Master Thesis - Model draft n°4

Maxime Brun

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1 Introduction

1.1 Approach

We replicate Galí and Monacelli (2008) in a two-country setup.
We add features to the model

- We relax the function form assumptions
- We add a country size parameter
- We add a labor disutility shock

1.2 References

Below are the references we used to build the model:

- ENSAE MiE 2 course : AE332, Monetary Economics, Olivier Loisel
- Galí and Monacelli, Optimal monetary and fiscal policy in a currency union, *Journal of International Economics*, 2008
- Marcos Antonio C. da Silveira, Two-country new Keynesian DSGE model : a small open economy as limit case, *Ipea*, 2006
- Cole et al., One EMU fiscal policy for the Euro, *Macroeconomic Dynamics*, 2019
- Forlati, Optimal monetary and fiscal policy in the EMU : does fiscal policy coordination matter?,*Center for Fiscal Policy, EPFL, Chair of International Finance (CFI) Working Paper No. 2009-04*, 2009
- Schäfer, Monetary union with sticky prices and direct spillover channels, *Journal of Macroeconomics*, 2016

2 A currency union model

We model a currency union as a closed system made up of two economies : *Home* and *Foreign*.
Variables without asterix (e.g. X) denote *Home* variables and variables with an asterix (e.g. X_t^*) denote *Foreign* variables.

Home is inhabited by a continuum of identical households indexed by j where $j \in [0, h]$ with $0 \leq h \leq 1$. *Foreign* is inhabited by a continuum of identical households indexed by j where $j \in [h, 1]$.

2.1 Households

2.1.1 Objective

Home j-th household seeks to maximizes

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t^j, N_t^{sj}, G_t),$$

where U is the instantaneous utility function, N_t^{sj} is the number of work hours supplied by *Home* j-th household, C_t^j is a composite index of *Home* j-th household's consumption, and G_t is an index of *Home*'s government consumption.

2.1.2 Aggregate composite consumption index

More precisely, C_t^j is given by

$$C_t^j \equiv \left[(1-\alpha)^{\frac{1}{\eta}} (C_{H,t}^j)^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} (C_{F,t}^j)^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}},$$

where

- $C_{H,t}^j$ is an index of *Home* j-th household's consumption of *Home*-made goods,
- $C_{F,t}^j$ is an index of *Home* j-th household's consumption of *Foreign*-made goods,
- $\alpha \in [0, 1]$ is a measure of *Home*'s **openess** and $1 - \alpha$ is a measure of *Home*'s **home bias**,
- η is *Home*'s elasticity of substitution between *Home*-made goods and *Foreign*-made goods.

2.1.3 Regional consumption indexes

$C_{H,t}^j$ is defined by the CES function

$$C_{H,t}^j \equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}} \int_0^h C_{H,t}^j(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where

- $C_{H,t}^j(i)$ is *Home* j-th household's consumption of *Home*-made good i,
- $\varepsilon > 1$ is the elasticity of substitution between *Home*-made goods,
- h measures the relative size of *Home*'s economy.

Similarly, $C^j_{F,t}$ is defined by the CES function

$$C^j_{F,t} \equiv \left[\left(\frac{1}{1-h}\right)^{\frac{1}{\varepsilon}} \int_h^1 C^j_{F,t}(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where

- $C^j_{F,t}(i)$ is *Home* j-th household's consumption of *Foreign*-made good i,
- $\varepsilon > 1$ is the elasticity of substitution between *Foreign*-made goods,
- $1 - h$ measures the relative size of *Foreign*'s economy.

2.1.4 Household budget constraints

Home j-th household faces a sequence of budget constraints

$$\forall t \geq 0, \int_0^h P_{H,t}(i) C^j_{H,t}(i)\mathrm{d}i + \int_h^1 P_{F,t}(i) C^j_{F,t}(i)\mathrm{d}i + \mathbb{E}_t\{Q_{t,t+1} D^j_{t+1}\} \leq D^j_t + W_t N^{sj}_t + T_t,$$

where

- $P_{H,t}(i)$ is *Home*'s price of *Home*-made good i,
- $P_{F,t}(i)$ is *Home*'s price of *Foreign*-made good i,
- D^j_{t+1} is the quantity of one-period nominal bonds held by *Home* j-th household,
- W_t is *Home*'s nominal wage
- T_t denotes *Home*'s lump sum taxes.

2.1.5 Optimal allocation of consumption across goods

Given $C^j_{H,t}$ and $C^j_{F,t}$, a first step is to find the optimal allocations $(C^j_{H,t}(i))_{i\in[0,h]}$ and $(C^j_{F,t}(i))_{i\in[h,1]}$ that minimize the regional expenditures.
Home j-th household's optimal consumption of *Home*-made good i is given by

$$C^j_{H,t}(i) = \frac{1}{h}\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} C^j_{H,t},$$

where $P_{H,t} \equiv \left[\frac{1}{h}\int_0^h P_{H,t}(i)^{1-\varepsilon}\mathrm{d}i\right]^{\frac{1}{1-\varepsilon}}$ is *Home*'s price index of *Home*-made goods.
Similarly, *Home* j-th household's optimal consumption of *Foreign*-made good i is given by

$$C^j_{F,t}(i) = \frac{1}{1-h}\left(\frac{P_{F,t}(i)}{P_{F,t}}\right)^{-\varepsilon} C^j_{F,t},$$

where $P_{F,t} \equiv \left[\frac{1}{1-h}\int_h^1 P_{F,t}(i)^{1-\varepsilon}\mathrm{d}i\right]^{\frac{1}{1-\varepsilon}}$ is *Home*'s price index of *Foreign*-made goods.

2.1.6 Optimal allocation of consumption across regions

Given C_t^j, a second step is to find the optimal allocation$(C_{H,t}^j, C_{F,t}^j)$ that minimizes total expenditures.
Home j-th household's optimal consumption of *Home*-made goods is given by

$$C_{H,t}^j = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t^j,$$

and *Home* j-th household's optimal consumption of *Foreign*-made goods is given by

$$C_{F,t}^j = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t^j,$$

where $P_t \equiv \left[(1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}\right]^{\frac{1}{1-\eta}}$ is *Home*'s consumer price index (CPI).

2.1.7 Rewrite household's budget constraints

Combining all the previous results, *Home* j-th household's expenditures in *Home*-made goods writes

$$\int_0^h P_{H,t}(i) C_{H,t}^j(i)\mathrm{d}i = C_{H,t}^j P_{H,t}^{\varepsilon} \frac{1}{h}\int_0^h P_{H,t}(i)^{1-\varepsilon}\mathrm{d}i = P_{H,t}C_{H,t}^j.$$

The same formula applies to *Home* j-th household's expenditures in *Foreign*-made goods. We can write *Home* j-th household's total expenditures as

$$\begin{aligned}
\int_0^h P_{H,t}(i) C_{H,t}^j(i)\mathrm{d}i + \int_h^1 P_{H,t}(i) C_{F,t}^j(i)\mathrm{d}i &= P_{H,t}C_{H,t}^j + P_{F,t}C_{F,t}^j \\
&= (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} P_{H,t}C_t^j + \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t^j \\
&= P_t C_t^j
\end{aligned}$$

Therefore, conditional on an optimal allocation across goods and regions, *Home* j-th household's budget constraints can be rewritten as

$$\forall t \geq 0, P_t C_t^j + \mathbb{E}_t\{Q_{t,t+1}D_{t+1}^j\} \leq D_t^j + W_t N_t^{sj} + T_t.$$

2.1.8 Household's intratemporal and intertemporal FOCs

Now, we can derive the first order conditions for *Home* j-th household's optimal consumption level C_t^j as well as for *Home* j-th household's optimal number of hours worked N_t^{sj}.
Home j-th household's **intratemporal** FOC is

$$-\frac{U_{n,t}^j}{U_{c,t}^j} = \frac{W_t}{P_t},$$

and *Home* j-th household's **intertemporal** FOC is

$$\mathbb{E}_t\{Q_{t,t+1}\} = \beta\mathbb{E}_t\left\{\frac{U_{c,t+1}^j}{U_{c,t}^j}\frac{P_t}{P_{t+1}}\right\}.$$

2.1.9 Functional form of the instantaneous utility function

We assume that the instantaneous utility takes the specific form

$$U(C_t^j, N_t^{sj}, G_t) = \frac{(C_t^j)^{1-\sigma}}{1-\sigma} - \Xi_t \frac{(N_t^{sj})^{1+\varphi}}{1+\varphi} + \chi \frac{(G_t)^{1-\gamma}}{1-\gamma}$$

where $\chi \in [0, 1]$, $\varphi > 0$ and where Ξ_t is a labor disutility shock.

