École Normale Supérieure de Lyon

Latvijas Universitāte

M1 Internship Report

Complexity of recognizing Dyck languages of bounded height with quantum query algorithms.

Key words: *Quantum Query Complexity, Dyck Words, Quantum Algorithms, Regular Languages, Star-Free Languages, Adversary Methods.*

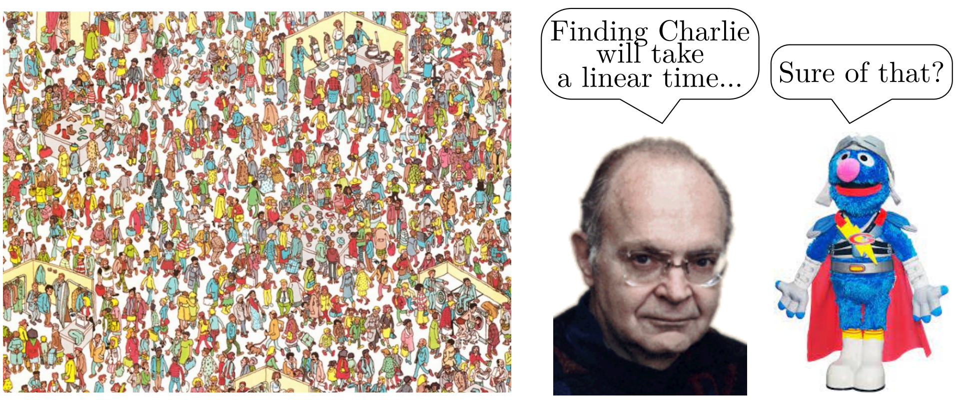

Figure 1: Historical discussion between Knuth and Grover.

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1 Introduction

Context of the internship

As part of the first year of Master at the École Normale Supérieure de Lyon, I was able to do a 12 weeks research internship in a laboratory.

My research for an internship in Quantum Algorithmic had brought me to the Faculty of Computing at the University of Latvia and my supervisor Andris Ambainis. My research also brought me to discuss with Kamil Khadiev from Kazan Federal University who became my co-supervisor. We discussed by email to find an interesting subject of research on which I liked to work on. I thank them for their help, their supervision and the time they gave to me during this 12 weeks.

During the internship, I have been integrated to the life of the Center for Quantum Computing Science. I thanks members of the team for the great discussions we had after the seminar.

I also want to thank Omar Fawzi for having introduced me to quantum computing and its fascinating possibilities.

The team's research area is quantum algorithms and complexity theory. More precisely, the team works on establishing new quantum algorithms with better complexity and proving new bounds to the quantum complexity for many different types of problem belonging from graph theory to cryptography passing by language recognition theory. My work on the recognition of restricted Dyck words integrated itself great in the team work as it has already been studied by the team for few years [4] and further by Kamil Khadiev [8].

My internship, named "Complexity of recognizing Dyck language with a quantum computer", had for goal to reduce the gap between the lower and the upper bound for the quantum query complexity of recognizing Dyck words of bounded height. The best known lower and upper bounds are describe in [4] by Andris Ambainis team.

In the end of the introduction, the field of research will be presented more precisely. After that, technical preliminaries, which are useful to understand the current and the new results, will be detailed. Finally, the last two sections present my new results for the quantum query complexity of bounded height Dyck word and the different tries to improve both upper and lower bounds.

1.1 History of quantum computing

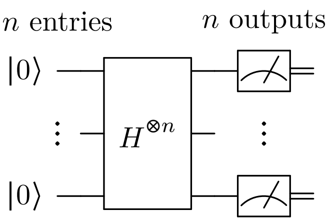
The history of quantum computing has started in 1980 when Paul Benioff, an american physicist, proposed a quantum mechanical model of Turing machines [5]. His machine uses some properties of the matter that has been discovered by quantum physicists. After that, some computer scientists suggested that the quantum model of Turing machines may be more expressive that the classical model. Few years after, the first bricks of the quantum circuit have been introduced by Richard Feynman [7]. The first quantum computers have started to arrived in the middle of 1990s. During the last 20 years, the funds given to the creation of the first quantum computer have skyrocketed, as the number of start-ups and companies dedicated to it. This emulation has made from the quantum computer field one of the most active field of research today. On the algorithmic side, the first astonishing result is the algorithm designed by Peter Shor (1994) [12]. The algorithm improves a lot the complexity of factorizing integers, enough to break our cryptographic protocols when quantum computer will be powerful enough. Since 1994, the quantum algorithm area has evolved almost independently from the quantum computers and has developed many beautiful theories and interesting results. But first, how does a quantum circuit works?

1.2 The quantum circuit, and quantum query model and complexity

In classical computer science, the piece of information is represented with 0 and 1. This two states can be easily obtained using electricity because 0 can be represented by 0V and 1 by 5V, it is easy to propagate electricity through wires and to stock its level into capacitor. Moreover a little piece of hardware, named transistor, has allowed to do some computations using logical gates which once include in a complex machine create our so-called "computers".

For quantum computers, the story isn't so different. First, the 0 and 1 are now represented using particles like electrons or photons. For example, an electron with a spin of $+\frac{1}{2}$ (noted $|1\rangle$, pronounced ket 1) represents a 1 and an other one with a spin of $-\frac{1}{2}$ (noted $|0\rangle$, pronounced ket 0) represents a 0. But the use of particles is motivated by their properties and mainly by a quantum property called superposition. A quantum state is not only $|0\rangle$ or $|1\rangle$, but can be both in the same time, i.e. $\lambda_0|0\rangle + \lambda_1|1\rangle$ for every $\lambda_{0,1} \in \mathbb{C}$ such that $|\lambda_0|^2 + |\lambda_1|^2 = 1$. A quantum bit, called qubit, correspond to a quantum state that is a superposition of two value. As before, the computations are done by gates, here quantum gates, which transform the quantum state of a qubit into another quantum state. At the end, to get the result of a computation, it is mandatory to measure the state of the quantum system, which breaks the quantum superposition. A quantum state of n qubits can be represented with a vector in a 2^n dimensional space whose norm is equal to 1, and a quantum gate by a linear unitary transformation on a 2^n dimensional space. A transformation is said unitary if it preserved the lengths.

A quantum circuit is a precise configuration of quantum gates on a finite number of qubits. The following Figure 2 represents the quantum circuit that computes a uniform randomize on $\{0,1\}^n$.



Legend:

- — ═ are respectively a quantum wire and a classical wire.
- $H^{\otimes n}$ is the Hadamard's gates on each input qubit.
- is the measure of 1 qubit in the algorithm.

Figure 2: A quantum circuit computing the uniform random on $\{0,1\}^n$.

The quantum query circuit is a quantum circuit used to compute function on an entry $x = x_1 \dots x_N$ that belongs to an entry space. The black box model [2] of a quantum query circuit is composed of an input state $|\psi_{start}\rangle$ and a sequence $U_0, Q, U_1, \dots, Q, U_T$ of linear unitary transformations such that $|\psi_{start}\rangle$ and all U_i do not depend on entry x unlike the Q_i which depend on x. The quantum state $|\psi_{start}\rangle$ belongs to a d-dimensional space generated by $|1\rangle, |2\rangle, \dots, |d\rangle$. To define Q, the basis vectors first need to be renamed from $|1\rangle, \dots, |d\rangle$ to $|i,j\rangle$ with $i \in [\![0, N]\!]$ and $j \in [\![1, d_i]\!]$ for some d_i such that $d_1 + d_2 + \dots + d_N = d$. Next, Q is define such that

$$Q(|i,j\rangle) := \begin{cases} |0,j\rangle & \text{if } i = 0 \\ |i,j\rangle & \text{if } i > 0 \text{ and } x_i = 0 \\ -|i,j\rangle & \text{if } i > 0 \text{ and } x_i = 1. \end{cases}$$

The gates Q are doing the queries to input x by flipping some of the vectors. Finally, to get the output of the quantum query algorithm it is necessary to measure the output. A quantum query algorithm can the summarized with the following quantum circuit.

The quantum query complexity of an algorithm corresponds to the number of calls to the Q gate. Often, this number of calls is depending on the size of the entry. The quantum query

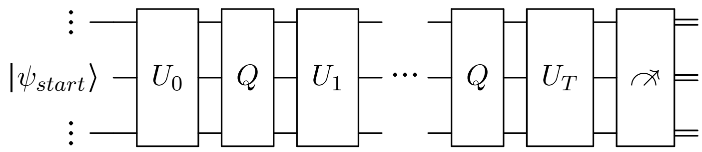


Figure 3: Structure of a quantum query algorithm.

complexity of a problem corresponds to the highest possible bound for which it is certain there is no quantum query algorithm with a lower quantum query complexity solving the problem.

1.3 Dyck Languages of height k

First, the Dyck language corresponds to the set of correct and balanced words of parenthesis. Often, computer scientist used another more graphical and convenient definition where parenthesis words are replaced with discrete paths onto a 2D space. Indeed, every path starts at coordinate $(0, 0)$ and is composed of 2 types of steps: The first one is an increasing step that is represented by adding to the end of the current path the vector $\overrightarrow{(1, 1)}$. The second one is an decreasing step, it works similarly but the add vector is $\overrightarrow{(1, -1)}$. Moreover, to be a Dyck path, the path should start with an increasing step, it should never cross the abscise axis and it should finish on it. A Dyck word of length 12 is presented in Figure 4. The Dyck language is a context free language as it can easily be recognized using a context free grammar. The work done by Andris Ambainis' team [4] focus on a restriction on Dyck language with bounded height k. More precisely, a Dyck word is of height at most k if, in every of its prefix, the difference between the number of opening and closing parenthesis does not exceeds k. In the path representation, a path is said to be of height at most k if the path never cross the line $y = k$. Figure 4 illustrates two different Dyck words with only one of height at most 3. The restricted Dyck language with bounded height k is noted DYCK_k and is interesting because it belong to the already well studied class of star free languages (Detail in subsubsection 2.1.2).

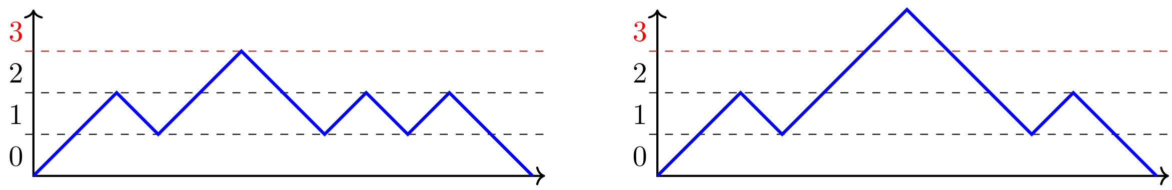


Figure 4: **On the left**, a valid Dyck word of height at most 3. **On the right**, an invalid Dyck word of height at most 3.

1.4 State of the art

This state of the art is not too precise as the understanding of the bibliography took almost the first half the internship and is more detail in section 2. First, few years ago Aaronson, Grier and Schaeffer [1, 2019] worked on quantum query complexity of recognizing regular languages as they can model a lot of tasks. They concluded that there are 3 different cases depending of the language:

- $O(1)$ if it is sufficient to read constant number of letters at the beginning and the end.
- $\tilde{\Theta}(\sqrt{n})$ if a Grover's search[1] is the best way to recognize the language. The tilde refers to the existence of a constant c_{te} such that the quantum query complexity is equal to

[1] The Grover search is a quantum query algorithm that allows to search for a marked element in a set of n elements, where p of them are marked, with a quantum query complexity of $O(\sqrt{\frac{n}{p}})$.

