta\hat{g}_t) - (1+\varphi)a^*_t]. \tag{13}$$

INTERPRET.

3.2.3 NKPCs when *Foreign* is a small open economy

In the limit case where *Foreign* is a small open economy (i.e. $1-h=0$), we have $\Omega_{\bar{\alpha},1-h} = 1$ and *Foreign*'s nominal marginal cost becomes

$$\hat{mc}^*_t = (\tilde{\sigma}_{\bar{\alpha}} + \varphi)\hat{y}^*_t - \tilde{\sigma}_{\bar{\alpha}}\delta\hat{g}^*_t + (\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}})(\hat{y}_t - \delta\hat{g}_t) - (1+\varphi)a^*_t.$$

In addition, when $\delta = 0$ we recover the results of a small open economy without government spending.

INTERPRET LIMIT CASE.

3.3 Summary sticky price equilibrium

Given the exogeneous sequence $(a_t, a^*_t)_{t\in\mathbb{N}}$ and the sequence $(\hat{i}^{CU}_t, \hat{g}_t, \hat{g}^*_t)_{t\in\mathbb{N}}$, the endogeneous sequence $(\hat{y}_t, \pi_{H,t}; \hat{y}^*_t, \pi^*_{F,t}; s_t)_{t\in\mathbb{N}}$ is given by

- *Home* and *Foreign* IS equations in log-deviation form (8-9),
- *Home* and *Foreign* NKPC in log-deviation form (12-13),
- the IRS condition at equilibrium in log-deviation form (7).

3.4 National accounting identities

We check that national accounting identities hold.

We must have

$$\mathrm{GDP}_t = P_t C_t + P_{H,t} G_t + \mathrm{EX}_t - \mathrm{IM}_t,$$

where GDP_t, IM_t and EX_t are respectively *Home*'s gross domestic product, *Home*'s imports and *Home*'s exports.

In the model, we have

$$\begin{aligned} \mathrm{GDP}_t &= P_{H,t} Y_t \\ \mathrm{EX}_t &= P^*_{H,t} C^*_{H,t} = P_{H,t} C^*_{H,t} \\ \mathrm{IM}_t &= P_{F,t} C_{F,t}. \end{aligned}$$

We must have

$$Y_t = \frac{P_t}{P_{H,t}} C_t - \frac{P_{F,t}}{P_{H,t}} C_{F,t} + C^*_{H,t} + G_t.$$

Note that

$$\begin{aligned} \frac{P_{H,t}}{P_t} C_{H,t} + \frac{P_{F,t}}{P_{H,t}} C_{F,t} &= (1-\alpha) g(S_t)^{\eta-1} C_t + \alpha g(S_t)^{\eta-1} S_t^{1-\eta} C_t \\ &= \frac{(1-\alpha) + \alpha S_t^{1-\eta}}{g(S_t)^{1-\eta}} C_t \\ &= C_t. \end{aligned}$$

Therefore,

$$\frac{P_t}{P_{H,t}} C_t - \frac{P_{F,t}}{P_{H,t}} C_{F,t} = C_{H,t}.$$

As a consequence

$$\begin{aligned} \mathrm{GDP}_t &= P_t C_t + P_{H,t} G_t + \mathrm{EX}_t - \mathrm{IM}_t \\ &= Y_t P_{H,t}. \end{aligned}$$

4 The efficient allocation

4.1 The social planner's problem

4.1.1 Planner's objective

In this section, we characterize the efficient allocation chosen by a benevolent social planner.

Equivalent to the original problem formulated in section B.7.1, the benevolent social planner seeks to maximize

$$\max_{\frac{C_{H,t}}{h}, \frac{C_{F,t}}{h}, \frac{N_t}{h}, \frac{G_t}{h}, \frac{C^*_{H,t}}{1-h}, \frac{C^*_{F,t}}{1-h}, \frac{N^*_t}{1-h}, \frac{G^*_t}{1-h}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big[hU(\frac{C_t}{h}, \frac{N_t}{h}, \frac{G_t}{h}) + (1-h) U(\frac{C^*_t}{1-h}, \frac{N^*_t}{1-h}, \frac{G^*_t}{1-h}) \Big]$$

subject to

$$
\begin{aligned}
&\frac{C_t}{h} = \left[(1-\alpha)^{\frac{1}{\eta}}(\frac{C_{H,t}}{h})^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}(\frac{C_{F,t}}{h})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}},\\
&\frac{C_t^*}{1-h} = \left[(\alpha^*)^{\frac{1}{\eta}}(\frac{C_{H,t}^*}{1-h})^{\frac{\eta-1}{\eta}} + (1-\alpha^*)^{\frac{1}{\eta}}(\frac{C_{F,t}^*}{1-h})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}},\\
&\frac{C_{H,t}}{h} + \frac{1-h}{h}\frac{C_{H,t}^*}{1-h} + \frac{G_t}{h} - A_t\frac{N_t}{h} \leq 0,\\
&\frac{h}{1-h}\frac{C_{F,t}}{h} + \frac{C_{F,t}}{1-h}^* + \frac{G_t^*}{1-h} - A_t^*\frac{N_t^*}{1-h} \leq 0.
\end{aligned}
$$

4.1.2 Planner's FOCs

The FOCs of the planner problem write

$$
\begin{aligned}
\chi_C(1-\alpha)^{\frac{1}{\eta}}\left(\frac{C_{H,t}}{h}\right)^{-\frac{1}{\eta}}\left(\frac{C_t}{h}\right)^{\frac{1}{\eta}-\sigma} &= \chi_G\left(\frac{G_t}{h}\right)^{-\gamma},\\
\chi_C(\alpha)^{\frac{1}{\eta}}\left(\frac{C_{F,t}}{h}\right)^{-\frac{1}{\eta}}\left(\frac{C_t}{h}\right)^{\frac{1}{\eta}-\sigma} &= \chi_G\left(\frac{G_t^*}{1-h}\right)^{-\gamma},\\
\chi_C(1-\alpha^*)^{\frac{1}{\eta}}\left(\frac{C_{F,t}^*}{1-h}\right)^{-\frac{1}{\eta}}\left(\frac{C_t^*}{1-h}\right)^{\frac{1}{\eta}-\sigma} &= \chi_G\left(\frac{G_t^*}{1-h}\right)^{-\gamma},\\
\chi_C(\alpha^*)^{\frac{1}{\eta}}\left(\frac{C_{H,t}^*}{1-h}\right)^{-\frac{1}{\eta}}\left(\frac{C_t^*}{1-h}\right)^{\frac{1}{\eta}-\sigma} &= \chi_G\left(\frac{G_t}{h}\right)^{-\gamma},\\
\left(\frac{N_t}{h}\right)^{\varphi} &= A_t\chi_G\left(\frac{G_t}{h}\right)^{-\gamma},\\
\left(\frac{N_t^*}{1-h}\right)^{\varphi} &= A_t^*\chi_G\left(\frac{G_t^*}{1-h}\right)^{-\gamma}.
\end{aligned}
\tag{14}
$$

4.1.3 The efficient steady state

The efficient steady state is given in section B.7.2.

4.1.4 Planner's FOCs log-linearized

Log-linearizing planner's FOCs (14), the resource constraints and the composite indexes around the efficient steady state gives a system of 10 equations that summarizes the efficient allocation in log-deviation form.

Precisely, given the exogeneous sequence $(a_t, a_t^*)_{t\in\mathbb{N}}$, and denoting with an exponent ^{e} the efficient log-deviations, the endogeneous sequence $(\hat{c}_t^e, \hat{c}_{H,t}^e, \hat{c}_{F,t}^e, \hat{y}_t^e, \hat{g}_t^e; \hat{c}_t^{*e}, \hat{c}_{H,t}^{*e}, \hat{c}_{F,t}^{*e}, \hat{y}_t^{*e}, \hat{g}_t^{*e})_{t\in\mathbb{N}}$

is given by

$$
\begin{aligned}
\hat{c}^e_{H,t} &= \eta\gamma\hat{g}^e_t + (1-\sigma\eta)\hat{c}^e_t, \\
\hat{c}^e_{F,t} &= \eta\gamma\hat{g}^{*e}_t + (1-\sigma\eta)\hat{c}^e_t, \\
\hat{c}^{*e}_{F,t} &= \eta\gamma\hat{g}^{*e}_t + (1-\sigma\eta)\hat{c}^{*e}_t, \\
\hat{c}^{*e}_{H,t} &= \eta\gamma\hat{g}^e_t + (1-\sigma\eta)\hat{c}^{*e}_t, \\
\varphi\hat{y}^e_t &= (1+\varphi)a_t - \gamma\hat{g}^e_t, \\
\varphi\hat{y}^{*e}_t &= (1+\varphi)a^*_t - \gamma\hat{g}^{*e}_t, \\
\hat{y}^e_t &= (1-\alpha)(1-\delta)\hat{c}^e_{H,t} + \alpha(1-\delta)\hat{c}^{*e}_{H,t} + \delta\hat{g}^e_t, \\
\hat{y}^{*e}_t &= \alpha^*(1-\delta)\hat{c}^e_{F,t} + (1-\alpha^*)(1-\delta)\hat{c}^{*e}_{F,t} + \delta\hat{g}^{*e}_t, \\
\hat{c}^e_t &= (1-\alpha)\hat{c}^e_{H,t} + \alpha\hat{c}^e_{F,t}, \\
\hat{c}^{*e}_t &= \alpha^*\hat{c}^{*e}_{H,t} + (1-\alpha^*)\hat{c}^{*e}_{F,t}.
\end{aligned}
\tag{15}
$$

4.2 Decentralization of the efficient allocation under flexible prices

4.2.1 Steady state and monopolistic distortion

In section B.8, we show that the steady state of the economy coincides with the efficient steady state if $\tau = \frac{1}{\varepsilon}$ and if governments behave efficiently at steady state (i.e. $\left(\frac{N}{h}\right)^{\varphi}\frac{1}{\chi_C}\left(\frac{C}{h}\right)^{\sigma} = 1$).

4.2.2 Marginal cost under flexible prices

In the previous section, we showed that the economy will reach the efficient steady state. Therefore, we made sure that the log-deviation chosen by the planner are comparable to the log-deviation of the economy.

We denote $\bar{x}_t$ the log natural level of the variable X_t. Also $\hat{\bar{x}}_t$ denotes the natural log deviations of the variable X_t from its steady state value X. Natural values are the values taken by variables under flexible prices (i.e. $\theta \Rightarrow 0$).

When prices are fully flexible, we have

$$\bar{mc}_t = \bar{mc}^*_t = -\mu.$$

Therefore,

$$
\begin{aligned}
-\mu &= \sigma\bar{c}_t + \varphi\bar{y}_t + \alpha\bar{s}_t - (1+\varphi)a_t + \log(1-\tau) - (\varphi+\sigma)\log(h), \\
-\mu &= \sigma\bar{c}^*_t + \varphi\bar{y}^*_t - \alpha^*\bar{s}_t - (1+\varphi)a_t + \log(1-\tau) - (\varphi+\sigma)\log(1-h).
\end{aligned}
$$

Therefore, log-deviation of the natural variables must satisfy

$$0 = \sigma\hat{\bar{c}}_t + \varphi\hat{\bar{y}}_t + \alpha\bar{s}_t - (1+\varphi)a_t, \tag{16}$$

$$0 = \sigma\hat{\bar{c}}^*_t + \varphi\hat{\bar{y}}^*_t - \alpha^*\bar{s}_t - (1+\varphi)a^*_t, \tag{17}$$

and the good-market clearing conditions

$$\tilde{\sigma}(\hat{\bar{y}}_t - \delta\hat{\bar{g}}_t) = \sigma\hat{\bar{c}}_t + \bar{\alpha}(1-h)w_{\bar{\alpha}}\bar{s}_t, \tag{18}$$

$$\tilde{\sigma}(\hat{\bar{y}}^*_t - \delta\hat{\bar{g}}^*_t) = \sigma\hat{\bar{c}}^*_t - \bar{\alpha}hw_{\bar{\alpha}}\bar{s}_t, \tag{19}$$

and the IRS condition at equilibrium

$$\bar{s}_t = \tilde{\sigma}_{\bar{\alpha}}[\hat{\bar{y}}_t - \hat{\bar{y}}_t^* - \delta(\hat{\bar{g}}_t - \hat{\bar{g}}_t^*)]. \tag{20}$$

Given the exogeneous sequence $(a_t, a_t^*)_{t\in\mathbb{N}}$, we have a system of 5 equations and 7 unknowns. The system lacks two expressions.

4.2.3 Government spending under flexible prices

Now, we need to define natural log-deviation of government spending so that the efficient equilibrium is decentralized in the flexible price economy.

Let $\hat{\bar{g}}_t$ and $\hat{\bar{g}}_t^*$ be defined by

$$\gamma\hat{\bar{g}}_t = \sigma\hat{\bar{c}}_t + \alpha\bar{s}_t, \tag{21}$$

$$\gamma\hat{\bar{g}}_t^* = \sigma\hat{\bar{c}}_t^* - \alpha^*\bar{s}_t. \tag{22}$$

It is easy to show that these definitions are necessary and sufficient for the flexible price equilibrium to be equivalent to the efficient equilibrium.
NOTE THAT WHEN $\bar{\alpha} = 0$ and $\eta = 1$ we recover the formula obtain by Beetsma and Jensen (2002) (see appendix A). Using our notation, they find $-\gamma ba\hat{r}h_t = \varphi[(1-h)(1-\delta)\bar{s}_t + (1-\delta)\bar{c}_t^{CU} + \delta\hat{\bar{g}}_t] - (1+varphi)a$A ATTENTION SLIGHT DIFFERENCE FOR THE PRODUCTIVITY SHOCK. TRY TO EXPLAIN.

4.2.4 Summary of the flexible price equilibrium

Given the exogeneous sequence $(a_t, a_t^*)_{t\in\mathbb{N}}$, the endogeneous sequence $(\hat{\bar{y}}_t, \hat{\bar{c}}_t, \hat{\bar{g}}_t; \hat{\bar{y}}_t^*, \hat{\bar{c}}_t^*, \hat{\bar{g}}_t^*; \bar{s}_t)_{t\in\mathbb{N}}$ is given by

- *Home* and *Foreign* conditions on marginal cost in log-deviation form (16-17),
- *Home* and *Foreign* good-market clearing conditions in log-deviation form (18-19),
- the IRS condition at equilibrium in log-deviation form (20),
- *Home* and *Foreign* conditions on government spending in log-deviation form (21-22).

4.2.5 Formula for the natural level of output

As show in section B.9, natural output writes

$$\begin{aligned}
\hat{\bar{y}}_t &= \Gamma^g_{\bar{\alpha},h}\delta\hat{\bar{g}}_t + \Gamma^a_{\bar{\alpha},h}a_t + \Gamma^{\text{ext}}_{\bar{\alpha},h}(\hat{\bar{y}}_t^* - \delta\hat{\bar{g}}_t^*) \\
\hat{\bar{y}}_t^* &= \Gamma^g_{\bar{\alpha},1-h}\delta\hat{\bar{g}}_t^* + \Gamma^a_{\bar{\alpha},1-h}a_t^* + \Gamma^{\text{ext}}_{\bar{\alpha},1-h}(\hat{\bar{y}}_t - \delta\hat{\bar{g}}_t)
\end{aligned}$$

where

$$\begin{aligned}
\Gamma^g_{\bar{\alpha},h} &= \frac{\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}}{\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h} + \varphi} \\
\Gamma^a_{\bar{\alpha},h} &= \frac{1+\varphi}{\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h} + \varphi} \\
\Gamma^{\text{ext}}_{\bar{\alpha},h} &= -\frac{\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}}{\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h} + \varphi}.
\end{aligned}$$

INTERPRET.