2.1.10 Rewrite household's intratemporal and intertemporal FOCs under the functional form assumptions

Under the functional forms assumptions, *Home* j-th household **intratemporal** FOC becomes

$$\Xi_t (N_t^{sj})^{\varphi} (C_t^j)^{\sigma} = \frac{W_t}{P_t},$$

and *Home* j-th household's **intertemporal** FOC becomes

$$\mathbb{E}_t\{Q_{t,t+1}\} = \beta \mathbb{E}_t \left\{ \left(\frac{C_{t+1}^j}{C_t^j} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right\}.$$

2.2 Aggregating optimal allocation

Home's optimal consumption of *Home*-made good i and of *Foreign*-made good i are given by

$$C_{H,t}(i) \equiv \int_0^h C_{H,t}^j(i) \mathrm{d}j = \frac{1}{h} \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\varepsilon} C_{H,t},$$

$$C_{F,t}(i) \equiv \int_0^h C_{F,t}^j(i) \mathrm{d}j = \frac{1}{1-h} \left(\frac{P_{F,t}(i)}{P_{F,t}} \right)^{-\varepsilon} C_{F,t},$$

whereas, *Home*'s optimal consumption of *Home*-made goods and of *Foreign*-made goods are given by

$$C_{H,t} \equiv \int_0^h C_{H,t}^j \mathrm{d}j = (1-\alpha) \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} C_t,$$

$$C_{F,t} \equiv \int_0^h C_{F,t}^j \mathrm{d}j = \alpha \left(\frac{P_{F,t}}{P_t} \right)^{-\eta} C_t,$$

where, the composite index of *Home*'s consumption is given by

$$C_t \equiv \int_0^h C_t^j \mathrm{d}j = h C_t^j,$$

since all *Home* households are identical.
Similarly, we define the number of work hours supplied by *Home* by

$$N_t^s \equiv \int_0^h N_t^{sj} \mathrm{d}j = h N_t^{sj}.$$

2.3 Aggregating optimal intratemporal and intertemporal FOCs

Using the previous results, we can re-write the intratemporal and intertemporal choices at the aggregate level.
At the aggregate level, **intratemporal** FOC becomes

$$\frac{1}{h^{\varphi+\sigma}}\Xi_t(N_t^s)^{\varphi}(C_t)^{\sigma} = \frac{W_t}{P_t},$$

and **intertemporal** FOC becomes

$$\mathbb{E}_t\{Q_{t,t+1}\} = \beta\mathbb{E}_t\left\{\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right\}.$$

2.3.1 Aggregate FOCs in log-linearized form

Home RH's **intratemporal** FOC in log form is

$$w_t - p_t = -(\varphi+\sigma)\log(h) + \xi_t + \sigma c_t + \varphi n_t^s,$$

where $\xi_t \equiv \log(\Xi_t)$, and *Home* RH's **intertemporal** FOC in log form is

$$c_t = \mathbb{E}_t\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{t+1}\} - \bar{i}),$$

where $i_t \equiv log\left(\frac{1}{\mathbb{E}_t\{Q_{t,t+1}\}}\right)$ is referred to as the **short-term nominal interest rate**, $\pi_t \equiv p_t - p_{t-1}$ is **CPI inflation**, and $\bar{i} \equiv -\log(\beta)$.

2.3.2 Summary of household's optimal allocation

Analogous results hold for the *Foreign* country. Similarly, we denote α^* the measure of *Foreign*'s degree of openness.

Table 1: Summary optimal allocation

Variable	Home	Foreign
j-th household's composite consumption index	$C_t^j \equiv \left[(1-\alpha)^{\frac{1}{\eta}}(C_{H,t}^j)^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}(C_{F,t}^j)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$	$C_t^{j*} \equiv \left[(\alpha^*)^{\frac{1}{\eta}}(C_{H,t}^{j*})^{\frac{\eta-1}{\eta}} + (1-\alpha^*)^{\frac{1}{\eta}}(C_{F,t}^{j*})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$
j-th household's composite consumption of *Home*-made good	$C_{H,t}^j \equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}} \int_0^h C_{H,t}^j(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$	$C_{H,t}^{j*} \equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}} \int_0^h C_{H,t}^{j*}(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$
j-th household's composite consumption of *Foreign*-made good	$C_{F,t}^j \equiv \left[\left(\frac{1}{1-h}\right)^{\frac{1}{\varepsilon}} \int_h^1 C_{F,t}^j(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$	$C_{F,t}^{j*} \equiv \left[\left(\frac{1}{1-h}\right)^{\frac{1}{\varepsilon}} \int_h^1 C_{F,t}^{j*}(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$
j-th household's optimal consumption of *Home*-made good i	$C_{H,t}^j(i) = \frac{1}{h}\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} C_{H,t}^j$	$C_{H,t}^{j*}(i) = \frac{1}{h}\left(\frac{P_{H,t}^*(i)}{P_{H,t}^*}\right)^{-\varepsilon} C_{H,t}^{j*}$
Price index of *Home*-made goods	$P_{H,t} \equiv \left[\frac{1}{h}\int_0^h P_{H,t}(i)^{1-\varepsilon}\mathrm{d}i\right]^{\frac{1}{1-\varepsilon}}$	$P_{H,t}^* \equiv \left[\frac{1}{h}\int_0^h P_{H,t}^*(i)^{1-\varepsilon}\mathrm{d}i\right]^{\frac{1}{1-\varepsilon}}$
j-th household's optimal consumption of *Foreign*-made good i	$C_{F,t}^j(i) = \frac{1}{1-h}\left(\frac{P_{F,t}(i)}{P_{F,t}}\right)^{-\varepsilon} C_{F,t}^j$	$C_{F,t}^{j*}(i) = \frac{1}{1-h}\left(\frac{P_{F,t}^*(i)}{P_{F,t}^*}\right)^{-\varepsilon} C_{F,t}^{j*}$
Price index of *Foreign*-made goods	$P_{F,t} \equiv \left[\frac{1}{1-h}\int_h^1 P_{F,t}(i)^{1-\varepsilon}\mathrm{d}i\right]^{\frac{1}{1-\varepsilon}}$	$P_{F,t}^* \equiv \left[\frac{1}{1-h}\int_h^1 P_{F,t}^*(i)^{1-\varepsilon}\mathrm{d}i\right]^{\frac{1}{1-\varepsilon}}$
j-th household's optimal consumption of *Home*-made goods	$C_{H,t}^j = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t^j$	$C_{H,t}^{j*} = \alpha^*\left(\frac{P_{H,t}^*}{P_t^*}\right)^{-\eta} C_t^{j*}$
j-th household's optimal consumption of *Foreign*-made goods	$C_{F,t}^j = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t^j$	$C_{F,t}^{j*} = (1-\alpha^*)\left(\frac{P_{F,t}^*}{P_t^*}\right)^{-\eta} C_t^{j*}$
Consumer price index (CPI)	$P_t \equiv \left[(1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}\right]^{\frac{1}{1-\eta}}$	$P_t^* \equiv \left[\alpha^*(P_{H,t}^*)^{1-\eta} + (1-\alpha^*)(P_{F,t}^*)^{1-\eta}\right]^{\frac{1}{1-\eta}}$
Optimal consumption of *Home*-made good i	$C_{H,t}(i) \equiv \int_0^h C_{H,t}^j(i)\mathrm{d}j = \frac{1}{h}\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} C_{H,t}$	$C_{H,t}^*(i) \equiv \int_h^1 C_{H,t}^{j*}(i)\mathrm{d}j = \frac{1}{h}\left(\frac{P_{H,t}^*(i)}{P_{H,t}^*}\right)^{-\varepsilon} C_{H,t}^*$
Optimal consumption of *Foreign*-made good i	$C_{F,t}(i) \equiv \int_0^h C_{F,t}^j(i)\mathrm{d}j = \frac{1}{1-h}\left(\frac{P_{F,t}(i)}{P_{F,t}}\right)^{-\varepsilon} C_{F,t}$	$C_{F,t}^*(i) \equiv \int_h^1 C_{F,t}^{j*}(i)\mathrm{d}j = \frac{1}{1-h}\left(\frac{P_{F,t}^*(i)}{P_{F,t}^*}\right)^{-\varepsilon} C_{F,t}^*$
Optimal consumption of *Home*-made goods	$C_{H,t} \equiv \int_0^h C_{H,t}^j\mathrm{d}j = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t$	$C_{H,t}^* \equiv \int_h^1 C_{H,t}^{j*}\mathrm{d}j = \alpha^*\left(\frac{P_{H,t}^*}{P_t^*}\right)^{-\eta} C_t^*$
Optimal consumption of *Foreign*-made goods	$C_{F,t} \equiv \int_0^h C_{F,t}^j\mathrm{d}j = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t$	$C_{F,t}^* \equiv \int_h^1 C_{F,t}^{j*}\mathrm{d}j = (1-\alpha^*)\left(\frac{P_{F,t}^*}{P_t^*}\right)^{-\eta} C_t^*$
Composite consumption index	$C_t \equiv \int_0^h C_t^j\mathrm{d}j = hC_t^j$	$C_t^* \equiv \int_h^1 C_t^{j*}\mathrm{d}j = hC_t^{j*}$
Number of work hours supplied	$N_t^s \equiv \int_0^h N_t^{sj}\mathrm{d}j = hN_t^{sj}$	$N_t^{s*} \equiv \int_h^1 N_t^{sj*}\mathrm{d}j = hN_t^{sj*}$
Intratemporal FOC	$w_t - p_t = -(\varphi+\sigma)\log(h) + \xi_t + \sigma c_t + \varphi n_t^s$	$w_t^* - p_t^* = -(\varphi+\sigma)\log(1-h) + \xi_t^* + \sigma c_t^* + \varphi n_t^{s*}$
Intertemporal FOC	$c_t =$	$c_t^* =$

2.4 Definitions, identities and international risk sharing

2.4.1 The law of one price

Since we are in a currency union, the law of one price (LOP) states that $P_{H,t}(i) = P^*_{H,t}(i)$ and $P_{F,t}(i) = P^*_{F,t}(i)$. As a consequence, $P_{H,t} = P^*_{H,t}$ and $P_{F,t} = P^*_{F,t}$.

2.4.2 Terms of trade

We derive the relationship between inflation, terms of trade and real exchange rate.
Home's terms of trade is defined as

$$S_t \equiv \frac{P_{F,t}}{P_{H,t}},$$

and *Foreign*'s terms of trade is defined as

$$S^*_t \equiv \frac{P^*_{H,t}}{P^*_{F,t}}.$$

The terms of trade is simply the relative price of imported goods in terms of domestic goods.
Using the LOP, we have

$$S^*_t = \frac{1}{S_t}.$$

2.4.3 Home bias

It is crucial to understand the role of the parameter α. We follow Da Silveira (2006) and we assume that α and α^* are linked to h by

$$\begin{aligned} \alpha &= \bar{\alpha}(1-h) \\ \alpha^* &= \bar{\alpha}h \end{aligned}$$

where $\bar{\alpha}$ is exogeneously given. See Da Silveira page 16.