$\Theta\left(\sqrt{n}(\log_2(n))^{c_{te}}\right)$.

- $\Theta(n)$ if recognizing the language is the same as counting modulo some value that can be computed with $O(n)$.

Further more, it is proved that being in the second classes is equivalent to being a star free language. Andris Ambainis' team decided to work on $\textsc{Dyck}_k$ as this language is a beautiful example of star free languages. In [4, 2020], the team focused on finding the value of c_{te}, the power of the logarithm, in order to find the exact quantum query complexity. They first proved by reduction that the quantum query complexity of $\textsc{Dyck}_k$ called $Q\left(\textsc{Dyck}_k\right)$ is in $\Omega\left(c^k\sqrt{n}\right)$ where c is a constant greater than 1. They also gave an algorithm for $\textsc{Dyck}_k$ with a quantum query complexity of $O\left(\sqrt{n}(\log(n))^{0.5k}\right)$. Since then, no better lower bound or algorithm have been found.

1.5 Goals of the internship

The problem on the quantum query complexity of $\textsc{Dyck}_k$ is still open, my internship has for goal to reduce the gap between the lower bound and the best known algorithm. My researches have been organized on two main axes:

- **Increasing the lower bound.** To do this, it is necessary to understand the bibliography on the adversary method in order to try to find a new adversary with better property. An other way is to use already existing lower bounds and translates them to $\textsc{Dyck}_k$ with reductions.

- **Lowering the upper bound.** It is sufficient to find new algorithms with a better quantum query complexity than the previous ones. As for lower bounds, reduction method can also provides interesting new upper bounds.

Finally, the overall goal would be to made the two bounds match in order to get the exact quantum query complexity of $\textsc{Dyck}_k$.

1.6 Results

The main results of my internship are presented in section 3. The first one is a small revision of the original quantum query algorithm that recognize $\textsc{Dyck}_k$ [4] and the second one is a new quantum query algorithm for $\textsc{Dyck}_2$ and its modifications to improve more the already revised original algorithm.

2 Preliminaries

In order to understand the $\tilde{\Theta}(\sqrt{n})$ quantum query complexity, it is mandatory to learn the logic behind the trichotomy theorem [1] and its dependency on regular languages and star free languages. After that, it will be necessary to presents the tools that computer scientists have developed in order to find new bounds on the quantum query complexity. These tools can be group into 3 main categories: reductions, algorithms, and adversary methods[13].

2.1 Quantum query for regular languages.

In the article [1], Aaronson, Grier and Schaeffer use a really interesting algebraic characterization of regular languages based on monoids and syntactic congruence. This definition was unknown to me and I spend a lot of time to understand it correctly.

2.1.1 Regular languages

Usually, the set of regular languages $\mathcal{R}$ on the alphabet Σ is defined as the smallest fixed point of the function F (the function that computes concatenations, unions, and Kleene stars) including $\{\varnothing\} \cup \{\varepsilon\} \cup (\cup_{l \in \Sigma}\{l\})$ with

$$\begin{aligned} F(X) = &\{AB \ \forall (A,B) \in X^2\} \\ &\cup \{A \cup B \ \forall (A,B) \in X^2\} \\ &\cup \{A \cap B \ \forall (A,B) \in X^2\} \text{ \# optional} \\ &\cup \{A^* \ \forall A \in X\}. \end{aligned}$$

However in [1], the more convenient way to characterize regular languages is to use their algebraic characterization. More precisely, for every regular language L it exists a finite monoid M, a subset S of this monoid, and a monoid homomorphism δ from Σ^* to M such that the regular language is exactly the pre-image of the subset S by the monoid homomorphism δ (i.e. $\delta^{-1}(S)$). Lets takes apart this characterization: A monoid is a 3-tuple of a set M, an internal associative binary operation and finally the identity element associated to the operation. A monoid homomorphism is a map from a monoid to another that preserves the operation and the identity. Now, to get every elements of the characterization, the first step is to compute the syntactic monoid. The syntactic monoid is obtained by dividing Σ^* by the following equivalence relation called syntactic congruence

$$x \sim_L y \Leftrightarrow \forall (u,v) \in \left(\Sigma^*\right)^2, (uxv \in L \Leftrightarrow uyv \in L).$$

This equivalence relation is a congruence relation as the equivalence class can be multiplied (i.e. if $x \sim_L y$ and $u \sim_L v$ then $xu \sim_L yv$). With this syntactic monoid, it is now possible to define a monoid homomorphism. For that it is sufficient to take the homomorphism that map an element to its congruence class. Moreover, using a subset of the syntactic monoid is sufficient as in a congruence class, none or every element of the class is in L. Indeed, if $x \sim_L y$ then for all (u,v) in $\left(\Sigma^*\right)^2, (uxv \text{ in } L \Leftrightarrow uyv \text{ in } L)$ thus x in L is equivalent to y in L. So, it exists a subset of equivalent class that represent every word of L and no word not in L. Now, the last hard things to show is that the syntactical monoid is of finite size[2]. Finally, every regular language can be recognized by finite monoid.

2.1.2 Star free languages

The set of star free languages is a really well studied subset of the regular languages. Its definition differs a little from regular languages' one as the Kleene star is replaced by the complement operation (noted $\overline{L}$). So star free languages are defined as the smallest fixed point of the function F' (the function that computes concatenations, unions and complements) and such that it includes $\{\varnothing\} \cup \{\varepsilon\} \cup (\cup_{l \in \Sigma}\{l\})$. This restriction does not imply that every star free language is finite. For example, Σ^* can be written $\overline{\varnothing}$ and the language on $\Sigma = \{1,2,3\}$ described with the regular expression $\Sigma^* 2 0^* 2 \Sigma^*$ can be written as following in the star free way $\overline{\varnothing}2\overline{\overline{\varnothing}\Sigma\backslash\{0\}\overline{\varnothing}}2\overline{\varnothing}$.

As for regular languages, it exists an algebraic characterization for star free languages. Let M be a monoid, M is said to be aperiodic if for every x in M it exits a positive integer n such that $x^n = x^n + 1$. A theorem proved by Schützenberger [11] states that a language is recognized by a finite aperiodic monoid if and only if it is star free.

[2] More precisely, the finiteness of the syntactic monoid is the main property that characterize every regular languages. A good intuition to understand is first that it is always possible to construct finite automata from a finite monoid. For the second way, it is more delicate but the work done by Brzozozowski, Szykula and Ye [6, 2018] summarized a lot of result on the influence of the size of the minimal automata size on the size of the smaller syntactic monoid.

Good examples of star free languages are the Dyck word languages with bounded heights. First, it is easy to have a finite automaton that recognize Dyck word of height at most k by putting one state for each integer from 0 up to k. However, the belonging to the star free regular languages is more delicate to prove. It has been done by an italian researcher in [14, 1978].

2.1.3 Trichotomy theorem

Theorem 2.1 (Aaronson, Grier and Schaeffer [1]). *Every regular language has quantum query complexity $0, \Theta(1), \tilde{\Theta}(\sqrt{n})$, or $\Theta(n)$ according to the smallest class in the following hierarchy that contains the language.*

- *Degenerate: One of the four languages $\varnothing, \varepsilon, \Sigma^*$, or Σ^+.*
- *Trivial: The set of languages which have trivial [3] regular expressions.*
- *Star free: The set of languages which have star-free regular expressions.*
- *Regular: The set of languages which have regular expressions.*

This theorem is really important as it give a good idea on the quantum query complexity of many language recognitions. Moreover, the classes are now clearly defined so it is now easier to know where is a problem compared to the classes in the introduction. However, it does not give the exact quantum query complexity because for star free languages the result is given using $\tilde{\Theta}$. The $\tilde{\Theta}(\sqrt{n})$ means that the quantum query complexity of any star free regular language is in $\Theta(\sqrt{n}(\log_2(n))^p)$ for some p a non negative constant. As it is known that $\textsc{Dyck}_k$ languages are star free, it is an interesting problem to find the power of $\log_2(n)$ depending on the value of k.

2.2 The bounds for Dyck$_{k,n}$ problem

The trichotomy theorem state that $\textsc{Dyck}_{k,n}$ language has a quantum query complexity in $\tilde{\Theta}(\sqrt{n})$. The Θ means that the best possible algorithm is both a $\tilde{O}(\sqrt{n})$ and a $\tilde{\Omega}(\sqrt{n})$. So, a common method to find the power of $\log_2(n)$ depending on k is the squeeze theorem. More precisely, the quantum query complexity of $\textsc{Dyck}_{k,n}$ is trapped between a lower and an upper bound for quantum query complexity. So it is possible to deduce the quantum query complexity from an increasing sequence of lower bound and a decreasing sequence of upper bound such that they share the same limit l with l equals to $Q(\textsc{Dyck}_{k,n})$. How to find this sequence ?

- **For the lower bound sequence.** Some of the most important tools to compute lower bounds are called **quantum adversary methods** and have been invented by Ambainis [3]. This tools can compute lower bounds more or less tight and some of this adversary methods have really useful property such as being compatible with a certain type of composition of problems. This lead to the **reduction methods** that can compute a lower bounds from different but easier problem's ones.

- **For the upper bound sequence.** The main method is to **find quantum query algorithms** that are more and more efficient. An other way is also **by reduction** to a more difficult problem with a good enough upper bound.

- **For the same limit.** The idea is to do an iterative process where each step improves successively each bound until only one will continue to be improved. Finally, both bounds may end up matching. However, before my internship, both $\textsc{Dyck}_{k,n}$'s lower and upper bounds where stuck to $\Omega\left(c^k\sqrt{n}\right)$ for some constant c greater than 1 and $O\left(\sqrt{n}\,(\log_2(n))^{0.5k}\right)$.

[3] A language L is said to be trivial if and only if it exists 2 finite size alphabets L_1 and L_2 such that $L = L_1\Sigma^*L_2$.

2.2.1 Lower bound on the quantum query complexity of $\mathbf{Dyck}_{k,n}$

The quantum adversary methods: This part as for goal to present some quantum adversary methods and their usages.

The adversary method. The adversary method is a classical way to prove lower bound in classical algorithmic. Indeed, the idea behind the classical adversary is to prove that for every algorithm, it exist an entry such that the algorithm cannot decide in less unit of complexity than the lower bounds. In general, during the algorithm execution, the adversary modifies entry values that have not been already used in order to increase the execution time. This classical adversary work great for classical algorithm but is not really useable on quantum algorithms.

