Model draft n°1 - Replication of Galí and Monacelli (2008)

1 Introduction

Approach

We replicate Galí and Monacelli (2008) in a two-country setup.

We keep the same functional form. We allow for different home bias, but log-linearizations are made in the case where the home bias are identical.

We allow for different price stickiness.

References

Below are the references we used to build the model:

- ENSAE MiE 2 course : AE332, Monetary Economics, Olivier Loisel
- Galí and Monacelli, Optimal monetary and fiscal policy in a currency union, *Journal of International Economics*, 2008
- Marcos Antonio C. da Silveira, Two-country new Keynesian DSGE model : a small open economy as limit case, *Ipea*, 2006
- Cole et al., One EMU fiscal policy for the Euro, *Macroeconomic Dynamics*, 2019
- Forlati, Optimal monetary and fiscal policy in the EMU : does fiscal policy coordination matter?,*Center for Fiscal Policy, EPFL, Chair of International Finance (CFI) Working Paper No. 2009-04*, 2009
- Schäfer, Monetary union with sticky prices and direct spillover channels, *Journal of Macroeconomics*, 2016

2 A currency union model

We model a currency union as a closed system made up of two economies : *Home* and *Foreign*.

Variables without asterix (e.g. X) denote *Home* variables and variables with an asterix (e.g. X_t^*) denote *Foreign* variables.

2.1 Households

Objective

Home's representative household (RH) seeks to maximizes

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t, G_t),$$

where U is the instantaneous utility function, N_t is the number of work hours supplied by *Home*'s RH, C_t is a composite index of *Home*'s consumption, and G_t is an index of *Home*'s government consumption.

Aggregate composite consumption index

More precisely, C_t is given by

$$C_t \equiv \frac{(C_{H,t})^{1-\alpha}(C_{F,t})^{\alpha}}{(1-\alpha)^{(1-\alpha)}\alpha^{\alpha}},$$

where

- $C_{H,t}$ is an index of *Home*'s consumption of *Home*-made goods,
- $C_{F,t}$ is an index of *Home*'s consumption of *Foreign*-made goods,
- $\alpha \in [0,1]$ is a measure of *Home*'s **openess** and $1-\alpha$ is a measure of *Home*'s **home bias.**

Regional consumption indexes

$C_{H,t}$ is defined by the CES function

$$C_{H,t} = \left[\int_0^1 C_{H,t}(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where

- $C_{H,t}(i)$ is *Home*'s consumption of *Home*-made good i, and
- $\varepsilon > 1$ is the elasticity of substitution between *Home*-made goods.

Similarly, $C_{F,t}$ is defined by the CES function

$$C_{F,t} = \left[\int_0^1 C_{F,t}(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where

- $C_{F,t}(i)$ is *Home*'s consumption of *Foreign*-made good i, and
- $\varepsilon > 1$ is the elasticity of substitution between *Foreign*-made goods.

RH's Budget constraints

Home's RH faces a sequence of budget constraints

$$\forall t \geq 0, \int_0^1 P_{H,t}(i) C_{H,t}(i) \mathrm{d}i + \int_0^1 P_{F,t}(i) C_{F,t}(i) \mathrm{d}i + \mathbb{E}_t\{Q_{t,t+1} D_{t+1}\} \leq D_t + W_t N_t + T_t,$$

where

- $P_{H,t}(i)$ is *Home*'s price of *Home*-made good i,
- $P_{F,t}(i)$ is *Home*'s price of *Foreign*-made good i,
- D_{t+1} is the quantity of one-period nominal bonds held by Home's RH,
- W_t is *Home*'s nominal wage,
- T_t denotes *Home*'s lump sum taxes.

Optimal allocation of consumption across goods

Given $C_{H,t}$ and $C_{F,t}$, a first step is to find the optimal allocations $(C_{H,t}(i))_{i\in[0,1]}$ and $(C_{F,t}(i))_{i\in[0,1]}$ that minimize the regional expenditures.

Home's optimal consumption of *Home*-made good i is given by

$$C_{H,t}(i) = \left[\frac{P_{H,t}(i)}{P_{H,t}} \right]^{-\varepsilon} C_{H,t},$$

where $P_{H,t} \equiv \left[\int_0^1 P_{H,t}(i)^{1-\varepsilon} \mathrm{d}i \right]^{\frac{1}{1-\varepsilon}}$ is *Home*'s price index of *Home*-made goods.