2.4.4 Price level and inflation identities

Using the definitions of P_t, P^*_t, S_t, and S^*_t, we get

$$\begin{aligned} \frac{P_t}{P_{H,t}} &= \left[(1-\alpha) + \alpha(S_t)^{1-\eta}\right]^{\frac{1}{1-\eta}} \equiv g(S_t) \\ \frac{P_t}{P_{F,t}} &= \frac{P_t}{P_{H,t}}\frac{P_{H,t}}{P_{F,t}} = \frac{g(S_t)}{S_t} \equiv h(S_t) \\ \frac{P^*_t}{P^*_{H,t}} &= \left[\alpha^* + (1-\alpha^*)(S_t)^{1-\eta}\right]^{\frac{1}{1-\eta}} \equiv g^*(S_t) \\ \frac{P^*_t}{P^*_{F,t}} &= \frac{P^*_t}{P^*_{H,t}}\frac{P^*_{H,t}}{P^*_{F,t}} = \frac{g^*(S_t)}{S_t} \equiv h^*(S_t). \end{aligned}$$

Log-linearizing around the symmetric where $S_t = 1$, we get

$$
\begin{aligned}
p_t - p_{H,t} &= \alpha s_t \\
p_t - p_{F,t} &= -(1-\alpha)s_t \\
p_t^* - p_{H,t}^* &= (1-\alpha^*)s_t \\
p_t^* - p_{F,t}^* &= -\alpha^* s_t.
\end{aligned}
$$

Using the expression of home bias as a function of $\bar{\alpha}$ and h, we get

$$
\begin{aligned}
\pi_t &= \pi_{H,t} + \bar{\alpha}(1-h)\Delta s_t \\
\pi_t^* &= \pi_{F,t}^* - \bar{\alpha} h \Delta s_t,
\end{aligned}
$$

where *Home* and *Foreign* inflation of domestic price indexes are respectively given by $\pi_{H,t} = p_{H,t} - p_{H,t-1}$ and $\pi_{F,t}^* = p_{F,t}^* - p_{F,t-1}^*$.

2.4.5 Real exchange rate

Using the LOP, *Home*'s real exchange rate denoted $\mathcal{Q}_t$ is given by

$$\mathcal{Q}_t \equiv \frac{P_t^*}{P_t} = \frac{g^*(S_t)}{g(S_t)}.$$

A first order approximation around the steady state where $S_t = 1$ gives

$$\mathcal{Q}_t \simeq 1 + (1-\alpha^*-\alpha)(S_t - 1).$$

Therefore, around the steady state where $S_t = 1$ and $\mathcal{Q}_t = 1$, we have

$$q_t = (1-\bar{\alpha})s_t.$$

2.4.6 International risk sharing (not detailed)

The international risk sharing (IRS) condition implies that

$$C_t = \frac{h}{1-h}\vartheta \mathcal{Q}_t^{\frac{1}{\sigma}} C_t^*.$$

We assume the same initial conditions for *Home* and *Foreign* households, so that $\vartheta = 1$. In log form, the IRS condition writes

$$c_t = \log(\frac{h}{1-h}) + \frac{1}{\sigma} q_t + c_t^*.$$

2.5 Government

2.5.1 Government consumption index

Home's public consumption index is given by the CES function

$$G_t \equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}} \int_0^h G_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where $G_t(i)$ is the quantity of *Home*-made good i purchased *Home*'s government.

2.5.2 Government demand schedules

For any level of public consumption G_t, the government demand schedules are analogous to those obtain for private consumption, namely

$$G_t(i) = \frac{1}{h}\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} G_t.$$

2.5.3 Summary government results

Table 2: Summary government

Variable	Home	Foreign
Government consumption index	$G_t \equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}} \int_0^h G_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$	$G_t^* \equiv \left[\left(\frac{1}{1-h}\right)^{\frac{1}{\varepsilon}} \int_h^1 G_t^*(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$
Optimal government consumption of domestically made good	$G_t(i) = \frac{1}{h}\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} G_t$	$G_t^*(i) = \frac{1}{1-h}\left(\frac{P_{H,t}^*(i)}{P_{H,t}^*}\right)^{-\varepsilon} G_t^*$

2.6 Firms

Each country has a continuum of firms represented by the interval $[0, h]$ for *Home* and by the interval $[h, 1]$ for *Foreign*. Each firm produces a differentiated good.

2.6.1 Technology

All *Home* firms use the same technology, represented by the production function

$$Y_t(i) = A_t N_t(i),$$

where A_t is *Home*'s productivity.

2.6.2 Labor demand

The technology constraint implies that *Home* i-th firm's labor demand is given by

$$N_t(i) = \frac{Y_t(i)}{A_t}.$$

2.6.3 Aggregate labor demand

Home's aggregate labor demand is defined as

$$N_t \equiv \int_0^h N_t(i)\mathrm{d}i = \frac{Y_t Z_t}{A_t}$$

where

$$Y_t \equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}} \int_0^h Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

is the aggregate production index while $Z_t \equiv \int_0^h \frac{Y_t(i)}{Y_t} \mathrm{d}i$ is a measure of the dispersion of *Home* firms' output.

2.6.4 Aggregate production function

In log form, we have a relationship between *Home*'s aggregate employment and *Home*'s output

$$y_t = a_t + n_t,$$

because the variation of $z_t \equiv \log(Z_t)$ around the steady state are of second order. (Admitted for now)

2.6.5 Marginal cost

Home's nominal marginal cost is given by

$$MC_t^n = \frac{(1-\tau)W_t}{MPN_t},$$

where MPN_t is *Home*'s average marginal product of labor at t defined as

$$MPN_t \equiv \frac{1}{h} \int_0^h \frac{\partial Y_t(i)}{\partial N_t(i)} \mathrm{d}i = A_t,$$

and where τ is *Home*'s (constant) employment subsidy. This subsidy will be used latter to offset the monopolistic distortion.
The real marginal cost (express in terms of domestic goods) is the same across firms in any given country.
Home firms' real marginal cost is given by

$$MC_t \equiv \frac{MC_t^n}{P_{H,t}} = \frac{(1-\tau)W_t}{A_t P_{H,t}}.$$

In log form, we get

$$mc_t = \log(1-\tau) + w_t - p_{H,t} - a_t.$$

2.6.6 Firm's problem : price setting

We assume a price setting *à la Calvo*. At each date t, all *Home* firms resetting their prices will choose the same price denoted $\bar{P}_{H,t}$ because they face the same problem.
Home firms' resetting price problem is

$$\max_{\bar{P}_{H,t}} \sum_{k=0}^{+\infty} \theta^k \mathbb{E}_t \Big\{ Q_{t,t+k} \Big[\bar{P}_{H,t} Y_{t+k|t} - \Psi_{t+k}(Y_{t+k|t}) \Big] \Big\},$$

where

- $Q_{t,t+k} \equiv \beta^k \frac{C_t}{C_{t+k}} \frac{P_t}{P_{t+k}}$ is *Home* firms' stochastic discount factor for nominal payoffs between t and $t+k$,
- $Y_{t+k|t}$ is output at $t+k$ for a firm that last resetted its price at t,
- $\Psi_t(\cdot)$ is *Home*'s nominal cost function at t,

subject to $Y_{t+k|t} = \left(\frac{\bar{P}_{H,t}}{P_{H,t+k}} \right)^{-\varepsilon} (C_{H,t+k} + C^*_{H,t+k} + G_{t+k})$ for $k \in \mathbb{N}$, taking $(C_{t+k})_{k\in\mathbb{N}}$ and $(P_{t+k})_{k\in\mathbb{N}}$ as given.

2.6.7 Firm's FOC

Noticing that $\frac{\partial Y_{t+k|t}}{\partial \bar{P}_{H,t}} = -\varepsilon \frac{Y_{t+k|t}}{\bar{P}_{H,t}}$, *Home* firms' FOC is

$$\max_{\bar{P}_{H,t}} \sum_{k=0}^{+\infty} \theta^k \mathbb{E}_t \Big\{ Q_{t,t+k} Y_{t+k|t} \Big[\bar{P}_{H,t} - \mathcal{M} \psi_{t+k|t} \Big] \Big\} = 0,$$

where $\psi_{t+k|t} \equiv \Psi'_{t+k}(Y_{t+k|t})$ denotes the nominal marginal cost at $t+k$ for a firm that last reset its price at t, and $\mathcal{M} \equiv \frac{\varepsilon}{\varepsilon-1}$.
Under flexible prices ($\theta = 0$), *Home* firms' FOC collapses to $\bar{P}_{H,t} = \mathcal{M}\psi_{t|t}$, so that $\mathcal{M}$ is the "desired" (or frictionless) markup.
Dividing by $P_{H,t-1}$, we get

$$\max_{\bar{P}_{H,t}} \sum_{k=0}^{+\infty} \theta^k \mathbb{E}_t \Big\{ Q_{t,t+k} Y_{t+k|t} \Big[\frac{\bar{P}_{H,t}}{P_{H,t-1}} - \mathcal{M} MC_{t+k|t} \Pi_{t-1,t+k} \Big] \Big\} = 0,$$

where $\Pi_{t-1,t+k} \equiv \frac{P_{H,t+k}}{P_{H,t-1}}$ and $MC_{t+k|t} \equiv \frac{\psi_{t+k|t}}{P_{H,t+k}}$ is the real marginal cost at $t+k$ for a *Home* firm whose price was last set at t.