The super basic quantum adversary method. In order to recognize a language it is mandatory to be able to distinguish between a valid word v and an invalid word w. So at the output of the quantum query algorithm, the states $|\psi_v^T\rangle$ and $|\psi_w^T\rangle$ should be distinguishable thus $|\langle\psi_v^T|\psi_w^T\rangle| < \frac{2}{3}$. How do they will differs? The quantum query algorithm start with $|\psi_v^0\rangle = |\psi_w^0\rangle = |\psi_{start}\rangle$. Moreover, the inner product of both states isn't affected by U_i gates because there are unitary. However, the Q_i gates affect this inner product as the Q_i's behaviors are depending on the input. Let define the progress measure $\mathcal{P}$ such that $\mathcal{P}(t) := \langle\psi_v^t|\psi_w^t\rangle$ where $|\psi^t\rangle$ represents the state after Q_t. Let suppose it exists d such that for all $0 \leq t \leq T-1$, $|\mathcal{P}(t+1) - \mathcal{P}(t)| \leq d$. It implies the following inequality

$$\mathcal{P}(T) = \underbrace{\mathcal{P}(0)}_{=1} + \sum_{i=0}^{T-1} \underbrace{\mathcal{P}(i+1) - \mathcal{P}(i)}_{\geq -d} \geq \mathcal{P}(0) - T \times d.$$

Moreover, $\mathcal{P}(T)$ should be lower than $\frac{2}{3}$ so it implies that $1 - T \times d$ should also be lower than $\frac{2}{3}$ and finally that T should be a $O(\frac{1}{d})$. This give a general idea of how to determined a lower bound but it isn't precise enough to get a theorem. In its courses, Ryan O'Donnell [9] explained a simple to understand adversary method called the super basic adversary method. This adversary allow to compute a lower bound by using the following method:

Theorem 2.2. *Let define YES the set equal to $f^{-1}(accepted)$ and NO the set equal to $f^{-1}(rejected)$. If it exist two subset $Y \subseteq YES$ and $Z \subseteq NO$ such that:*

- *For each y in Y, there are at least m strings z in Z with $dist(y,z) = 1$.*
- *For each z in Z, there are at least m' strings y in Y with $dist(y,z) = 1$.*

Then the quantum query complexity $Q(f)$ is in $\Omega(\sqrt{mm'})$.

The proof of the theorem can be found in Appendix A. One important result of the adversary method is the lower bound on the quantum query complexity of $Ex_{2m}^{m|m+1}$ in $\Omega(m)$ where the problem $Ex_{2m}^{m|m+1}$ consists to recognize between $|x| = 2m \wedge |x|_1 = m$, and $|x| = 2m \wedge |x|_1 = m+1$. More generally, the method of an adversary define a process to find the lower d possible. The super-basic adversary method as its name let suggest is simple but does not give a tight lower bounds. For this, the method should be more complicated which results in longer days on the board to find the parameters of the methods.

Let $f : D \to \{0,1\}$ be the function whose quantum query complexity is unknown. Two other adversary methods are described as following:

- **The Basic adversary method by Ambainis [3].**

  Theorem 2.3. *Let define YES the set equal to $f^{-1}(accepted)$ and NO the set equal to $f^{-1}(rejected)$. Moreover, let $Y \subseteq YES$, $Z \subseteq NO$, and let $\mathcal{R} \subseteq Y \times Z$ be a set of "hard to distinguish" pairs, such that:*

– *For each y in Y, there are at least m strings z in Z with (y,z) in $\mathcal{R}$.*
– *For each z in Z, there are at least m' strings y in Y with (y,z) in $\mathcal{R}$.*

Also, let define $\mathcal{R}_i$ as a restriction from $\mathcal{R}$ such that (y,z) is in $\mathcal{R}_i$ if and only if (y,z) is in $\mathcal{R}$ and $y_i \neq z_i$. In addition,

– *For each y in Y and i, there are at most l strings z in Z with (y,z) in $\mathcal{R}_i$*
– *For each z in Z and i, there are at most l' strings y in Y with (y,z) in $\mathcal{R}_i$*

Then the quantum query complexity $Q(f)$ is in $\Omega(\sqrt{\frac{mm'}{ll'}})$.

- **The general adversary method by Reichardt [10].**

  A symmetric matrix Γ is an adversary matrix for f if the rows and cols of Γ can be indexed by input x in D such that $\Gamma_{x,y} = 0$ if $f(x) = f(y)$. $\Gamma^{(i)}$ is defined from Γ and is a similarly sized matrix such that $\Gamma^{(i)}_{x,y} = \begin{cases} \Gamma_{x,y} & \text{if } x_i \neq y_i \\ 0 & \text{otherwise} \end{cases}$. This two objects allow to define the following notion of adversary plus minus

  $$Adv^{\pm}(f) = \max_{\substack{\Gamma \text{ - an adversary} \\ \text{matrix for } f}} \frac{||\Gamma||}{\max_i ||\Gamma^{(i)}||}.$$

  Theorem 2.4. *The adversary plus minus is such that $Q(f) = O(Adv^{\pm}(f))$.*

  In [10], Reichardt gave the proof that the general adversary method is not only giving a lower bound to the quantum query complexity as the method return the directly the quantum query complexity of f.

All this adversary methods are really interesting to compute lower bounds, but sometime it is useful to reuse already computed ones.

The reduction method: For some problems, it looks obvious that some are easier than the others. Computer scientists have developed tools to handle more formally this notion of difficulty comparison. One of the main tool is named reduction. A reduction is the process to solve a first problem using an algorithm for a second one. Because the second problem is able to solve the first one, it is said to be harder, which is written $P_1 \leq P_2$.

Theorem 2.5. *Reduction and $Adv^{\pm}$*

$$P_1 \leq P_2 \implies Adv^{\pm}(P_1) \leq Adv^{\pm}(P_2)$$

Finally, problems can also be composed. Lets take $P_1 : \{0,1\}^n \to \{0,1\}$ and $P_2 : \{0,1\}^m \to \{0,1\}$. Then $P_1 \circ P_2$ is defined as

$$(P_1 \circ P_2)(x_1 \dots x_{nm}) := P_1\left(\underbrace{P_2(x_1 \dots x_m), \dots, P_2(x_{(n-1)m+1} \dots x_{nm})}_{n}\right).$$

This composition has a great behavior with the adversary plus minus.

Theorem 2.6. *Composition and $Adv^{\pm}$*

$$Adv^{\pm}(P_1 \circ P_2) \geq Adv^{\pm}(P_1) \times Adv^{\pm}(P_1)$$

Andris' team found that the $Ex_{2m}^{m|m+1}$ problem can be reduce using the OR and AND problems to $\textsc{Dyck}_k$ problem with the reduction described in [4]. They finally get a lower bound in $\Omega(c^k\sqrt{n})$ for the quantum query complexity of $\textsc{Dyck}_{k,n}$.

The team also gets another result (not published), they have founded an upper bound using a reduction from $\textsc{Dyck}_{k,n}$ to a problem about the connectivity into a 2d directed grid with missing edges. This upper bound of $O(\sqrt{n}(\log_2(n))^{0.5(k-1)})$ is interesting as it is an upper bound in $\tilde{\Theta}(\sqrt{n})$.

2.2.2 Best known algorithm to recognize Dyck$_{k,n}$

Before checking the algorithm for $\textsc{Dyck}_{k,n}$, it is necessary to define some terms useful for later.

Preliminary definition:

- **The height function h:** This first utilarity function allow the computation of final height of a string. It is define as following

$$h(x_1 \dots x_n) := \sum_{i=1}^{n} (-1)_i^x$$

- **$\pm k$-strings:** A string $x_1 \dots x_n$ is said to be a $+k$-string (resp. $-k$-string) if

$$\max_{1 \le i \le j \le n} h(x_i \dots x_j) = k \left(\text{resp.} \min_{1 \le i \le j \le n} h(x_i \dots x_j) = -k \right).$$

- **minimal $\pm k$-strings:** A $\pm k$-string $x_1 ... x_n$ is said to be minimal if it doesn't exist i, j such that $1 \le i \le j \le n$, $(i,j) \ne (1,n)$ with $x_i \dots x_j$ a $\pm k$-string.

Dyck$_{k,n}$ characterization: In order to recognize $\textsc{Dyck}_k$ language, multiple approach are possible. The most method used in [4] by Ambainis's team is to search for a substring that cannot be seen into a Dyck word of height at most k. A natural way to reject a word w from $\textsc{Dyck}_k$ is to search for a $\pm k+1$-string into $1^k w 0^k$. Lets detail this technic more precisely. First, a Dyck word is always above the abscise axe, so it cannot exist i such that $h(w_1 \dots w_i) = -1$ thus it cannot exist i such that $h((1^k w 0^k)_1 \dots (1^k w 0^k)_i) = -k-1$ and finally it cannot exist a $-(k+1)$-string into $1^k w 0^k$. After that, a Dyck word always end on the abscise axis, which means that it doesn't exist i such that $h(w_i \dots w_n) = 1$ which implies that $1^k w 0^k$ cannot contain any $+(k+1)$-string. Moreover, for a Dyck word of height at most k, it cannot exist i such that $h(w_1 \dots w_i) = k+1$ so the bounded height constraint is already taken into account by the impossibility of having a $+(k+1)$-string into $1^k w 0^k$. Finally, this give us that the belonging of a $\pm(k+1)$-string into $1^k w 0^k$ is sufficient to reject every non Dyck word of height at most k.

$\pm k+1$-strings recognition: In order to recognize $\textsc{Dyck}_k$, the main point is now to find efficiently a $\pm(k+1)$-string. Let detailed a little bit how it is done.

- **For k=1:** To reject a non-$\textsc{Dyck}_1$ word w, it is sufficient to find ± 2-strings. However, the only every minimal ± 2)-strings is of size 2, so every $\pm(2)$-strings can be found by searching for 11 or 00 using 2 grover search. This methods implies a quantum query complexity of $O(\sqrt{n})$ from its two calls to Grover's one.

- **For k=2 (naive approach):** To reject, the goal is to find a $\pm(3)$-string. Unfortunately, there are an infinite number of minimal $\pm(3)$-strings as they form the language $1(10)^*11 + 0(01)^*00$. So trying every possible minimal $\pm(3)$-string for an input string of size n require

$O(n)$ calls to Grover with a final quantum query complexity of $O(n\sqrt{n})$. This algorithm has its complexity already above the known upper bounds of the trichotomy theorem. In order to stay into the trichotomy theorem, the algorithm should be improved.

- **For any k+1:** In order to have a faster algorithm, Ambainis' team has found an inductive algorithm on the depth. Indeed, a minimal $\pm(k+1)$-string can be share into two smaller $\pm k$-strings as shown in Figure 5. The main ideas of the induction step are:

  1. Chose an upper bound d in $\{2, 4, 8, \dots, 2^{\lfloor \log_2(n) \rfloor}\}$ for the length of the $\pm(k+1)$-string.
  2. Chose an indices t in $\{1, 2, 3, \dots, n\}$ that have to be in the $\pm(k+1)$-string.
  3. Try to find two $\pm(k)$-string in an interval of length at most d that include t
     (a) Try to find a $\pm(k)$-string that include t of length at most d-1.
     (b) If it exists, find an other $\pm(k)$-string on the left or on the right, if it fail return `NULL`.
     (c) If it does not exist, try to find $\pm(k)$-string on the left, and another $\pm(k)$-string on the right, if it fail return `NULL`.
     (d) Test if both $\pm(k)$-string are of the same sign and if the string that include both if of length lower than d.
     (e) If test is good, return the $\pm(k+1)$-string.
     (f) Otherwise return `NULL`.

