

# Complexity of recognizing Dyck languages of bounded height with quantum query algorithms.

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# Sommaire

- 1 Introduction
  - Quantum query model and complexity
  - Dyck languages of bounded height
  - History and state of the art
- 2 The progress to reduce Quantum Query Complexity of bounded height Dyck word
- 3 New idea to get better quantum query complexity bounds

# Interacting with qubits is more complex.

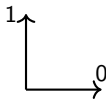


Figure: A classical bit

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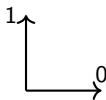


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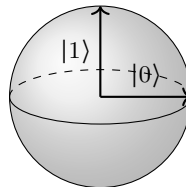


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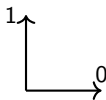


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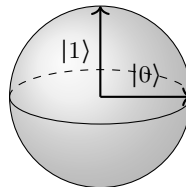


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$A$	$B$	$A \oplus B$
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0	1	1
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Figure: Truth table on 2 bits.

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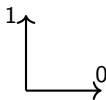


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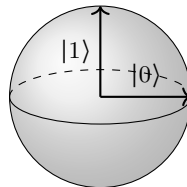


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$$H^{\otimes 2} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Figure: Unitary matrix on 2 qubits.

# Quantum query algorithm is just a quantum circuit.

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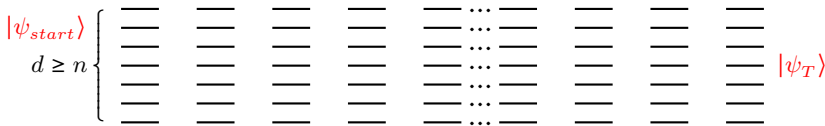


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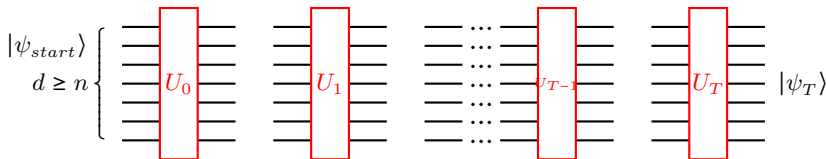


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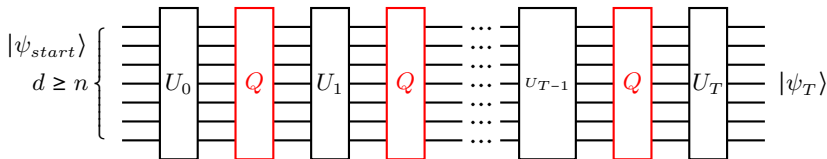


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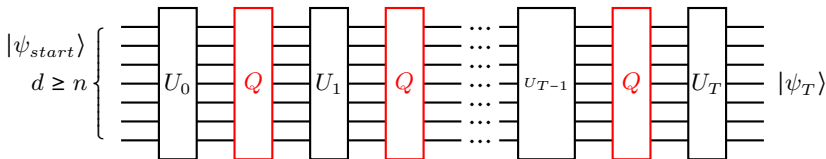


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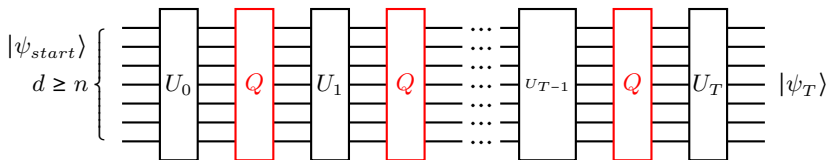
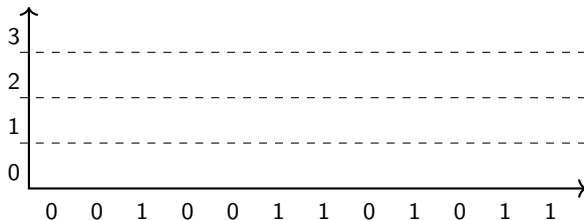


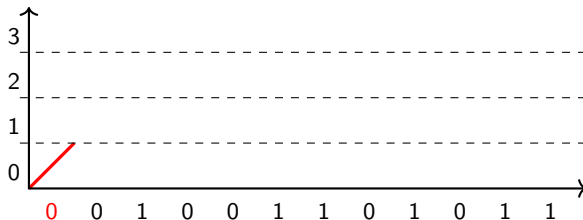
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$$Q(f) = T \quad Q(\text{GROVER}) = O(\sqrt{n})$$