Similarly, *Home*'s optimal consumption of *Foreign*-made good i is given by

$$C_{F,t}(i) = \left[\frac{P_{F,t}(i)}{P_{F,t}} \right]^{-\varepsilon} C_{F,t},$$

where $P_{F,t} \equiv \left[\int_0^1 P_{F,t}(i)^{1-\varepsilon} \mathrm{d}i \right]^{\frac{1}{1-\varepsilon}}$ is *Home*'s price index of *Foreign*-made goods.

Optimal allocation of consumption across regions

Given C_t, a second step is to find the optimal allocation $(C_{H,t}, C_{F,t})$ that minimizes total expenditures.

Home's optimal consumption of *Home*-made goods is given by

$$C_{H,t} = (1-\alpha) \left(\frac{P_{H,t}}{P_t} \right)^{-1} C_t,$$

and *Home*'s optimal consumption of *Foreign*-made goods is given by

$$C_{F,t} = \alpha \left(\frac{P_{F,t}}{P_t} \right)^{-1} C_t,$$

where $P_t \equiv \left(P_{H,t}\right)^{1-\alpha} \left(P_{F,t}\right)^{\alpha}$ is *Home*'s consumer price index (CPI).

Summary optimal allocation

Analogous results hold for the *Foreign* country. Similarly, we denote α^* the measure of *Foreign*'s degree of openess.

Variable	*Home*	*Foreign*
Composite consumption index	$C_t \equiv \frac{(C_{H,t})^{1-\alpha}(C_{F,t})^{\alpha}}{(1-\alpha)^{(1-\alpha)}\alpha^{\alpha}}$	$C_t^* \equiv \frac{(C_{H,t}^*)^{\alpha^*}(C_{F,t}^*)^{1-\alpha^*}}{\alpha^{*\alpha*}(1-\alpha^*)^{(1-\alpha^*)}}$
Composite consumption of *Home*-made good	$C_{H,t} = \left[\int_0^1 C_{H,t}(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \right]^{\frac{\varepsilon}{\varepsilon-1}}$	$C_{H,t}^* = \left[\int_0^1 C_{H,t}^*(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \right]^{\frac{\varepsilon}{\varepsilon-1}}$
Composite consumption of *Foreign*-made good	$C_{F,t} = \left[\int_0^1 C_{F,t}(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \right]^{\frac{\varepsilon}{\varepsilon-1}}$	$C_{F,t}^* = \left[\int_0^1 C_{F,t}^*(i) \frac{\varepsilon-1}{\varepsilon} \mathrm{d}i \right]^{\frac{\varepsilon}{\varepsilon-1}}$
Optimal consumption of *Home*-made good i	$C_{H,t}(i) = \left[\frac{P_{H,t}(i)}{P_{H,t}} \right]^{-\varepsilon} C_{H,t}$	$C_{H,t}^*(i) = \left[\frac{P_{H,t}^*(i)}{P_{H,t}^*} \right]^{-\varepsilon} C_{H,t}^*$
Price index of *Home*-made goods	$P_{H,t} \equiv \left[\int_0^1 P_{H,t}(i)^{1-\varepsilon} \mathrm{d}i \right]^{\frac{1}{1-\varepsilon}}$	$P_{H,t}^* \equiv \left[\int_0^1 P_{H,t}^*(i)^{1-\varepsilon} \mathrm{d}i \right]^{\frac{1}{1-\varepsilon}}$
Optimal consumption of *Foreign*-made good i	$C_{F,t}(i) = \left[\frac{P_{F,t}(i)}{P_{F,t}} \right]^{-\varepsilon} C_{F,t}$	$C_{F,t}^*(i) = \left[\frac{P_{F,t}^*(i)}{P_{F,t}^*} \right]^{-\varepsilon} C_{F,t}^*$
Price index of *Foreign*-made goods	$P_{F,t} \equiv \left[\int_0^1 P_{F,t}(i)^{1-\varepsilon} \mathrm{d}i \right]^{\frac{1}{1-\varepsilon}}$	$P_{F,t}^* \equiv \left[\int_0^1 P_{F,t}^*(i)^{1-\varepsilon} \mathrm{d}i \right]^{\frac{1}{1-\varepsilon}}$
Optimal consumption of *Home*-made goods	$C_{H,t} = (1-\alpha) \left(\frac{P_{H,t}}{P_t} \right)^{-1} C_t$	$C_{H,t}^* = \alpha^* \left(\frac{P_{H,t}^*}{P_t^*} \right)^{-1} C_t^*$
Optimal consumption of *Foreign*-made goods	$C_{F,t} = \alpha \left(\frac{P_{F,t}}{P_t} \right)^{-1} C_t$	$C_{F,t}^* = (1-\alpha^*) \left(\frac{P_{F,t}^*}{P_t} \right)^{-1} C_t^*$
Consumer price index (CPI)	$P_t \equiv \left(P_{H,t}\right)^{1-\alpha} \left(P_{F,t}\right)^{\alpha}$	$P_t^* \equiv \left(P_{H,t}^*\right)^{\alpha^*} \left(P_{F,t}^*\right)^{1-\alpha^*}$