2.6.8 Zero-inflation steady state

At the zero-inflation-rate steady state (ZIRSS),

- $\bar{P}_{H,t}$ and $P_{H,t}$ are equal to each other and constant over time,
- therefore, all *Home* firms produce the same quantity of output,
- this quantity is constant over time, as the model features no deterministic trend,
- therefore,

$$\begin{aligned} \frac{\bar{P}_{H,t}}{P_{H,t}} &= 1, & \Pi_{t-1,t+k} &= 1, \\ Q_{t,t+k} &= \beta^k, & Y_{t+k|t} &= Y, \\ MC_{t+k|t} &= MC = \frac{1}{\mathcal{M}}. \end{aligned}$$

2.6.9 Log-linearized firm's FOC

Log-linearization of *Home* firms' FOC around the ZIRSS yields

$$\bar{p}_{H,t} = (1-\beta\theta) \sum_{k=0}^{+\infty} (\beta\theta)^k \mathbb{E}_t \big\{ \mu + mc_{t+k|t} + p_{H,t+k} \big\},$$

where $\bar{p}_{H,t}$ denotes the (log) of newly set prices in *Home* (same for all firms reoptimizing), and $\mu \equiv \log(\frac{\varepsilon}{\varepsilon-1})$.

2.6.10 Aggregate price level dynamics

As only a fraction $1-\theta$ of firms adjusts price each period, we have

$$P_{H,t} = \left[\theta (P_{H,t-1})^{1-\varepsilon} + (1-\theta)(\bar{P}_{H,t})^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}.$$

Log-linearizing around the ZIRSS, we get

$$\pi_{H,t} = (1-\theta)(\bar{p}_{H,t} - p_{H,t}).$$

2.6.11 Rewrite log-linearized firms' FOC

Because of the constant returns to scale, we have

$$\begin{aligned}
\forall k \in \mathbb{N}, mc_{t+k|t} &= \log(1-\tau) + (w_{t+k} - p_{H,t+k}) - mpn_{t+k|t} \\
&= \log(1-\tau) + (w_{t+k} - p_{H,t+k}) - a_{t+k} \\
&= mc_{t+k}.
\end{aligned}$$

Note also that we have

$$\begin{aligned}
(1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k \mathbb{E}_t\big\{p_{H,t+k} - p_{H,t-1}\big\} &= (1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k \sum_{s=0}^{k}\mathbb{E}_t\big\{\pi_{H,t+s}\big\} \\
&= \sum_{s=0}^{+\infty}\mathbb{E}_t\big\{\pi_{H,t+s}\big\}(1-\beta\theta)\sum_{k=s}^{+\infty}(\beta\theta)^k \\
&= \sum_{s=0}^{+\infty}(\beta\theta)^s\mathbb{E}_t\big\{\pi_{H,t+s}\big\}.
\end{aligned}$$

Using the previous result, *Home* firms' FOC can be rewritten as

$$\begin{aligned}
\bar{p}_{H,t} - p_{H,t-1} &= (1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big\{\mu + mc_{t+k} + (p_{H,t+k} - p_{H,t-1})\big\} \\
&= (1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big\{\mu + mc_{t+k}\big\} + \sum_{k=0}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big\{\pi_{H,t+k}\big\} \\
&= (1-\beta\theta)(\mu + mc_t) + \pi_{H,t} + (1-\beta\theta)\sum_{k=1}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big\{\mu + mc_{t+k}\big\} + \sum_{k=1}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big\{\pi_{H,t+k}\big\} \\
&= (1-\beta\theta)(\mu + mc_t) + \pi_{H,t} + \beta\theta\Bigg[(1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big\{\mu + mc_{t+1+k}\big\} + \sum_{k=1}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big\{\pi_{H,t+1} \\
&= (1-\beta\theta)(\mu + mc_t) + \pi_{H,t} + \beta\theta\mathbb{E}_t\Bigg\{(1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k\mathbb{E}_{t+1}\big\{\mu + mc_{t+1+k}\big\} + \\
&\sum_{k=1}^{+\infty}(\beta\theta)^k\mathbb{E}_{t+1}\big\{\pi_{H,t+1+k}\big\}\Bigg\} \\
&= (1-\beta\theta)(\mu + mc_t) + \pi_{H,t} + \beta\theta\mathbb{E}_t\{\bar{p}_{H,t+1} - p_{H,t}\}
\end{aligned}$$

Using the aggregate price level dynamics equation, we get

$$\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \lambda(\mu + mc_t)$$

where $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$.

2.6.12 Summary firm results

Table 3: Firm results

Variable	Home	Foreign
i-th firm's production function	$Y_t(i) = A_t N_t(i)$	$Y_t^*(i) = A_t^* N_t^*(i)$
i-th firm's labor demand	$N_t(i) = \frac{Y_t(i)}{A_t}$	$N_t^*(i) = \frac{Y_t^*(i)}{A_t^*}$
Aggregate labor demand	$N_t \equiv \int_0^h N_t(i)\mathrm{d}i = \frac{Y_t Z_t}{A_t}$	$N_t^* \equiv \int_h^1 N_t^*(i)\mathrm{d}i = \frac{Y_t^* Z_t^*}{A_t^*}$
Aggregate production index	$Y_t \equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}} \int_0^h Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$	$Y_t^* \equiv \left[\left(\frac{1}{1-h}\right)^{\frac{1}{\varepsilon}} \int_h^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}$
Output dispersion	$Z_t \equiv \int_0^h \frac{Y_t(i)}{Y_t}\mathrm{d}i$	$Z_t \equiv \int_h^1 \frac{Y_t^*(i)}{Y_t^*}\mathrm{d}i$
Aggregate production function	$y_t = a_t + n_t$	$y_t^* = a_t^* + n_t^*$
Real marginal cost	$mc_t = \log(1-\tau) + w_t - p_{H,t} - a_t$	$mc_t^* = \log(1-\tau) + w_t^* - p_{F,t}^* - a_t^*$
Aggregate price level dynamics	$\pi_{H,t} = (1-\theta)(\bar{p}_{H,t} - p_{H,t})$	$\pi_{F,t}^* = (1-\theta^*)(\bar{p}_{F,t}^* - p_{F,t}^*)$
Firms' FOC	$\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \lambda(\mu + mc_t)$ where $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$	$\pi_{F,t}^* = \beta\mathbb{E}_t\{\pi_{F,t+1}^*\} + \lambda^*(\mu + mc_t^*)$ where $\lambda^* \equiv \frac{(1-\theta^*)(1-\beta\theta^*)}{\theta^*}$

3 Equilibrium dynamics

3.1 Aggregate demand and output determination

3.1.1 Labor market

At equilibrium, labor supply equals labor demand

$$N_t^s = N_t \Rightarrow n_t^s = n_t.$$

3.1.2 Good markets

The world demand of *Home*-made good i is given by

$$\begin{aligned} Y_t^d(i) &\equiv C_{H,t}(i) + C_{H,t}^*(i) + G_t(i) \\ &= \frac{1}{h}\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon}(C_{H,t} + C_{H,t}^* + G_t). \end{aligned}$$

The market of all *Home* and *Foreign* goods clear in equilibrium so that

$$Y_t(i) = Y_t^d(i), \forall i \in [0,1].$$

Using *Home* RH's optimal allocations, identities and the international risk condition, we get

$$\begin{aligned}
Y_t &\equiv \left[\left(\frac{1}{h}\right)^{\frac{1}{\varepsilon}} \int_0^h Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \left[\frac{1}{h}\int_0^h \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{1-\varepsilon} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}} (C_{H,t} + C_{H,t}^* + G_t) \\
&= C_{H,t} + C_{H,t}^* + G_t \\
&= (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha^*\left(\frac{P_{H,t}^*}{P_t^*}\right)^{-\eta} C_t^* + G_t \\
&\overset{LOP}{=} \left(\frac{P_{H,t}}{P_t}\right)^{-\eta}\left[(1-\alpha)C_t + \alpha^*\left(\frac{P_t}{P_t^*}\right)^{-\eta} C_t^*\right] + G_t \\
&\overset{IRS}{=} \left(\frac{P_{H,t}}{P_t}\right)^{-\eta}\left[(1-\alpha) + \alpha^*\left(\frac{P_t}{P_t^*}\right)^{-\eta}\frac{1-h}{h}\mathcal{Q}_t^{-\frac{1}{\sigma}}\right] C_t + G_t \\
&= \left(\frac{P_{H,t}}{P_t}\right)^{-\eta}\left[(1-\alpha) + \alpha^*\frac{1-h}{h}\mathcal{Q}_t^{\eta-\frac{1}{\sigma}}\right] C_t + G_t.
\end{aligned}$$

Because $\alpha^* = \frac{h}{1-h}\alpha$, we have

$$Y_t = \left(\frac{P_{H,t}}{P_t}\right)^{-\eta}\left[(1-\alpha) + \alpha\mathcal{Q}_t^{\eta-\frac{1}{\sigma}}\right] C_t + G_t.$$

3.1.3 Log-linearization of the good markets clearing condition

We define $\hat{x}_t \equiv x_t - x$ the log-deviation of the variable x_t from its steady state value. Also, $\delta \equiv \frac{G}{Y}$ be the steady state share of government spending.
Log-linearizing around the symmetric steady state where $\mathcal{Q}_t = 1$, we get

$$\begin{aligned}
\frac{1}{1-\delta}(\hat{y}_t - \delta\hat{g}_t) &= \hat{c}_t + \frac{w_{\bar{\alpha}} + \bar{\alpha} - 1}{\sigma} s_t, \\
\frac{1}{1-\delta}(\hat{y}_t^* - \delta\hat{g}_t^*) &= \hat{c}_t^* - \frac{w_{\bar{\alpha}}^*}{\sigma} s_t,
\end{aligned}$$

where

$$\begin{aligned}
w_{\bar{\alpha}} &= 1 - \bar{\alpha}h + (1-h)\bar{\alpha}(2-\bar{\alpha})(\sigma\eta - 1) > 0, \\
w_{\bar{\alpha}}^* &= \bar{\alpha}h[1 + (2-\bar{\alpha})(\sigma\eta - 1)] > 0.
\end{aligned}$$

We keep the same notation as Da Silveira (2006).