  Let explain the quantum query complexities and the idea behind each step. It has been shown before that searching for every minimal $\pm(k+1)$-string isn't a solution has it is too slow. So the idea is to bound the size of minimal $\pm(k+1)$-string the function is currently searching for. This is interesting as it allows to use a function describes in step 3 named $\textsc{FindAtLeftMost}_{k+1}$, whose main parameters are d and t, and which is able to find a minimal $\pm(k+1)$-string if and only if it include the index t and the size of the minimal $\pm(k+1)$-string is bounded between $\frac{d}{2}$ and d. These constraints imply two things: First, the parameter t implies a call to Grover (as it may be possible to have a $\pm(k+1)$-string not including t), but there is $O(d)$ value of t such that $\textsc{FindAtLeftMost}_{k+1}$ returns a minimal $\pm(k+1)$-string so it is possible to cut early the Grover search to get its quantum query complexity in $O\left(\sqrt{\frac{n}{d}}\right)$. This Grover search corresponds to item 2 named $\textsc{FindFixedLength}_{k+1}$. After that for d, in order not to miss any minimal $\pm(k+1)$-string, it is necessary to call $\textsc{FindFixedLength}_{k+1}$ for every d in $\{2, 4, 8, \dots, 2^{\lfloor \log_2(n) \rfloor}\}$ (item 1) which implies a call to Grover in $O\left(\sqrt{\log(n)}\right)$. This is done in item 1 by the function named $\textsc{FindAny}_{k+1}$. Finally, the quantum query complexity of $\textsc{FindAtLeftMost}_{k+1}$ is $O\left(\sqrt{d}(\log_2(d))^{0.5(k-1)}\right)$, it comes from complex calls to many subroutines. In steps 3a and 3b, at most three calls to $\textsc{FindAtLeftMost}_k$ are done with a quantum query complexity of $O\left(\sqrt{d}(\log_2(d))^{0.5(k-2)}\right)$. In step 3b and 3c, at most 4 calls to $\textsc{FindFirst}_k$ are done. The subroutines $\textsc{FindFirst}_k$ find the closer $\pm k$-strings from the end r or the beginning l of a specified interval. This function finally do a binary search using calls to $\textsc{FindAny}_k$ and $\textsc{FindFixedPos}_k$ in a quantum query complexity of $O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-1)}\right)$. This last subroutine $\textsc{FindFixedPos}_k$ searches only for $\pm k$-strings which include the index t with a quantum query complexity of $O\left(\sqrt{n}(\log_2(n))^{0.5(k-1)}\right)$.

Finally, this complex algorithm has a quantum query complexity of $O\left(\sqrt{n}(\log(n))^{0.5k}\right)$ which is a $\tilde{\Theta}\left(\sqrt{n}\right)$. This description of the existing algorithm stay at the surface in order to stay simple, every the pseudo code of every subroutines can be found in Appendix B. No proof of their quantum query complexity is provided as a proof for Theorem C.1, almost identical, is already provided in Appendix C. However, this algorithm can be improved, indeed the first upper

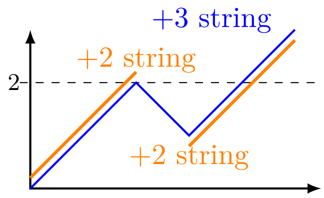


Figure 5: Decomposition of a +3-string into two $\pm(k+1)$-strings.

bound by reduction is still better than the upper bound from algorithm. Now, two personal improvements are described in section 3.

3 A better algorithm for Dyck$_{k,n}$

3.1 A better Complexity Analysis of the original algorithm

In the article [4], Andris Ambainis give us a quantum algorithm to recognize the belonging of a n length bit string in $\textsc{Dyck}_{k,n}$ using $O(\sqrt{n}(\log_2(n))^{0.5k})$ quantum queries. But the quantum query complexity for $k = 1$ ($O(\sqrt{n}\log_2(n))$) is not as good as a Grover's search $O(\sqrt{n})$ which is sufficient (seen in subsubsection 2.2.2). More precisely, the logarithmic search done by $\textsc{FindAny}_{k+1}$ for $k = 1$ is useless as there is exactly one minimal ± 2 -string so it is sufficient to add a new initial case to $\textsc{FindAny}_{k+1}$ for $k = 1$ in order to use a Grover search for 00 and 11 in $1w0$ instead of $\textsc{FindFixedLength}_2$. This lowers the quantum query complexity for $k = 1$ of the function to $O(\sqrt{n})$ instead of $O(\sqrt{n\log_2(n)})$. This give us this following algorithm for $\textsc{FindAny}_k$.

Algorithm 1 $\textsc{FindAny}_k$(l,r,s)

Require: $0 \le l < r$ and $s \subseteq \{1, -1\}$
  if $k > 2$ **then**
    Find d in $\{2^{\lceil \log_2(k) \rceil}, 2^{\lceil \log_2(k)+1 \rceil}, \dots, 2^{\lceil \log_2(r-l) \rceil}\}$ such that
      $v_d \leftarrow \textsc{FindFixedLength}_k(l, r, d, s)$ is **not** $\textsc{Null}$
    return v_d or $\textsc{Null}$ if none
  else
    Find t in $\{l, l+1, \dots, r\}$ such that
      $v_t \leftarrow \textsc{FindAtLeftMost}_2(l, r, t, 2, s)$ is **not** $\textsc{Null}$
    return v_t of $\textsc{Null}$ if none

The same improvement can be done on $\textsc{FindFixedPos}_k$ because if $k = 2$ the logarithmic search is useless. So $\textsc{FindFixedPos}_k$ can be redefined as in $\textsc{Algorithm}$ 2. For $k = 2$, the complexity is lowered from $O(\sqrt{\log_2(l-r)})$ to $O(1)$.

Algorithm 2 $\textsc{FindFixedPos}_k(l, r, t, s)$

Require: $0 \le l < r$, $l \le t \le r$ and $s \subseteq \{1, -1\}$
  if $k > 2$ **then**
    Find d in $\{2^{\lceil \log_2(k) \rceil}, 2^{\lceil \log_2(k)+1 \rceil}, \dots, 2^{\lceil \log_2(r-l) \rceil}\}$ such that
      $v_d \leftarrow \textsc{FindAtLeftMost}_k(l, r, t, d, s)$ is **not** $\textsc{Null}$
    return v_d or $\textsc{Null}$ if none
  else $v \leftarrow \textsc{FindAtLeftMost}_k(l, r, t, 2, s)$ is **not** $\textsc{Null}$
    return v_d or $\textsc{Null}$ if none

This small improvements on the initial cases will improve the global quantum query complexity of each subroutine and finally the quantum query complexity for $\textsc{Dyck}_{k,n}$.

Theorem 3.1. $\text{Dyck}_{k,n}$'s *algorithm correctness* *The new definition of* FindAny *and* FindFixedPos *does not change the behavior the original algorithm as other subroutines (Appendix B) stay unchanged.*

Proof Theorem 3.1. The behavior of the $\text{Dyck}_{k,n}$ algorithm with the new subroutines is the same than the older one as FindAny (resp. FindFirst) has the same sub-behavior on every entry than its older definition.

□

Theorem 3.2. $\text{Dyck}_{k,n}$'s *Subroutines complexity* *The subroutines' quantum query complexity for k are the following.*

1. $Q(\text{Dyck}_{k,n}) = O\left(\sqrt{n}(\log_2(n))^{0.5(k-1)}\right)$ *for* $k \geq 1$

2. $Q(\text{FindAny}_{k+1}(l,r,s)) = O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-1)}\right)$ *for* $k \geq 1$

3. $Q(\text{FindFixedLength}_{k+1}(l,r,d,s)) = O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-2)}\right)$ *for* $k \geq 2$

4. $Q(\text{FindAtLeftMost}_{k+1}(l,r,t,d,s)) = \begin{cases} O\left(\sqrt{d}(\log_2(d))^{0.5(k-2)}\right) & \text{for } k \geq 2 \\ O(1) & \text{for } k = 1 \end{cases}$

5. $Q(\text{FindFirst}_{k}(l,r,s,left)) = O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-2)}\right)$ *for* $k \geq 2$

6. $Q(\text{FindFixedPos}_{k}(l,r,t,s)) = \begin{cases} O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-2)}\right) & \text{for } k \geq 3 \\ O(1) & \text{for } k = 2 \end{cases}$

Proof Theorem 3.2. The idea is that only the $O(\sqrt{n})$ comes from the initial cases for $k = 1$ and for each of the $k-1$ level of the recursion, the quantum query complexity is increased by a $O\left(\sqrt{\log_2(n)}\right)$ factor. The $O\left(\sqrt{\log_2(n)}\right)$ factor is proven by Andris Ambainis' team in [4] while the $O(\sqrt{n})$ for $k = 1$ comes from the new version of FindAny_k (Algorithm 2). The complete proof for the theorem is given in Appendix C.

□

Unfortunately, the improvements done on the initial cases of some of the subroutines are not sufficient to get a significant improvement for the quantum query complexity of $\text{Dyck}_{k,n}$ algorithm. In order to improve more the query complexity, an other algorithm using a different strategy should be found.

3.2 A new algorithm for $\text{Dyck}_{2,n}$

First, we would like to find an algorithm with a quantum query complexity near to match the lower bound, $\exists c > 1$ such that $Q(\text{Dyck}_{k,n}) = \Omega\left(\sqrt{n}c^k\right)$, describes by Andris Ambainis' team in [4]. So the searched algorithm must have a quantum query complexity of $O(\sqrt{n})$.

For $k = 1$, the query complexity comes only from a call to Grover's search because rejecting is easily by finding a 00 or a 11 substrings inside the entry. For $k = 2$ it no more possible as the substrings that reject are of the form 00(10)*0 or of the form 11(01)*1. It implies that the number of calls to Grover's search in the naive approach is in $O(n)$ so the quantum query complexity finally becomes $O(n\sqrt{n})$. In order to keep it in $O(\sqrt{n})$, the algorithm must do a constant number of calls to Grover's search.

For that, we define a new alphabet that can express every even length binary strings and that have convenient property for a Grover's search. Let $\mathcal{A} = \{a,b,c,d\}$ the alphabet where a corresponds to 00, b to 11, c to 01, and d to 10. So every string of size 2 has its letter in $\mathcal{A}$ thus every even length bit string is expressed in $\mathcal{A}^*$. This alphabet allow us to prove the following theorem.

Theorem 3.3. ***Substrings rejection for Dyck word of height at most 2.*** *A word on the alphabet $\mathcal{A}$ embodies a Dyck word of height at most 2 if and only if it does not contain $aa, ac, bb, bd, cb, cd, da,$ or dc as substrings.*

Proof Theorem 3.3. First, this alphabet $\mathcal{A}$ is important because each letter has a height variation in $\{-2, 0, 2\}$. Indeed, a has a 2 height variation, b a -2, c a 0, and d a 0. This means that after each letter in a word, the current height will be even. Moreover, for a valid Dyck word of height at most 2, after every letter the height will be 0 or 2 which are respectively the lower and upper bound for the height. It means that no letter can cross a border after its first bit.

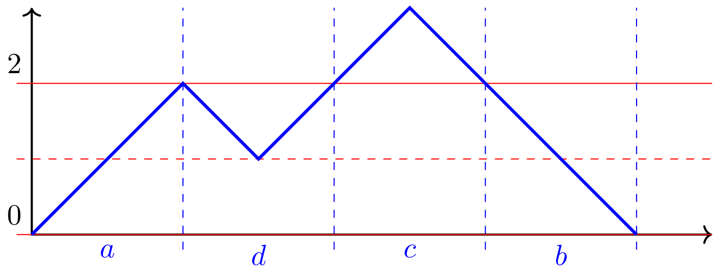


Figure 6: Illustration of the letters of $\mathcal{A}$ using Dyck's representation.

