Seminar: Uncertainty Quantification and Approximation Theory of Neural Networks

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What is Uncertainty Quantification?

Uncertainty Quantification (UQ) is a currently active and hot topic in both classical and numerical analysis. In practice, the input of a fixed model, e.g. right-hand side or geometry, is uncertain, possibly due to inaccurate measurements. The main goal of UQ is to determine how the uncertainty of the input propagates through the model. As a standard model problem, one can think of finding u such that

$$-\nabla\cdot(a(x,\boldsymbol{y})\nabla u) = f \qquad x\in D, \tag{1}$$

where $\boldsymbol{y}\in[-1,1]^{\mathbb{N}}$ can be understood as a parametrization of the uncertainty. Equivalently, problem (1) can be seen as a stochastic partial differential equation (SPDE).

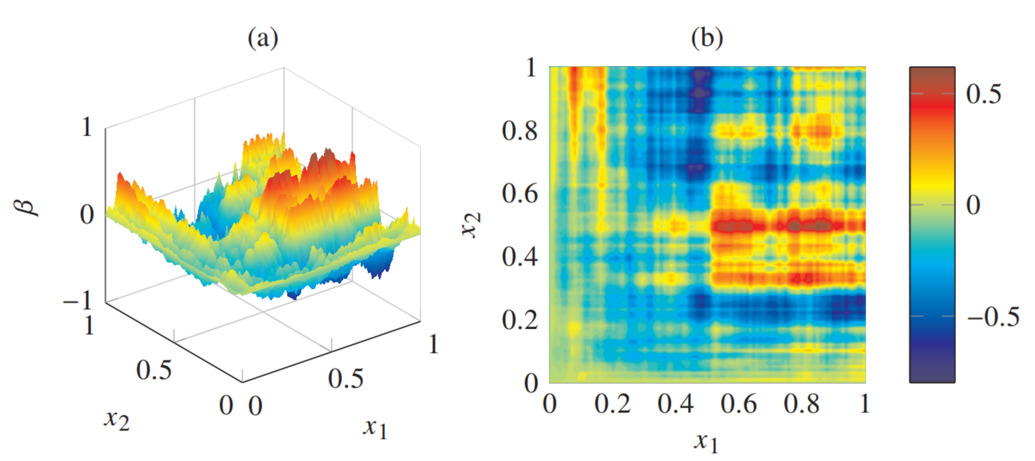
Numerical methods in UQ exploit the fact, that under certain regularity assumptions on $a(\cdot,\cdot)$ the parameter-to-solution map $\boldsymbol{y}\mapsto u(\cdot,\boldsymbol{y})$ admits certain regularity. Sometimes, however, one is only concerned with calculating a quantity of interest (QoI), i.e., approximating the parameter-to-QoI map $\boldsymbol{y}\mapsto G(u(\cdot,\boldsymbol{y}))$, for some linear functional G.

Neural Networks and UQ

In recent years there was and still is quite a *hype* surrounding terms like *artificial intelligence*, *machine learning* and *neural networks*. This seminar is mostly interested in neural networks (NNs) from an approximation theoretical point of view, while still keeping in mind that they belong to a much larger class of machine learning algorithms. Most generally they can be seen as universal function approximators.

Due to the expressive power of NNs, cutting-edge research is currently concerned with the question if solutions to certain problems arising in UQ and high dimensional analysis can be efficiently approximated using neural networks. While many questions regarding the mathematical justification of NNs are still open, some results are already available and include the approximation of the following quantities by NNs:

- The parameter-to-QoI map $\boldsymbol{y}\mapsto G(u(\cdot,\boldsymbol{y}))$ for problem (1).
- Solutions to high dimensional parabolic PDEs, which arise for example in quantitative finance when pricing large basket options.



Realization of a two dimensional Brownian sheet β.
See [LPS14, Section 7].

Prerequisites

This seminar is intended for bachelor, master and doctoral students. Minimum requirements however should be Functional Analysis 1, Differential Equations 1 and Numerical Analysis.