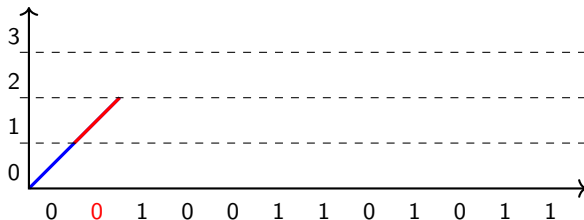
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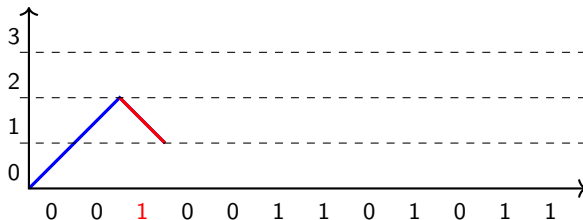
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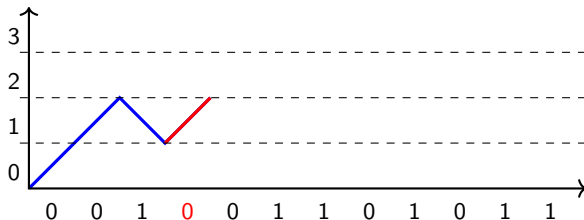


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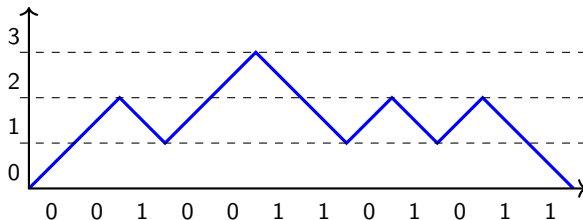




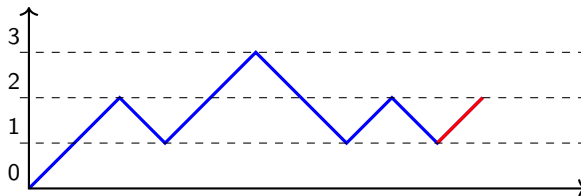
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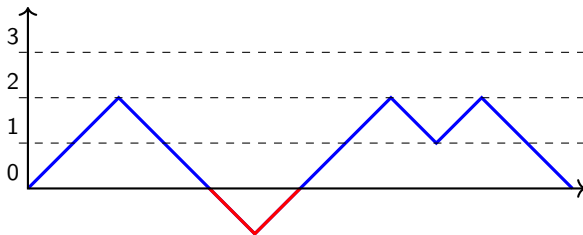
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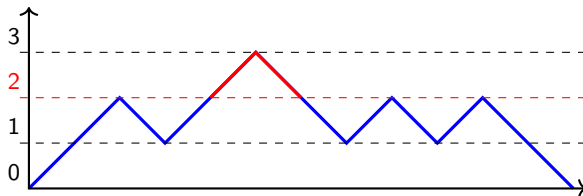
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$\text{DYCK}_k$

A result which does not answer fully the  $Q(\text{DYCK}_k)$  problem.

**The Trichotomy theorem:**(Aaronson, Grier and Schaeffer [1, 2019])

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**Implication:**

$$Q(\text{DYCK}_k) = \Theta(\sqrt{n} \log_2(n)^{p(k)})$$



# First step, one try to have a good upper bounds.

- **Algorithms:** A. Ambainis and al. [2, 2020]

$$Q(\text{DYCK}_k) = O\left(\sqrt{n}(\log_2(n))^{0.5k}\right)$$

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- **Reductions to:**

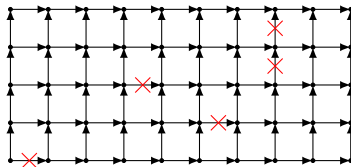


Figure: A reduction to 2D directed grid connectivity.

$$Q(\text{DYCK}_k) = O\left(\sqrt{n}(\log_2(n))^{0.5(k-1)}\right)$$

## Second step, one try to prove the optimality with a matching lower bound.

- **Adversary methods:**

$$\text{EX}_{2m}^{m|m+1}(x) = 0 \Leftrightarrow |x|_0 - |x|_1 = 2$$

$$\text{EX}_{2m}^{m|m+1}(x) = 1 \Leftrightarrow |x|_0 - |x|_1 = 0$$

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- **Reduction from:**  $\text{EX}_{2m}^{m|m+1} \leq \text{DYCK}_{k,n}$

$$Q(\text{DYCK}_k) = \Omega(\sqrt{nc}^k)$$

A natural goal is to made the bounds match.