3.1.4 IS equation

Combining the intratemporal household condition, the inflation identities and the good-market clearing condition, we obtain a version of the IS equation

$$
\begin{aligned}
& c_t = \mathbb{E}_t\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{t+1}\} - \bar{i}) \\
& \Rightarrow \hat{c}_t = \mathbb{E}_t\{\hat{c}_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{t+1}\} - \bar{i}) \\
& \Rightarrow \hat{c}_t = \mathbb{E}_t\{\hat{c}_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{H,t+1} - \bar{\alpha}(1-h)\Delta s_{t+1}\} - \bar{i}) \\
& \Rightarrow \hat{c}_t = \mathbb{E}_t\{\hat{c}_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \bar{i}) - \frac{\bar{\alpha}(1-h)}{\sigma}\mathbb{E}_t\{\Delta s_{t+1}\} \\
& \Rightarrow \hat{y}_t = \mathbb{E}_t\{\hat{y}_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \bar{i}) + \frac{1 - \bar{\alpha}h - w_{\bar{\alpha}}}{\sigma}\mathbb{E}_t\{\Delta s_{t+1}\} - \delta\mathbb{E}_t\{\Delta\hat{g}_{t+1}\}.
\end{aligned}
$$

Similarly,

$$
\hat{y}^*_t = \mathbb{E}_t\{\hat{y}^*_{t+1}\} - \frac{1}{\sigma}(i^*_t - \mathbb{E}_t\{\pi^*_{F,t+1}\} - \bar{i}) + \frac{w^*_{\bar{\alpha}} - \bar{\alpha}h}{\sigma}\mathbb{E}_t\{\Delta s_{t+1}\} - \delta\mathbb{E}_t\{\Delta\hat{g}^*_{t+1}\}.
$$

Compare with Da Silveira (2006).

3.1.5 IRS condition at equilibrium

We can use the good market clearing condition to re-write the IRS condition as

$$
\begin{aligned}
& c_t = \log\left(\frac{h}{1-h}\right) + \frac{1}{\sigma}q_t + c^*_t \\
& \Rightarrow \hat{c}_t = \frac{1}{\sigma}q_t + \hat{c}^*_t \\
& \Rightarrow \frac{1}{\sigma}(1-\bar{\alpha})s_t = \hat{c}_t - \hat{c}^*_t \\
& \Rightarrow \frac{1}{\sigma}(1-\bar{\alpha})s_t = \frac{1}{1-\delta}[\hat{y}_t - \hat{y}^*_t - \delta(\hat{g}_t - \hat{g}^*_t)] - \frac{w_{\bar{\alpha}} + \bar{\alpha} - 1}{\sigma}s_t - \frac{w^*_{\bar{\alpha}}}{\sigma}s_t \\
& \Rightarrow \frac{1-\bar{\alpha}}{\sigma}s_t = \frac{1}{1-\delta}[\hat{y}_t - \hat{y}^*_t - \delta(\hat{g}_t - \hat{g}^*_t)] - \frac{w_{\bar{\alpha}} + w^*_{\bar{\alpha}}}{\sigma}s_t + \frac{1-\bar{\alpha}}{\sigma}s_t \\
& \Rightarrow s_t = \frac{\sigma}{(1-\delta)(w_{\bar{\alpha}} + w^*_{\bar{\alpha}})}[\hat{y}_t - \hat{y}^*_t - \delta(\hat{g}_t - \hat{g}^*_t)].
\end{aligned}
$$

3.2 The supply side: marginal cost and inflation dynamics

3.2.1 Marginal cost

Using *Home* RH's intratemporal FOC, *Home*'s aggregate production function and *Home*'s price level identities, we have

$$
\begin{aligned}
mc_t &= w_t - p_{H,t} - a_t + \log(1-\tau) \\
&= w_t - p_t + (p_t - p_{H,t}) - a_t + \log(1-\tau) \\
&= -(\varphi+\sigma)\log(h) + \xi_t + \sigma c_t + \varphi n_t + (p_t - p_{H,t}) - a_t + \log(1-\tau) \\
&= \xi_t + \sigma c_t + \varphi(y_t - a_t) + (p_t - p_{H,t}) - a_t + \log(1-\tau) - (\varphi+\sigma)\log(h) \\
&= \xi_t + \sigma c_t + \varphi y_t + (p_t - p_{H,t}) - (1+\varphi)a_t + \log(1-\tau) - (\varphi+\sigma)\log(h) \\
&= \xi_t + \sigma c_t + \varphi y_t + \alpha s_t - (1+\varphi)a_t + \log(1-\tau) - (\varphi+\sigma)\log(h).
\end{aligned}
$$

Re-expressing in log-deviation form, we get

$$
\hat{mc}_t = \xi_t + \sigma\hat{c}_t + \varphi\hat{y}_t + \alpha s_t - (1+\varphi)a_t
$$

where $\hat{mc}_t = mc_t + \mu$.
From the good market clearing condition we have

$$
\sigma\hat{c}_t = \frac{\sigma}{1-\delta}(\hat{y}_t - \delta\hat{g}_t) - (w_{\bar{\alpha}} + \bar{\alpha} - 1)s_t.
$$

Therefore,

$$
\begin{aligned}
\hat{mc}_t &= \xi_t + \frac{\sigma}{1-\delta}(\hat{y}_t - \delta\hat{g}_t) - (w_{\bar{\alpha}} + \bar{\alpha} - 1)s_t + \varphi\hat{y}_t + \alpha s_t - (1+\varphi)a_t \\
&= \xi_t + (\frac{\sigma}{1-\delta} + \varphi)\hat{y}_t - \frac{\sigma\delta}{1-\delta}\hat{g}_t + (1 - \bar{\alpha} - w_{\bar{\alpha}} + \alpha)s_t - (1+\varphi)a_t \\
&= \xi_t + (\frac{\sigma}{1-\delta} + \varphi)\hat{y}_t - \frac{\sigma\delta}{1-\delta}\hat{g}_t + (1 - \bar{\alpha}h - w_{\bar{\alpha}})s_t - (1+\varphi)a_t
\end{aligned}
$$

Similarly,

$$
\hat{mc}^*_t = \xi^*_t + (\frac{\sigma}{1-\delta} + \varphi)\hat{y}^*_t - \frac{\sigma\delta}{1-\delta}\hat{g}^*_t + (w^*_{\bar{\alpha}} - \bar{\alpha}h)s_t - (1+\varphi)a^*_t.
$$

3.2.2 NKPC

Combining the previous results with the *Home* and *Foreign* firms' FOCs, we obtain a version of the NKPC

$$
\begin{aligned}
\pi_{H,t} &= \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \lambda\xi_t + \lambda(\frac{\sigma}{1-\delta} + \varphi)\hat{y}_t - \lambda\frac{\sigma\delta}{1-\delta}\hat{g}_t + \lambda(1 - \bar{\alpha}h - w_{\bar{\alpha}})s_t - \lambda(1+\varphi)a_t \\
\pi^*_{F,t} &= \beta\mathbb{E}_t\{\pi^*_{F,t+1}\} + \lambda^*\xi^*_t + \lambda^*(\frac{\sigma}{1-\delta} + \varphi)\hat{y}^*_t - \lambda^*\frac{\sigma\delta}{1-\delta}\hat{g}^*_t + \lambda^*(w_*\bar{\alpha} - \bar{\alpha}h)s_t - \lambda^*(1+\varphi)a^*_t.
\end{aligned}
$$

See Gali et Monacelli (2008) eq. (32). See Da Silveira (2006) eq. (91-94)

3.3 Summary sticky price equilibrium

Given the exogeneous sequence $(a_t, a^*_t, \xi_t, \xi^*_t)_{t\in\mathbb{N}}$ and the sequence $(i_t, i^*_t, \hat{g}_t, \hat{g}^*_t)_{t\in\mathbb{N}}$, the endogeneous sequence $(\hat{y}_t, \pi_{H,t}; \hat{y}^*_t, \pi^*_{F,t}; s_t)_{t\in\mathbb{N}}$ made of 5 variables is determined by a system of equilibrium conditions made of 5 linear equations:

$$\hat{y}_t = \mathbb{E}_t\{\hat{y}_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{H,t+1}\} - \bar{i}) + \frac{1 - \bar{\alpha}h - w_{\bar{\alpha}}}{\sigma}\mathbb{E}_t\{\Delta s_{t+1}\} - \delta\mathbb{E}_t\{\Delta\hat{g}_{t+1}\} \quad \text{(IS)}$$

$$\pi_{H,t} = \beta\mathbb{E}_t\{\pi_{H,t+1}\} + \lambda\xi_t + \lambda(\frac{\sigma}{1-\delta} + \varphi)\hat{y}_t - \lambda\frac{\sigma\delta}{1-\delta}\hat{g}_t + \lambda(1 - \bar{\alpha}h - w_{\bar{\alpha}})s_t - \lambda(1+\varphi)a_t \quad \text{(NKPC)}$$

$$\hat{y}^*_t = \mathbb{E}_t\{\hat{y}^*_{t+1}\} - \frac{1}{\sigma}(i^*_t - \mathbb{E}_t\{\pi^*_{F,t+1}\} - \bar{i}) + \frac{w^*_{\bar{\alpha}} - \bar{\alpha}h}{\sigma}\mathbb{E}_t\{\Delta s_{t+1}\} - \delta\mathbb{E}_t\{\Delta\hat{g}^*_{t+1}\} \quad \text{(IS*)}$$

$$\pi^*_{F,t} = \beta\mathbb{E}_t\{\pi^*_{F,t+1}\} + \lambda^*\xi^*_t + \lambda^*(\frac{\sigma}{1-\delta} + \varphi)\hat{y}^*_t - \lambda^*\frac{\sigma\delta}{1-\delta}\hat{g}^*_t + \lambda^*(w_*\bar{\alpha} - \bar{\alpha}h)s_t - \lambda^*(1+\varphi)a^*_t \quad \text{(NKPC*)}$$

$$s_t = \frac{\sigma}{(1-\delta)(w_{\bar{\alpha}} + w^*_{\bar{\alpha}})}[\hat{y}_t - \hat{y}^*_t - \delta(\hat{g}_t - \hat{g}^*_t)] \quad \text{(IRS)}$$