Rewrite RH's budget constraints

Combining all the previous results, *Home*'s expenditures in *Home*-made goods writes

$$\int_0^1 P_{H,t}(i) C_{H,t}(i) \mathrm{d}i = C_{H,t} P_{H,t}^{-\varepsilon} \int_0^1 P_{H,t}(i)^{1-\varepsilon} \mathrm{d}i \quad = P_{H,t} C_{H,t}.$$

The same formula applies to *Home*'s expenditures in *Foreign*-made goods.

We can write *Home* RH's total consumption expenditures as

$$P_t C_t = P_{H,t} C_{H,t} + P_{F,t} C_{F,t}.$$

Therefore, conditional on an optimal allocation across goods and regions, *Home* RH's budget constraints can be rewritten as

$$\begin{gathered} \forall t \geq 0, P_t C_t + \mathbb{E}_t\{Q_{t,t+1} D_{t+1}\} \\ \leq D_t + W_t N_t + T_t. \end{gathered}$$

Intratemporal and intertemporal FOCs

Now, we can derive the first order conditions for *Home*'s optimal consumption level C_t as well as for *Home*'s optimal number of hours worked N_t.

Home's **intratemporal** FOC is

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t},$$

and *Home*'s **intertemporal** FOC is

$$\mathbb{E}_t\{Q_{t,t+1}\} = \beta\mathbb{E}_t\left\{\frac{U^i_{c,t+1}}{U^i_{c,t}}\frac{P^i_t}{P^i_{t+1}}\right\}.$$

Functional form of the instantaneous utility function

We assume that the instantaneous utility takes the specific form

$$U(C_t, N_t, G_t) = (1-\chi)log(C_t) + \chi log(G_t) - \frac{(N_t)^{1+\varphi}}{1+\varphi}$$

where $\chi \in [0,1]$ and $\varphi > 0$. The parameter χ measures the weight attached to public consumption (relative to private consumption).

Rewrite the intratemporal and intertemporal FOCs under the functional form assumptions

Under the functional forms assumptions, *Home* RH's **intratemporal** FOC becomes

$$\frac{(N_t)^{\varphi}}{\frac{1-\chi}{C_t}} = \frac{W_t}{P_t} \Rightarrow C_t(N_t)^{\varphi} = (1-\chi)\frac{W_t}{P_t},$$

and *Home* RH's **intertemporal** FOC becomes

$$\mathbb{E}_t\{Q_{t,t+1}\} = \beta\mathbb{E}_t\left\{\frac{C_t}{C_{t+1}}\frac{P_t}{P_{t+1}}\right\}.$$

FOCs in log-linearized form

Home RH's **intratemporal** FOC in log form is

$$w_t - p_t = c_t + \varphi n_t - log(1-\chi),$$

and *Home* RH's **intertemporal** FOC in log form is

$$c_t = \mathbb{E}_t\{c_{t+1}\} - (i_t - \mathbb{E}_t\{\pi_{t+1}\} - \bar{i}),$$

where $i_t \equiv log\left(\frac{1}{\mathbb{E}_t\{Q_{t,t+1}\}}\right)$ is referred to as the **short-term nominal interest rate,** $\pi_t \equiv p_t - p_{t-1}$ is **CPI inflation**, and $\bar{i} \equiv -log(\beta)$.

Summary RH's FOCs

Variable	*Home*	*Foreign*
Intratemporal FOC	$w_t - p_t = c_t + \varphi n_t - log(1-\chi)$	$w^*_t - p^*_t = c^*_t + \varphi n^*_t - log(1-\chi^*)$
Intertemporal FOC	$c_t = \mathbb{E}_t\{c_{t+1}\} - (i_t - \mathbb{E}_t\{\pi_{t+1}\} - \bar{i})$	$c^*_t = \mathbb{E}_t\{c^*_{t+1}\} - (i^*_t - \mathbb{E}_t\{\pi^*_{t+1}\} - \bar{i})$

2.2 Definitions, identities and international risk sharing

The law of one price

Using the law of one price we have $P_{H,t}(i) = P^*_{H,t}(i)$ and $P_{F,t}(i) = P^*_{F,t}(i)$.