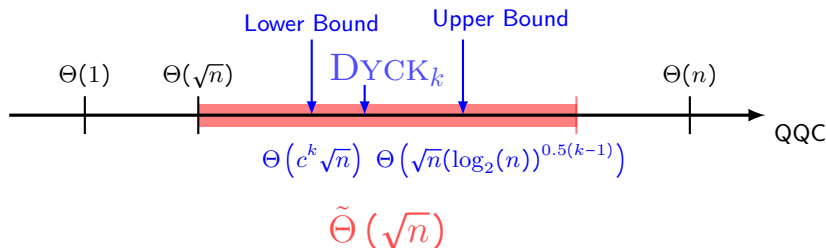


Figure: Representation of the different bounds.

# Sommaire

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- 2 The progress to reduce Quantum Query Complexity of bounded height Dyck word
  - The problem is not only a grover search.
  - Original algorithm and small updates
  - A new algorithm for  $k=2$
- 3 New idea to get better quantum query complexity bounds

# Every $k$ is not as simple as 1.

- $k = 1$ :

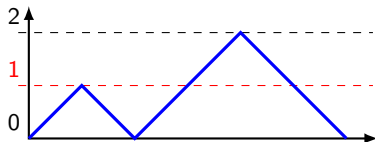


Figure: A dyck word of height 2.

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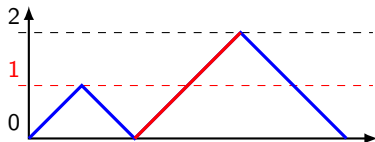


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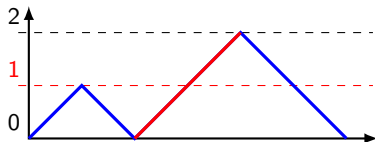


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•  $k = 2$ :

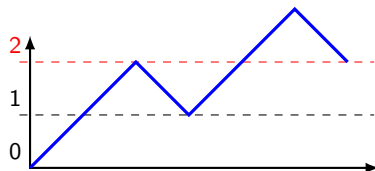


Figure: A substring of height 3.

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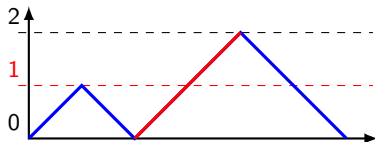


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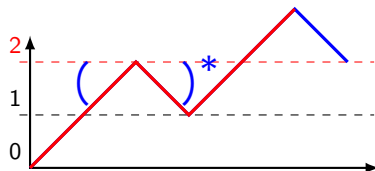


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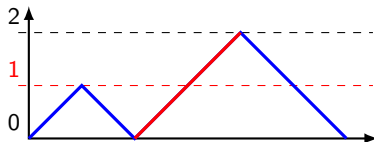


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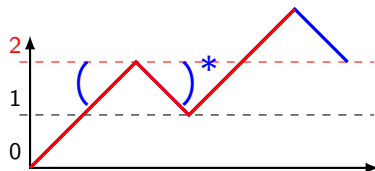


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# Small definitions and intuitions.

- $\pm k$  strings:

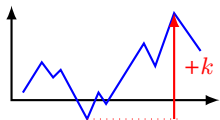


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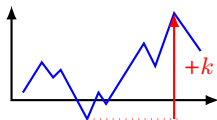


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- Minimal decomposition:

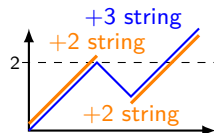


Figure: A  $+3$  string decomposition.

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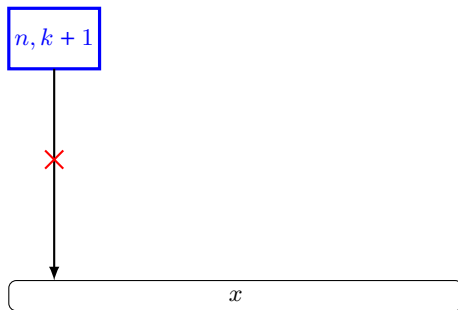
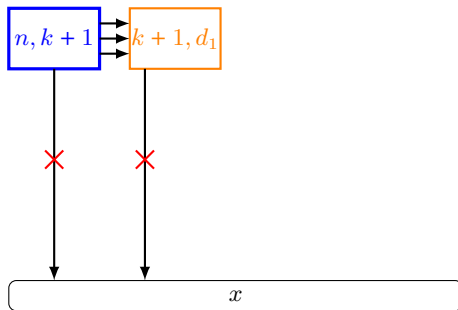