4 Flexible price equilibrium

4.1 Flexible price economy

We denote $\bar{x}_t$ the log natural level of the variable x_t. Natural values are the values taken by variables under flexible prices (i.e. $\theta \Rightarrow 0$).
When prices are fully flexible, we have

$$\bar{mc}_t = \bar{mc}^*_t = -\mu.$$

Therefore,

$$-\mu = \xi_t + \sigma\bar{c}_t + \varphi\bar{y}_t + \alpha\bar{s}_t - (1+\varphi)a_t + \log(1-\tau) - (\varphi+\sigma)\log(h)$$
$$-\mu = \xi^*_t + \sigma\bar{c}^*_t + \varphi\bar{y}^*_t - \alpha^*\bar{s}_t - (1+\varphi)a_t + \log(1-\tau) - (\varphi+\sigma)\log(1-h).$$

Substracting the two equations and using the IRS conditions gives an expression for the Home natural terms of trade in log form

$$0 = (\xi_t - \xi^*_t) + \sigma(\bar{c}_t - \bar{c}^*_t) + \varphi(\bar{y}_t - \bar{y}^*_t) + (\alpha + \alpha^*)\bar{s}_t - (1+\varphi)(a_t - a^*_t) - (\varphi+\sigma)\log(\frac{h}{1-h})$$
$$\Rightarrow 0 = (\xi_t - \xi^*_t) + \sigma(\bar{c}_t - \bar{c}^*_t) + \varphi(\bar{y}_t - \bar{y}^*_t) + \bar{\alpha}\bar{s}_t - (1+\varphi)(a_t - a^*_t) - (\varphi+\sigma)\log(\frac{h}{1-h})$$
$$\Rightarrow 0 = (\xi_t - \xi^*_t) + \varphi(\bar{y}_t - \bar{y}^*_t) + \bar{s}_t - (1+\varphi)(a_t - a^*_t) - \varphi\log(\frac{h}{1-h})$$
$$\Rightarrow \bar{s}_t = (1+\varphi)(a_t - a^*_t) - (\xi_t - \xi^*_t) - \varphi(\bar{y}_t - \bar{y}^*_t) + \varphi\log(\frac{h}{1-h})$$

We denote $\hat{\bar{x}}_t$ the natural log deviations of the variable x_t from its steady state value x.
The flexible price equilibrium expressed in log deviations writes

$$0 = \xi_t + \sigma\hat{\bar{c}}_t + \varphi\hat{\bar{y}}_t + \alpha\bar{s}_t - (1+\varphi)a_t$$
$$0 = \xi^*_t + \sigma\hat{\bar{c}}^*_t + \varphi\hat{\bar{y}}^*_t - \alpha^*\bar{s}_t - (1+\varphi)a^*_t$$
$$s_t = (1+\varphi)(a_t - a^*_t) - (\xi_t - \xi^*_t) - \varphi(\hat{\bar{y}}_t - \hat{\bar{y}}^*_t),$$

together with the two good-market equilibrium conditions.
Given the exogenous processes, we are provided with a flexible price equilibrium made of 7 unknowns and 5 equations.
In the next section, we characterize the optimal level of government consumption when prices are flexible.

4.2 Fiscal policy under flexible prices

4.2.1 Definition of government spending under flexible prices

Kirsanova et al. (2007) finds that the natural deviation of government spending is a linear combination of the natural deviation of Home and Foreign variables.
To obtain this relationship, we minimize a welfare loss function with respect to $\hat{\bar{g}}_t$ and $\hat{\bar{g}}_t^*$ subject to the equilibrium dynamics under flexible prices.

4.2.2 Planner's problem

To obtain a welfare loss function, we first need to identify the efficient steady state around which we will approximate the aggregate utility function.
The benevolent social planner seeks to maximize

$$\max_{C_t^j, N_t^j, \frac{G_t}{h}, C_t^{j*}, N_t^{j*}, \frac{G_t^*}{1-h}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\int_0^h U(C_t^j, N_t^j, \frac{G_t}{h}) \mathrm{d}j + \int_h^1 U(C_t^{j*}, N_t^{j*}, \frac{G_t^*}{1-h}) \mathrm{d}j \right]$$

subject to

$$\begin{aligned}
&C_{H,t} + C_{H,t}^* + G_t - A_t N_t \leq 0 && C_{F,t} + C_{F,t}^* + G_t^* - A_t^* N_t^* \leq 0 \\
&C_{H,t} = (1-\alpha) C_t && C_{F,t} = \alpha C_t \\
&C_{H,t}^* = \alpha^* C_t^* && C_{F,t}^* = (1-\alpha^*) C_t^* \\
&C_t^j = \frac{C_t}{h} && C_t^{j*} = \frac{C_t^*}{1-h} \\
&N_t^j = \frac{N_t}{h} && N_t^{j*} = \frac{N_t^*}{1-h}.
\end{aligned}$$

Equivalently, the benevolent social planner seeks to maximize

$$\max_{\frac{C_t}{h}, \frac{N_t}{h}, \frac{G_t}{h}, \frac{C_t^*}{1-h}, \frac{N_t^*}{1-h}, \frac{G_t^*}{1-h}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big[hU(\frac{C_t}{h}, \frac{N_t}{h}, \frac{G_t}{h}) + (1-h) U(\frac{C_t^*}{1-h}, \frac{N_t^*}{1-h}, \frac{G_t^*}{1-h}) \Big]$$

subject to

$$\begin{aligned}
&(1-\alpha) C_t + \alpha^* C_t^* + G_t - A_t N_t \leq 0, && \alpha C_t + (1-\alpha^*) C_t^* + G_t^* - A_t^* N_t^* \leq 0 \\
\Leftrightarrow &(1-\alpha) \frac{C_t}{h} + \alpha \frac{C_t^*}{1-h} + \frac{G_t}{h} - A_t \frac{N_t}{h} \leq 0, && \alpha^* \frac{C_t}{h} + (1-\alpha^*) \frac{C_t^*}{1-h} + \frac{G_t^*}{1-h} - A_t^* \frac{N_t^*}{1-h} \leq 0.
\end{aligned}$$

FOC of the social planner

$$
\begin{aligned}
h\left(\frac{C_t}{h}\right)^{-\sigma} &= (1-\alpha)\lambda_{1,t} + \alpha^*\lambda_{2,t}, \\
h\Xi_t\left(\frac{N_t}{h}\right)^{\varphi} &= A_t\lambda_{1,t}, \\
h\chi\left(\frac{G_t}{h}\right)^{-\gamma} &= \lambda_{1,t}, \\
(1-h)\left(\frac{C_t^*}{1-h}\right)^{-\sigma} &= \alpha\lambda_{1,t} + (1-\alpha^*)\lambda_{2,t}, \\
(1-h)\Xi_t^*\left(\frac{N_t^*}{1-h}\right)^{\varphi} &= A_t^*\lambda_{2,t}, \\
(1-h)\chi\left(\frac{G_t^*}{1-h}\right)^{-\gamma} &= \lambda_{2,t}.
\end{aligned}
$$

Therefore, the efficient allocation is summarized by 6 equations

$$
\begin{aligned}
\left(\frac{C_t}{h}\right)^{-\sigma} &= (1-\alpha)\frac{\Xi_t}{A_t}\left(\frac{N_t}{h}\right)^{\varphi} + \alpha\frac{\Xi_t^*}{A_t^*}\left(\frac{N_t^*}{1-h}\right)^{\varphi}, \\
\chi\left(\frac{G_t}{h}\right)^{-\gamma} &= \frac{\Xi_t}{A_t}\left(\frac{N_t}{h}\right)^{\varphi}, \\
\left(\frac{C_t^*}{1-h}\right)^{-\sigma} &= \alpha^*\frac{\Xi_t}{A_t}\left(\frac{N_t}{h}\right)^{\varphi} + (1-\alpha^*)\frac{\Xi_t^*}{A_t^*}\left(\frac{N_t^*}{1-h}\right)^{\varphi}, \\
\chi\left(\frac{G_t^*}{1-h}\right)^{-\gamma} &= \frac{\Xi_t^*}{A_t^*}\left(\frac{N_t^*}{1-h}\right)^{\varphi}, \\
A_t\frac{N_t}{h} &= (1-\alpha)\frac{C_t}{h} + \alpha\frac{C_t^*}{1-h} + \frac{G_t}{h}, \\
A_t^*\frac{N_t^*}{1-h} &= \alpha^*\frac{C_t}{h} + (1-\alpha^*)\frac{C_t^*}{1-h} + \frac{G_t^*}{1-h},
\end{aligned}
$$

where we used the fact that $\alpha = \frac{1-h}{h}\alpha^*$.
Evaluated at steady state, these equations become

$$
\begin{aligned}
\left(\frac{C}{h}\right)^{-\sigma} &= (1-\alpha)\left(\frac{Y}{h}\right)^{\varphi} + \alpha\left(\frac{Y^*}{1-h}\right)^{\varphi}, \\
\chi\left(\frac{G}{h}\right)^{-\gamma} &= \left(\frac{Y}{h}\right)^{\varphi}, \\
\left(\frac{C^*}{1-h}\right)^{-\sigma} &= \alpha^*\left(\frac{Y}{h}\right)^{\varphi} + (1-\alpha^*)\left(\frac{Y^*}{1-h}\right)^{\varphi}, \\
\chi\left(\frac{G^*}{1-h}\right)^{-\gamma} &= \left(\frac{Y^*}{1-h}\right)^{\varphi}, \\
\frac{Y}{h} &= (1-\alpha)\frac{C}{h} + \alpha\frac{C^*}{1-h} + \frac{G}{h}, \\
\frac{N^*}{1-h} &= \alpha^*\frac{C}{h} + (1-\alpha^*)\frac{C^*}{1-h} + \frac{G^*}{1-h}.
\end{aligned}
$$