This property is important as it implies that every ± 3 strings uses at least two letters. So by checking if a pair of letter as a substring of a word make it not a Dyck word, $\mathcal{A}^2$ can be split into two sets described in Table 1.

Table 1: Partition of $\mathcal{A}$ into $\mathcal{X}, \mathcal{V}$.

$\mathcal{X}$	*aa ac bb bd cb cd da dc*
$\mathcal{V}$	*ab ad ba bc ca cc db dd*

- The set $\mathcal{X}$. First, every couple of letter which contains a ± 3 strings is in $\mathcal{X}$. This first condition explains the belonging of $aa, ac, dc, da, cb, bb, bd,$ and cd. Next, cd and dc belong to $\mathcal{X}$ because of the following property: For any valid Dyck word of height at most 2, the current height is bounded between 0 and 2, moreover after each letter the current height is even so both couple cd and dc start and finish on the same bound. Futhermore, cd and dc are going above and below the height at which they start so both are going outside off the bounds, thus a word which contains cd or dc can not be a Dyck Word of height a most 2. The Figure 7 shows each couple of $\mathcal{X}$.

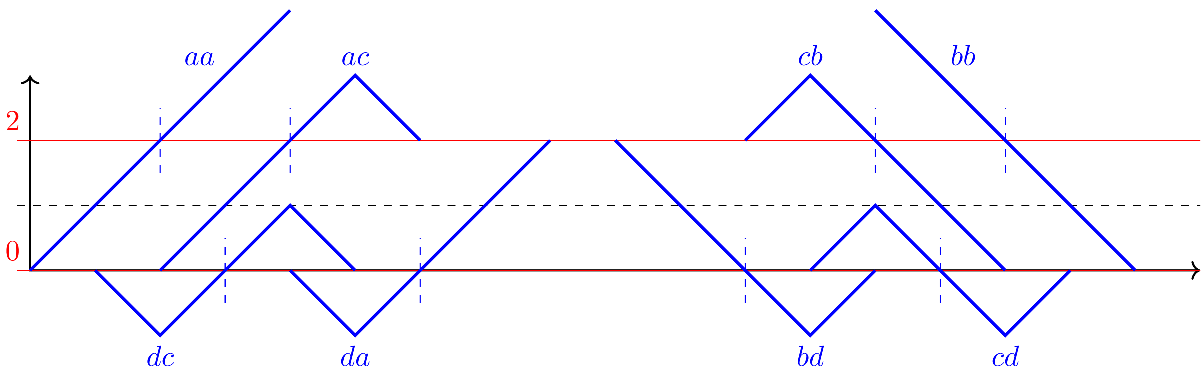


Figure 7: Every 2 letters configuration that implies the word, whom the configuration is a substring, is not a Dyck word of height at most 2.

- The set $\mathcal{V}$. The couples of $\mathcal{A}$ do not imply that the word is not a Dyck word of height at most 2 because each couple can fit inside the height bounds. The Figure 8 shows that every couple not in $\mathcal{X}$ (ie. $ab, ad, ba, bc, ca, cc, db, dd$) fit between height 0 and 2.

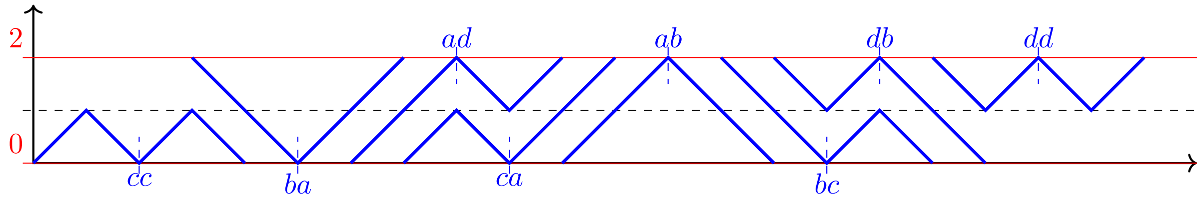


Figure 8: Every 2 letters configuration that can be found in a valid Dyck word of height at most 2.

So a word, whose letter representation has a substring in $\mathcal{X}$, cannot be a Dyck word of height two. But does every non Dyck word of height at most 2 have a substring in $\mathcal{X}$?

A word is not a dyck word of height at most 2 if it include a ± 3 strings. But how are represented ± 3 strings using the letters? There are 8 different cases which are 2 by 2 symmetrical so Figure 9 and Figure 10 show only the cases for $+3$ strings. In Figure 9, every $+3$ string of size 3 is include in aa or ac so it is sufficient to search for this two couple. In Figure 10 every $+3$ strings of length greater than 3 are composed of 2 minimal $+2$ strings. This implies that one must be a a while the other must be da of dc. Because da or dc are rejecting substrings, it is sufficient to search for them.

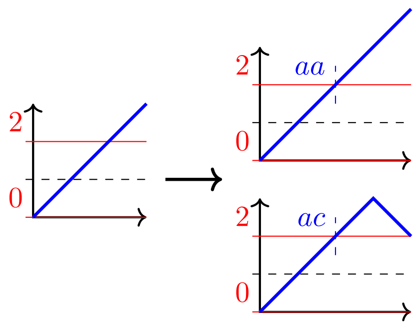


Figure 9: Configuration for a $+3$ strings of size 3.

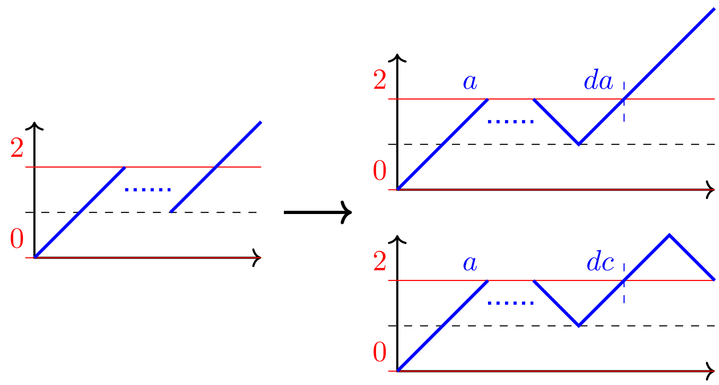


Figure 10: Configurations for a $+3$ string of size greater than 3.

□

Finally, a word on the alphabet $\mathcal{A}$ embodies a Dyck word of height at most 2 if and only if it does not contain $aa, ac, bb, bd, cb, cd, da, dc$ as substrings. The following Algorithm 3 for $\text{Dyck}_{2,n}$ comes from the direct application of the theorem.

Theorem 3.4. ***The quantum query complexity of*** **DyckFast$_{2,n}$.** *The* $\text{DyckFast}_{2,n}$ *algorithm has a quantum query complexity of* $O\left(\sqrt{n}\right)$.

Proof Theorem 3.4. The algorithm is doing at most 8 Grover's search on the modified input string $11x00$. So the total quantum query complexity is the folling.

$$Q(\text{DyckFast}_{2,n}) = 8 \times O\left(\sqrt{n+4}\right) = O\left(\sqrt{n}\right)$$

□

Algorithm 3 $\textsc{DyckFast}_{2,n}$

Require: $n \geq 0$, x such that $|x| = 2n$
  $x \leftarrow 11x00$
  $\texttt{t} \leftarrow \textsc{Null}$
  for $\texttt{reject\_symbol} \in \{aa, ac, bb, bd, cb, cd, da, dc\}$ **do**
    if $t == \textsc{Null}$ **then**
      Find $\texttt{t}$ in $[\![0, n]\!]$ such that
      $x[2\texttt{t} + 1, \ldots, 2\texttt{t} + 4] = \texttt{reject\_symbol}$
  return $\texttt{t} == \textsc{Null}$

3.3 A simplification for $\textbf{Dyck}_{2,n}$ algorithm

The algorithm describes in subsection 3.2 is complex to explain but it turns out that a simpler version of it exists and is way easier to understand that the first one.

Algorithm 4 $\textsc{DyckFastEasy}_{2,n}$

Require: $n \geq 0$, x such that $|x| = 2n$
  $x \leftarrow 11x00$
  Find $\texttt{t}$ in $[\![0, n]\!]$ such that
      $x[2\texttt{t}, 2\texttt{t} + 1] \in \{00, 11\}$
  return $\texttt{t} == \textsc{Null}$

Lets explain how it works. A Dyck word of height at most 2 can be recognized by the following grammar on the alphabet 0 and 1:

$$S \mapsto 0I1$$
$$I \mapsto 01I|10I|\varepsilon$$

Indeed, a Dyck word of height at most 2 $w_1 \ldots w_n$ can only start with a zero else it gets a negative height. After, if it goes up, it reach the height 2 so it can only go down and if it goes down, it reach height 0 so it can only go up. This implies that after every letter of odd indices, the height is equal to one, and that the possible next two letters can only be in $\{01, 10\}$. So, finding a $\{00, 11\}$ starting on an even index is equivalent to not being a dyck word of height at most two. Finally, using two Grover searches for 00 and 11 starting on an even index is now sufficient to recognize $\textsc{Dyck}_{2,n}$.

3.4 A final improvement to $\textbf{Dyck}_k$ algorithm

The last algorithm describes is useful to recognize $\textsc{Dyck}_2$. However, it cannot be plug into the huge one because it does not return the ± 3-string that allow him to reject. Indeed, the algorithm only find the second ± 2-string of the ± 3-string, so in order to reduce the complexity of the main algorithm it is required to find the first ± 2-strings. To do that, it is sufficient to made a call to the already existing subroutines $\textsc{FindFirst}_2$ described in 9 by asking for the closest ± 2-string on the left of the already known $\pm 2-$string. This is done with a quantum query complexity of $O(\sqrt{n})$ (Computation is similar as the one done in the proof in Appendix C). Finally, the algorithm for $\textsc{Dyck}_2$ now return the ± 3-string when it rejects. So, it can be plug into the main algorithm, the computation of the quantum query complexity are almost identical to the one done in Appendix C but instead of having a initial case for $k = 2$ it now for $k = 3$. The consequence is the removing of a recursion step which implies the loss of a $\sqrt{\log_2(n)}$ which bring the quantum query complexity of the algorithm that recognizing $\textsc{Dyck}_k$ to $O(\sqrt{n}(\log_2(n))^{0.5(k-2)})$.

4 Multiple tries to improve the quantum query complexity upper and lower bounds

4.1 A try to expand the new algorithm's first version to every k.

The second version of the algorithm is a lot more simple than the previous one, but it doesn't make the first version of the algorithm obsolete. Indeed, the first algorithm was think with the goal to be easily modified to work with higher value of k. Unfortunately, the key property that make the algorithm works (i.e. after each letter with an even index, the height is equal to 0 or 2, both being the bounds for the height) no more hold. So I didn't succeed to find a way to generalized to greater value of k.

4.2 A try for a new algorithm for any k.

In order to reduce the quantum query complexity of $\textsc{Dyck}_k$, one idea was to reduce the number of recursive calls during the execution. For that, two different possibilities seemed possible, both were using the same main principle: The recursive calls could be associated to nested loops, whose variables would correspond to the choice of d for each recursion level. Indeed, in every recursion level, $\textsc{FindAny}_k$ is called with l and r such that $r - l \leq d_{k+1}$ with d_{k+1} the value give to d in the call of $\textsc{FindAny}_{k+1}$. It is in reality more complicated but the goal is just to check if, first it is possible to compute the quantum query complexity of this model, and after to translate the method to get a better algorithm to recognize $\textsc{Dyck}_k$.