As a consequence, $P_{H,t} = P^*_{H,t}$ and $P_{F,t} = P^*_{F,t}$.

Terms of trade

We derive the relationship between inflation, terms of trade and real exchange rate.

Home's terms of trade is defined as

$$S_t \equiv \frac{P_{F,t}}{P_{H,t}},$$

and *Foreign*'s terms of trade is defined as

$$S^*_t \equiv \frac{P^*_{H,t}}{P^*_{F,t}}.$$

The terms of trade is simply the relative price of imported goods in terms of domestic goods.

Using the LOP, we have

$$S_t^* = \frac{1}{S_t}.$$

Price level and inflation identities

Using the definitions of P_t, P_t^*, S_t, and S_t^*, we get

$$\begin{aligned}\frac{P_t}{P_{H,t}} &= (S_t)^\alpha \\ \frac{P_t}{P_{F,t}} &= \left(\frac{1}{S_t}\right)^{1-\alpha} \\ \frac{P_t^*}{P_{H,t}^*} &= (S_t)^{1-\alpha^*} \\ \frac{P_t^*}{P_{F,t}^*} &= \left(\frac{1}{S_t}\right)^{\alpha^*}.\end{aligned}$$

In log, we get

$$\begin{aligned}p_t - p_{H,t} &= \alpha s_t \\ p_t - p_{F,t} &= -(1-\alpha)s_t \\ p_t^* - p_{H,t}^* &= (1-\alpha^*)s_t \\ p_t^* - p_{F,t}^* &= -\alpha^* s_t.\end{aligned}$$

In the case where $\alpha^* = \alpha$, we get

$$\begin{aligned}\pi_t &= \pi_{H,t} + \alpha \Delta s_t \\ \pi_t^* &= \pi_{F,t}^* - \alpha \Delta s_t,\end{aligned}$$

where *Home* and *Foreign* inflation of domestic price indexes are respectively given by $\pi_{H,t} = p_{H,t} - p_{H,t-1}$ and $\pi_{F,t}^* = p_{F,t}^* - p_{F,t-1}^*$.

Real exchange rate

Home's real exchange rate denoted $\mathcal{Q}_t$ is given by

$$\mathcal{Q}_t \equiv \frac{P_t^*}{P_t} = \frac{(S_t)^{1-\alpha^*}}{(S_t)^\alpha} = (S_t)^{1-\alpha^*-\alpha}.$$

In the case where $\alpha^* = \alpha$, we get

$$q_t = (1-2\alpha)s_t.$$

International risk sharing (not detailed)

The international risk sharing (IRS) condition implies that

$$C_t = \vartheta \mathcal{Q}_t C_t^*.$$

We assume the same initial conditions for *Home* and *Foreign* households, so that $\vartheta = 1$.

In log, the IRS condition writes

$$c_t = q_t + c_t^*.$$

2.3 Government

Government consumption index

Home's public consumption index is given by the CES function

$$G_t \equiv \left[\int_0^1 G_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\,\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where $G_t(i)$ is the quantity of *Home*-made good i purchased *Home*'s government.

Government demand schedules

For any level of public consumption G_t, the government demand schedules are analogous to those obtain for private consumption:

$$G_t(i) = \left[\frac{P_{H,t}(i)}{P_{H,t}}\right]^{-\varepsilon} G_t.$$

Summary government results

Variable	*Home*	*Foreign*
Government consumption index	$G_t \equiv \left[\int_0^1 G_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\,\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$	$G_t^* \equiv \left[\int_0^1 G_t^*(i)^{\frac{\varepsilon-1}{\varepsilon}}\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$
Optimal government consumption of domestically made good i	$G_t(i) = \left[\frac{P_{H,t}(i)}{P_{H,t}}\right]^{-\varepsilon} G_t.$	$G_t^*(i) = \left[\frac{P_{F,t}^*(i)}{P_{F,t}^*}\right]^{-\varepsilon} G_t^*.$

2.4 Firms

Each country has a continuum of firms represented by the interval $[0, 1]$. Each firm produces a differentiated good.

Technology

All *Home* firms use the same technology, represented by the production function

$$Y_t(i) = A_t N_t(i),$$

where A_t is *Home*'s productivity shifter, a stochastic exogeneous factor.