Preliminary Meeting

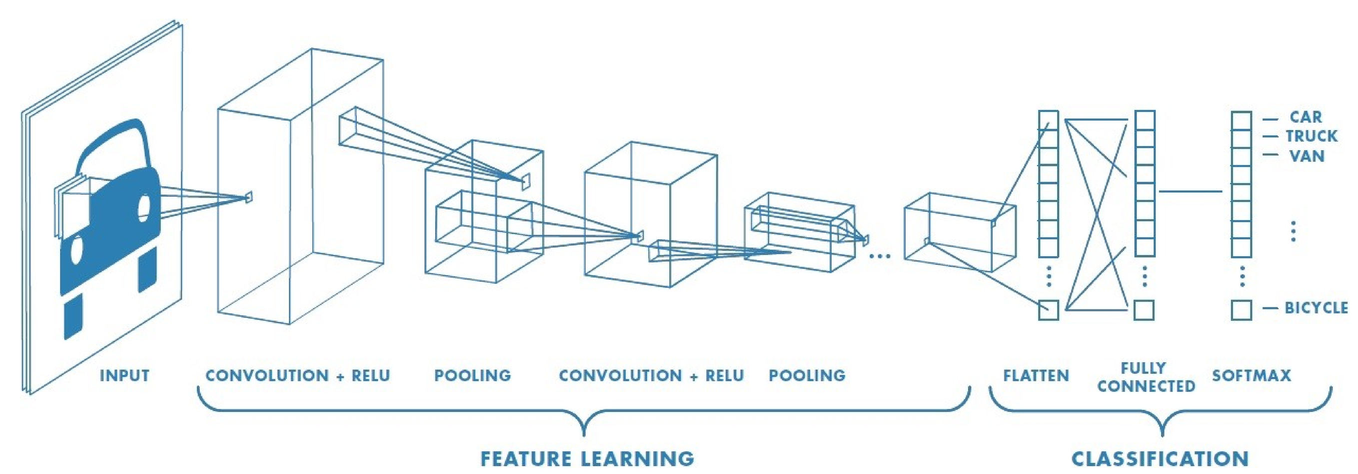
Tuesday 9.10.2018 11:00, seminar room DC green 03 C, Freihaus 3rd floor.

Roadmap

To fix both notation and set the mathematical foundation for all further topics, introductory talks will be given by the organizers in the first few sessions of the seminar. Afterwards student talks will be held. Possible topics include

- Introduction to UQ (**UQ**)
- Introduction to random fields (**UQ**)
- Tensor numerical methods for stochastic right-hand side $f(x,\omega)$ in (1) (**UQ**)
- Multi Level Monte Carlo (MLMC) for stochastic diffusion coefficient $a(x,\omega)$ (**UQ**)
- Quasi Monte Carlo (QMC) for stochastic diffusion coefficient $a(x,\omega)$ (**UQ**)
- Numerical methods for SDEs (stochastic ODEs) (not really **UQ**)
- Parameter uncertainty in option pricing - quantitative finance (**UQ**)
- Introduction to NN (**NN**)
- Linear and non-linear approximation theory (**NN**)
- Classical and new results concerning approximation properties of NNs (**NN**)
- Topological properties of NNs of fixed architecture (**NN**)
- Representation of the solution surface of problem (1) using NNs (**UQ** & **NN**)
- Solving high dimensional option pricing problems using NNs (**UQ** & **NN**)

The above list can be extended or shortened taking the number of participants into account.



Schematic depiction of a convolutional neural network (CNN).
Image taken from Raghav Prabhu Medium Blog.

Examination modalities

For a positive certificate a (approximately) 90-minute talk on the topic is mandatory. Furthermore bachelor students should prepare a written seminar paper. We encourage speakers to also implement certain numerical methods in their favorite programming language. Other listeners are always welcome.