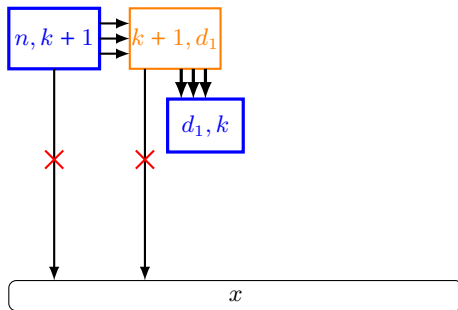
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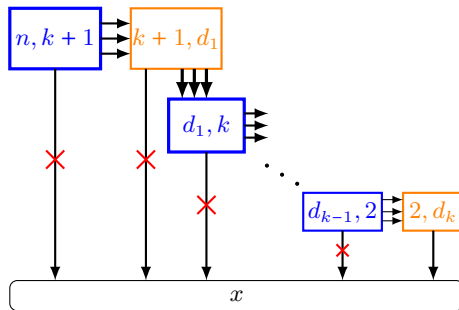
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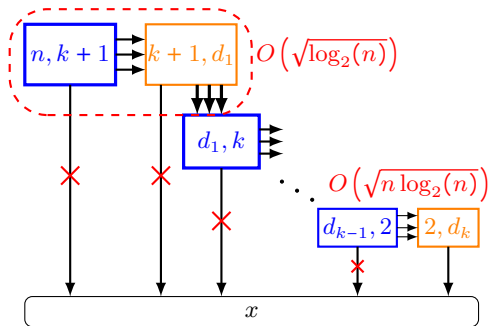


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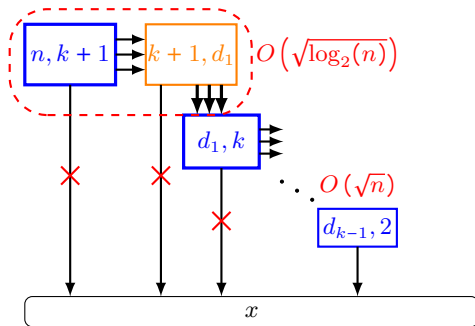


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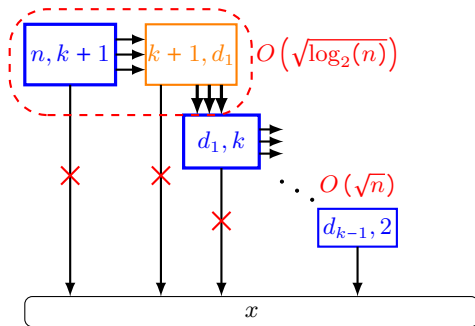
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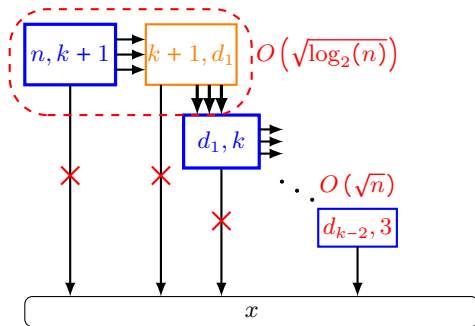
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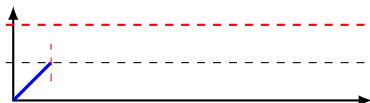
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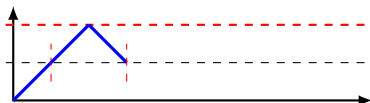
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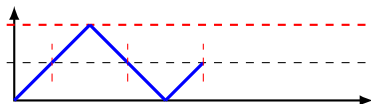
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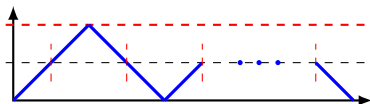
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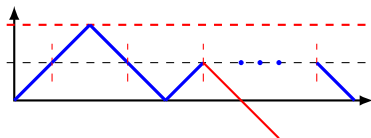
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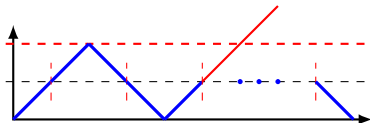
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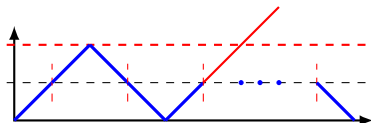
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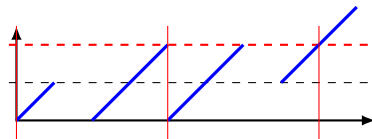