Labor demand

The technology constraint implies\ that *Home* i-th firm's labor demand is given by

$$N_t(i) = \frac{Y_t(i)}{A_t}.$$

Aggregate labor demand

Home's aggregate labor demand is defined as

$$N_t \equiv \int_0^1 N_t(i)\mathrm{d}i = \frac{Y_t Z_t}{A_t}$$

where

$$Y_t \equiv \left[\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\,\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

is the aggregate production while $Z_t \equiv \int_0^1 \frac{Y_t(i)}{Y_t}\mathrm{d}i$ is a measure of the dispersion of *Home* firms' output.

Aggregate production function

In log form, we get a relationship between *Home*'s aggregate employment and *Home*'s output

$$y_t = a_t + n_t,$$

because the variation of $z_t \equiv log(Z_t)$ around the steady state are of second order. (Admitted)

Marginal cost

Home's nominal marginal cost is given by

$$MC_t^m = \frac{(1-\tau)W_t}{MPN_t},$$

where MPN_t is *Home*'s average marginal product of labor at t defined as

$$MPN_t \equiv \int_0^1 \frac{\partial Y_t(i)}{\partial N_t(i)} \mathrm{d}i = A_t,$$

and where τ is *Home*'s (constant) employment subsidy. This subsidy will be used latter to offset the monopolistic distortion.

The real marginal cost (express in terms of domestic goods) is the same across firms in any given country.

Home firms' real marginal cost is given by

$$MC_t \equiv \frac{MC_t^n}{P_{H,t}} = \frac{(1-\tau)W_t}{A_t P_{H,t}}.$$

In log terms, we get

$$mc_t = log(1-\tau) + w_t - p_{H,t} - a_t.$$

Firm's problem : price setting

We assume a price setting *à la Calvo*. At each date t, all *Home* firms resetting their prices will choose the same price denoted $\bar{P}_{H,t}$ because they face the same problem.

Home firms' resetting price problem is

$$\max_{\bar{P}_{H,t}} \sum_{k=0}^{+\infty} \theta^k \mathbb{E}_t \left\{ Q_{t,t+k} \left[\bar{P}_{H,t} Y_{t+k|t} - \Psi_{t+k}(Y_{t+k|t}) \right] \right\},$$

where

- $Q_{t,t+k} \equiv \beta^k \frac{C_t}{C_{t+k}} \frac{P_t}{P_{t+k}}$ is *Home* firms' stochastic discount factor for nominal payoffs between t and $t+k$,
- $Y_{t+k|t}$ is output at $t+k$ for a firm that last resetted its price at t,
- $\Psi_t(\cdot)$ is *Home*'s nominal cost function at t,

subject to $Y_{t+k|t} = \left[\frac{\bar{P}_{H,t}}{P_{H,t+k}} \right]^{-\varepsilon} [C_{H,t+k} + C^*_{H,t+k} + G_{t+k}]$ for $k \in \mathbb{N}$,

taking $(C_{t+k})_{k\in\mathbb{N}}$ and $(P_{t+k})_{k\in\mathbb{N}}$ as given.

Firm's FOC

Noticing that $\frac{\partial Y_{t+k|t}}{\partial \bar{P}_{H,t}} = -\varepsilon \frac{Y_{t+k|t}}{\bar{P}_{H,t}}$, the *Home* firms' FOC is

$$\max_{\bar{P}_{H,t}} \sum_{k=0}^{+\infty} \theta^k \mathbb{E}_t \left\{ Q_{t,t+k} Y_{t+k|t} \left[\bar{P}_{H,t} - \mathcal{M} \psi_{t+k|t} \right] \right\} = 0,$$

where $\psi_{t+k|t} \equiv \Psi'_{t+k}(Y_{t+k|t})$ denotes the nominal marginal cost at $t+k$ for a firm that last reset its price at t, and $\mathcal{M} \equiv \frac{\varepsilon}{\varepsilon-1}$.

Under flexible prices ($\theta = 0$), *Home* firms' FOC collapses to $\bar{P}_{H,t} = \mathcal{M}\psi_{t|t}$, so that $\mathcal{M}$ is the "desired" (or frictionless) markup.

Dividing by $P_{H,t-1}$, we get

$$\max_{\bar{P}_{H,t}} \sum_{k=0}^{+\infty} \theta^k \mathbb{E}_t \left\{ Q_{t,t+k} Y_{t+k|t} \left[\frac{\bar{P}_{H,t}}{P_{H,t-1}} - \mathcal{M} MC_{t+k|t} \Pi_{t-1,t+k} \right] \right\} = 0,$$

where $\Pi_{t-1,t+k} \equiv \frac{P_{H,t+k}}{P_{H,t-1}}$ and $MC_{t+k|t} \equiv \frac{\psi_{t+k|t}}{P_{H,t+k}}$ is the real marginal cost at $t+k$ for a *Home* firm whose price was last set at t.