4.2.6 Natural output when *Foreign* is a small open economy

In the limit case where *Foreign* is a small open economy (i.e. $1 - h = 0$), we have $\Omega_{\bar{\alpha},1-h} = 1$ and the coefficients entering *Foreign*'s natural output expression become

$$\Gamma^{g}_{\bar{\alpha},0} = \frac{\tilde{\sigma}_{\bar{\alpha}}}{\tilde{\sigma}_{\bar{\alpha}} + \varphi}$$
$$\Gamma^{a}_{\bar{\alpha},0} = \frac{1+\varphi}{\tilde{\sigma}_{\bar{\alpha}} + \varphi}$$
$$\Gamma^{\text{ext}}_{\bar{\alpha},0} = -\frac{\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}}{\tilde{\sigma}_{\bar{\alpha}} + \varphi}.$$

With $\delta = 0$, we replicate the results of Gali and Monacelli (2005).
INTERPRET LIMIT CASE.

5 Sticky price and policy trade-off

5.1 Model equation in gap form

In this section we combine the sticky price equilibrium and the flexible price equilibrium, to rewrite the equilibrium in gap form. With this representation, we aim to highlight the trade-offs between union stabilization and national stabilization.

5.1.1 Definitions : gap, fiscal stance and *Union*'s variables

We first provide some definitions. Let $\tilde{x}_t \equiv \hat{x}_t - \hat{\bar{x}}_t = x_t - \bar{x}_t$ be the log-deviation of the variable X_t from its natural level $\bar{X}_t$.

As Gali and Monacelli (2008), we also introduce the variable $\tilde{f}_t$ defined as

$$\tilde{f}_t \equiv \tilde{g}_t - \tilde{y}_t = log(G_t/Y_t) - log(\bar{G}_t/\bar{Y}_t) \simeq \frac{\delta_t - \bar{\delta}_t}{\bar{\delta}_t}$$

where $\delta_t \equiv \frac{G_t}{Y_t}$. If $\tilde{f}_t = 1\%$ it means that *Home*'s government consumption share in output at time t is 1% above its natural level. As Gali and Monacelli (2008) show, this variable is essential to understand how fiscal policy helps absorb productivity shocks. In the next section, we will include this variable in the equilibrium equations.

Let also define *Union*'s output $Y_t^{CU} \equiv Y_t + Y_t^*$ and *Union*'s fiscal stance $F_t^{CU} \equiv F_t + F_t^* = \frac{G_t}{Y_t} + \frac{G_t^*}{Y_t^*}$. Log-linearization around the steady state under both sticky and flexible price gives an expression of the *Union*'s output gap and fiscal stance gap

$$\tilde{y}_t^{CU} = h\tilde{y}_t + (1-h)\tilde{y}_t^*,$$
$$\tilde{f}_t^{CU} = h\tilde{f}_t + (1-h)\tilde{f}_t^*.$$

5.1.2 Model in gap form

Using section B.10 and *Union* gap definitions, we can rewrite Home's IS and NKPC as

$$\tilde{y}_t = \mathbb{E}_t\{\tilde{y}_{t+1}\} - \frac{\delta}{1-\delta}\mathbb{E}_t\{\Delta\tilde{f}_{t+1}\} - \frac{1}{\sigma_{\bar{\alpha}}}(\tilde{i}_t - \mathbb{E}_t\{\pi_{H,t+1}\}) + \bar{\alpha}\Theta_{\bar{\alpha}}\mathbb{E}_t\{\Delta\tilde{y}^{CU}_{t+1} - \frac{\delta}{1-\delta}\Delta\tilde{f}^{CU}_{t+1}\}, \tag{23}$$

$$\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \lambda\big[(\sigma_{\bar{\alpha}} + \varphi)\tilde{y}_t - \sigma_{\bar{\alpha}}\frac{\delta}{1-\delta}\tilde{f}_t + \sigma_{\bar{\alpha}}\bar{\alpha}\Theta_{\bar{\alpha}}(\tilde{y}^{CU}_t - \frac{\delta}{1-\delta}\tilde{f}^{CU})\big], \tag{24}$$

and Foreign's IS and NKPC as

$$\tilde{y}^*_t = \mathbb{E}_t\{\tilde{y}^*_{t+1}\} - \frac{\delta}{1-\delta}\mathbb{E}_t\{\Delta\tilde{f}^*_{t+1}\} - \frac{1}{\sigma_{\bar{\alpha}}}(\tilde{i}^*_t - \mathbb{E}_t\{\pi^*_{F,t+1}\}) + \bar{\alpha}\Theta_{\bar{\alpha}}\mathbb{E}_t\{\Delta\tilde{y}^{CU}_{t+1} - \frac{\delta}{1-\delta}\Delta\tilde{f}^{CU}_{t+1}\}, \tag{25}$$

$$\pi^*_{F,t} = \beta\mathbb{E}_t\{\pi^*_{F,t+1}\} + \lambda^*\big[(\sigma_{\bar{\alpha}} + \varphi)\tilde{y}^*_t - \sigma_{\bar{\alpha}}\frac{\delta}{1-\delta}\tilde{f}^*_t + \sigma_{\bar{\alpha}}\bar{\alpha}\Theta_{\bar{\alpha}}(\tilde{y}^{CU}_t - \frac{\delta}{1-\delta}\tilde{f}^{CU})\big] \tag{26}$$

where $\sigma_{\bar{\alpha}} \equiv (1-\delta)\tilde{\sigma}_{\bar{\alpha}} = \frac{\sigma}{1+\bar{\alpha}\Theta_{\bar{\alpha}}}$ and

$$\begin{aligned} \tilde{i}_t &= \hat{i}^{CU}_t - \bar{r}_t, & \tilde{i}^*_t &= \hat{i}^{CU}_t - \bar{r}^*_t, \\ \bar{r}_t &= (1+\varphi)\mathbb{E}_t\{\Delta a_{t+1}\} + \varphi E_t\{\Delta\hat{\bar{y}}_{t+1}\} & \bar{r}^*_t &= (1+\varphi)\mathbb{E}_t\{\Delta a^*_{t+1}\} + \varphi E_t\{\Delta\hat{\bar{y}}^*_{t+1}\} \end{aligned}$$

while $\bar{r}_t$ and $\bar{r}^*_t$ denote respectively Home and Foreign natural rates (see section B.10 for computation).
WHY TRADEOFF?

5.2 Welfare loss approximation

As we saw in the previous section, in a sticky price economy, trade-off arise. They appear in the welfare loss caused by fluctuation around the natural allocation.

5.2.1 Union welfare criterion

We did not derive the second order approximation of the planner objective (SEE PLANNER PROBLEM). Instead, for simplicity, we decided to rely on the approximation made for analogous models. We found that the criterion proposed by Beetsma and Jensen (2002,2004) was relevant and well-formulated to encapsulate all the *Union*'s trade offs. Precisely, we define the instantaneous loss at time t at the Union level by

$$\begin{aligned} l^{CU}_t(h, \bar{\alpha}, \theta, \theta^*) \equiv\ & \xi_c \times (\tilde{c}^{CU}_t)^2 + \bar{\alpha}h(1-h) \times \xi_s \times (\tilde{s}_t)^2 \\ & + h \times \xi_g \times (\tilde{g}_t)^2 + h \times \xi_\pi \times (\pi_{H,t})^2 \\ & + (1-h) \times \xi_g \times (\tilde{g}^*_t)^2 + (1-h) \times \xi^*_\pi \times (\pi^*_{F,t})^2 \\ & + \xi_{c,g} \times \tilde{c}^{CU}_t \tilde{g}^{CU}_t + \bar{\alpha}h(1-h) \times \xi_{s,g^R} \times \tilde{s}_t(\tilde{g}_t - \tilde{g}^*_t). \end{aligned}$$

where

$$\begin{aligned} &\xi_c \equiv (1-\delta)\big(\sigma + (1-\delta)\varphi\big), && \xi_s \equiv (1-\delta)\big(1 + \varphi(1-\delta)\big), && \\ &\xi_g \equiv \delta(\gamma + \varphi\delta), && \xi_\pi \equiv \frac{\varepsilon}{\lambda}, && \xi^*_\pi \equiv \frac{\varepsilon}{\lambda^*}, \\ &\xi_{c,g} \equiv 2(1-\delta)\varphi, && \xi_{s,g^R} \equiv 2(1-\delta)\delta\varphi. && \end{aligned}$$

The interpretation of this quadratic loss given in Beetsma and Jensen (2002) can be summarized as follow. First, it features inflation rates and the terms-of-trade gap as cause dispersion in relative goods' prices both among and across *Home* and *Foreign*. Secondly, it involves a welfare loss associated with fluctuations in private and public consumption that are increasing utility parameters σ, γ and φ. Thirdly, it is increasing in the co-movement of *Union*'s private and public consumption which add undesirable work effort.

Note that we have slightly modified the author's original formulation. Indeed, in Beetsma and Jensen (2002), $\bar{\alpha} = 1$ and $\eta = 1$. To gain in accuracy, we multiply all the terms involving $\tilde{s}_t$ by $\bar{\alpha}$. However, we do not incorporate η in the welfare loss function in order no to depart too much from the original version.

In the next section, we will allow *Union* to consider a loss that does not take h as an argument but any weight $w_H \in (0, 1)$.
Therefore, we let the expected discounted future *Union*'s loss at time $t = 0$ be defined by

$$\mathcal{L}_0^{CU}(w_H, \bar{\alpha}, \theta, \theta^*) \equiv \mathbb{E}_0\left\{ \sum_{t=0}^{\infty} \beta^t l_t(w_H, \bar{\alpha}, \theta, \theta^*)^{CU} \right\}.$$

In the simulation, this discounted loss will serve as an objective to be minimized with respect to the policy variables.

5.2.2 Regional welfare criterion

Before continuing to the simulation section, we want to define domestic welfare criteria for *Home* and *Foreign*. It seems indeed relevant to have a measure of individual country welfare from their own point of view. A domestic welfare criterion is myopic to the *Union* fluctuations and only features domestic fluctuations.

Formally, we define *Home* domestic criterion by

$$\begin{aligned} l_t^H(h, \bar{\alpha}, \theta, \theta^*) \equiv\ & \xi_c \times (\tilde{c}_t)^2 + \bar{\alpha}(1-h) \times \xi_s \times (\tilde{s}_t)^2 \\ & + \xi_g \times (\tilde{g}_t)^2 + \xi_\pi \times (\pi_{H,t})^2 \\ & + \xi_{c,g}\tilde{c}_t\tilde{g}_t, \end{aligned}$$

and *Foreign*'s domestic criterion by

$$\begin{aligned} l_t^F(h, \bar{\alpha}, \theta, \theta^*) \equiv\ & \xi_c \times (\tilde{c}_t^*)^2 + \bar{\alpha}h \times \xi_s \times (\tilde{s}_t^*)^2 \\ & + \xi_g \times (\tilde{g}_t^*)^2 + \xi_\pi^* \times (\pi_{F,t}^*)^2 \\ & + \xi_{c,g}\tilde{c}_t^*\tilde{g}_t^*, \end{aligned}$$

We interpret domestic criteria as the objective national governments would like to minimize if they had no requirement from *Union*'s authorities on how to conduct fiscal policy.

As for *Union*, we define country i expected future discounted loss at time $t = 0$ by

$$\mathcal{L}_0^i(h, \bar{\alpha}, \theta, \theta^*) \equiv \mathbb{E}_0\left\{ \sum_{t=0}^{\infty} \beta^t l_t^i(h, \bar{\alpha}, \theta, \theta^*) \right\} \text{ where } i = \{H, F\},$$

and where H stands for *Home* and F stands for *Foreign*.

6 Simulations

At this stage, we have defined the equilibrium of the economy around the steady state and we have introduced metrics to measure welfare loss caused by disturbances around the natural allocation, both at the national and union level. We are therefore equipped to simulate the model and evaluate different policy regimes.

Throughout this section, we study the Impulse Response Functions (IRFs) of the model variables to a 1% negative productivity shock affecting *Foreign*.

We begin with detailing and justifying our choice of calibration. Then we explain how we define the different policy setup and how we declared them on Dynare to configure different monetary and fiscal policy regimes. After, we simulate the model under different policy configurations and for different *Union*'s features. Precisely, we carefully interpret the IRFs and derive general principle governing optimal fiscal and monetary policy. We also investigate if optimal dynamics can be replicated by simple implementable rules and we report the welfare loss associated. Finally, we propose to investigate a subcase where *Foreign*'s fiscal policy is constrained. Within this subcase, we allow *Home* to pursue a domestic-oriented criterion instead of a *Union*-oriented criterion, and we report the welfare losses associated. Therefore, we propose a way to quantify *Home*'s political incentive to cooperate with *Union*'s fiscal authorities when *Foreign*'s fiscal policy is constrained.

6.1 Calibration

Table 1: Calibration

Parameter	Value
Elasticity of substitution among goods produced in the same country	$\varepsilon = 6$
Intertemporal elasticity of substitution of the private goods	$\sigma^{-1} = 1/3$
Intertemporal elasticity of substitution of the public goods	$\gamma^{-1} = 1$
Elasticity of substitution between home and foreign private goods	$\eta = 4.5$
Elasticity of substitution of labor	$\varphi = 1$
Preferences discount factor	$\beta = 0.99$
Steady state government spending share	$\delta = 0.25$
Autocorrelation of shocks	$\rho_a = 0.95$
Foreign's price stickiness	$\theta^* = 0.75$
Home's size	$h = \{0.5, 0.75\}$
Degree of openness for a small open economy	$\bar{\alpha} = \{0.4, 0.6\}$
Home's price stickiness	$\theta = \{0.5, 0.75\}$

Our model calibration is given in Table 1. We follow Forlati (2006) to choose the parameters entering the utility function, the elasticity of substitution among goods produced in the same country and for the degree of price stickiness in *Foreign*. It is important to note that under our calibration households are assumed to be more adverse to risk in private consumption fluctuations that in public consumption fluctuations ($\sigma > \gamma$). This will have welfare consequences since σ and γ enter the welfare loss criteria. Besides, we follow Gali and Monacelli (2008) in the choice of the steady state government consumption share in output by setting = 0.25.

In the simulations, we want to simulate the model responses under both different policy regimes and economies features. Therefore, we will vary the parameters *Union* along three dimensions: the level of asymmetry between *Home* and *Foreign* (h), the (limit) degree of openness ($\bar{\alpha}$) and *Home*'s degree of nominal rigidity. This parameter space will allow us test the robustness of the monetary and fiscal policy implications.

6.2 Policy setups and methodology

6.2.1 Policy Regimes and Dynare commands

Before continuing to the simulation part, we need to define the different policy regimes we will as study and explain how to use Dynare commands to declare them. In section (GIVE NUMBER OF MODEL SECTION) we setup a model which needs three more equations to be complete. These equations correspond to *Home*'s fiscal policy, *Foreign*'s fiscal policy and *Union*'s monetary policy. We will use Dynare marker[2] to complete the model with fiscal and monetary equations under different configuration.