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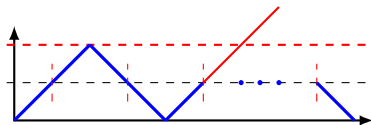


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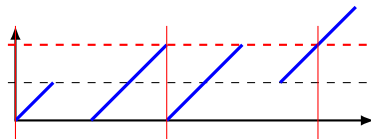


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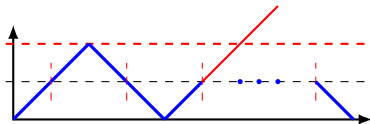
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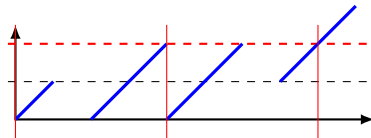
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- Highest +2-substring:



- Lowest +2-substring:



$$Q(\text{DYCK}_{2,n}) = O(\sqrt{n}) \implies Q(\text{DYCK}_{k,n}) = O(\sqrt{n}(\log_2(n))^{0.5(k-2)})$$

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- 2 The progress to reduce Quantum Query Complexity of bounded height Dyck word
- 3 New idea to get better quantum query complexity bounds
  - For lower bounds
  - For upper bounds:
  - Conclusion

# A better reduction can increase the lower bounds.

- Tightness of  $EX_{2m}^{m|m+1}$ 's one.
- New problem for a new reduction:

$$EX_{2m}^{m|m+1} \leq \text{new problem} \leq \text{DYCK}_k.$$



Optimizing the recursion step of Ambainis and all. algorithm can decrease the upper bound.

```

for  $1 \leq d_1 \leq \log_2(n)$  do
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Figure: Inexact simplification of the original algorithm.

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Figure: Inexact simplification of the original algorithm.

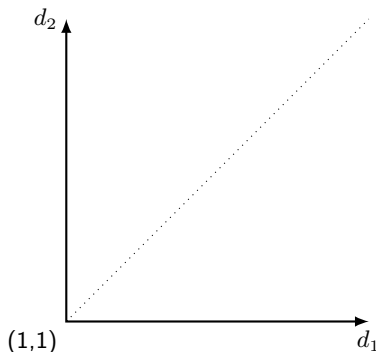


Figure: Graph for  $k = 4$ .

Optimizing the recursion step of Ambainis and all. algorithm can decrease the upper bound.

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for  $1 \leq d_1 \leq \log_2(n)$  do
  for  $1 \leq d_2 < d_1$  do
     $\vdots$ 
    for  $1 \leq d_{k-1} < d_{k-2}$  do
      Find a  $\pm(k+1)$ -string
  
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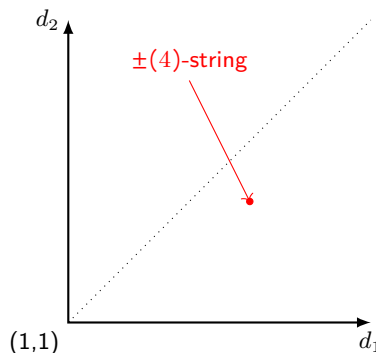


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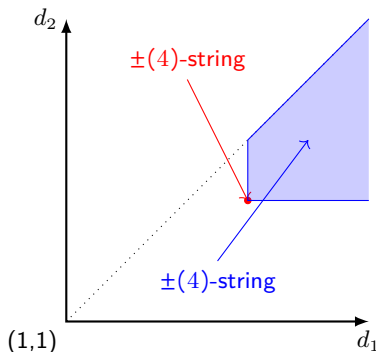


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# To conclude:

## What has been done:

- Trichotomy
- Adversary methods
- Reduction methods
- New algorithm:  $O\left(\sqrt{n}(\log_2(n))^{0.5k}\right) \rightarrow O\left(\sqrt{n}(\log_2(n))^{0.5(k-2)}\right)$

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### Possible idea to go further:

- Prove that new upper bound approach cannot work
- New algorithm
- General Adversary method

# Thanks for your attention.

## Question Time



Scott Aaronson, Daniel Grier, and Luke Schaeffer.

A quantum query complexity trichotomy for regular languages, 2018.



Andris Ambainis, Kaspars Balodis, Jānis Iraids, Kamil Khadiev, Vladislavs Kļevickis, Krišjānis Prūsis, Yixin Shen, Juris Smotrovs, and Jevgēnijs Vihrovs.

Quantum lower and upper bounds for 2d-grid and dyck language.

*Leibniz International Proceedings in Informatics*, 170, 2020.