A natural graph structure on this (k-2)-tuples. The initial cases of the recursive algorithm are for k equal to 2, 3, so there is only $k-2$ nested loops, thus d_i associated variable, one for each recursive level. This $k-2$ variables can be consider together as $k-2$-tuples. In every tuple, the following inequality is verified $d_{k+1} \geq d_k \geq \ldots \geq d_4$. Each tuple is a vertex in a graph (V, E). Edge E are defined as following.

$$(t_u, t_v) \in E \Leftrightarrow \exists! i \in \{1, k-2\}, ((t_u)_i \neq (t_v)_i \wedge (t_v)_i = 2(t_u)_i)$$

A vertex of the graph is said to me marked if and only if

> it exists a $\pm(k+1)$-strings of length at most $(t_u)_1$ composed of 2 $\pm k$-strings of size at most $(t_u)_2$ composed them selves of 2 $\pm(k-1)$-strings of size at most $(t_u)_3$, ... and composed themselves of 2 ± 3-strings of size at most $(t_u)_k - 2$.

This graph has an interesting property $\mathcal{P}$ for every vertex t_u. If t_u is marked then every descendent t_v of t_u in the graph is also marked. Without using this property, the $k-2$ nested loops are equivalent to a DFS that search for a marked vertex. Then the quantum query complexity of this algorithm would be the one of the quantum DFS multiplied by the one to check a marked vertex. This give us a quantum query complexity in $O\left(\sqrt{\log_2(n)^{k-2}} \times Q(\text{mark checking})\right)$. The quantum query complexity of checking if a vertex is marked depends a lot on what is allowed to do and the final quantum query complexity required. For the nested loops, the goal is to have an algorithm that improve the quantum query complexity of $O\left(\sqrt{n}\log_2(n)^{0.5(k-2)}\right)$. Thus, the quantum query complexity of checking a vertex should be at most $O(\sqrt{n})$. What does it mean to check a vertex? This is difficult to define as the current model of nested loop is quite far from the original algorithm. But, one can do some hypothesis and state that as for the original algorithm, it is only possible to check easily $O(\sqrt{n})$ a second type of marks define as:

> it exists a $\pm(k+1)$-strings of length between $(t_u)_1/2$ and $(t_u)_1$ composed of 2 $\pm k$-strings of size between $(t_u)_2/2$ and $(t_u)_2$ composed them selves of 2 $\pm(k-1)$-strings of size between $(t_u)_3/2$ and $(t_u)_3$, ... and composed themselves of 2 ± 3-strings of size between $(t_u)_{k-2}$ and $(t_u)_{k-2}$.

With this statement, it is not possible to use an algorithm whose probability to check a node can be zero else as for the original one, it is not possible to be sure of the answer. The only solutions are graph traversal that have a continuous border between explored and non explored vertices (DFS or BFS for example). So, in order to go around this constraint, it may be possible to use slower algorithm to check the first type of mark if it implies a non negligible improvement on the quantum query complexity of the exploration as it is not really clear what would be the quantum query complexity of checking a mark.

In order to present the the main way to reduce the cost of the traversal, it is useful to order the vertex of the graph onto a discrete finite space of dimension $k-2$ were every axis has a logarithmic scale (base 2) starting from 1 to $2^{\lfloor \log_2(n) \rfloor}$.

In this space, a vertex t_u has its coordinate on axis i equal to $(t_u)_i$. The goal here is to do a binary search on each dimension of the $k-2$-tuple, in order to find a marked vertex. In fact, the property $\mathcal{P}$ implies that if a vertex t_u is marked then every vertex t_v is marked if t_u and t_v differ only on the j-th dimension and $(t_u)_j \leq (t_v)_j$, t_v. So on every of the $k-2$ dimensions, it is possible to do a binary search because it exists a value, depending on the value of all the first dimension such that no vertex is marked before and all vertex are marked after. Thus, the quantum query complexity of this multidimensional binary search would be

$$O\left(\sqrt{\log_2(\log_2(n))}^{k-2} \times Q(\text{marking checking})\right).$$

The improvement is quite spectacular if it is possible to find an algorithm that check the mark of a vertex with a quantum query complexity that does not counter balance. With more time, it would have been possible to define more precisely what it means to check a mark and may be an algorithm could have been found. It may also be possible to prove that such an algorithm cannot exits.

4.3 A new adversary plus minus for Dyck$_{k,n}$

The idea was first to search for a basic adversary (Theorem 2.3) for small value of k. The efforts were focus on $k = 2$. The goal was to find two sets of word from $\{0,1\}^n$ that satisfy the requirement of the basic adversary. Moreover, this two sets should give a lower bound of the form $O(\sqrt{n}f(n))$ (for some function $f(n)$) in order to increase the current lower bound of $O(\sqrt{n}c^2)$ (for some constant c). This basic adversary returns lower bounds of the form $\sqrt{\frac{mm'}{ll'}}$ where m, m', l, and l' (depending on n) specify properties on both sets. I do not have found such sets, more precisely, it is already difficult to think about a way to define two sets such that their sizes increase nicely according to a function in n. So, having to deal in the same time with their associated values m, m', l, and l' made the task really difficult.

4.4 A new reduction from easier problems

In order to find a new lower bounds, one idea was to try new reduction. For $k = 2$, some languages have been tested but an optimal algorithm have been found. For $k = 3$, one of the most interesting language is defined by $c^*(ac^*bc^*)^*$. In order to reject a word w, it is sufficient to search if a substring v from bwa can be part of the language $ac^*a + bc^*b$ so this language is also a star free language (the same arguments than the one for Dyck$_k$ would work here). This means that its quantum query complexity is in $\tilde{\Theta}(\sqrt{n})$. Moreover, it is possible to reduce this problem to Dyck$_3$ simply by doing the rewritings $a \mapsto 00, b \mapsto 11$ and $c \mapsto 01$. So, finding the quantum query complexity of this problem can be useful in order to get a lower bound to the quantum query complexity of Dyck$_3$.

5 Conclusion

During the internship, the work I have led in order to get the final quantum query complexity of $\textsc{Dyck}_k$, didn't allow to merge both upper and lower bounds. However, the multiple approaches used to improve both bounds have been half successful. Indeed, every try to use new reductions or to compute new adversary lower bounds didn't succeed. Nonetheless, the upper bound have been improved two times, from $O(\sqrt{n}(\log_2(n))^{0.5k})$ to $O(\sqrt{n}(\log_2(n))^{0.5(k-1)})$ and finally to $O(\sqrt{n}(\log_2(n))^{0.5(k-2)})$, thanks to a small revision of the original algorithm and to a faster algorithm for $\textsc{Dyck}_2$.

For the future, there is still a lot of things to try. First, it may be possible to prove that the approach (detailed in subsection 4.2) to simplified the quantum query algorithm cannot work. Next, it could be possible that the next breakthrough for the quantum query complexity of $\textsc{Dyck}_k$ comes from a totally different quantum query algorithm. Finally, Andris team's members have found a potential candidate solution for the general adversary of $\textsc{Dyck}_k$, unfortunately they didn't succeed to compute its adversary bounds.

This internship has allowed me to experience the life in an other country and in a new laboratory. The discussions with different colleagues about their research and mine were really interesting and have helped me to understand more precisely the scope and the main problems of quantum computing.

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6 Appendix

List of Figures

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The frame of the intership

A Proof of the super-basic adversary method

As a little reminder here is the theorem.

Theorem A.1. *Let define YES the set equal to $f^{-1}(accepted)$ and NO the set equal to $f^{-1}(rejected)$. If it exist two subset $Y \subseteq YES$ and $Z \subseteq NO$ such that:*

- *For each y in Y, there are at least m strings z in Z with dist$(y,z) = 1$.*
- *For each z in Z, there are at least m' strings y in Y with dist$(y,z) = 1$.*

Then the quantum query complexity $Q(f)$ is in $\Omega(\sqrt{m \times m'})$

Understanding this proof is important as it will give the intuitions to understand the general adversary.

Proof Theorem A.1. First, let define $\mathcal{R} := \{(y,z)$ such that $y \in Y, z \in Z$, and $\text{dist}(y,z) = 1\}$. Now, it is necessary to redefine the progress notion such that $\mathcal{P}(t) := \sum_{(y,z)\in\mathcal{R}} |\langle\psi_y^t|\psi_z^t\rangle|$ where

$|\psi_x^t\rangle$ is the state of the algorithm after t query to the entry x. So, $\mathcal{P}(0) = |\mathcal{R}|$ and $\mathcal{P}(T) = |\frac{2}{3}\mathcal{R}|$. In order to prove the theorem, lets try to prove that for all t,

$$\mathcal{P}(t) - \mathcal{P}(t+1) \leq \frac{2}{\sqrt{mm'}}|\mathcal{R}|$$

as it is a correct value of d such that the lower bound is $O(\sqrt{mm'})$. Moreover, there are relations between size of the sets:

$$|\mathcal{R}| \geq m|Y|, |\mathcal{R}| \geq m'|Z| \implies |2\mathcal{R}| \geq m|Y| + m'|Z|.$$

Thus, instead of proving $\mathcal{P}(t) - \mathcal{P}(t+1) \leq \frac{2}{\sqrt{mm'}}|\mathcal{R}|$, lets try to prove

$$\mathcal{P}(t) - \mathcal{P}(t+1) \leq \frac{1}{\sqrt{mm'}}(m|Y| + m'|Z|) = \sqrt{\frac{m}{m'}}|Y| + \sqrt{\frac{m'}{m}}|Z|.$$

Now, lets take any a couple (y, z) in $\mathcal{R}$, it exists a unique i^* such that $y_{i^*} \neq z_{i^*}$. This small variation between y and z can be well translated to the progress after quantum query gate. Lets write $|\psi_y^t\rangle$ and $|\psi_z^t\rangle$:

$$|\psi_y^t\rangle = \sum_{\substack{i\in[\![0,N]\!]\\ j\in[\![1,d_i]\!]}} \alpha_{i,j}|i,j\rangle \qquad |\psi_z^t\rangle = \sum_{\substack{i\in[\![0,N]\!]\\ j\in[\![1,d_i]\!]}} \beta_{i,j}|i,j\rangle.$$

Now by applying Q_t, the states become

$$|\psi_y^t\rangle = \sum_{\substack{i\in[\![0,N]\!]\\ j\in[\![1,d_i]\!]}} (-1)^{(0y)_{i+1}}\alpha_{i,j}|i,j\rangle \qquad |\psi_z^t\rangle = \sum_{\substack{i\in[\![0,N]\!]\\ j\in[\![1,d_i]\!]}} (-1)^{(0z)_{i+1}}\beta_{i,j}|i,j\rangle.$$