Zero-inflation steady state

At the zero-inflation-rate steady state (ZIRSS),

- $\bar{P}_{H,t}$ and $P_{H,t}$ are equal to each other and constant over time
- therefore, all *Home* firms produce the same quantity of output,
- this quantity is constant over time, as the model features no deterministic trend,
- therefore,

$$\begin{aligned} \frac{\bar{P}_{H,t}}{P_{H,t}} &= 1, & \Pi_{t-1,t+k} &= 1, \\ Q_{t,t+k} &= \beta^k, & Y_{t+k|t} &= Y, \\ MC_{t+k|t} = MC &= \frac{1}{\mathcal{M}}. \end{aligned}$$

Log-linearized firm's FOC

Log-linearization of *Home* firms' FOC around the ZIRSS yields

$$\bar{p}_{H,t} = (1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k \mathbb{E}_t\{\mu + mc_{t+k|t} + p_{H,t+k}\},$$

where $\bar{p}_{H,t}$ denotes the (log) of newly set prices in *Home* (same for all firms reoptimizing), and $\mu \equiv log(\frac{\varepsilon}{\varepsilon-1})$.

Aggregate price level dynamics

As only a fraction $1-\theta$ of firms adjusts price each period, we have

$$P_{H,t} = \left[\theta(P_{H,t-1})^{1-\varepsilon} + (1-\theta)(\bar{P}_{H,t})^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}.$$

Log-linearizing around the ZIRSS, we get

$$\pi_{H,t} = (1-\theta)(\bar{p}_{H,t} - p_{H,t}).$$

Rewrite log-linearized firms' FOC

Because of the constant returns to scale, we have

$$\begin{aligned} \forall k \in \mathbb{N}, mc_{t+k|t} &= log(1-\tau) + (w_{t+k} - p_{H,t+k}) - mpn_{t+k|t} \\ &= log(1-\tau) + (w_{t+k} - p_{H,t+k}) - a_{t+k} \\ &= mc_{t+k}. \end{aligned}$$

Note also that we have

$$\begin{aligned} (1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k \mathbb{E}_t\{p_{H,t+k} - p_{H,t-1}\} &= (1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k \sum_{s=0}^{k}\mathbb{E}_t\{\pi_{H,t+s}\} \\ &= \sum_{s=0}^{+\infty}\mathbb{E}_t\{\pi_{H,t+s}\}(1-\beta\theta)\sum_{k=s}^{+\infty}(\beta\theta)^k \\ &= \sum_{s=0}^{+\infty}(\beta\theta)^s \mathbb{E}_t\{\pi_{H,t+s}\}. \end{aligned}$$

Using the previous result, *Home* firms' FOC can be rewritten as

$$
\begin{aligned}
\bar{p}_{H,t} - p_{H,t-1} &= (1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k \mathbb{E}_t\{\mu + mc_{t+k} + (p_{H,t+k} - p_{H,t-1})\} \\
&= (1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k \mathbb{E}_t\{\mu + mc_{t+k}\} + \sum_{k=0}^{+\infty}(\beta\theta)^k \mathbb{E}_t\{\pi_{H,t+k}\} \\
&= (1-\beta\theta)(\mu + mc_t) + \pi_{H,t} + (1-\beta\theta)\sum_{k=1}^{+\infty}(\beta\theta)^k \mathbb{E}_t\{\mu + mc_{t+k}\} + \sum_{k=1}^{+\infty}(\beta\theta)^k \mathbb{E}_t\{\pi_{H,t+k}\} \\
&= (1-\beta\theta)(\mu + mc_t) + \pi_{H,t} + \beta\theta\left[(1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k \mathbb{E}_t\{\mu + mc_{t+1+k}\} + \sum_{k=1}^{+\infty}(\beta\theta)^k \mathbb{E}_t\{\pi_{H,t+1+k}\}\right] \\
&= (1-\beta\theta)(\mu + mc_t) + \pi_{H,t} + \beta\theta\mathbb{E}_t\Bigg\{(1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k \mathbb{E}_{t+1}\{\mu + mc_{t+1+k}\} + \\
&\sum_{k=1}^{+\infty}(\beta\theta)^k \mathbb{E}_{t+1}\{\pi_{H,t+1+k}\}\Bigg\} \\
&= (1-\beta\theta)(\mu + mc_t) + \pi_{H,t} + \beta\theta\mathbb{E}_t\{\bar{p}_{H,t+1} - p_{H,t}\}
\end{aligned}
$$

Using the aggregate price level dynamics equation, we get

$$
\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \lambda(\mu + mc_t)
$$

where $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$.