In the first configuration, *Union*'s fiscal authorities minimize an objective by choosing monetary and fiscal instruments subject to equilibrium condition of the economy. Precisely, this is an optimal policy under commitment where at time $t = 0$ *Union*'s authorities choose a state-contingent policy $(\tilde{g}_t, \tilde{g}_t^*, \tilde{i}_t)$ that minimizes any objective. As we have already teased, we allow *Union*'s authorities to choose between the population weighted criterion where $w_H = h$ and the Equally-weighted criterion where $w_H = 0.5$ regardless the value of h. If *Union*'s fiscal authorities opt for $w_H = h$, they consider *Union* as a continuum of individuals and assign the same weight to any *Home* and *Foreign* household. If instead *Union*'s fiscal authorities opt for $w_H = 0.5$, they consider the *Union* as a sum of two country and disregard their relative size.[3]. In Dynare, this configuration is declared and run using `planner_objective` (instantaneous objective to minimize), `ramsey_model` (policy instruments and discount rate) and `stoch_simul` (run simulations) commands. Henceforth, we refer this configuration as the Ramsey policy setup.

In a second configuration, *Union*'s authorities are constrained to assign to *Home* and *Foreign* governments optimal simple implementable rules. Precisely, *Union*'s fiscal authorities optimize the parameters entering simple national fiscal rules so as to minimize a linear quadratic objective, taking as given the rule governing *Union*'s nominal interest rate. In other words, we have an independent central bank which follows a simple interest rate rule (e.g. a Taylor rule), while *Union*'s fiscal authorities communicate to national policy makers optimal fiscal parameters entering their national fiscal policy rule.

[2]For more details, refer to Dynare Reference Manual, 4.19 Optimal Policy

[3]The population weighted criterion is the one chosen by the benevolent social planner. Yet, we argue that, in reality, a currency union may assign the same welfare weight to the fluctuations of small country as those of a big country. Indeed, from financial to political reasons that go beyond this model, fluctuations in a small country can have detrimental spillover welfare effects if they are not taken into account and left under treated. This argument does not contradict our assumption that in a flexible price economy the benevolent social planner follows the population weighted criterion. We are just allowing *Union*'s fiscal authorities to behave differently when they observe price stickiness, by assigning a biased weight to national welfare

In Dynare, this configuration is declared and run using `optim_param` (parameters to optimize), `optim_weights` (objective to minimize), `osr_bounds` (parameters constrainst) and `osr` (run simulations) commands.[4] Henceforth, we refer this configuration as the OSR policy setup.

For each configuration, we will also consider a sub-case where *Foreign*'s fiscal policy is constrained. This constraint just add one equation to the model without modifying the tools to declare and run the simulations.

SUMMARIZE THE DIFFERENT SCENARIO BELOW.

6.2.2 Comparing welfare

To assess the performance of the OSR setup compared to the Ramsey setup, we measure the welfare loss in consumption equivalence term (CEV).

We first measure the welfare differential between the OSR and the Ramsey setup, both at the national and union level.

The loss associated with the change of policy regimes at *Union*'s level writes

$$\Delta\mathcal{L}^{CU}(w_H, h, \bar{\alpha}, \theta, \theta^* \equiv \mathcal{L}^{CU}(\text{OSR fluctuations}; w_H, h, \bar{\alpha}, \theta, \theta^*) - \mathcal{L}^{CU}(\text{RAMSEY fluctuations}; w_H, h, \bar{\alpha}, \theta$$

while for country $i \in \{H, F\}$ it writes

$$\Delta\mathcal{L}^{i}(w_H, h, \bar{\alpha}, \theta, \theta^*) \equiv \mathcal{L}^{i}(\text{OSR fluctuations}; h, \bar{\alpha}, \theta, \theta^*) - \mathcal{L}^{i}(\text{RAMSEY fluctuations}; w_H, h, \bar{\alpha}, \theta, \theta^*),$$

because domestic welfare criterion is not affected by w_H.
Note that the weight attached to consumption in the discounted welfare loss ξ_c is the same across area. We define the consumption equivalence as the permanent percentage deviation from the natural allocation that would perfectly equalize the loss incurred by change in the policy regime.
Therefore, when $\Delta\mathcal{L}^i \geq 0$, we solve

$$\frac{\xi_c}{1-\beta}\left(\frac{CEV^i}{100}\right)^2 = \Delta\mathcal{L}^i \Rightarrow CEV^i \equiv 100\sqrt{\frac{1-\beta}{\xi_c}\Delta\mathcal{L}^i}$$

Though Ramsey fluctuations will always be preferable to OSR fluctuations at *Union*'s level (by definition), this may not hold at the national level. In particular, we will find some situations where *Home* prefers OSR fluctuations to Ramsey fluctuations, i.e. $\Delta\mathcal{L}^H < 0$. When $\Delta\mathcal{L}^H < 0$, we will report

$$CEV^i \equiv -100\sqrt{\frac{1-\beta}{\xi_c}\Delta\mathcal{L}^i},$$

meaning that we would have remove loss from the Ramsey setup to match OSR welfare loss.

[4] Note also that before computing the OSR parameters, we conduct a sensitivity analysis to check determinacy and explosiveness issues. To do so, we use `estimated_params` and `dynare_sensitivity` commands.

6.3 IRFs under flexible price

Before analyzing optimal fiscal and monetary policy in a sticky world economy, we provide an overview of the natural fluctuations that occurs in a flexible price economy following a negative 1% productivity shock in *Foreign*.

We consider a symmetric economy where $h = 0.5$ with $\bar{\alpha} = 0.4$ which implies that the degree of openess of Home and Foreign is $\alpha = \alpha^* = 0.2$. When *Foreign* is hit by a negative productivity shock, Home becomes more competitive than *Foreign*. As a consequence, relative consumption baskets shift toward *Home*-made goods which boosts Home's output and depresses *Foreign*'s output. However, total consumption drops for both countries as the shift of consumption toward *Home*-made goods does not fully compensate the drop in Foreign-made goods. Yet, the drop in total consumption is steeper in *Foreign* which is more affected by the loss in competitivity due to its home bias. Government consumption is set so as to equalize the marginal utility of consumption and therefore drops in both country. Nevertheless, the fiscal stance differs across countries. *Foreign* fiscal stance is positive which means that the government share in output is above its steady state level. Meanwhile, *Home* fiscal stance is significantly negative. This phenomenon can be explain as follows. In Home, government consumption drops and output rise so the *Home*'s fiscal stance turns out to be negative so as to allow private (which is more valued than public consumption by households) consume *Home*-made goods. In *Foreign* the phenomenon is more ambiguous since both government consumption and output drop. However, it is clear from the good market clearing condition that the drop in *Foreign* private consumption and exports must be compensated by a combination of a drop in output and a rise in the fiscal stance. Here the fiscal stance rise so as to substitute for the massive decrease in the consumption of *Foreign*-made goods. Therefore, the role play by a fiscal stance is opposed in Home and Foreign. Indeed, in *Home* it contracts massively to allow a higher weight of private consumption while in Foreign it rise slightly.

6.4 Simulations when *Foreign* fiscal policy is unconstrained

We begin the simulation considering *Foreign* policy as unconstrained which means that *Foreign* fiscal tools can be used by *Union*'s fiscal authorities to stabilize the *Union*'s economy. In section (SECTION NUMBER), we will investigate when Foreign is constrained.

In this section, we study monetary and fiscal policy when *Foreign* fiscal policy is unconstrained. In first time, analyze what are the responses in a Ramsey setup and how they are modified by changes in the parameters. In a second time, we investigate how simple rule perform in replicating responses of a Ramsey setup.

6.4.1 Monetary and Fiscal policy in a Ramsey setup

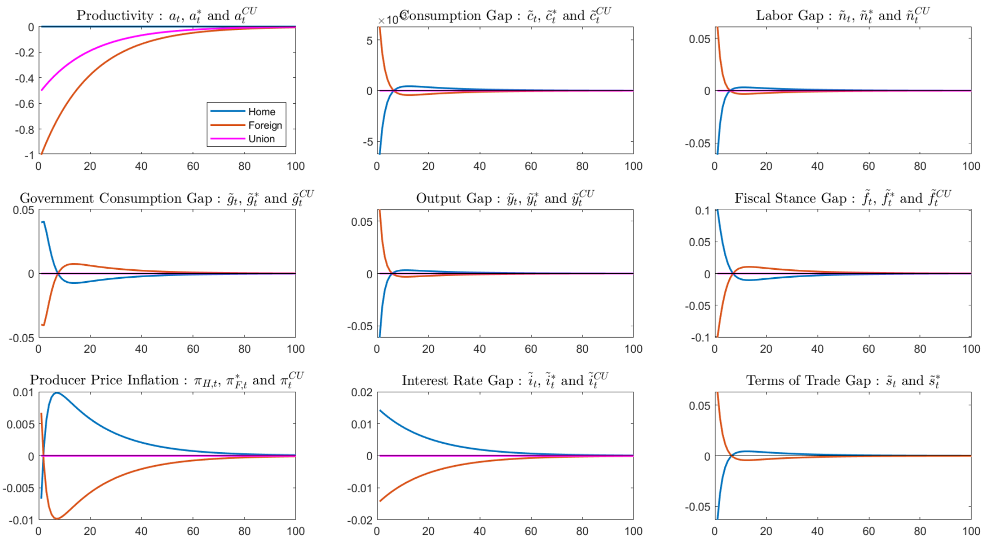
We analyze how optimal fiscal and monetary policy is set in the symmetric union with sticky prices where $\theta = \theta^* = 0.75$. The IRFs are reported in Figure 7. Under sticky price, the terms of trade does not adjust instantaneously and efficiently, leading to a negative *Home* terms of trade (meaning *Home* competitivity is below its natural level) which distorts national economies. However, it is optimal for monetary and fiscal policy to fully close the interest gap, the fiscal stance gap and the output gap, in response to asymmetric movements in productivity. This is a common result in the literature (see Gali and Monacelli, 2008). However, as prices are sticky, stabilization tradeoff arise. In

Home (Foreign) the optimal policy mix is such that the productivity change is absorbed through a combination of a fall (rise) in the output gap and a rise (fall) in the fiscal stance gap. In a first time, this policy mix leads to a rise (fall) in Home (*Foreign*) inflation. As soon as gaps domestic gaps are closed, domestic nominal interest rate gaps and inflation progressively go back to their steady state level.

Proposition 1 *In a symmetric Union where* $h = 0.5$ *and* $\theta = \theta^*$*, it is optimal to fully close Union's gaps.*

From this analysis we can draw general monetary and fiscal principles. Importantly, we find that it is optimal for government spending gap to be countercyclical with respect to output gap. This is also a common result in the literature (see Beetsma and Jensen, 2002). Also the terms of trade gap is countercyclical with respect to output gap. This means that, if output is above its natural level, government share in output should be below its natural level. INTERPRET WITH RESPECT TO CONSUMPTION??

Figure 1: *Foreign* 1% negative productivity shock - Ramsey policy - *Foreign* unconstrained - $h = 0.5$, $\bar{\alpha} = 0.4$ and $\theta = 0.75$.

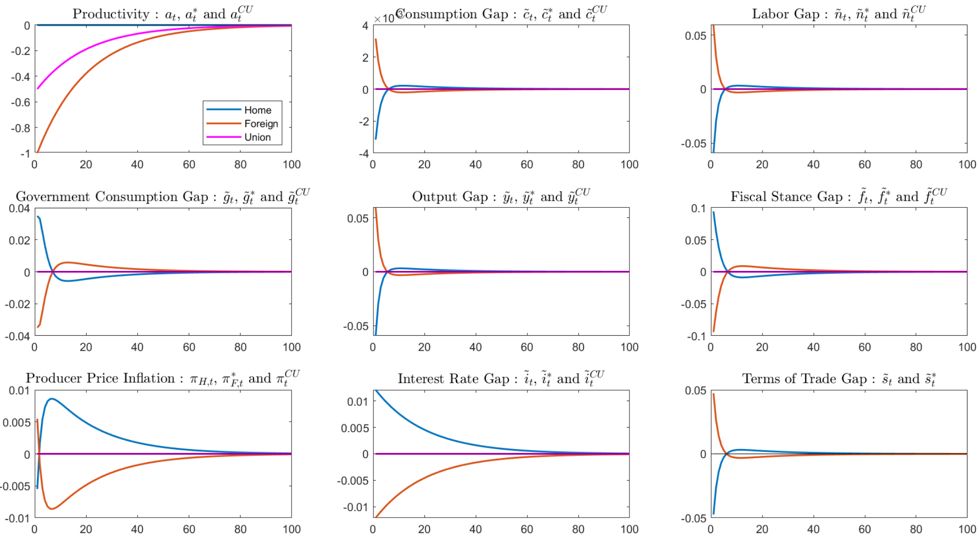


POP WEIGHT, FOREIGN UNCONSTRAINED, RAMSEY, $h = 0.5$, $\bar{\alpha}$ =0.4, $\theta = 0.75$

We will now analyze how IRFs are affected by parameter changes.

A higher degree of openness ($\bar{\alpha} = 0.4 \rightarrow \bar{\alpha} = 0.6$)
The shape of the IRFs are not modified when the degree of openness increases and the previous proposition is still holding. The modification appear in the magnitude of the responses, particularly for consumption. With a higher $\bar{\alpha}$, national home biases are decreased and *Home* and *Foreign* are more interdependent. Therefore, the cost of consumption adjustment is less important with a higher $\bar{\alpha}$ which reduce the consumption gap. Because of the increase in the inter-dependency, national fluctuations have more impact abroad. Therefore, the magnitude of the deviations from the natural allocation can be reduced.

Figure 2: *Foreign* 1% negative productivity shock - Ramsey policy - *Foreign* unconstrained - $h = 0.5$, $\underline{\bar{\alpha} = \mathbf{0.6}}$ and $\theta = 0.75$.



POP WEIGHT, FOREIGN UNCONSTRAINED, RAMSEY, $h = 0.5$, $\bar{\alpha} = 0.6$, $\theta = 0.75$

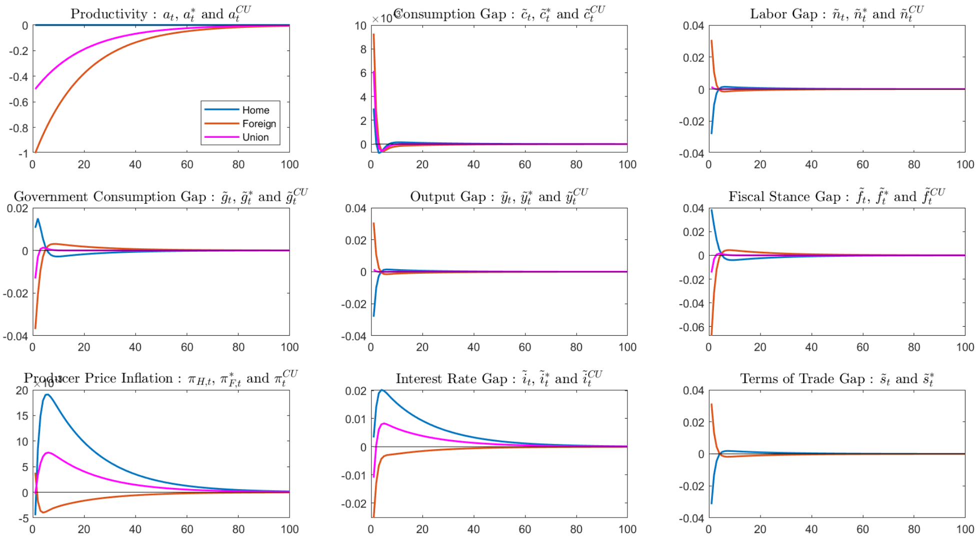
A lower degree of Home price stickiness ($\theta = 0.75 \rightarrow \theta = 0.5$)

As shown in Figure 9 when *Home* has a lower degree of nominal rigidity than *Foreign*, fiscal and monetary policies implication are altered. Before analyzing the changes, it is important to understand the implication of a lower degree of nominal rigidity in *Home* from a theoretical point of view. In a limit case where $\theta = 0$, policy makers effort would (almost) entirely shift toward *Foreign* stabilization. Therefore, a lower degree of nominal rigidity in *Foreign* simplifies the stabilization problem for *Union* fiscal and monetary authorities. Here, we observe part of this phenomenon. First, we note that *Union* nominal interest rate and inflation gaps are not closed while other *Union*'s gaps are closed after just a few quarters. Indeed, *Union*'s nominal interest rate is set above its natural level so as to be (almost) exclusively oriented towards closing *Foreign*'s nominal interest gap. As $\xi_c < \xi_c^*$, *Foreign*'s inflation fluctuations are more welfare detrimental than those of *Home*.