However, y and z only differ on the index i^*, so the restricted progress measure to y and z at t and $t+1$ are equal to

$$\mathcal{P}_{y,z}(t) = \sum_{\substack{i\in[\![0,N]\!]\\ j\in[\![1,d_i]\!]}} \overline{\alpha_{i,j}}\beta_{i,j} \qquad \begin{aligned}\mathcal{P}_{y,z}(t+1) &= \sum_{\substack{i\in[\![0,N]\!]\\ j\in[\![1,d_i]\!]}} (-1)^{(0y)_{i+1}+(0z)_{i+1}}\overline{\alpha_{i,j}}\beta_{i,j}\\ &= \sum_{\substack{i\in[\![0,N]\!]\\ j\in[\![1,d_i]\!]}} \overline{\alpha_{i,j}}\beta_{i,j} - 2\sum_{j\in[\![1,d_{i^*}]\!]} \overline{\alpha_{i^*,j}}\beta_{i^*,j}\\ &= \mathcal{P}_{y,z}(t) - 2\sum_{j\in[\![1,d_{i^*}]\!]} \overline{\alpha_{i^*,j}}\beta_{i^*,j}.\end{aligned}$$

So, for restricted progress measure there is the inequality

$$\begin{aligned}|\mathcal{P}_{y,z}(t)| - |\mathcal{P}_{y,z}(t+1)| &\leq |\mathcal{P}_{y,z}(t) - \mathcal{P}_{y,z}(t+1)|\\ &\leq |2\sum_{j\in[\![1,d_{i^*}]\!]} \overline{\alpha_{i^*,j}}\beta_{i^*,j}|\\ &\leq 2\sum_{j\in[\![1,d_{i^*}]\!]} |\alpha_{i^*,j}||\beta_{i^*,j}|.\end{aligned}$$

Moreover, the math trick for all p, q real numbers and h non negative real number $2pq \leq hp^2 + \frac{1}{h}q^2$ allow to rewrite the previous inequality to

$$\begin{aligned}|\mathcal{P}_{y,z}(t)| - |\mathcal{P}_{y,z}(t+1)| &\leq 2\sum_{j\in[\![1,d_{i^*}]\!]} |\alpha_{i^*,j}||\beta_{i^*,j}|\\ &\leq \sum_{j\in[\![1,d_{i^*}]\!]} h|\alpha_{i^*,j}|^2 + \frac{1}{h}|\beta_{i^*,j}|^2.\end{aligned}$$

By chosing h equal to $\sqrt{\frac{m}{m'}}$, it becomes

$$|\mathcal{P}_{y,z}(t)| - |\mathcal{P}_{y,z}(t+1)| \leq \sum_{j\in[\![1,d_{i^*}]\!]} \sqrt{\frac{m}{m'}}|\alpha_{i^*,j}|^2 + \sqrt{\frac{m'}{m}}|\beta_{i^*,j}|^2.$$

Now, it is time to sum on every couple (y,z) of $\mathcal{R}$. But i^* is depending on y, z as much as $\alpha_{i,j}$s and $\beta_{i,j}$s so lets give them y and z as argument

$$\begin{aligned}\mathcal{P}(t) - \mathcal{P}(t+1) &= \sum_{(y,z)\in\mathcal{R}} \left|\mathcal{P}_{y,z}(t)\right| - \sum_{(y,z)\in\mathcal{R}} \left|\mathcal{P}_{y,z}(t+1)\right| \\ &\leq \sum_{\substack{(y,z)\in\mathcal{R} \\ j\in[\![1,d_{i^*_{y,z}}]\!]}} \sqrt{\frac{m}{m'}}|\alpha^y_{i^*_{y,z},j}|^2 + \sqrt{\frac{m'}{m}}|\beta^z_{i^*_{y,z},j}|^2.\end{aligned}$$

Now, it is useful to do the first summation in two part in order to find a upped bound to the sum

$$\begin{aligned}\mathcal{P}(t) - \mathcal{P}(t+1) \leq &\sqrt{\frac{m}{m'}} \sum_{\substack{y \ st \ \exists z \\ (y,z)\in\mathcal{R}}} \sum_{\substack{z \ \text{width} \\ (y,z)\in\mathcal{R}}} \sum_{j\in[\![1,d_{i^*_{y,z}}]\!]} |\alpha^y_{i^*_{y,z},j}|^2 \\ &+ \sqrt{\frac{m'}{m}} \sum_{\substack{z \ st \ \exists y \\ (y,z)\in\mathcal{R}}} \sum_{\substack{y \ \text{width} \\ (y,z)\in\mathcal{R}}} \sum_{j\in[\![1,d_{i^*_{y,z}}]\!]} |\beta^z_{i^*_{y,z},j}|^2.\end{aligned}$$

Now, at y fixed, $z \mapsto i^*_{y,z}$ is an injective function from $\{z, (y,z) \in \mathcal{R}\}$ to $[\![1,N]\!]$ as y and z are at distance 1. Thus $(z,j) \mapsto (i^*_{y,z}, j)$ is also an injective function from $\{(z,j),(y,z) \in \mathcal{R}$ and $j \in [\![1, d_{i^*_{y,z}}]\!]\}$ to $[\![1,N]\!] \times [\![1, d_{i^*_{y,z}}]\!]$. So, it implies that

$$\bigsqcup_{\substack{z \ \text{width} \\ (y,z)\in\mathcal{R}}} \{i^*_{y,z}\} \times [\![1, d_{i^*_{y,z}}]\!] \subseteq \bigsqcup_{n\in[\![0,N]\!]} \{n\} \times [\![1, d_n]\!]$$

and finally that

$$\sum_{\substack{z \ \text{width} \\ (y,z)\in\mathcal{R}}} \sum_{j\in[\![1,d_{i^*_{y,z}}]\!]} |\alpha^y_{i^*_{y,z},j}|^2 \leq \sum_{i\in[\![0,N]\!]} \sum_{j\in[\![1,d_i]\!]} |\alpha^y_{i,j}|^2 \leq \left\langle \psi^t_y | \psi^t_y \right\rangle \leq 1.$$

By symmetry it also works for the second sum, so the difference of progress is now the following

$$\begin{aligned}
\mathcal{P}(t) - \mathcal{P}(t+1) \leq & \sqrt{\frac{m}{m'}} \sum_{\substack{y \ st \ \exists z \\ (y,z)\in\mathcal{R}}} \sum_{\substack{z \text{ width} \\ (y,z)\in\mathcal{R}}} \sum_{j\in[\![1,d_{i^*_{y,z}}]\!]} |\alpha^y_{i^*_{y,z},j}|^2 \\
& + \sqrt{\frac{m'}{m}} \sum_{\substack{z \ st \ \exists y \\ (y,z)\in\mathcal{R}}} \sum_{\substack{y \text{ width} \\ (y,z)\in\mathcal{R}}} \sum_{j\in[\![1,d_{i^*_{y,z}}]\!]} |\beta^z_{i^*_{y,z},j}|^2 \\
\leq & \sqrt{\frac{m}{m'}} \sum_{\substack{y \ st \ \exists z \\ (y,z)\in\mathcal{R}}} 1 + \sqrt{\frac{m'}{m}} \sum_{\substack{z \ st \ \exists y \\ (y,z)\in\mathcal{R}}} 1 \\
\leq & \sqrt{\frac{m}{m'}} |Y| + \sqrt{\frac{m'}{m}} |Z| \\
\leq & \frac{2}{\sqrt{mm'}} |\mathcal{R}|.
\end{aligned}$$

To conclude, this give a lower bound $\Omega(\sqrt{mm'})$ for the quantum query complexity of f.

$\square$

B The algorithm for $\mathbf{Dyck}_{k,n}$

All the subroutines' pseudo code can be found from ALGORITHM 5 to ALGORITHM 10.

Algorithm 5 $\textsc{Dyck}_{k,n}$

Require: $n \geq 0$ and $k \geq 1$
Ensure: $|x| = n$
  $x \leftarrow 1^k x 0^k$
  $v \leftarrow \textsc{FindAny}_{k+1}(0, n + 2 * k - 1, \{1, -1\})$
  return v = $\textsc{Null}$

Algorithm 6 $\textsc{FindAny}_k(l, r, s)$

Require: $0 \leq l < r$ and $s \subseteq \{1, -1\}$
  Find d in $\{2^{\lceil \log_2(k) \rceil}, 2^{\lceil \log_2(k)+1 \rceil}, \dots, 2^{\lceil \log_2(r-l) \rceil}\}$ such that
    $v_d \leftarrow \textsc{FindFixedLength}_k(l, r, d, s)$ is **not** $\textsc{Null}$
  return v_d or $\textsc{Null}$ if none

Algorithm 7 $\textsc{FindFixedLength}_k(l, r, d, s)$

Require: $0 \leq l < r$, $1 \leq d \leq r - l$ and $s \subseteq \{1, -1\}$
  Find t in $\{l, l + 1, \dots, r\}$ such that
    $v_t \leftarrow \textsc{FindAtLeftMost}_k(l, r, t, d, s)$ is **not** $\textsc{Null}$
  return v_t of $\textsc{Null}$if none

Algorithm 8 $\textsc{FindAtLeftMost}_k(l, r, d, t, s)$

Require: $0 \leq l < r$, $l \leq r \leq r$, $1 \leq d \leq r - l$ and $s \subseteq \{1, -1\}$
  $v = (i_1, j_1, \sigma_1) \leftarrow \textsc{FindAtLeftMost}_{k-1}(l, r, t, d - 1, \{1, -1\})$
  if $v \neq \textsc{Null}$ **then**
    $v' = (i_2, j_2, \sigma_2) \leftarrow \textsc{FindAtRightMost}_{k-1}(l, r, i_1 - 1, d - 1, \{1, -1\})$
    if $v' = \textsc{Null}$ **then**
      $v' = (i_2, j_2, \sigma_2) \leftarrow \textsc{FindFirst}_{k-1}(\max(l, j_1 - d + 1), i_1 - 1, \{1, -1\}, left)$
    if $v' \neq \textsc{Null}$ and $\sigma_2 \neq \sigma_1$ **then** $v' \leftarrow \textsc{Null}$
    if $v' = \textsc{Null}$ **then**
      $v' = (i_2, j_2, \sigma_2) \leftarrow \textsc{FindAtLeftMost}_{k-1}(l, r, j_1 + 1, d - 1, \{1, -1\})$
    if $v' = \textsc{Null}$ **then**
      $v' = (i_2, j_2, \sigma_2) \leftarrow \textsc{FindFirst}_{k-1}(j_1 + 1, \max(r, i_1 + d - 1), \{1, -1\}, right)$
    if $v' = \textsc{Null}$ **then return** $\textsc{Null}$
  else
    $v = (i_1, j_1, \sigma_1) \leftarrow \textsc{FindFirst}_{k-1}(t, min(t + d - 1, r), \{1, -1\}, right)$
    if v = $\textsc{Null}$ **then return** $\textsc{Null}$
    $v' = (i_2, j_2, \sigma_2) \leftarrow \textsc{FindFirst}_{k-1}(max(t - d + 1, l), t, \{1, -1\}, left)$
    if $v' = \textsc{Null}$ **then return** $\textsc{Null}$
  if $\sigma_1 = \sigma_2$ and $\sigma_1 \in s$ and $\max(j_1, j_2) - \min(i_1, i_2) + 1 \leq d$ **then**
    return $(\min(i_1, i_2), \max(j_1, j_2), \sigma_1)$
  else return $\textsc{Null}$

