Graded Homework 9 (due Wednesday, May 6)

Exercise 1. Someone proposes you to play the following game: start with an initial amount of $S_0 > 0$ francs, of your choice. Then toss a coin: if it falls on heads, you win $S_0/2$ francs; while if it falls on tails, you lose $S_0/2$ francs. Call S_1 your amount after this first coin toss. Then the game goes on, so that your amount after coin toss number $n \geq 1$ is given by

$$S_n = \begin{cases} S_{n-1} + \frac{S_{n-1}}{2} & \text{if coin number } n \text{ falls on heads} \\ \\ S_{n-1} - \frac{S_{n-1}}{2} & \text{if coin number } n \text{ falls on tails} \end{cases}$$

We assume moreover that the coin tosses are independent and fair, i.e., with probability 1/2 to fall on each side. Nevertheless, you should *not* agree to play such a game: explain why!

Hints:

First, to ease the notation, define $X_n = +1$ if coin n falls on heads and $X_n = -1$ if coin n falls on tails. That way, the above recursive relation may be rewritten as $S_n = S_{n-1}\,(1 + \frac{X_n}{2})$ for $n \geq 1$.

a) Compute recursively $\mathbb{E}(S_n)$; if it were only for expectation, you could still consider playing such a game, but…

b) Define now $Y_n = \log(S_n/S_0)$, and use the central limit theorem to approximate $\mathbb{P}(\{Y_n > t\})$ for a fixed value of $t \in \mathbb{R}$ and a relatively large value of n. Argue from there why it is definitely not a good idea to play such a game! (computing for example an approximate value of $\mathbb{P}(\{S_{100} > S_0/10\})$)

Exercise 2. Let $\lambda > 0$ be fixed. For a given $n \geq \lceil \lambda \rceil$, let $X_1^{(n)}, \ldots, X_n^{(n)}$ be i.i.d. Bernoulli(λ/n) random variables and let $S_n = X_1^{(n)} + \ldots + X_n^{(n)}$.

a) Compute $\mathbb{E}(S_n)$ and $\mathrm{Var}(S_n)$ for a fixed value of $n \geq \lceil \lambda \rceil$.

b) Deduce the value of $\mu = \lim_{n\to\infty} \mathbb{E}(S_n)$ and $\sigma^2 = \lim_{n\to\infty} \mathrm{Var}(S_n)$.

c) Compute the limiting distribution of S_n (as $n \to \infty$).

Hint: Use characteristic functions. You might also have a look at tables of characteristic functions of some well known distributions in order to solve this exercise.

For a given $n \geq 1$, let now $Y_1^{(n)}, \ldots, Y_n^{(n)}$ be i.i.d. Bernoulli$(1/n)$ random variables and let

$$T_n = Y_1^{(n)} + \ldots + Y_{\lceil \lambda n \rceil}^{(n)}$$

where $\lambda > 0$ is the same as above.

d) Compute the limiting distribution of T_n (as $n \to \infty$).

e) Is it also the case that either S_n or T_n converge almost surely or in probability towards a limit? Justify your answer!

Exercise 3. (application of Lindeberg's principle to non i.i.d. random variables)
Let $(\sigma_n,\, n \geq 0)$ be a sequence of (strictly) positive numbers and $(X_n,\, n \geq 1)$ be a sequence of independent random variables such that $\mathbb{E}(X_n) = 0$, $\mathrm{Var}(X_n) = \sigma_n^2$ and $\mathbb{E}(|X_n|^3) \leq K\, \sigma_n^3$ for every $n \geq 1$ (note that the constant K is uniform over all values of n).

For $n \geq 1$, define also $V_n = \mathrm{Var}(X_1 + \ldots + X_n) = \sigma_1^2 + \ldots + \sigma_n^2$.

Using Lemma 9.12 in the lecture notes (equivalently, Lemma 2 in the video lecture 9.2b), find a sufficient condition on the sequence $(\sigma_n,\, n \geq 1)$ guaranteeing that

$$\frac{1}{\sqrt{V_n}}\,(X_1 + \ldots + X_n) \underset{n\to\infty}{\xrightarrow{d}} Z \sim \mathcal{N}(0,1)$$

Note: From the course, you already know a sufficient condition: $\sigma_n = 1$ for all $n \geq 1$, but this is too strong! The aim here is to find a sufficient condition which is most general possible.

b) Which of the following sequences $(\sigma_n,\, n \geq 1)$ satisfy the condition you have found in a)?

b1) $\sigma_n = n$ b2) $\sigma_n = \frac{1}{n}$ b3) $\sigma_n = 2^n$

Hint: The following might be useful: $$\sum_{j=1}^{n} j^\alpha = \begin{cases} \Theta(n^{\alpha+1}) & \text{if } \alpha > -1 \\ \Theta(\log(n)) & \text{if } \alpha = -1 \\ \Theta(1) & \text{if } \alpha < -1 \end{cases}$$

Exercise 4. Let $(X_n,\, n \geq 1)$ be a sequence of independent random variables such that

$$\mathbb{P}\left(\{X_n = +1/\sqrt{n}\}\right) = \mathbb{P}\left(\{X_n = -1/\sqrt{n}\}\right) = \frac{1}{2}$$

and let, for $n \geq 1$,

$$Y_n = X_1 + \ldots + X_n \quad \text{and} \quad Z_n = X_{n+1} + \ldots + X_{2n}$$

a) Run multiple times the process Y and draw a histogram of Y_n for $n = 100$, $n = 1'000$ and $n = 10'000$, respectively. Draw also the graphs of the empirical mean and standard deviation of Y_n as a function of n. Do you observe that the histogram of Y_n converges as n grows large (i.e., that Y_n converges in distribution)?

b) Same questions for the process Z.

c) In the case(s) where you observed convergence in distribution, prove that the sequence of random variables indeed converges to a limit, using characteristic functions. What is the limiting distribution?

Hint: You may use approximations here, as well as the following:

$$\sum_{j=n_1+1}^{n_2} j^\alpha \simeq \int_{n_1}^{n_2} dx\, x^\alpha$$

as n_2 gets large (and n_1 is either fixed or getting large also).