At steady state, the IRS condition implies

$$\frac{C}{h} = \frac{C^*}{1-h}.$$

Therefore, at the efficient steady state, the following relationship holds

$$\left(\frac{C}{h}\right)^{-\sigma} = \left(\frac{C^*}{1-h}\right)^{-\sigma} = \chi\left(\frac{G}{h}\right)^{-\gamma} = \chi\left(\frac{G^*}{1-h}\right)^{-\gamma} = \left(\frac{Y}{h}\right)^{\varphi} = \left(\frac{Y^*}{1-h}\right)^{\varphi}.$$

We know that the efficient steady state coincides with the flexible price when steady state monopolistic distortions at steady state are removed by means of an appropriate labor subsidy $\tau = \frac{1}{\varepsilon}$.

4.2.3 Approximation of the planner objective

We know turn to the second-order approximation of the planner objective.

$$\begin{aligned}
\frac{\left(\frac{C_t}{h}\right)^{1-\sigma}}{1-\sigma} &\simeq \frac{\left(\frac{C}{h}\right)^{1-\sigma}}{1-\sigma} + \left(\frac{C}{h}\right)^{-\sigma}\left(\frac{C_t}{h} - \frac{C}{h}\right) - \frac{\sigma}{2}\left(\frac{C}{h}\right)^{-\sigma-1}\left(\frac{C_t}{h} - \frac{C}{h}\right)^2 \\
&\simeq \frac{\left(\frac{C}{h}\right)^{1-\sigma}}{1-\sigma} + \left(\frac{C}{h}\right)^{1-\sigma}\hat{c}_t + \left(\frac{C}{h}\right)^{1-\sigma}\frac{1-\sigma}{2}\hat{c}_t^2.
\end{aligned}$$

Similarly,

$$\chi\frac{\left(\frac{G_t}{h}\right)^{1-\gamma}}{1-\gamma} \simeq \frac{\chi\left(\frac{G}{h}\right)^{1-\gamma}}{1-\gamma} + \chi\left(\frac{G}{h}\right)^{1-\gamma}\hat{g}_t + \chi\left(\frac{G}{h}\right)^{1-\gamma}\frac{1-\gamma}{2}\hat{g}_t^2.$$

When there is no larbor disutility shock, labor disutility is approximated by

$$\frac{\left(\frac{N_t}{h}\right)^{1+\varphi}}{1+\varphi} \simeq \frac{\left(\frac{N}{h}\right)^{1+\varphi}}{1+\varphi} + \left(\frac{N}{h}\right)^{1+\varphi}\hat{n}_t + \left(\frac{N}{h}\right)^{1+\varphi}\frac{1+\varphi}{2}\hat{n}_t^2$$

But,

$$\hat{n}_t = \hat{y}_t - a_t + z_t.$$

We can show that

$$z_t \simeq \frac{\varepsilon}{2h}\mathrm{var}_i\big(p_{H,t}(i)\big).$$

Also, following to Woordford (2001) it is possible to show that

$$\sum_{t=0}^{\infty}\beta^t\mathrm{var}_i\big(p_{H,t}(i)\big) \simeq \frac{1}{\lambda}\sum_{t=0}^{\infty}\beta^t\pi_{H,t}^2.$$

In addition, note that

$$\begin{aligned}
\left(\frac{C}{h}\right)^{1-\sigma} &= (1-\delta)\left(\frac{Y}{h}\right)^{1+\varphi}, \\
\chi\left(\frac{G}{h}\right)^{1-\gamma} &= \delta\left(\frac{Y}{h}\right)^{1+\varphi},
\end{aligned}$$

where $\delta \equiv G/Y$ is the steady state share of governement spending in output.
TO BE CONTINUED

4.2.4 Welfare loss under flexible prices

The previous equation allows us write an approximation of the welfare loss under flexible prices

$$\bar{\mathcal{L}} = \frac{1}{2}\sum_{t=0}^{\infty}\beta^t\Big[h\big(\varphi\hat{\bar{y}}_t^2 + \gamma\delta\hat{\bar{g}}_t^2 + \sigma(1-\delta)\hat{\bar{c}}_t^2\big) + (1-h)\big(\varphi\hat{\bar{y}}_t^{*2} + \gamma\delta\hat{\bar{g}}_t^{*2} + \sigma(1-\delta)\hat{\bar{c}}_t^{*2}\big)\Big].$$

We assume that $\hat{\bar{g}}_t$ and $\hat{\bar{g}}_t^*$ write

$$\begin{aligned}\hat{\bar{g}}_t &= \begin{bmatrix}a_t & \xi_t & \hat{\bar{y}}_t & \hat{\bar{c}}_t & a_t^* & \xi_t^* & \hat{\bar{y}}_t^* & \hat{\bar{c}}_t^*\end{bmatrix}\bar{\omega}\\ \hat{\bar{g}}_t^* &= \begin{bmatrix}a_t & \xi_t & \hat{\bar{y}}_t & \hat{\bar{c}}_t & a_t^* & \xi_t^* & \hat{\bar{y}}_t^* & \hat{\bar{c}}_t^*\end{bmatrix}\bar{\omega}^*\end{aligned}$$

where $\bar{\omega}$ and $\bar{\omega}$ are vector of coefficients that minimize subject to the flexible price equilibrium condition.

4.2.5 Summary flexible price equilibrium

Expressed in deviation from the steady state, the flexible price equilibrium conditions writes

$$\begin{aligned}0 &= \xi_t + (\frac{\sigma}{1-\delta} + \varphi)\hat{\bar{y}}_t - \frac{\sigma\delta}{1-\delta}\hat{\bar{g}}_t + (1-\bar{\alpha}h - w_{\bar{\alpha}})\bar{s}_t - (1+\varphi)a_t,\\ 0 &= \xi_t^* + (\frac{\sigma}{1-\delta} + \varphi)\hat{\bar{y}}_t^* - \frac{\sigma\delta}{1-\delta}\hat{\bar{g}}_t^* + (w_{\bar{\alpha}}^* - \bar{\alpha}h)\bar{s}_t - (1+\varphi)a_t^*,\\ \bar{s}_t &= (1+\varphi)(a_t - a_t^*) - (\xi_t - \xi_t^*) - \varphi(\hat{\bar{y}}_t - \hat{\bar{y}}_t^*),\\ \hat{\bar{g}}_t &= \begin{bmatrix}a_t & \xi_t & \hat{\bar{y}}_t & \hat{\bar{c}}_t & a_t^* & \xi_t^* & \hat{\bar{y}}_t^* & \hat{\bar{c}}_t^*\end{bmatrix}\bar{\omega},\\ \hat{\bar{g}}_t^* &= \begin{bmatrix}a_t & \xi_t & \hat{\bar{y}}_t & \hat{\bar{c}}_t & a_t^* & \xi_t^* & \hat{\bar{y}}_t^* & \hat{\bar{c}}_t^*\end{bmatrix}\bar{\omega}^*.\end{aligned}$$

5 Simulation

5.1 Calibration

To be done.

5.2 Government spending and monetary policy

Under optimal fiscal and monetary policies, $\hat{g}_t$, $\hat{g}_t^*$ and i_t are obtained by minimizing the welfare criterion

$$\mathcal{L}_0 = \frac{1}{2}\sum_{t=0}^{\infty}\beta^t\Big[h\big(\frac{\varepsilon}{\lambda}\pi_{H,t}^2 + \varphi\tilde{y}_t^2 + \gamma\delta\tilde{g}_t^2 + \sigma(1-\delta)\tilde{c}_t^2\big) + (1-h)\big(\frac{\varepsilon}{\lambda^*}\pi_{F,t}^2 + \varphi\tilde{y}_t^{*2} + \gamma\delta\tilde{g}_t^{*2} + \sigma(1-\delta)\tilde{c}_t^{*2}\big)\Big].$$

When fiscal policy is derived following optimal simple rules, we assume that i_t is obtained by minimizing the welfare criterion conditional on closed government spending gaps

$$\mathcal{L}_1 = \frac{1}{2}\sum_{t=0}^{\infty}\beta^t\Big[h\big(\frac{\varepsilon}{\lambda}\pi_{H,t}^2 + \varphi\tilde{y}_t^2 + \sigma(1-\delta)\tilde{c}_t^2\big) + (1-h)\big(\frac{\varepsilon}{\lambda^*}\pi_{F,t}^2 + \varphi\tilde{y}_t^{*2} + \sigma(1-\delta)\tilde{c}_t^{*2}\big)\Big].$$