Second, we note that the symmetry observed in the fiscal policies in Figure 7 is now broken as *Foreign* absolute fiscal stance deviations are bigger than those of *Home*. This asymmetry is linked to output gap asymmetry : because *Foreign*'s output gap deviates more than in Home, *Foreign*'s fiscal effort is more important than in *Foreign*.

Finally, we remark that fiscal and monetary policies perform better when $\theta < \theta^*$ than when $\theta = \theta^*$ as fluctuations have smaller magnitude and gaps are closed more rapidly.

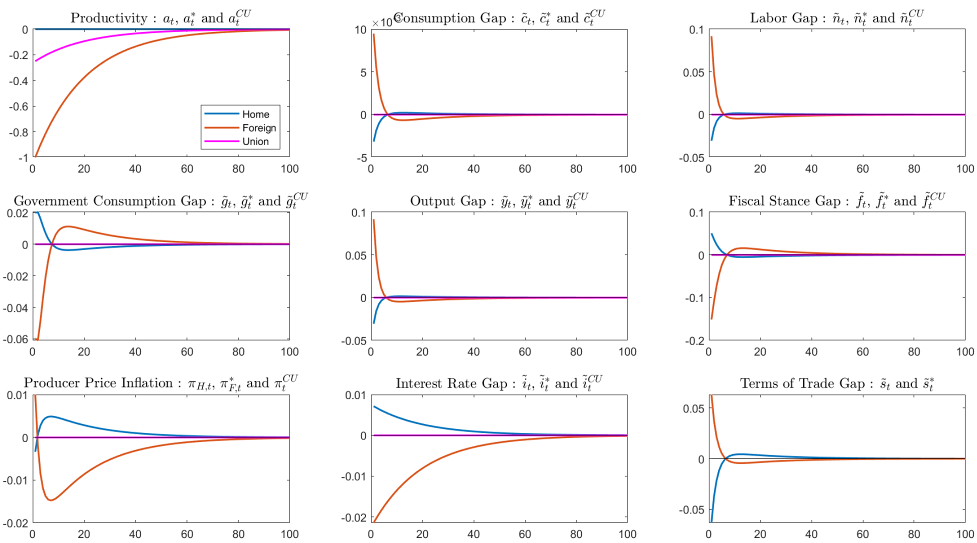
Figure 3: *Foreign* 1% negative productivity shock - Ramsey policy - *Foreign* unconstrained - $h = 0.5$, $\bar{\alpha} = 0.4$ and $\underline{\theta = \mathbf{0.5}}$.



POP WEIGHT, FOREIGN UNCONSTRAINED, RAMSEY, $h = 0.5$, $\bar{\alpha} = 0.4$, $\theta = 0.5$

A bigger Home economy $(h = 0.5 \rightarrow h = 0.75)$
When Home economy is bigger than Foreign economy and when *Union*'s authotities follow a population-weighted objective ($w_H = h$, the policy implications are analogous to those obtain in Figure 7. The only difference are to be found in the magnitude : *Home*'s deviation's are about three times smaller than those of *Foreign*. This is due to the fact that, with a population-weighted objective, *Union*'s authorities prefer to control for the fluctuations from *Home* than from Foreign.

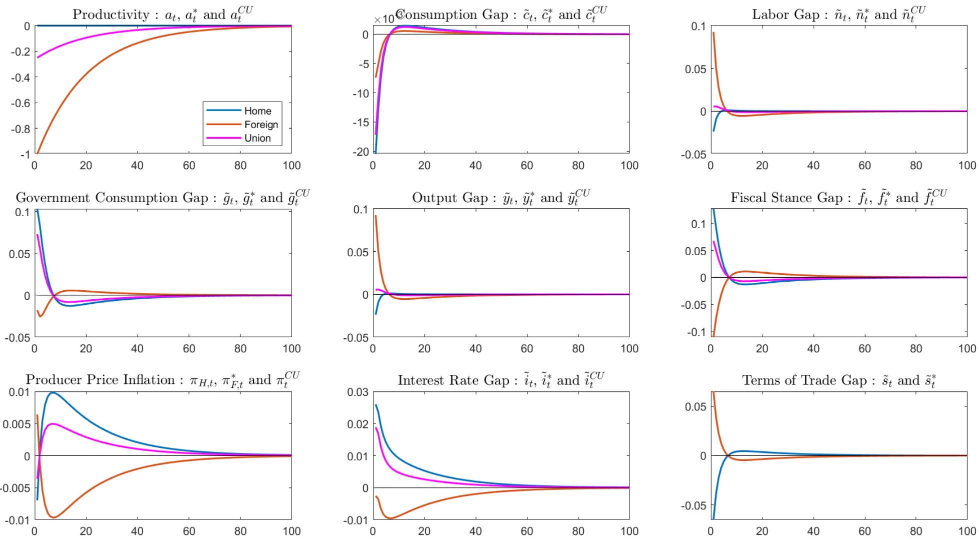
Figure 4: *Foreign* 1% negative productivity shock - Ramsey policy - *Foreign* unconstrained - $\underline{h = \mathbf{0.75}}$, $\bar{\alpha} = 0.4$ and $\theta = 0.75$.



POP WEIGHT, FOREIGN UNCONSTRAINED, RAMSEY, $h = 0.75$, $\bar{\alpha} = 0.4$, $\theta = 0.75$

However, when *Union*'s authorities pursue an equally-weighted objective ($w_H = 0.5$), monetary and fiscal policies differ significantly from Figure 10. While output gap responses follow the same trajectory as for a population-weighted objective, fiscal stance and inflation gaps now are symmetric. This means that, as soon as *Union*'s authorities pursue an equally-weighted objective, they will prescribe a symmetric fiscal effort to *Home* and Foreign. Besides, compared to Figure 10, *Union*'s nominal interest gap is positive so has not share the inflation burden accross countries.

Figure 5: *Foreign* 1% negative productivity shock - Ramsey policy - *Foreign* unconstrained - **Equally-weighted objective** - **$\underline{h = 0.75}$**, $\bar{\alpha} = 0.4$ and $\theta = 0.75$.



EQUAL WEIGHT, FOREIGN UNCONSTRAINED, RAMSEY, $h = 0.75$, $\bar{\alpha} = 0.4$, $\theta = 0.75$

We conclude this section with general principles on how monetary and fiscal policies should be conduct in currency union. COUNTERCYCLICAL SYMMETRY OF EFFORT ALTERED BY PARAMETERS.

In this section we have studies monetary and fiscal policy in a Ramsey setup. Nevertheless, it is hard to draw policy recommendation from Ramsey solutions which are purely theoretical. Instead, we need to investigate how simple rule could perform in replicating Ramsey fluctuations while limiting the welfare loss. This is the objective of the next section.

6.4.2 Monetary and Fiscal policy in an OSR setup

We begin with defining the monetary and fiscal policy rules.

For *Union*'s monetary policy we consider two different rules. The first one is a standard Taylor rule which writes

$$\tilde{i}_t^{CU} = 1.5 \times \pi_t^{CU} + 0.5 \times \tilde{y}_t^{CU}, \tag{27}$$

The second one follows Blanchard (2015) and allow nominal interest rate inertia, it writes

$$\tilde{i}_t^{CU} = 0.7 \times \tilde{i}_{t-1}^{CU} + 2.5 \times \pi_t^{CU} + 0.125 \times \tilde{y}_t^{CU}. \tag{28}$$

We encountered issues in the choice of the fiscal rule. Indeed, we first tried the fiscal rules we found in the literature (Beetsma and Jensen, 2002 ; Kirsanova et al., 2007). However, these policies often lead to a limited set of parameters leading to saddle-path solution. We decided to retain the rule proposed by Vieira. We call this rule the "government consumption rule" and it writes

$$\tilde{g}_t = \rho_g \times \tilde{g}_{t-1} + \Phi_y \times (\tilde{y}_{t-1} - \tilde{y}^{CU}_{t-1}) + \Phi_\pi \times \pi_{H,t-1}, \tag{29}$$

$$\tilde{g}^*_t = \rho_g \times \tilde{g}^*_{t-1} + \Phi^*_y \times (\tilde{y}^*_{t-1} - \tilde{y}^{CU}_{t-1}) + \Phi^*_\pi \times \pi^*_{F,t-1}. \tag{30}$$

The government consumption rule features both inertia and lag. The coefficient ρ_g is calibrated according to Blanchard (2015) so that $\rho_g = 0.92$. We consider a lagged in the response to count for delays in reporting and parliamentary timing restrictions.
We report the welfare loss associated with the OSR setup in Table 2 when monetary follow a rule (EQUATION) and in Table 3.
The greener the cell, the lower the consumption equivalence, the lower the welfare loss differential between the OSR setup and the Ramsey setup.
We observe that for any parameter configuration, consumption equivalence at Union's is always below 1%. This means that *Union*'s loss from conducting fiscal policy in the OSR setup instead of the Ramsey setup represent less than the welfare loss a permanent 1% consumption deviation from its natural level. This result is particularly important since it brings a convincing argument that there is room for policy coordination, even in a OSR setup which is the policy relevant setup. From Table 2, we also learn that switching for the OSR setup is less costly when $\theta < \theta^*$. Also, while the CEV are equal when the economies are of the same size (h=0.5), *Foreign* is more affected than *Home* by the OSR setup when *Home*'s economy is bigger ($h > 0.5$). In addition, when $h = 0.75$ *Home*'s cost in the OSR setup is lower and independent of θ. This finding support policy coordination in an asymmetric currency union.

Table 2: Consumption equivalence OSR vs. Ramsey when *Foreign* is unconstrained - Monetary policy follows (27) - Fiscal policy follows (29-30)

			Population-weighted objective			Equally-weighted objective		
h	$\bar{\alpha}$	θ	F	H	CU	F	H	CU
0.5	0.4	0.5	0.53	0.57	0.55			
		0.75	0.99	1	1.01			
	0.6	0.5	0.47	0.51	0.49			
		0.75	0.9	0.9	0.91			
0.75	0.4	0.5	1	0.47	0.65	0.68	0.72	0.66
		0.75	1.5	0.49	0.87	1.02	0.98	1.01
	0.6	0.5	0.89	0.42	0.58	0.56	0.66	0.58
		0.75	1.36	0.44	0.79	0.89	0.9	0.91

We check test the robustness of our findings to the monetary policy rule. The CEV when monetary policy is set according to (28) are shown in Table 3. We note that at *Union*'s level CEVs are higher when monetary follows (28) instead of (27) but are below 1.2 %

when *Union*'s fiscal authorities follow a population-weighted objective. Contrary to the standard Taylor rule, a
Again, we find that when the degree of price stickiness is the same in *Home* and *Foreign* ($\theta = \theta^*$), CEVs are the same at the domestic level

Table 3: Consumption equivalence OSR vs. Ramsey when *Foreign* is unconstrained - Monetary policy follows (28) - Fiscal policy follows (29-30)

			Population-weighted objective			Equally-weighted objective		
h	$\bar{\alpha}$	θ	F	H	CU	F	H	CU
0.5	0.4	0.5	1.91	-0.93	1.18			
		0.75	1	0.99	1.01			
	0.6	0.5	1.67	-0.79	1.04			
		0.75	0.9	0.9	0.91			
0.75	0.4	0.5	2.74	-0.86	1.15	2.78	-0.84	1.85
		0.75	1.5	0.49	0.87	2.22	-1.14	1.34
	0.6	0.5	2.4	-0.74	1.02	2.43	-0.7	1.62
		0.75	1.36	0.44	0.79	1.96	-0.98	1.19

Table 4: OSR coefficients when *Foreign* is unconstrained - Monetary policy follows (27) - Fiscal policy follows (29-30)

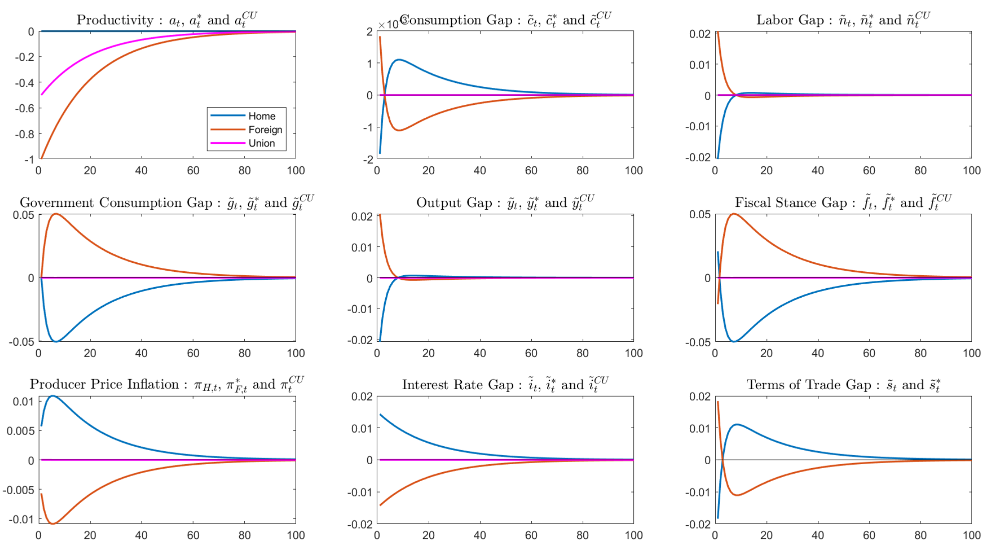
			Population-weighted objective				Equally-weighted objective			
			Foreign		Home		Foreign		Home	
h	$\bar{\alpha}$	θ	Φ_y^*	Φ_π^*	Φ_y	Φ_π	Φ_y^*	Φ_π^*	Φ_y	Φ_π
0.5	0.4	0.5	-0.05	0.16	1.96	-0.28				
		0.75	1.04	-0.25	1.04	-0.25				
	0.6	0.5	-0.03	0.17	2.26	-0.38				
		0.75	1.24	-0.41	1.24	-0.41				
0.75	0.4	0.5	0.48	0.06	2.46	-0.63	0.08	0.13	3.49	-0.51
		0.75	1.04	-0.25	1.03	-0.25	0.7	-0.13	2.06	-0.64
	0.6	0.5	0.55	0.04	2.91	-0.84	0.11	0.13	3.98	-0.63
		0.75	1.24	-0.41	1.24	-0.41	0.77	-0.22	2.66	-0.96

Table 5: OSR coefficients when *Foreign* is unconstrained - Monetary policy follows (28) - Fiscal policy follows (29-30)

			Population-weighted objective				Equally-weighted objective			
			Foreign		Home		Foreign		Home	
h	$\bar{\alpha}$	θ	Φ_y^*	Φ_π^*	Φ_y	Φ_π	Φ_y^*	Φ_π^*	Φ_y	Φ_π
0.5	0.4	0.5	0.84	-0.03	1.24	-0.22				
		0.75	1.04	-0.25	1.04	-0.25				
	0.6	0.5	1.03	-0.1	1.45	-0.33				
		0.75	1.24	-0.41	1.24	-0.41				
0.75	0.4	0.5	0.96	-0.14	1.28	-0.34	0.42	0.03	2.89	-0.53
		0.75	1.04	-0.25	1.04	-0.25	0.45	-0.04	2.82	-0.65
	0.6	0.5	1.15	-0.24	1.5	-0.49	0.51	0	3.41	-0.82
		0.75	1.24	-0.41	1.24	-0.41	0.51	-0.11	3.42	-1.03

THESE COEFFICIENT LEADS TO THESE IRFS.