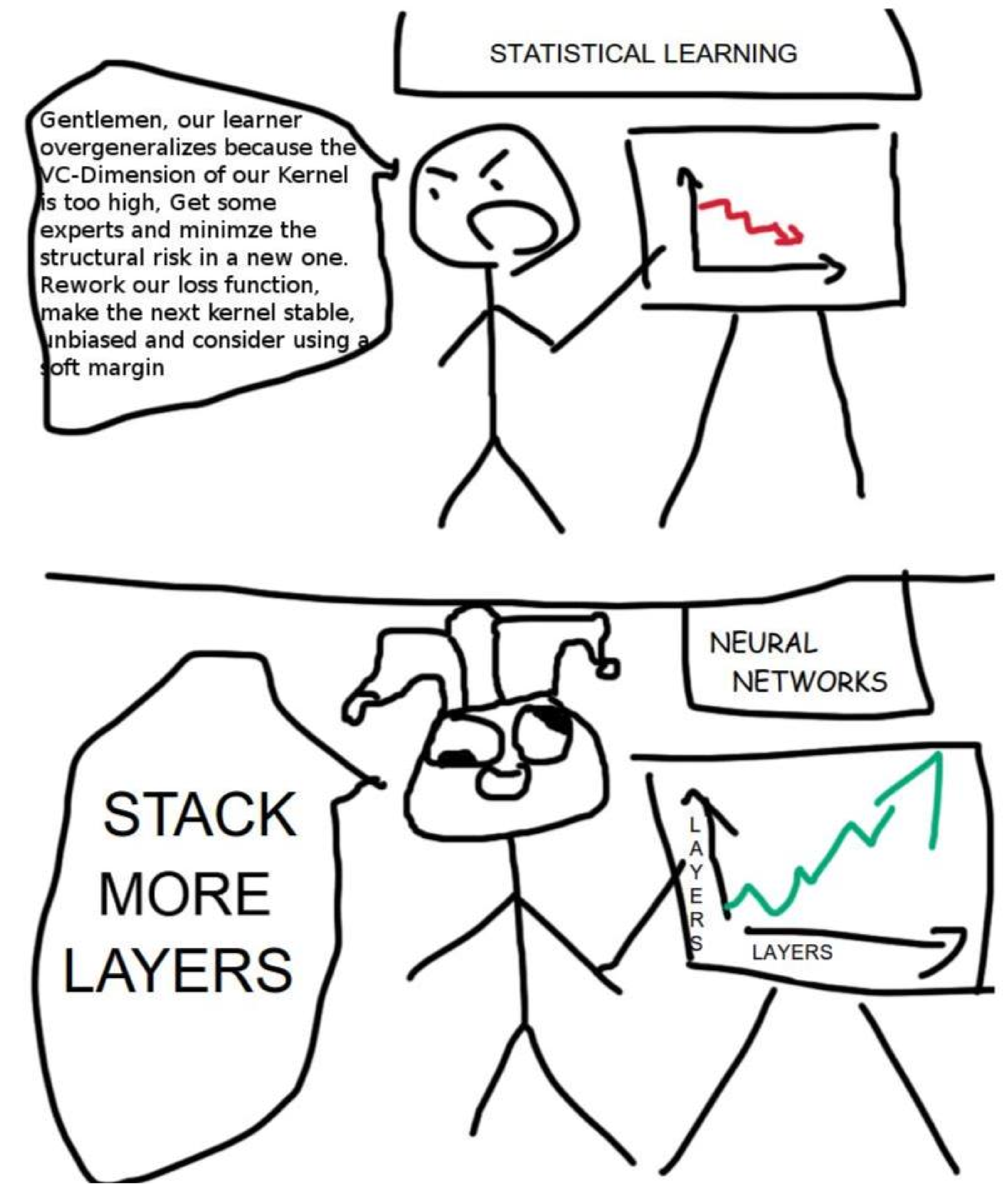
Some References

[GBC16] Ian Goodfellow, Yoshua Bengio, and Aaron Courville. *Deep Learning*. MIT Press, 2016. `http://www.deeplearningbook.org`.

[LPS14] Gabriel J. Lord, Catherine E. Powell, and Tony Shardlow. *An Introduction to Computational Stochastic PDEs*. Cambridge Texts in Applied Mathematics. Cambridge University Press, 2014.

[SZ18] Christoph Schwab and Jakob Zech. Deep learning in high dimension. Technical Report 2017-57 (revised), Seminar for Applied Mathematics, ETH Zürich, 2018.

[Yar17] Dmitry Yarotsky. Error bounds for approximations with deep relu networks. *Neural Networks*, 94:103 – 114, 2017.



Schematic depiction of the difference between classical statistical and deep learning.
Image taken from Reddit.