Summary firm results

Variable	*Home*	*Foreign*
i-th firm's production function	$Y_t(i) = A_t N_t(i)$	$Y_t(i)^* = A_t^* N_t^*(i)$
i-th firm's labor demand	$N_t(i) = \frac{Y_t(i)}{A_t}$	$N_t^*(i) = \frac{Y_t^*(i)}{A_t^*}$
Aggregate labor demand	$N_t \equiv \int_0^1 N_t(i)\mathrm{d}i = \frac{Y_t Z_t}{A_t}$	$N_t^* \equiv \int_0^1 N_t^*(i)\mathrm{d}i = \frac{Y_t^* Z_t^*}{A_t^*}$
Aggregate output	$Y_t \equiv \left[\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$	$Y_t^* \equiv \left[\int_0^1 Y_t^*(i)^{\frac{\varepsilon-1}{\varepsilon}}\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$
Output dispersion	$Z_t \equiv \int_0^1 \frac{Y_t(i)}{Y_t}\mathrm{d}i$	$Z_t^* \equiv \int_0^1 \frac{Y_t^*(i)}{Y_t^*}\mathrm{d}i$
Aggregate production function in log form	$y_t = a_t + n_t$	$y_t^* = a_t^* + n_t^*$
Real marginal cost	$mc_t = log(1-\tau) + w_t - p_{H,t} - a_t$	$mc_t^* = log(1-\tau^*) + w_t^* - p_{F,t}^* - a_t^*$
Aggregate price level dynamics	$\pi_{H,t} = (1-\theta)(\bar{p}_{H,t} - p_{H,t})$	$\pi_{F,t}^* = (1-\theta^*)(\bar{p}_{F,t}^* - p_{F,t}^*)$
Firms' FOC	$\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \lambda(\mu + mc_t)$ where $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$	$\pi_{F,t}^* = \beta\mathbb{E}_t\{\pi_{F,t+1}^*\} + \lambda^*(\mu + mc_t^*)$ where $\lambda^* \equiv \frac{(1-\theta^*)(1-\beta\theta^*)}{\theta^*}$

3 Equilibrium dynamics

3.1 Aggregate demand and output determination

Good markets

The world demand of *Home*-made good i is given by

$$
\begin{aligned}
Y_t^d(i) &\equiv C_{H,t}(i) + C_{H,t}^*(i) + G_t(i) \\
&= \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon}(C_{H,t} + C_{H,t}^* + G_t).
\end{aligned}
$$

The market of all *Home* and *Foreign* goods clear in equilibrium so that

$$
Y_t(i) = Y_t^d(i), \forall i \in [0,1].
$$

Using *Home* RH's optimal allocations, identities and the international risk condition, we get

$$
\begin{aligned}
Y_t &= C_{H,t} + C^*_{H,t} + G_t \\
&= (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-1} C_t + \alpha^* \left(\frac{P^*_{H,t}}{P^*_t}\right)^{-1} C^*_t + G_t \\
&\overset{LOP}{=} \left(\frac{P_{H,t}}{P_t}\right)^{-1} \left[(1-\alpha)C_t + \alpha^* \left(\frac{P_t}{P^*_t}\right)^{-1} C^*_t\right] + G_t \\
&\overset{IRS}{=} \left(\frac{P_{H,t}}{P_t}\right)^{-1} \left[(1-\alpha) + \alpha^* \left(\frac{P_t}{P^*_t}\right)^{-1} \mathcal{Q}_t^{-1}\right] C_t + G_t \\
&= \left(\frac{P_{H,t}}{P_t}\right)^{-1} \left[(1-\alpha) + \alpha^*\right] C_t + G_t \\
&= (S_t)^{\alpha} \left[(1-\alpha) + \alpha^*\right] C_t + G_t
\end{aligned}
$$

When $\alpha^* = \alpha$, we get

$$Y_t = (S_t)^{\alpha} C_t + G_t.$$

Log-linearization of the good markets clearing condition

Express the good markets clearing condition in log-linearized form. It involves the deviation of output from its steady state value ($\hat{y}_t \equiv y_t - y$), the deviation of government consumption from its steady state value ($\hat{g}_t \equiv g_t - g$), the deviation of private consumption from its steady state value ($\hat{c}_t \equiv c_t - c$), and s_t.