Figure 6: *Foreign* 1% negative productivity shock - OSR - Monetary policy follows (28) - Fiscal policy follows (29-30) - Population-weighted objective - $h = 0.5$, $\bar{\alpha} = 0.4$ and $\theta = 0.75$.



POP WEIGHT, FOREIGN UNCONSTRAINED, OSR, TAYLOR, G GAP RULE, h = 0.5, $\bar{\alpha}$ =0.4, θ = 0.75

DESCRIBE THE RULE WE CONSIDER FOR MP AND FP. EXPLAIN IF THEY REPLICATE THE RAMSEY SOLUTION.
SHOW THE WELFARE LOSS ASSOCIATED.
WHAT IF THE MONETARY POLICY IS DIFFERENT.
PARTIAL CONCLUSION.
TRANSITION : IN REAL WORLD, PERIPHERY MAY BE CONSTRAINED. UNION AUTHORITY MAY ONLY RELY ON THE CORE PERIPHERY. IN THE NEXT SEC-

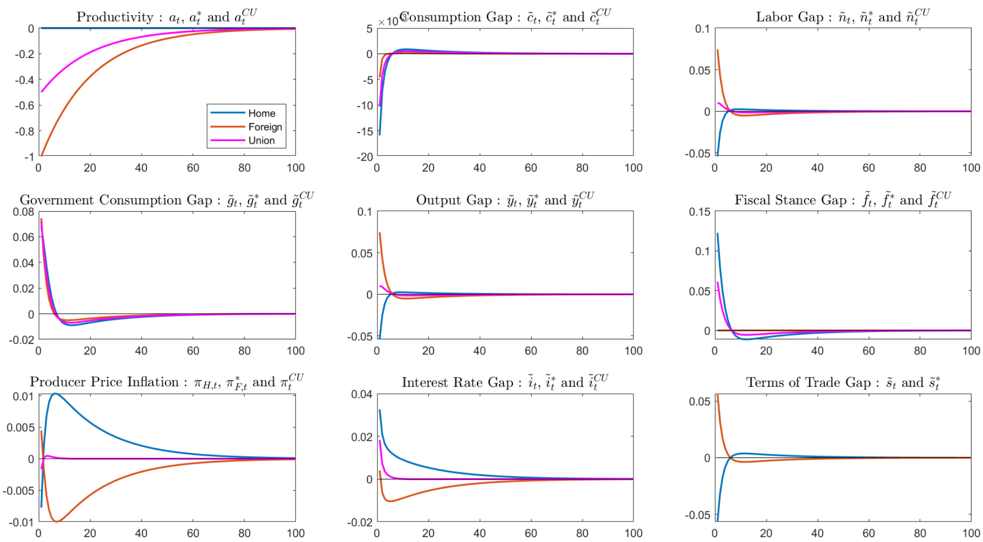
TION, WE REPLICATE THE PREVIOUS ANALYSIS. WE CONFRONT OUR RESULT.

6.5 Simulations when *Foreign* policy is constrained

WHEN FOREIGN IS CONSTRAINT, WHAT IS THE INCENTIVE FOR HOME TO FOLLOW A UNION-ORIENTED OBJECTIVE? WHAT WOULD BE THE COST FOR HOME TO MOVE FROM A HOME-ORIENTED OBJECTIVE TO A UNION-ORIENTED OBJECTIVE?
IN THIS SECTION WE DO NOT ONLY INVESTIGATE THE PERFORMANCE OF OSR BUT ALSO THE POLITICAL FEASIBILITY OF IMPOSING A UNION-WIDE OBJECTIVE TO HOME.
CONCLUDE THIS SECTION.

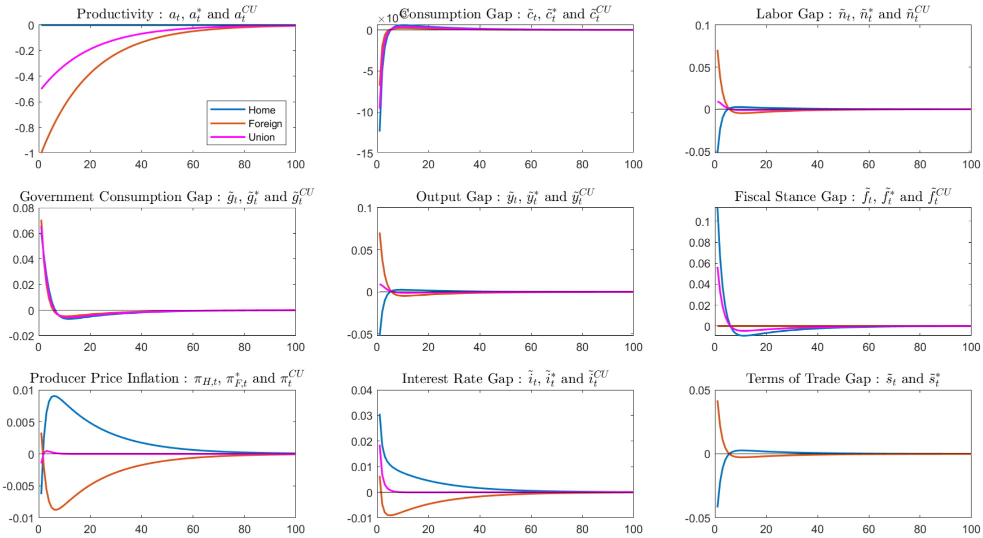
6.5.1 Ramsey setup

Figure 7: *Foreign* 1% negative productivity shock - Ramsey policy - ***Foreign* constrained** - $h = 0.5$, $\bar{\alpha} = 0.4$ and $\theta = 0.75$.



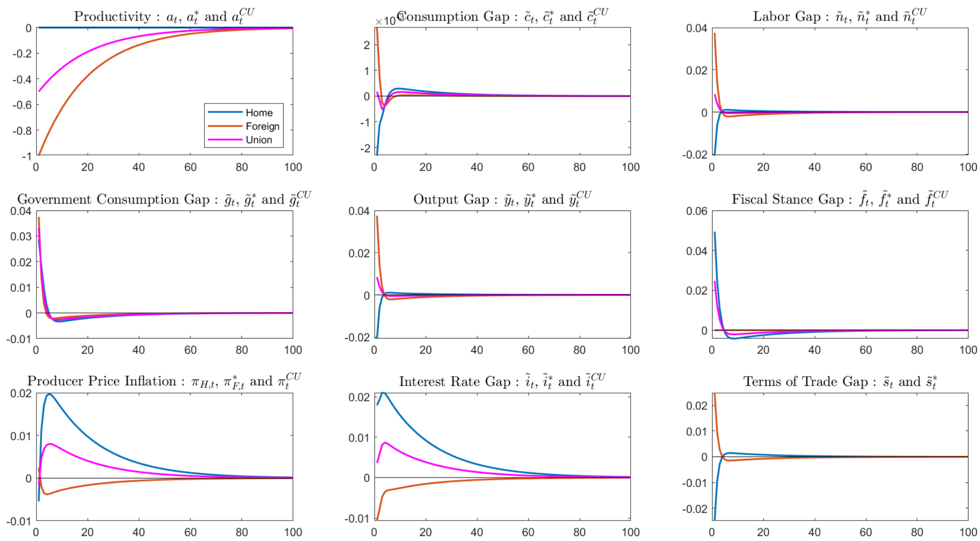
POP WEIGHT, FOREIGN CONSTRAINED, RAMSEY, $h = 0.5$, $\bar{\alpha}$ =0.4, $\theta = 0.75$

Figure 8: *Foreign* 1% negative productivity shock - Ramsey policy - ***Foreign* constrained** - $h = 0.5$, $\underline{\bar{\alpha} = \mathbf{0.6}}$ and $\theta = 0.75$.



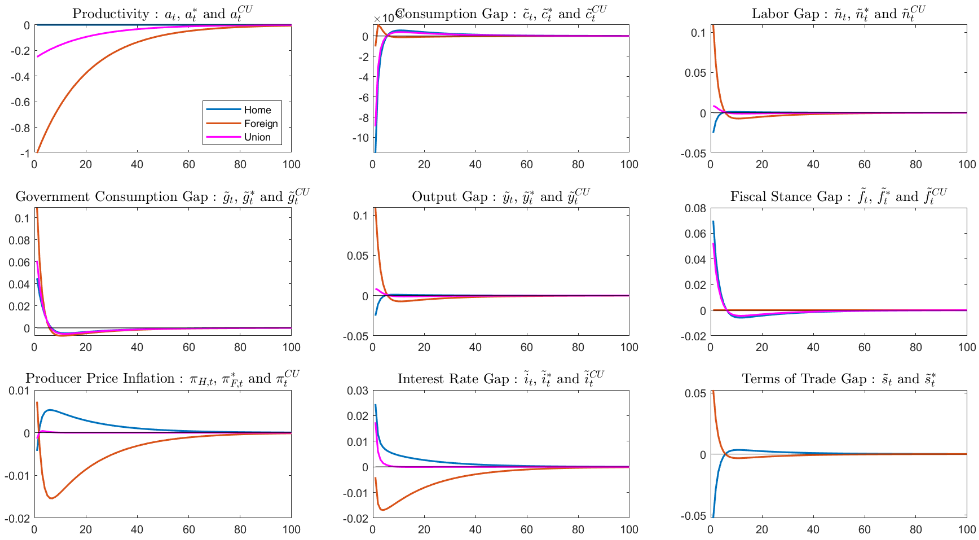
POP WEIGHT, FOREIGN CONSTRAINED, RAMSEY, $h = 0.5$, $\bar{\alpha}$ =0.6, $\theta = 0.75$

Figure 9: *Foreign* 1% negative productivity shock - Ramsey policy - ***Foreign* constrained** - $h = 0.5$, $\bar{\alpha} = 0.4$ and $\underline{\theta = \mathbf{0.5}}$.



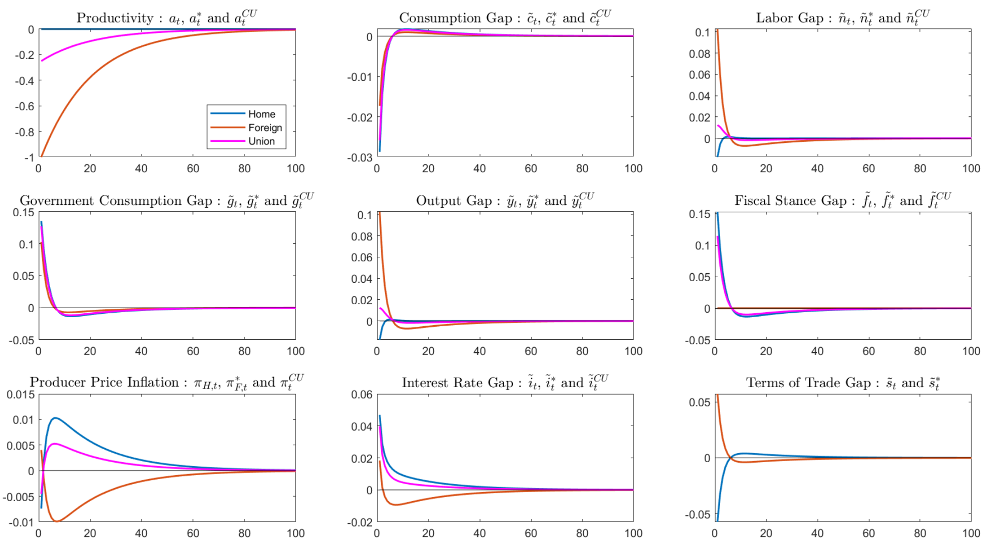
POP WEIGHT, FOREIGN CONSTRAINED, RAMSEY, $h = 0.5$, $\bar{\alpha}$ =0.4, $\theta = 0.5$

Figure 10: *Foreign* 1% negative productivity shock - Ramsey policy - ***Foreign* constrained** - $\underline{h = \mathbf{0.75}}$, $\bar{\alpha} = 0.4$ and $\theta = 0.75$.



POP WEIGHT, FOREIGN CONSTRAINED, RAMSEY, $h = 0.75$, $\bar{\alpha}$ =0.4, $\theta = 0.75$

Figure 11: *Foreign* 1% negative productivity shock - Ramsey policy - ***Foreign* constrained - Equally-weighted objective** - $\underline{h = \mathbf{0.75}}$, $\bar{\alpha} = 0.4$ and $\theta = 0.75$.