Algorithm 9 $\textsc{FindFirst}_k(l, r, s, left)$

Require: $0 \le l < r$ and $s \subseteq \{1, -1\}$
  $lBorder \leftarrow l, rBorder \leftarrow r, d \leftarrow 1$
  while $lBorder + 1 < rBorder$ **do**
    $mid \leftarrow \lfloor (lBorder + rBorder)/2 \rfloor$
    $v_l \leftarrow \textsc{FindAny}_k(lBorder, mid, s)$
    if $v_l \ne \textsc{Null}$ **then** $rBorder \leftarrow mid$
    else
      $v_{mid} \leftarrow \textsc{FindFixedPos}_k(lBorder, rBorder, mid, s, left)$
      if $v_{mid} \ne \textsc{Null}$ **then return** v_{mid}
      else $lBorder \leftarrow mid + 1$
    $d \leftarrow d + 1$
  return $\textsc{Null}$

Algorithm 10 $\textsc{FindFixedPos}_k(l, r, t, s)$

Require: $0 \le l < r$, $l \le t \le r$ and $s \subseteq \{1, -1\}$
  Find d in $\{2^{\lceil \log_2(k) \rceil}, 2^{\lceil \log_2(k)+1 \rceil}, \dots, 2^{\lceil \log_2(r-l) \rceil}\}$ such that
    $v_d \leftarrow \textsc{FindAtLeftMost}_k(l, r, t, d, s)$ is **not** $\textsc{Null}$
  return v_d or $\textsc{Null}$ if none

C The proof of the quantum query complexity for $\mathbf{Dyck}_{k,n}$ algorithm's subroutines

Theorem C.1. $\mathbf{Dyck}_{k,n}$*'s Subroutines complexity* *The subroutines' quantum query complexity for k are the following.*

1. $Q(\textsc{Dyck}_{k,n}) = O\left(\sqrt{n}(\log_2(n))^{0.5(k-1)}\right)$ *for* $k \ge 1$
2. $Q(\textsc{FindAny}_{k+1}(l, r, s)) = O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-1)}\right)$ *for* $k \ge 1$
3. $Q(\textsc{FindFixedLength}_{k+1}(l, r, d, s)) = O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-2)}\right)$ *for* $k \ge 2$
4. $Q(\textsc{FindAtLeftMost}_{k+1}(l, r, t, d, s)) = \begin{cases} O\left(\sqrt{d}(\log_2(d))^{0.5(k-2)}\right) & \text{for } k \ge 2 \\ O(1) & \text{for } k = 1 \end{cases}$
5. $Q(\textsc{FindFirst}_k(l, r, s, left)) = O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-2)}\right)$ *for* $k \ge 2$
6. $Q(\textsc{FindFixedPos}_k(l, r, t, s)) = \begin{cases} O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-2)}\right) & \text{for } k \ge 3 \\ O(1) & \text{for } k = 2 \end{cases}$

Proof Theorem C.1. The proof is done by induction on the height k of the Dyck word.

Initialization: For $k = 1$ and $k = 2$ we have the following initialization.

- For $k = 1$, only $\textsc{FindAtLeftMost}_2$, $\textsc{FindAny}_2$,and $\textsc{Dyck}_{1,n}$ are defined. The $O(1)$ quantum query complexity of $\textsc{FindAtLeftMost}_2$ comes directly from the definition of its initial case, as the $O(\sqrt{r-l})$ quantum query complexity of $\textsc{FindAny}_2$. Then the $O(\sqrt{n})$ quantum query complexity of $\textsc{Dyck}_{1,n}$ comes from the call to $\textsc{FindAny}_2$.

- For $k = 2$, the inductive part of the algorithm start and every subroutines is defined. The $O(1)$ quantum query complexity of $\textsc{FindFixedPos}_2$ comes from the call to $\textsc{FindAtLeftMost}_2$. The $O\left(\sqrt{r-l}\right)$ quantum query complexity of $\textsc{FindFirst}_2$ comes from the dichotomize search using $\textsc{FindAny}_2$ and $\textsc{FindFixedPos}_2$ because $\sum_{u=1}^{log_2(r-l)} 2u\left(O\left(\sqrt{\frac{r-l}{2^{u-1}}}\right)+O(1)\right) = O(\sqrt{r-l})$ (Detailed in the induction). The $O(\sqrt{d})$ quantum query complexity of $\textsc{FindAtLeftMost}_3$ comes from the constant amount of calls to $\textsc{FindFirst}_2$ and $\textsc{FindAtLeftMost}_2$ with entry of size d. The $O(\sqrt{r-l})$ quantum query complexity of $\textsc{FindFixedLength}_3$ comes from the $O\left(\sqrt{\frac{r-l}{d}}\right)$ calls to $\textsc{FindAtLeftMost}_3$. The $O\left(\sqrt{(r-l)\log_2(r-l)}\right)$ quantum query complexity of $\textsc{FindAny}_3$ comes from the $O\left(\sqrt{\log_2(r-l)}\right)$ calls to $\textsc{FindFixedLength}_3$. Finally, the $O\left(\sqrt{(r-l)\log_2(r-l)}\right)$ quantum query complexity of $\textsc{Dyck}_2$ comes from the call to $\textsc{FindAny}_3$.

Induction: Let suppose it exists k such that Theorem C.1 is true for k. Let prove that it is true for $k+1$.

First, the $O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-1)}\right)$ quantum query complexity of $\textsc{FindFixedPos}_{k+1}$ comes from the $O\left(\sqrt{\log(r-l)}\right)$ calls to $\textsc{FindAtLeftMost}_{k+1}$.

$$\begin{aligned}
Q(\textsc{FindFixedPos}_{k+1}(l,r,t,s)) &= O(\sqrt{\log(r-l)}) \times O\left(Q(\textsc{FindAtLeftMost}_{k+1}(l,r,t,d,s))\right)\\
&\overset{IH}{=} O\left(\sqrt{\log(r-l)} \times \sqrt{r-l}(\log_2(r-l))^{0.5(k-2)}\right)\\
&= O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-1)}\right)
\end{aligned}$$

Thus the $O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-2)}\right)$ quantum query complexity of $\textsc{FindFirst}_{k+1}$ comes from the dichotomize search using calls to $\textsc{FindAny}_{k+1}$ and $\textsc{FindFirst}_{k+1}$.

$$\begin{aligned}
Q(\textsc{FindFirst}_{k+1}(l,r,t,d,s)) = &\ \textstyle\sum_{u=1}^{\log_2(r-l)} 2u \times O\left(Q(\textsc{FindAny}_{k+1}(0,\frac{r-l}{2^{u-1}},s))\right)\\
&+ \textstyle\sum_{u=1}^{\log_2(r-l)} 2u \times O\left(Q(\textsc{FindFixedPos}_{k+1}(0,\frac{r-l}{2^{u-1}},_,s,left))\right)\\
&\overset{IH}{=} O\left(\sum_{u=1}^{\log_2(r-l)} 2u \times \sqrt{\frac{r-l}{2^{u-1}}}(\log_2(\frac{r-l}{2^{u-1}}))^{0.5(k-1)}\right)\\
&= O\left(\sum_{u=1}^{\log_2(r-l)} 2u \times \sqrt{\frac{r-l}{2^{u-1}}}(\log_2(r-l))^{0.5(k-1)}\right)\\
&= O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-1)} \sum_{u=1}^{\log_2(r-l)} u \times (\frac{1}{\sqrt{2}})^{u-1}\right)\\
&=^{a} O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-1)} \frac{\sqrt{2}^2}{(\sqrt{2}-1)^2}\right)\\
&= O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-1)}\right)
\end{aligned}$$

---

[a] $\sum_{u=1}^{+\infty}\left(\frac{\mathrm{d}}{\mathrm{d}x}(x^u)\right)\left(\frac{1}{\sqrt{2}}\right) \leq \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{u=1}^{+\infty}x^u\right)\right)\left(\frac{1}{\sqrt{2}}\right) \leq \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{1-x}\right)\right)\left(\frac{1}{\sqrt{2}}\right) \leq \left(\frac{1}{(1-x)^2}\right)\left(\frac{1}{\sqrt{2}}\right) \leq \frac{1}{(1-\frac{1}{\sqrt{2}})^2} \leq \frac{\sqrt{2}^2}{(\sqrt{2}-1)^2}$

Next, the $O\left(\sqrt{d}(\log_2(d))^{0.5(k-1)}\right)$ quantum query complexity comes of $\textsc{FindAtLeftMost}_{k+2}$from the constant amount of calls to $\textsc{FindAtLeftMost}_{k+1}$, $\textsc{FindAtRightMost}_{k+1}$,and $\textsc{FindFirst}_{k+1}$.

$$
\begin{aligned}
Q(\textsc{FindAtLeftMost}_{k+2}(l,r,t,d,s)) = \;& \begin{array}{l} 3 \times O\left(Q(\textsc{FindAtLeftMost}_{k+1}(l,r,t,d,\{1,-1\}))\right) \\ +4 \times O\left(Q(\textsc{FindFirst}_{k+1}(l,r,\{1,-1\},left))\right) \end{array} \\
&\overset{IH}{=} O\left(\sqrt{d}(\log_2(d))^{0.5(k-1)}\right)
\end{aligned}
$$

After that, the $O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-1)}\right)$ quantum query complexity of $\textsc{FindFixedLength}_{k+2}$ comes from the $O\left(\sqrt{\frac{r-l}{d}}\right)$ calls to $\textsc{FindAtLeftMost}_{k+2}$.

$$
\begin{aligned}
Q(\textsc{FindFixedLength}_{k+2}(l,r,d,s)) &= O\left(\sqrt{\frac{r-l}{d}}\right) \times O\left(Q(\textsc{FindAtLeftMost}_{k+2}(l,r,t,d,s))\right) \\
&= O\left(\sqrt{\frac{r-l}{d}} \times \sqrt{d}(\log_2(d))^{0.5(k-1)}\right) \\
&= O\left(\sqrt{r-l}(\log_2(d))^{0.5(k-1)}\right) \\
&= O\left(\sqrt{r-l}(\log_2(r-l))^{0.5(k-1)}\right)
\end{aligned}
$$

Hence the $O\left(\sqrt{r-l}(\log_2(r-l))^{0.5k}\right)$ quantum query complexity of $\textsc{FindAny}_{k+2}$ comes from the $O\left(\sqrt{\log_2(r-l)}\right)$ calls to $\textsc{FindFixedLength}_{k+2}$.

$$
\begin{aligned}
Q(\textsc{FindAny}_{k+2}(l,r,s)) &= O\left(\sqrt{\log(r-l)}\right) \times O\left(Q(\textsc{FindFixedLength}_{k+2}(l,r,d,s))\right) \\
&= O\left(\sqrt{\log(r-l)} \times \sqrt{r-l}(\log_2(r-l))^{0.5(k-1)}\right) \\
&= O\left(\sqrt{r-l}(\log_2(r-l))^{0.5k}\right)
\end{aligned}
$$

Finally, the $O\left(\sqrt{n}(\log_2(n))^{0.5k}\right)$ quantum query complexity of $\textsc{Dyck}_{k+1,n}$ comes from the call to $\textsc{FindAny}_{k+2}$.

$$
\begin{aligned}
Q(\textsc{Dyck}_{k+1,n}) &= O\left(Q(\textsc{FindAny}_{k+2}(0,n+2k+1,s))\right) \\
&= O\left(Q(\textsc{FindAny}_{k+2}(0,n,s))\right) \\
&= O\left(\sqrt{n}(\log_2(n))^{0.5k}\right)
\end{aligned}
$$

Conclusion: By the induction principle we get that the Theorem C.1 is true for $k \in \mathbb{N}^*$

□