A first version of the IS equation

Use the consumption Euler equation to derive a first version of the IS equation (which involves deviation of the variables from their steady state value).

3.2 The supply side: marginal cost and inflation dynamics

Using *Home* RH's intratemporal FOC, *Home*'s aggregate production function and *Home's* price level identities, we have

$$
\begin{aligned}
mc_t &= w_t - p_{H,t} - a_t + log(1-\tau) \\
&= c_t + \varphi n_t + (p_t - p_{H,t}) - a_t + log(1-\tau) - log(1-\chi) \\
&= c_t + \varphi(y_t - a_t) + (p_t - p_{H,t}) - a_t + log(1-\tau) - log(1-\chi) \\
&= c_t + \varphi y_t - (1+\varphi)a_t + (p_t - p_{H,t}) + log(1-\tau) - log(1-\chi) \\
&= c_t + \varphi y_t - (1+\varphi)a_t + \alpha s_t + log(1-\tau) - log(1-\chi).
\end{aligned}
$$

Find $\hat{mc}_t$ and use the goods market clearing condition to replace $\hat{c}_t$.

A first version of the NKPC

Plug the previous result in the firms' FOC and obtain a first version of the NKPC.

3.3 Flexible price equilibrium or natural equilibrium ($\theta \to 0$)

We consider the equilibrium under flexible price, i.e. when $\theta \to 0$.

Obtain the natural output ($\bar{y}_t$), government spending ($\bar{g}_t$) and terms of trade ($\bar{s}_t$).

3.3bis Efficient allocation

Alternatively, follow Galí and Monacelli (2008) and solve the social planner problem and obtain the efficient allocation.

Prove that the efficient allocation can be decentralized in a flexible price economy where monopolistic distortions are removed by means of a subsidy ($\tau = \frac{1}{\varepsilon}$).

In this case $\bar{x}_t$ would denote the efficient allocation of x_t.

3.4 Sticky price equilibrium

Re-express the first version of the IS equation and the NKPC using the output gap ($\tilde{y}_t \equiv y_t - \bar{y}_t$), the government consumption gap ($\tilde{g}_t \equiv g_t - \bar{g}_t$) and the terms of trade gap ($\tilde{s}_t \equiv s_t - \bar{s}_t$).

Summary sticky price equilibrium

Given the sequence $(a_t, a_t^*, g_t, g_t^*)_{t\in\mathbb{N}}$, the endogeneous sequence$(\tilde{y}_t, \tilde{g}_t, \pi_{H,t}, \pi_t, y_t, c_t, n_t, w_t - p_t; \tilde{y}_t^*, \tilde{g}_t^*, \pi_{F,t}^*, \pi_t^*, y_t^*, c_t^*, n_t^*, w_t^* - p_t^*; s_t, \tilde{s}_t, i_t, i_t^*)_{t\in\mathbb{N}}$ made of 20 variables is determined by a system of equilibrium conditions made of 20 linear equations:

- IS: 2 equations,
- NKPC: 2 equations,
- Definition of the output gap: 2 equations,
- Definition of the government consumption gap : 2 equations,
- Inflation identities (i.e. link PPI and CPI): 2 equations,
- Goods-market-clearing condition: 2 equations,
- Aggregate production function: 2 equations,
- Labor-consumption trade-off condition: 2 equations,
- International risk sharing condition: 1 equation,
- Definition of the terms of trade gap: 1 equation.
- Nominal interest rate rule : 1 equation,
- Common central bank (i.e. $i_t = i_t^*$) : 1 equation.

4 Optimal fiscal policy

Similar to the chapter 2 of Olivier Loisel and to Galí and Monacelli (2008).

Derive a welfare loss function.

Proceed as if the governments, at each date t,

- directly controlled not only g_t but also π_t and $\tilde{y}_t$,
- observed the history of the exogeneous shocks.

Simulations

5 Fiscal policy design

Similar to the chapter 3 of Olivier Loisel.

Assume that the governments, at each date t,

- directly controlled g_t,
- may have a limited observation set.

Is there a kind of Taylor rule for fiscal policy? How should core country react the periphery inflation or output gap?

Simulations

Has this rule an important effect on periphery stabilization?

6 Sensitivity analysis

Assess the sensitivity of the responses to calibrated parameters values.