EQUAL WEIGHT, FOREIGN CONSTRAINED, RAMSEY, $h = 0.75$, $\bar{\alpha}$ =0.4, $\theta = 0.75$

6.5.2 OSR setup

Table 6: Consumption equivalence OSR vs. Ramsey when *Foreign* is constrained - Monetary policy follows (27) - Fiscal policy follows (29-30)

			Population-weighted objective						Equally-weighted objective					
			Home domestically-oriented			Home Union-oriented			Home domestically-oriented			Home Union-oriented		
h	$\bar{\alpha}$	θ	F	H	CU	F	H	CU	F	H	CU	F	H	CU
0.5	0.4	0.5	2.65	-1.05	1.72	0.69	0.43	0.58						
		0.75	3.63	-1.24	2.42	0.5	1.77	1.31						
	0.6	0.5	2.26	-0.92	1.46	0.58	0.43	0.51						
		0.75	3.02	-1.13	1.99	0.6	1.4	1.08						
0.75	0.4	0.5	3.75	-0.98	1.67	0.99	0.49	0.66	3.95	-1.35	2.63	0.78	0.7	0.7
		0.75	3.61	-0.84	1.66	1.61	0.91	1.13	4.19	-1.86	2.66	0.92	1.32	1.15
	0.6	0.5	3.16	-0.85	1.4	0.91	0.41	0.58	3.34	-1.16	2.21	0.62	0.66	0.6
		0.75	2.98	-0.73	1.36	1.39	0.7	0.93	3.5	-1.6	2.2	0.84	1.07	0.97

Table 7: Consumption equivalence OSR vs. Ramsey when *Foreign* is constrained - Monetary policy follows (28) - Fiscal policy follows (29-30)

			Population-weighted objective						Equally-weighted objective					
			Home domestically-oriented			Home Union-oriented			Home domestically-oriented			Home Union-oriented		
h	$\bar{\alpha}$	θ	F	H	CU	F	H	CU	F	H	CU	F	H	CU
0.5	0.4	0.5	2.15	-1.04	1.33	1.93	-0.89	1.21						
		0.75	1.15	1.24	1.21	0.85	1.31	1.12						
	0.6	0.5	1.84	-0.9	1.14	1.68	-0.78	1.06						
		0.75	1.02	1.04	1.04	0.8	1.1	0.97						
0.75	0.4	0.5	3.15	-0.95	1.35	2.84	-0.81	1.23	3.39	-1.32	2.21	2.83	-0.71	1.91
		0.75	2.05	0.61	1.16	1.54	0.77	1.03	2.95	-1.54	1.8	2.26	-1.03	1.41
	0.6	0.5	2.67	-0.82	1.13	2.48	-0.73	1.06	2.88	-1.14	1.87	2.48	-0.65	1.67
		0.75	1.68	0.49	0.95	1.37	0.61	0.87	2.48	-1.34	1.49	2	-0.95	1.23

Table 8: OSR coefficients when *Foreign* is constrained - Monetary policy follows (27) - Fiscal policy follows (29-30)

			Population-weighted objective				Population-weighted objective			
			Home domestically-oriented		Home Union-oriented		Home domestically-oriented		Home Union-oriented	
h	$\bar{\alpha}$	θ	Φ_y	Φ_π	Φ_y	Φ_π	Φ_y	Φ_π	Φ_y	Φ_π
0.5	0.4	0.5	10	-4.39	2.03	-0.13				
		0.75	10	-8.46	2.23	-1.04				
	0.6	0.5	10	-4.47	2.37	-0.22				
		0.75	10	-8.14	2.68	-1.38				
0.75	0.4	0.5	10	-5.2	4	-0.67			4	-0.17
		0.75	10	-6.38	4.62	-2.31			4.62	-1.88
	0.6	0.5	10	-5.33	4.64	-0.95			4.64	-0.36
		0.75	10	-6.42	5.57	-2.98			5.57	-2.55

Table 9: OSR coefficients when *Foreign* is constrained - Monetary policy follows (28) - Fiscal policy follows (29-30)

			Population-weighted objective				Population-weighted objective			
			Home domestically-oriented		Home Union-oriented		Home domestically-oriented		Home Union-oriented	
h	$\bar{\alpha}$	θ	Φ_y	Φ_π	Φ_y	Φ_π	Φ_y	Φ_π	Φ_y	Φ_π
0.5	0.4	0.5	2.23	-0.61	2.23	-0.38				
		0.75	2.24	-0.98	2.24	-0.78				
	0.6	0.5	2.67	-0.81	2.67	-0.58				
		0.75	2.69	-1.34	2.69	-1.14				
0.75	0.4	0.5	10	-4.11	4.6	-1.41			4.6	-0.83
		0.75	10	-5.28	4.64	-1.96			4.64	-1.36
	0.6	0.5	10	-4.35	5.52	-1.98			5.52	-1.36
		0.75	10	-5.59	5.59	-2.7			5.59	-2.08

7 Conclusion

A Appendices

A.1 Summary of the results for *Home* and *Foreign*

A.1.1 Summary of household's optimal allocation

Table 10: Summary optimal allocation at the household level

Variable	Home	Foreign
j-th household's composite consumption index	$C_t^j \equiv \left[(1-\alpha)^{\frac{1}{\eta}}(C_{H,t}^j)^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}(C_{F,t}^j)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$	$C_t^{j*} \equiv \left[(\alpha^*)^{\frac{1}{\eta}}(C_{H,t}^{j*})^{\frac{\eta-1}{\eta}} + (1-\alpha^*)^{\frac{1}{\eta}}(C_{F,t}^{j*})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$
j-th household's composite consumption of *Home*-made good	$C_{H,t}^j \equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}} \int_0^h C_{H,t}^j(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$	$C_{H,t}^{j*} \equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}} \int_0^h C_{H,t}^{j*}(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$
j-th household's composite consumption of *Foreign*-made good	$C_{F,t}^j \equiv \left[\left(\frac{1}{1-h}\right)^{\frac{1}{\varepsilon}} \int_h^1 C_{F,t}^j(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$	$C_{F,t}^{j*} \equiv \left[\left(\frac{1}{1-h}\right)^{\frac{1}{\varepsilon}} \int_h^1 C_{F,t}^{j*}(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$
j-th household's optimal consumption of *Home*-made good $i \in [0,h]$	$C_{H,t}^j(i) = \frac{1}{h}\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} C_{H,t}^j$	$C_{H,t}^{j*}(i) = \frac{1}{h}\left(\frac{P_{H,t}^*(i)}{P_{H,t}^*}\right)^{-\varepsilon} C_{H,t}^{j*}$
Price index of *Home*-made goods	$P_{H,t} \equiv \left[\frac{1}{h}\int_0^h P_{H,t}(i)^{1-\varepsilon}\mathrm{d}i\right]^{\frac{1}{1-\varepsilon}}$	$P_{H,t}^* \equiv \left[\frac{1}{h}\int_0^h P_{H,t}^*(i)^{1-\varepsilon}\mathrm{d}i\right]^{\frac{1}{1-\varepsilon}}$
j-th household's optimal consumption of *Foreign*-made good $i \in (h,1]$	$C_{F,t}^j(i) = \frac{1}{1-h}\left(\frac{P_{F,t}(i)}{P_{F,t}}\right)^{-\varepsilon} C_{F,t}^j$	$C_{F,t}^{j*}(i) = \frac{1}{1-h}\left(\frac{P_{F,t}^*(i)}{P_{F,t}^*}\right)^{-\varepsilon} C_{F,t}^{j*}$
Price index of *Foreign*-made goods	$P_{F,t} \equiv \left[\frac{1}{1-h}\int_h^1 P_{F,t}(i)^{1-\varepsilon}\mathrm{d}i\right]^{\frac{1}{1-\varepsilon}}$	$P_{F,t}^* \equiv \left[\frac{1}{1-h}\int_h^1 P_{F,t}^*(i)^{1-\varepsilon}\mathrm{d}i\right]^{\frac{1}{1-\varepsilon}}$
j-th household's optimal consumption of *Home*-made goods	$C_{H,t}^j = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t^j$	$C_{H,t}^{j*} = \alpha^*\left(\frac{P_{H,t}^*}{P_t^*}\right)^{-\eta} C_t^{j*}$
j-th household's optimal consumption of *Foreign*-made goods	$C_{F,t}^j = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t^j$	$C_{F,t}^{j*} = (1-\alpha^*)\left(\frac{P_{F,t}^*}{P_t^*}\right)^{-\eta} C_t^{j*}$
Consumer price index (CPI)	$P_t \equiv \left[(1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}\right]^{\frac{1}{1-\eta}}$	$P_t^* \equiv \left[\alpha^*(P_{H,t}^*)^{1-\eta} + (1-\alpha^*)(P_{F,t}^*)^{1-\eta}\right]^{\frac{1}{1-\eta}}$

A.1.2 Summary of household's optimal allocation

Table 11: Summary optimal allocation at the aggregate level

Variable	Home	Foreign
Optimal consumption of *Home*-made good $i \in [0, h]$	$C_{H,t}(i) \equiv \int_0^h C_{H,t}^j(i)\mathrm{d}j = \frac{1}{h}\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} C_{H,t}$	$C_{H,t}^*(i) \equiv \int_h^1 C_{H,t}^{j*}(i)\mathrm{d}j = \frac{1}{h}\left(\frac{P_{H,t}^*(i)}{P_{H,t}^*}\right)^{-\varepsilon} C_{H,t}^*$
Optimal consumption of *Foreign*-made good $i \in (h, 1]$	$C_{F,t}(i) \equiv \int_0^h C_{F,t}^j(i)\mathrm{d}j = \frac{1}{1-h}\left(\frac{P_{F,t}(i)}{P_{F,t}}\right)^{-\varepsilon} C_{F,t}$	$C_{F,t}^*(i) \equiv \int_h^1 C_{F,t}^{j*}(i)\mathrm{d}j = \frac{1}{1-h}\left(\frac{P_{F,t}^*(i)}{P_{F,t}^*}\right)^{-\varepsilon} C_{F,t}^*$
Optimal consumption of *Home*-made goods	$C_{H,t} \equiv \int_0^h C_{H,t}^j\mathrm{d}j = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t$	$C_{H,t}^* \equiv \int_h^1 C_{H,t}^{j*}\mathrm{d}j = \alpha^*\left(\frac{P_{H,t}^*}{P_t^*}\right)^{-\eta} C_t^*$
Optimal consumption of *Foreign*-made goods	$C_{F,t} \equiv \int_0^h C_{F,t}^j\mathrm{d}j = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t$	$C_{F,t}^* \equiv \int_h^1 C_{F,t}^{j*}\mathrm{d}j = (1-\alpha^*)\left(\frac{P_{F,t}^*}{P_t^*}\right)^{-\eta} C_t^*$
Composite consumption index	$C_t \equiv \int_0^h C_t^j\mathrm{d}j = hC_t^j$	$C_t^* \equiv \int_h^1 C_t^{j*}\mathrm{d}j = hC_t^{j*}$
Number of work hours supplied	$N_t^s \equiv \int_0^h N_t^{sj}\mathrm{d}j = hN_t^{sj}$	$N_t^{s*} \equiv \int_h^1 N_t^{sj*}\mathrm{d}j = hN_t^{sj*}$
Intratemporal FOC	$w_t - p_t = -(\varphi+\sigma)\log(h) + \sigma c_t + \varphi n_t^s - \log(\chi_C)$	$w_t^* - p_t^* = -(\varphi+\sigma)\log(1-h) + \sigma c_t^* + \varphi n_t^{s*} - \log(\chi_C)$
Intertemporal FOC	$c_t = \mathbb{E}_t\{c_{t+1}\} - \frac{1}{\sigma}(i_t^{CU} - \mathbb{E}_t\{\pi_{t+1}\} - \bar{i})$	$c_t^* = \mathbb{E}_t\{c_{t+1}^*\} - \frac{1}{\sigma}(i_t^{CU} - \mathbb{E}_t\{\pi_{t+1}^*\} - \bar{i})$

A.1.3 Summary of the government allocation

Table 12: Summary government

Variable	Home	Foreign
Government consumption index	$G_t \equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}} \int_0^h G_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$	$G_t^* \equiv \left[\left(\frac{1}{1-h}\right)^{\frac{1}{\varepsilon}} \int_h^1 G_t^*(i)^{\frac{\varepsilon-1}{\varepsilon}}\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$
Optimal government consumption of domestically made good	$G_t(i) = \frac{1}{h}\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} G_t$	$G_t^*(i) = \frac{1}{1-h}\left(\frac{P_{H,t}^*(i)}{P_{H,t}^*}\right)^{-\varepsilon} G_t^*$

A.1.4 Summary of firm results

Table 13: Firm results

Variable	Home	Foreign
i-th firm's production function	$Y_t(i) = A_t N_t(i)$	$Y_t^*(i) = A_t^* N_t^*(i)$
i-th firm's labor demand	$N_t(i) = \frac{Y_t(i)}{A_t}$	$N_t^*(i) = \frac{Y_t^*(i)}{A_t^*}$
Aggregate labor demand	$N_t \equiv \int_0^h N_t(i)\mathrm{d}i = \frac{Y_t Z_t}{A_t}$	$N_t^* \equiv \int_h^1 N_t^*(i)\mathrm{d}i = \frac{Y_t^* Z_t^*}{A_t^*}$
Aggregate production index	$Y_t \equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}} \int_0^h Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$	$Y_t^* \equiv \left[\left(\frac{1}{1-h}\right)^{\frac{1}{\varepsilon}} \int_h^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$
Output dispersion	$Z_t \equiv \int_0^h \frac{Y_t(i)}{Y_t}\mathrm{d}i$	$Z_t \equiv \int_h^1 \frac{Y_t^*(i)}{Y_t^*}\mathrm{d}i$
Aggregate production function	$y_t = a_t + n_t$	$y_t^* = a_t^* + n_t^*$
Real marginal cost	$mc_t = \log(1-\tau) + w_t - p_{H,t} - a_t$	$mc_t^* = \log(1-\tau) + w_t^* - p_{F,t}^* - a_t^*$
Aggregate price level dynamics	$\pi_{H,t} = (1-\theta)(\bar{p}_{H,t} - p_{H,t})$	$\pi_{F,t}^* = (1-\theta^*)(\bar{p}_{F,t}^* - p_{F,t}^*)$
Firms' FOC	$\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \lambda(\mu + mc_t)$ where $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$	$\pi_{F,t}^* = \beta\mathbb{E}_t\{\pi_{F,t+1}^*\} + \lambda^*(\mu + mc_t^*)$ where $\lambda^* \equiv \frac{(1-\theta^*)(1-\beta\theta^*)}{\theta^*}$

A.1.5 National accounting

B Supplementary material

B.1 Rewrite household's budget constraints

Using the optimal allocation at the household level, *Home* j-th household's expenditures in *Home*-made goods writes

$$\int_0^h P_{H,t}(i) C_{H,t}^j(i)\mathrm{d}i = C_{H,t}^j P_{H,t}^{\varepsilon} \frac{1}{h} \int_0^h P_{H,t}(i)^{1-\varepsilon}\mathrm{d}i = P_{H,t} C_{H,t}^j.$$

The same formula applies to *Home* j-th household's expenditures in *Foreign*-made goods.

We can write *Home* j-th household's total expenditures as

$$\begin{aligned}
\int_0^h P_{H,t}(i) C_{H,t}^j(i)\mathrm{d}i + \int_h^1 P_{H,t}(i) C_{F,t}^j(i)\mathrm{d}i &= P_{H,t} C_{H,t}^j + P_{F,t} C_{F,t}^j \\
&= (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} P_{H,t} C_t^j + \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t^j \\
&= P_t C_t^j
\end{aligned}$$

Substituting these expressions in (1), we obtain (2).

B.2 Firms' FOC

B.2.1 Log-linearize firms' FOC

Dividing (3) by $P_{H,t-1}$, we get

$$\max_{\bar{P}_{H,t}} \sum_{k=0}^{+\infty} \theta^k \mathbb{E}_t \Big\{ Q_{t,t+k} Y_{t+k|t} \Big[\frac{\bar{P}_{H,t}}{P_{H,t-1}} - \mathcal{M} MC_{t+k|t} \Pi_{t-1,t+k} \Big] \Big\} = 0,$$

where $\Pi_{t-1,t+k} \equiv \frac{P_{H,t+k}}{P_{H,t-1}}$ and $MC_{t+k|t} \equiv \frac{\psi_{t+k|t}}{P_{H,t+k}}$ is the real marginal cost at $t+k$ for a *Home* firm whose price was last set at t.