5.3 Simulations under optimal policies and flexible prices

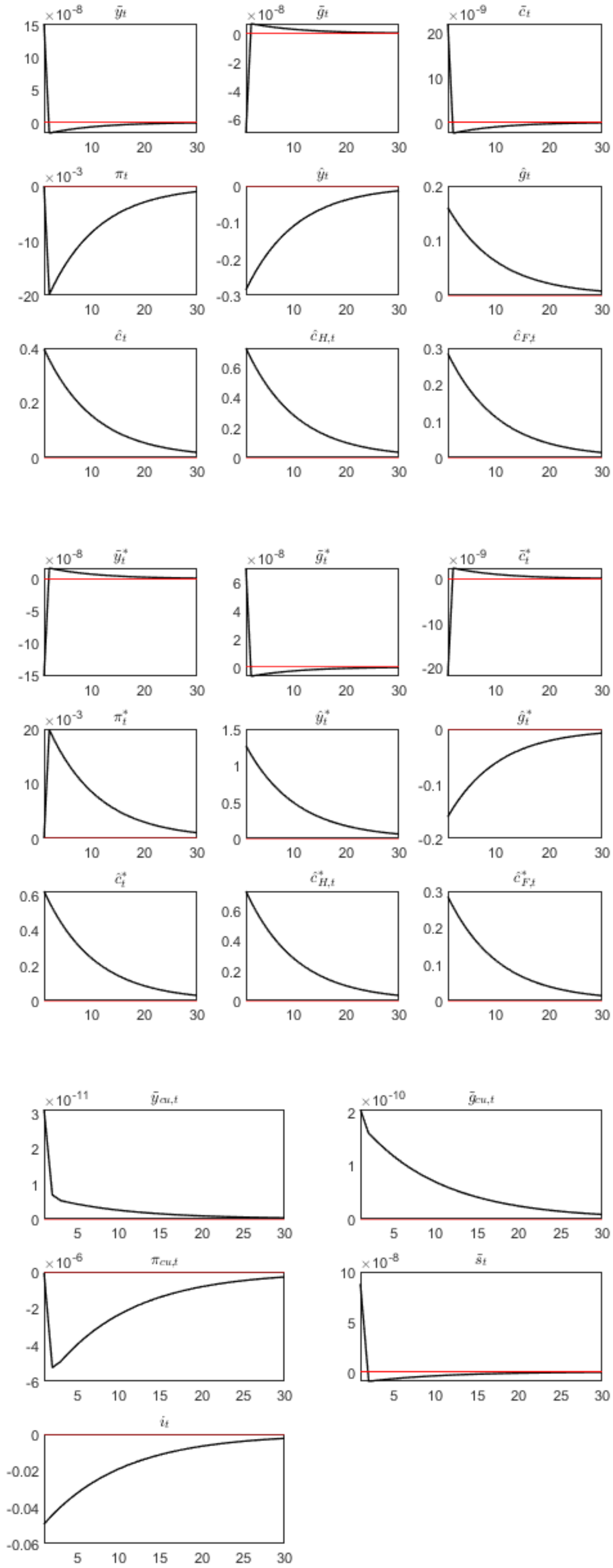

Figure 1: Caption

5.4 Simulations under optimal policies and sticky prices

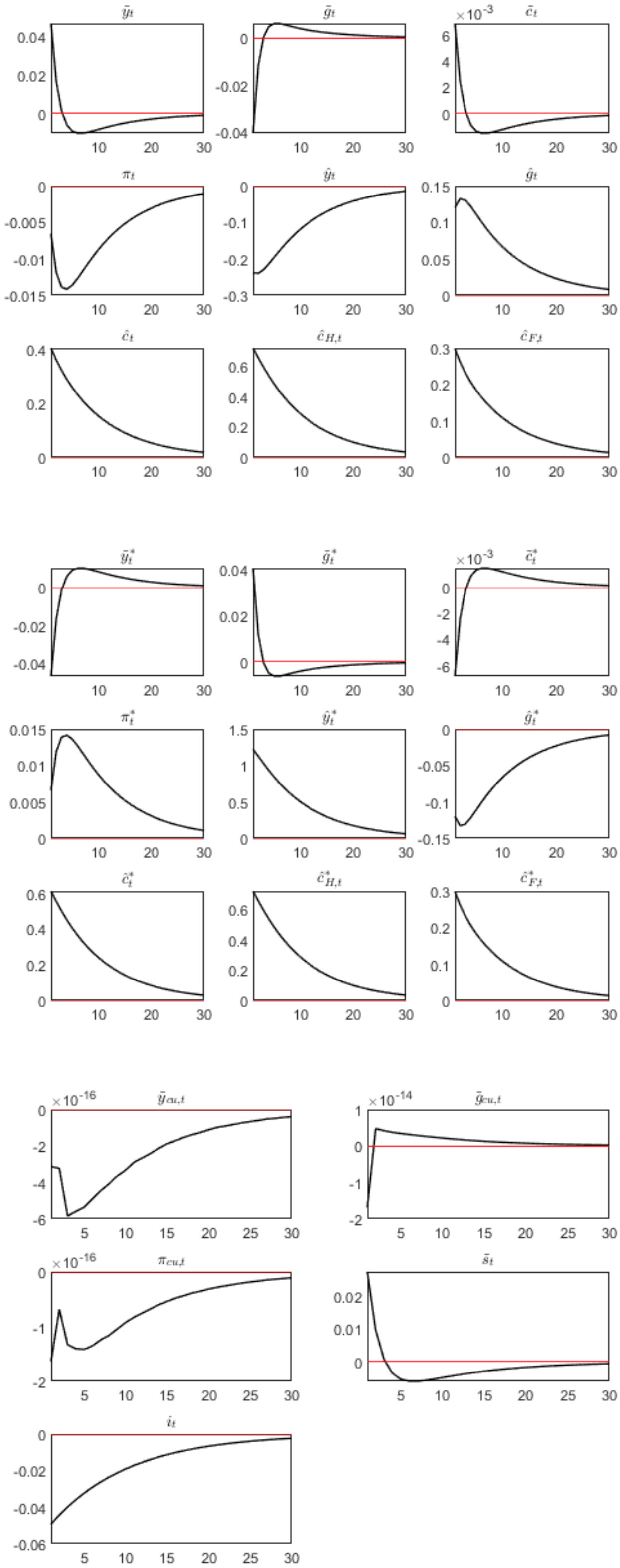

Figure 2: Caption

5.5 Simulations under suboptimal policies and sticky prices

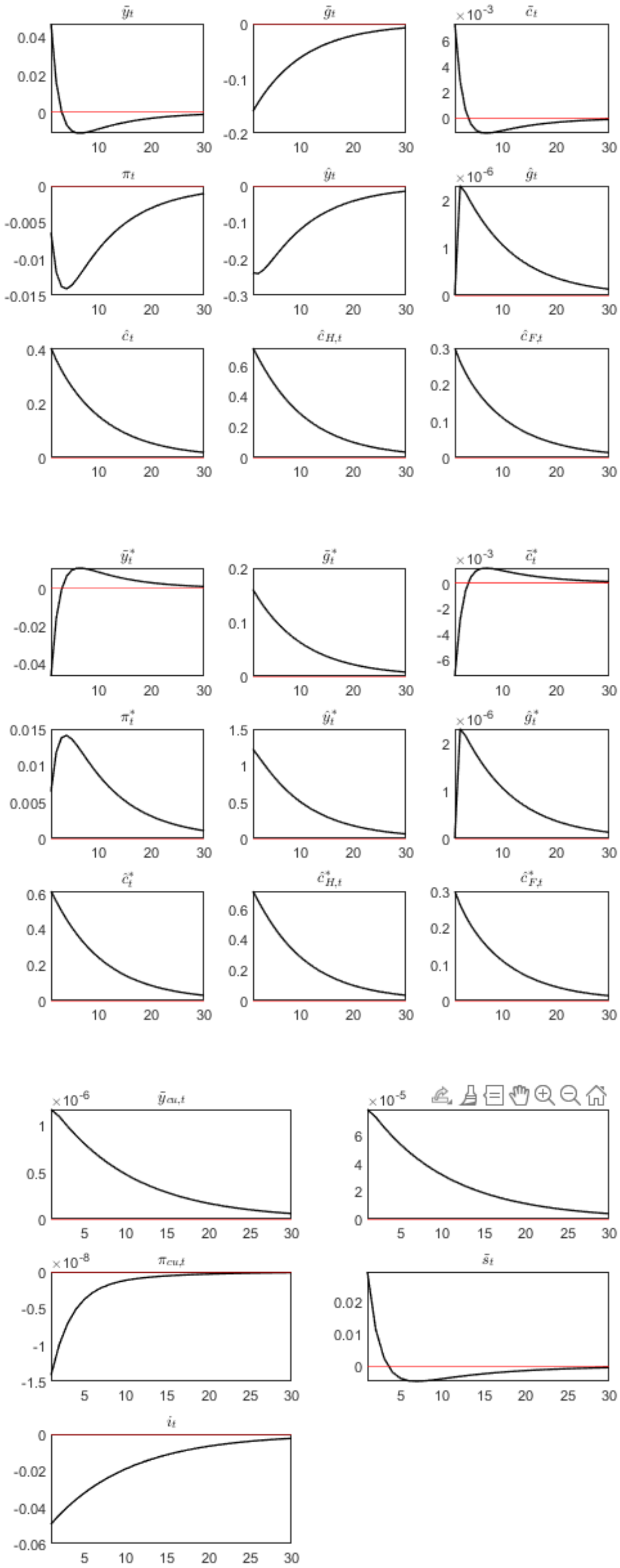

Figure 3: Caption