Note that at the zero-inflation-rate steady state (ZIRSS),

- $\bar{P}_{H,t}$ and $P_{H,t}$ are equal to each other and constant over time,
- therefore, all *Home* firms produce the same quantity of output,
- this quantity is constant over time, as the model features no deterministic trend,
- therefore,

$$\begin{aligned} \frac{\bar{P}_{H,t}}{P_{H,t}} &= 1, & \Pi_{t-1,t+k} &= 1, \\ Q_{t,t+k} &= \beta^k, & Y_{t+k|t} &= Y, \\ MC_{t+k|t} &= MC = \frac{1}{\mathcal{M}}. \end{aligned}$$

B.2.2 Rewrite log-linearized firms' FOC

Because of the constant returns to scale, we have

$$\begin{aligned} \forall k \in \mathbb{N}, mc_{t+k|t} &= \log(1-\tau) + (w_{t+k} - p_{H,t+k}) - mpn_{t+k|t} \\ &= \log(1-\tau) + (w_{t+k} - p_{H,t+k}) - a_{t+k} \\ &= mc_{t+k}. \end{aligned}$$

Note also that we have

$$\begin{aligned} (1-\beta\theta) \sum_{k=0}^{+\infty} (\beta\theta)^k \mathbb{E}_t \big\{ p_{H,t+k} - p_{H,t-1} \big\} &= (1-\beta\theta) \sum_{k=0}^{+\infty} (\beta\theta)^k \sum_{s=0}^{k} \mathbb{E}_t \big\{ \pi_{H,t+s} \big\} \\ &= \sum_{s=0}^{+\infty} \mathbb{E}_t \big\{ \pi_{H,t+s} \big\} (1-\beta\theta) \sum_{k=s}^{+\infty} (\beta\theta)^k \\ &= \sum_{s=0}^{+\infty} (\beta\theta)^s \mathbb{E}_t \big\{ \pi_{H,t+s} \big\}. \end{aligned}$$

Using the previous result, *Home* firms' FOC can be rewritten as

$$
\begin{aligned}
\bar{p}_{H,t} - p_{H,t-1} &= (1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big\{\mu + mc_{t+k} + (p_{H,t+k} - p_{H,t-1})\big\} \\
&= (1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big\{\mu + mc_{t+k}\big\} + \sum_{k=0}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big\{\pi_{H,t+k}\big\} \\
&= (1-\beta\theta)(\mu + mc_t) + \pi_{H,t} + (1-\beta\theta)\sum_{k=1}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big\{\mu + mc_{t+k}\big\} + \sum_{k=1}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big\{\pi_{H,t+k}\big\} \\
&= (1-\beta\theta)(\mu + mc_t) + \pi_{H,t} + \beta\theta\Bigg[(1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big\{\mu + mc_{t+1+k}\big\} + \\
&\sum_{k=1}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big\{\pi_{H,t+1+k}\big\}\Bigg] \\
&= (1-\beta\theta)(\mu + mc_t) + \pi_{H,t} + \beta\theta\mathbb{E}_t\Bigg\{(1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k\mathbb{E}_{t+1}\big\{\mu + mc_{t+1+k}\big\} + \\
&\sum_{k=1}^{+\infty}(\beta\theta)^k\mathbb{E}_{t+1}\big\{\pi_{H,t+1+k}\big\}\Bigg\} \\
&= (1-\beta\theta)(\mu + mc_t) + \pi_{H,t} + \beta\theta\mathbb{E}_t\{\bar{p}_{H,t+1} - p_{H,t}\}
\end{aligned}
$$

B.3 Good-market clearing condition

Using *Home* RH's optimal allocations, identities and the international risk condition, we get

$$
\begin{aligned}
Y_t &\equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}}\int_0^h Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \left[\frac{1}{h}\int_0^h\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{1-\varepsilon}\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}(C_{H,t} + C^*_{H,t} + G_t) \\
&= C_{H,t} + C^*_{H,t} + G_t \\
&= (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t + \alpha^*\left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta}C^*_t + G_t \\
&\overset{LOP}{=} \left(\frac{P_{H,t}}{P_t}\right)^{-\eta}\left[(1-\alpha)C_t + \alpha^*\left(\frac{P_t}{P^*_t}\right)^{-\eta}C^*_t\right] + G_t \\
&\overset{IRS}{=} \left(\frac{P_{H,t}}{P_t}\right)^{-\eta}\left[(1-\alpha) + \alpha^*\left(\frac{P_t}{P^*_t}\right)^{-\eta}\frac{1-h}{h}\mathcal{Q}_t^{-\frac{1}{\sigma}}\right]C_t + G_t \\
&= \left(\frac{P_{H,t}}{P_t}\right)^{-\eta}\left[(1-\alpha) + \alpha^*\frac{1-h}{h}\mathcal{Q}_t^{\eta-\frac{1}{\sigma}}\right]C_t + G_t.
\end{aligned}
$$

B.4 IRS condition at equilibrium

We can use the good-market clearing conditions to re-write the IRS condition as

$$
\begin{aligned}
c_t = \log(\frac{h}{1-h}) + \frac{1}{\sigma}q_t + c_t^* \Rightarrow \hat{c}_t &= \frac{1}{\sigma}q_t + \hat{c}_t^* \\
&\Rightarrow (1-\bar{\alpha})s_t = \sigma(\hat{c}_t - \hat{c}_t^*) \\
&\Rightarrow (1-\bar{\alpha})s_t = \tilde{\sigma}[\hat{y}_t - \hat{y}_t^* - \delta(\hat{g}_t - \hat{g}_t^*)] - \bar{\alpha}(1-h)w_{\bar{\alpha}}s_t - \bar{\alpha}hw_{\bar{\alpha}}s_t \\
&\Rightarrow (1-\bar{\alpha})s_t = \tilde{\sigma}[\hat{y}_t - \hat{y}_t^* - \delta(\hat{g}_t - \hat{g}_t^*)] - \bar{\alpha}w_{\bar{\alpha}}s_t \\
&\Rightarrow (1+\bar{\alpha}(w_{\bar{\alpha}}-1))s_t = \tilde{\sigma}[\hat{y}_t - \hat{y}_t^* - \delta(\hat{g}_t - \hat{g}_t^*)] \\
&\Rightarrow s_t = \frac{\tilde{\sigma}}{1+\bar{\alpha}\Theta_{\bar{\alpha}}}[\hat{y}_t - \hat{y}_t^* - \delta(\hat{g}_t - \hat{g}_t^*)] \\
&\Rightarrow s_t = \tilde{\sigma}_{\bar{\alpha}}[\hat{y}_t - \hat{y}_t^* - \delta(\hat{g}_t - \hat{g}_t^*)]
\end{aligned}
$$

where $\Theta_{\bar{\alpha}} \equiv w_{\bar{\alpha}} - 1$ and $\tilde{\sigma}_{\bar{\alpha}} \equiv \frac{\tilde{\sigma}}{1+\bar{\alpha}\Theta_{\bar{\alpha}}}$.

Also, note that

$$\mathbb{E}_t\{\Delta s_{t+1}\} = \tilde{\sigma}_{\bar{\alpha}}[\mathbb{E}_t\{\hat{y}_{t+1}\} - \hat{y}_t - \mathbb{E}_t\{\Delta\hat{y}_{t+1}^*\} - \delta\mathbb{E}_t\{\Delta\hat{g}_{t+1}\} + \delta\mathbb{E}_t\{\Delta\hat{g}_{t+1}^*\}],$$

or

$$\mathbb{E}_t\{\Delta s_{t+1}\} = \tilde{\sigma}_{\bar{\alpha}}[\mathbb{E}_t\{\Delta\hat{y}_{t+1}\} - \mathbb{E}_t\{\hat{y}_{t+1}^*\} + \hat{y}_t^* - \delta\mathbb{E}_t\{\Delta\hat{g}_{t+1}\} + \delta\mathbb{E}_t\{\Delta\hat{g}_{t+1}^*\}].$$

B.5 IS equations

Combining the intratemporal household condition, the inflation identities and the *Home*'s good-market clearing condition, we obtain

$$
\begin{aligned}
& c_t = \mathbb{E}_t\{c_{t+1}\} - \frac{1}{\sigma}(i_t^{CU} - \mathbb{E}_t\{\pi_{t+1}\} - \bar{i}) \\
& \Rightarrow \sigma\hat{c}_t = \mathbb{E}_t\{\sigma\hat{c}_{t+1}\} - (\hat{i}_t^{CU} - \mathbb{E}_t\{\pi_{t+1}\}) \\
& \Rightarrow \sigma\hat{c}_t = \mathbb{E}_t\{\sigma\hat{c}_{t+1}\} - (\hat{i}_t^{CU} - \mathbb{E}_t\{\pi_{H,t+1} + \bar{\alpha}(1-h)\Delta s_{t+1}\}) \\
& \Rightarrow \sigma\hat{c}_t = \mathbb{E}_t\{\sigma\hat{c}_{t+1}\} - (\hat{i}_t^{CU} - \mathbb{E}_t\{\pi_{H,t+1}\}) + \bar{\alpha}(1-h)\mathbb{E}_t\{\Delta s_{t+1}\} \\
& \Rightarrow \tilde{\sigma}\hat{y}_t = \tilde{\sigma}\mathbb{E}_t\{\hat{y}_{t+1}\} - (\hat{i}_t^{CU} - \mathbb{E}_t\{\pi_{H,t+1}\}) - \bar{\alpha}(1-h)\Theta_{\bar{\alpha}}\mathbb{E}_t\{\Delta s_{t+1}\} - \tilde{\sigma}\delta\mathbb{E}_t\{\Delta\hat{g}_{t+1}\}.
\end{aligned}
$$

Using the expression of $\mathbb{E}_t\{\Delta s_{t+1}\}$, we get

$$
\begin{aligned}
\hat{y}_t =& \mathbb{E}_t\{\hat{y}_{t+1}\} - \frac{1}{\tilde{\sigma}_{\bar{\alpha}}(1+\bar{\alpha}h\Theta_{\bar{\alpha}})}(\hat{i}_t^{CU} - \mathbb{E}_t\{\pi_{H,t+1}\}) - \delta\mathbb{E}_t\{\Delta\hat{g}_{t+1}\} \\
&+ \frac{\bar{\alpha}(1-h)\Theta_{\bar{\alpha}}}{1+\bar{\alpha}h\Theta_{\bar{\alpha}}}[\mathbb{E}_t\{\Delta\hat{y}_{t+1}^*\} - \delta\mathbb{E}_t\{\Delta\hat{g}_{t+1}^*\}].
\end{aligned}
$$

Similarly,

$$\hat{y}_t^* = \mathbb{E}_t\{\hat{y}_{t+1}^*\} - \frac{1}{\tilde{\sigma}_{\bar{\alpha}}(1+\bar{\alpha}(1-h)\Theta_{\bar{\alpha}})}(\hat{i}_t^{CU} - \mathbb{E}_t\{\pi_{F,t+1}^*\}) - \delta\mathbb{E}_t\{\Delta\hat{g}_{t+1}^*\}$$

Equations (8-9) follow.

B.6 NKPCs

Using *Home* RH's intratemporal FOC, *Home*'s aggregate production function and *Home*'s price level identities, we have

$$\begin{aligned}
mc_t &= w_t - p_{H,t} - a_t + \log(1-\tau) \\
&= w_t - p_t + (p_t - p_{H,t}) - a_t + \log(1-\tau) \\
&= -(\varphi+\sigma)\log(h) + \sigma c_t + \varphi n_t - \log(\chi_C) + (p_t - p_{H,t}) - a_t + \log(1-\tau) \\
&= \sigma c_t + \varphi(y_t - a_t) + (p_t - p_{H,t}) - a_t + \log(1-\tau) - (\varphi+\sigma)\log(h) - \log(\chi_C) \\
&= \sigma c_t + \varphi y_t + (p_t - p_{H,t}) - (1+\varphi)a_t + \log(1-\tau) - (\varphi+\sigma)\log(h) - \log(\chi_C) \\
&= \sigma c_t + \varphi y_t + \alpha s_t - (1+\varphi)a_t + \log(1-\tau) - (\varphi+\sigma)\log(h) - \log(\chi_C).
\end{aligned}$$

Re-expressing in log-deviation form, we get

$$\hat{mc}_t = \sigma\hat{c}_t + \varphi\hat{y}_t + \alpha s_t - (1+\varphi)a_t$$

where $\hat{mc}_t = mc_t + \mu$.

Using *Home*'s good-market clearing condition, we get

$$\begin{aligned}
\hat{mc}_t &= \tilde{\sigma}(\hat{y}_t - \delta\hat{g}_t) - \bar{\alpha}(1-h)w_{\bar{\alpha}}s_t + \varphi\hat{y}_t + \alpha s_t - (1+\varphi)a_t \\
&= (\tilde{\sigma}+\varphi)\hat{y}_t - \tilde{\sigma}\delta\hat{g}_t + (\alpha - \bar{\alpha}(1-h)w_{\bar{\alpha}})s_t - (1+\varphi)a_t \\
&= (\tilde{\sigma}+\varphi)\hat{y}_t - \tilde{\sigma}\delta\hat{g}_t - \bar{\alpha}(1-h)\Theta_{\bar{\alpha}}s_t - (1+\varphi)a_t
\end{aligned}$$

since $\alpha = \bar{\alpha}(1-h)$.

Similarly,

$$\hat{mc}_t^* = (\tilde{\sigma}+\varphi)\hat{y}_t^* - \tilde{\sigma}\delta\hat{g}_t^* + \bar{\alpha}h\Theta_{\bar{\alpha}}s_t - (1+\varphi)a_t^*.$$

Note that we have

$$\begin{aligned}
\tilde{\sigma} - \bar{\alpha}(1-h)\Theta_{\bar{\alpha}}\tilde{\sigma}_{\bar{\alpha}} &= \tilde{\sigma}_{\bar{\alpha}}(1 + \bar{\alpha}\Theta_{\bar{\alpha}} - \bar{\alpha}(1-h)\Theta_{\bar{\alpha}}) \\
&= \tilde{\sigma}_{\bar{\alpha}}(1 + \bar{\alpha}h\Theta_{\bar{\alpha}}) \\
&= \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\sigma} - \bar{\alpha}h\Theta_{\bar{\alpha}}\tilde{\sigma}_{\bar{\alpha}} &= \tilde{\sigma}_{\bar{\alpha}}(1 + \bar{\alpha}\Theta_{\bar{\alpha}} - \bar{\alpha}h\Theta_{\bar{\alpha}}) \\
&= \tilde{\sigma}_{\bar{\alpha}}(1 + \bar{\alpha}(1-h)\Theta_{\bar{\alpha}}) \\
&= \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h}
\end{aligned}$$

Using the IRS condition, we get

$$\begin{aligned}
\hat{mc}_t &= (\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h} + \varphi)\hat{y}_t - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}\delta\hat{g}_t + (\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h})(\hat{y}_t^* - \delta\hat{g}_t^*) - (1+\varphi)a_t, \\
\hat{mc}_t^* &= (\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h} + \varphi)\hat{y}_t^* - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h}\delta\hat{g}_t^* + (\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h})(\hat{y}_t - \delta\hat{g}_t) - (1+\varphi)a_t^*.
\end{aligned}$$

B.7 Planner's problem

B.7.1 Planner's objective

The benevolent social planner seeks to maximize

$$\max_{C_{H,t}^{j}, C_{F,t}^{j}, N_t^{j}, \frac{G_t}{h}, C_{H,t}^{j*}, C_{F,t}^{j*}, N_t^{j*}, \frac{G_t^*}{1-h}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\int_0^h U(C_t^j, N_t^j, \frac{G_t}{h}) \mathrm{d}j + \int_h^1 U(C_t^{j*}, N_t^{j*}, \frac{G_t^*}{1-h}) \mathrm{d}j \right]$$

subject to

$$C_t^j \equiv \left[(1-\alpha)^{\frac{1}{\eta}} (C_{H,t}^j)^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} (C_{F,t}^j)^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \qquad C_t^{j*} \equiv \left[(\alpha^*)^{\frac{1}{\eta}} (C_{H,t}^{j*})^{\frac{\eta-1}{\eta}} + (1-\alpha^*)^{\frac{1}{\eta}} (C_{F,t}^{j*})^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

$$C_{H,t} + C_{H,t}^* + G_t - A_t N_t \leq 0 \qquad C_{F,t} + C_{F,t}^* + G_t^* - A_t^* N_t^* \leq 0$$

$$C_{H,t} = h C_{H,t}^j \qquad C_{F,t} = h C_{F,t}^j$$

$$C_{H,t}^* = (1-h) C_{H,t}^{j*} \qquad C_{F,t}^* = (1-h) C_{F,t}^{j*}$$

$$N_t = h N_t^j \qquad N_t^* = (1-h) N_t^{j*}$$

$$C_t = h C_t^j \qquad C_t^* = (1-h) C_t^{j*}.$$

B.7.2 The efficient steady state

Evaluated at steady state, planner's FOCs and constraints become

$$\begin{aligned}
\chi_C \left[\frac{(1-\alpha) C}{C_H} \right]^{\frac{1}{\eta}} \left(\frac{C}{h} \right)^{-\sigma} &= \chi_G \left(\frac{G}{h} \right)^{-\gamma} \\
\chi_C \left[\frac{\alpha C}{C_F} \right]^{\frac{1}{\eta}} \left(\frac{C}{h} \right)^{-\sigma} &= \chi_G \left(\frac{G^*}{1-h} \right)^{-\gamma} \\
\chi_C \left[\frac{(1-\alpha^*) C^*}{C_F^*} \right]^{\frac{1}{\eta}} \left(\frac{C^*}{1-h} \right)^{-\sigma} &= \chi_G \left(\frac{G^*}{1-h} \right)^{-\gamma} \\
\chi_C \left[\frac{\alpha^* C^*}{C_H^*} \right]^{\frac{1}{\eta}} \left(\frac{C^*}{1-h} \right)^{-\sigma} &= \chi_G \left(\frac{G}{h} \right)^{-\gamma} \\
\left(\frac{N}{h} \right)^{\varphi} &= \chi_G \left(\frac{G}{h} \right)^{-\gamma} \\
\left(\frac{N^*}{1-h} \right)^{\varphi} &= \chi_G \left(\frac{G^*}{1-h} \right)^{-\gamma} \\
\frac{C}{h} &= \left[(1-\alpha)^{\frac{1}{\eta}} (\frac{C_H}{h})^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} (\frac{C_F}{h})^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \\
\frac{C^*}{1-h} &= \left[(\alpha^*)^{\frac{1}{\eta}} (\frac{C_H^*}{1-h})^{\frac{\eta-1}{\eta}} + (1-\alpha^*)^{\frac{1}{\eta}} (\frac{C_F^*}{1-h})^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \\
C_H + C_H^* + G - N &\leq 0 \\
C_F + C_F^* + G^* - N^* &\leq 0.
\end{aligned}$$

For a given value of $\delta \equiv \frac{G}{Y}$, we set

$$\chi_C = (1-\delta)^{\sigma} \text{ and } \chi_G = \delta^{\gamma},$$

so that the static efficient equilibrium is solved by

$$\begin{aligned}
&\frac{N}{h} = 1, && \frac{N^*}{1-h} = 1, && Y = N, && Y^* = N^*,\\
&C = (1-\delta)Y, && C^* = (1-\delta)Y^*, && G = \delta Y, && G^* = \delta Y^*,\\
&C_H = (1-\alpha)C, && C_F = \alpha C, && C_F^* = (1-\alpha^*)C^*, && C_H^* = \alpha^* C^*.
\end{aligned}$$

B.8 Steady state and monopolistic distortion

The economy will reach a steady state where there is no price dispersion across goods and across regions ($S = 1$). Therefore, the only source of distortion at steady state comes from the monopolistic competition in the goods market.

If the economy reaches the efficient steady state, we must have

$$\begin{aligned}
1 - \frac{1}{\epsilon} &= MC\\
&= (1-\tau)\frac{W}{P_H}\\
&= (1-\tau)\frac{W}{P}\frac{P}{P_H}\\
&= (1-\tau)\frac{W}{P}\\
&= \frac{1-\tau}{\chi_C}\left(\frac{N}{h}\right)^{\varphi}\left(\frac{C}{h}\right)^{\sigma}\\
&= \frac{1-\tau}{\chi_C}(1-\delta)^{\sigma}\\
&= 1-\tau
\end{aligned}$$

since $\chi_C = (1-\delta)^{\sigma}$.

Therefore, the condition $1 - \tau = 1 - \frac{1}{\epsilon}$ is necessary for the economy's steady state to reach the efficient steady state.

Therefore, if $\tau = \frac{1}{\varepsilon}$ and if governments behave efficiently at steady state (i.e. $\left(\frac{N}{h}\right)^{\varphi}\frac{1}{\chi_C}\left(\frac{C}{h}\right)^{\sigma} = 1$), the steady state of the economy coincides with the efficient steady state.

B.9 Natural level of output

The flexible price equilibrium is

$$\begin{aligned}
0 &= \sigma\hat{\bar{c}}_t + \varphi\hat{\bar{y}}_t + \alpha\bar{s}_t - (1+\varphi)a_t,\\
0 &= \sigma\hat{\bar{c}}_t^* + \varphi\hat{\bar{y}}_t^* - \alpha^*\bar{s}_t - (1+\varphi)a_t^*,\\
\tilde{\sigma}(\hat{\bar{y}}_t - \delta\hat{\bar{g}}_t) &= \sigma\hat{\bar{c}}_t + \bar{\alpha}(1-h)w_{\bar{\alpha}}s_t,\\
\tilde{\sigma}(\hat{\bar{y}}_t^* - \delta\hat{\bar{g}}_t^*) &= \sigma\hat{\bar{c}}_t^* - \bar{\alpha}hw_{\bar{\alpha}}s_t,\\
\bar{s}_t &= \tilde{\sigma}_{\bar{\alpha}}[\hat{\bar{y}}_t - \hat{\bar{y}}_t^* - \delta(\hat{\bar{g}}_t - \hat{\bar{g}}_t^*)],\\
\gamma\hat{\bar{g}}_t &= \sigma\hat{\bar{c}}_t + \alpha\bar{s}_t,\\
\gamma\hat{\bar{g}}_t^* &= \sigma\hat{\bar{c}}_t^* - \alpha^*\bar{s}_t.
\end{aligned}$$

Using the last two equations to remove $\hat{\bar{c}}_t$ and $\bar{c}_t^*$, we get

$$\begin{aligned}
0 &= \gamma\hat{\bar{g}}_t + \varphi\hat{\bar{y}}_t - (1+\varphi)a_t \\
0 &= \gamma\hat{\bar{g}}_t^* + \varphi\hat{\bar{y}}_t^* - (1+\varphi)a_t^* \\
\tilde{\sigma}(\hat{\bar{y}}_t - \delta\hat{\bar{g}}_t) &= \gamma\hat{\bar{g}}_t + \bar{\alpha}(1-h)\Theta_{\bar{\alpha}}\bar{s}_t \\
\tilde{\sigma}(\hat{\bar{y}}_t^* - \delta\hat{\bar{g}}_t^*) &= \gamma\hat{\bar{g}}_t^* - \bar{\alpha}h\Theta_{\bar{\alpha}}\bar{s}_t \\
\bar{s}_t &= \tilde{\sigma}_{\bar{\alpha}}[\hat{\bar{y}}_t - \hat{\bar{y}}_t^* - \delta(\hat{\bar{g}}_t - \hat{\bar{g}}_t^*)]
\end{aligned}$$

Replacing $\gamma\hat{\bar{g}}_t$ and $\gamma\bar{g}_t^*$ given the first two equations, we get

$$\begin{aligned}
\tilde{\sigma}(\hat{\bar{y}}_t - \delta\hat{\bar{g}}_t) &= -\varphi\hat{\bar{y}}_t + (1+\varphi)a_t + \bar{\alpha}(1-h)\Theta_{\bar{\alpha}}\bar{s}_t \\
\tilde{\sigma}(\hat{\bar{y}}_t^* - \delta\hat{\bar{g}}_t^*) &= -\varphi\hat{\bar{y}}_t^* + (1+\varphi)a_t^* - \bar{\alpha}h\Theta_{\bar{\alpha}}\bar{s}_t \\
\bar{s}_t &= \tilde{\sigma}_{\bar{\alpha}}[\hat{\bar{y}}_t - \hat{\bar{y}}_t^* - \delta(\hat{\bar{g}}_t - \hat{\bar{g}}_t^*)]
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\tilde{\sigma}+\varphi)\hat{\bar{y}}_t &= \tilde{\sigma}\delta\hat{\bar{g}}_t + \bar{\alpha}(1-h)\Theta_{\bar{\alpha}}\bar{s}_t + (1+\varphi)a_t \\
(\tilde{\sigma}+\varphi)\hat{\bar{y}}_t^* &= \tilde{\sigma}\delta\hat{\bar{g}}_t^* - \bar{\alpha}h\Theta_{\bar{\alpha}}\bar{s}_t + (1+\varphi)a_t^* \\
\bar{s}_t &= \tilde{\sigma}_{\bar{\alpha}}[\hat{\bar{y}}_t - \hat{\bar{y}}_t^* - \delta(\hat{\bar{g}}_t - \hat{\bar{g}}_t^*)]
\end{aligned}$$

Replacing the terms of trade,

$$\begin{aligned}
(\tilde{\sigma}+\varphi)\hat{\bar{y}}_t &= \tilde{\sigma}\delta\hat{\bar{g}}_t + (\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h})[\hat{\bar{y}}_t - \hat{\bar{y}}_t^* - \delta(\hat{\bar{g}}_t - \hat{\bar{g}}_t^*)] + (1+\varphi)a_t \\
(\tilde{\sigma}+\varphi)\hat{\bar{y}}_t^* &= \tilde{\sigma}\delta\hat{\bar{g}}_t^* + (\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h})[\hat{\bar{y}}_t - \hat{\bar{y}}_t^* - \delta(\hat{\bar{g}}_t - \hat{\bar{g}}_t^*)] + (1+\varphi)a_t^*
\end{aligned}$$

Using the fact that $\bar{\alpha}(1-h)\Theta_{\bar{\alpha}} = \tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}$ and $\bar{\alpha}h\Theta_{\bar{\alpha}} = \tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h}$, we can write

$$(\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}+\varphi)\hat{\bar{y}}_t = \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}\delta\hat{\bar{g}}_t + (1+\varphi)a_t - (\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h})(\hat{\bar{y}}_t^* - \delta\hat{\bar{g}}_t^*) \tag{31}$$

$$(\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h}+\varphi)\hat{\bar{y}}_t^* = \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h}\delta\hat{\bar{g}}_t^* + (1+\varphi)a_t^* - (\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h})(\hat{\bar{y}}_t - \delta\hat{\bar{g}}_t) \tag{32}$$

Therefore,

$$\begin{aligned}
\hat{\bar{y}}_t &= \Gamma^g_{\bar{\alpha},h}\delta\hat{\bar{g}}_t + \Gamma^a_{\bar{\alpha},h}a_t + \Gamma^{\text{ext}}_{\bar{\alpha},h}(\hat{\bar{y}}_t^* - \delta\hat{\bar{g}}_t^*) \\
\hat{\bar{y}}_t^* &= \Gamma^g_{\bar{\alpha},1-h}\delta\hat{\bar{g}}_t^* + \Gamma^a_{\bar{\alpha},1-h}a_t^* + \Gamma^{\text{ext}}_{\bar{\alpha},1-h}(\hat{\bar{y}}_t - \delta\hat{\bar{g}}_t)
\end{aligned}$$

where

$$\begin{aligned}
\Gamma^g_{\bar{\alpha},h} &= \frac{\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}}{\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}+\varphi} \\
\Gamma^a_{\bar{\alpha},h} &= \frac{1+\varphi}{\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}+\varphi} \\
\Gamma^{\text{ext}}_{\bar{\alpha},h} &= -\frac{\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}}{\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}+\varphi}.
\end{aligned}$$

B.10 Model in gap form

Combining the log-deviation of *Home* and *Foreign* real marginal cost under sticky price (10-11) with (31-32), we obtain an expression of the real marginal cost in gap form

$$\hat{mc}_t - 0 = \hat{mc}_t = (\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}+\varphi)\tilde{y}_t - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}\delta\tilde{g}_t + (\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h})(\tilde{y}_t^* - \delta\tilde{g}_t^*) - (1+\varphi)a_t,$$

$$\hat{mc}_t^* - 0 = \hat{mc}_t^* = (\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h}+\varphi)\tilde{y}_t^* - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h}\delta\tilde{g}_t^* + (\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},1-h})(\tilde{y}_t - \delta\tilde{g}_t) - (1+\varphi)a_t^*.$$

Given the exogeneous sequence $(a_t, a_t^*)_{t\in\mathbb{N}}$ and the sequence $(\hat{i}_t^{CU}, \tilde{g}_t, \tilde{g}_t^*)_{t\in\mathbb{N}}$, the endogeneous sequence $(\tilde{y}_t, \pi_{H,t}; \tilde{y}_t^*, \pi_{F,t}^*)_{t\in\mathbb{N}}$ is given by

$$\tilde{y}_t = \mathbb{E}_t\{\tilde{y}_{t+1} - \delta\Delta\tilde{g}_{t+1}\} - \frac{1}{\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}}(\hat{i}_t^{CU} - \mathbb{E}_t\{\pi_{H,t+1}\} - \bar{r}_t) + \frac{\bar{\alpha}(1-h)\Theta_{\bar{\alpha}}}{\Omega_{\bar{\alpha},h}}\mathbb{E}_t\{\Delta\tilde{y}_{t+1}^* - \delta\Delta\tilde{g}_{t+1}^*\},$$
$$\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \lambda[(\tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h} + \varphi)\tilde{y}_t - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}\delta\tilde{g}_t + (\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h})(\tilde{y}_t^* - \delta\tilde{g}_t^*)],$$

where *Home* natural rate is given by

$$\begin{aligned}\bar{r}_t &\equiv \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h}\mathbb{E}_t\big\{\Delta\hat{\bar{y}}_{t+1} - \delta\Delta\hat{\bar{g}}_{t+1}\big\} + (\tilde{\sigma} - \tilde{\sigma}_{\bar{\alpha}}\Omega_{\bar{\alpha},h})\mathbb{E}_t\big\{\Delta\hat{\bar{y}}_{t+1}^* - \delta\Delta\hat{\bar{g}}_{t+1}^*\big\} \\ &= (1+\varphi)\mathbb{E}_t\{\Delta a_{t+1}\} + \varphi E_t\{\Delta\hat{\bar{y}}_{t+1}\},\end{aligned}$$

where we used the expression of the real marginal cost in gap form to rewrite *Home*'s NKPC.

Analogous results can be obtain with *Foreign*'s variables.