# The Rokhlin Invariant and Homology Cobordism

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**Abstract**

The well-known basic relation between the Rokhlin invariant and homology cobordism will be established. We will consider the group $\Theta^3_{\mathbb{Z}}$ of integer homoloy spheres up to homology cobordism. Surprisingly, this group plays a key role in disproving the triangulation conjecture. We mostly follow [Sav02].

## 1 Homology Cobordism

**Definition 1.1** (H-cobordism)**.** Let $\Sigma_1$ and $\Sigma_2$ be two integer homology spheres. We call $\Sigma_1$ and $\Sigma_2$ homology cobordant, or $H$-cobordant, if there exists a smooth compact oriented manifold $W$ of dimension 4 such that $\partial W = \Sigma_1 \amalg -\Sigma_2$ and the inclusions $i_1 : \Sigma_1 \hookrightarrow W$, $i_2 : \Sigma_2 \hookrightarrow W$ induce an isomorphism in homology. We can choose $W$ to carry a spin structure.

*Remark* 1.2. This defines an equivalence relation on the set of all integer homology spheres.

**Definition 1.3** (H-cobordism group)**.** We denote $\Theta^3_{\mathbb{Z}}$ to be the set of all such equivalence classes. Under the operation of connected sum, this defines an abelian group. For an integer homology sphere $\Sigma$, we write $[\![\Sigma]\!]_H$ as its equivalence class in $\Theta^3_{\mathbb{Z}}$.

We need to check whether that is truly the case. If we show that the connected sum gives $\Theta^3_{\mathbb{Z}}$ a group structure, this group being abelian follows immediately. For that, we need some results.

**Lemma 1.4.** *Let $\Sigma$ be an integer homology sphere. $\Sigma$ is $H$-cobordant to $\mathbb{S}^3$ if and only if there exists a smooth 4-manifold $W$ with boundary $\Sigma$ and $H_\bullet(W;\mathbb{Z}) \cong H_\bullet(\mathbb{D}^4;\mathbb{Z})$.*

*Proof.* Assume there exists such a 4-manifold $W$. We can smoothly embed a $\mathbb{D}^4$ in the interior of $W$, $i : \mathbb{D}^4 \hookrightarrow W$. Then $W \setminus i(\mathbb{D}^4)$ is an $H$-cobordism of $\Sigma$ and $\mathbb{S}^3$, as removing $\mathbb{D}^4$ leaves one more boundary component, namely $\mathbb{S}^3$. Conversely, assume $\Sigma$ and $\mathbb{S}^3$ are $H$-cobordant. Attaching $\mathbb{D}^4$ to the boundary component $\mathbb{S}^3$ yields a 4-manifold $W$ with $\partial W = \Sigma$ and $H_\bullet(W;\mathbb{Z}) \cong H_\bullet(\mathbb{D}^4;\mathbb{Z})$. Figure 13 shows a sketch of the construction. $\square$

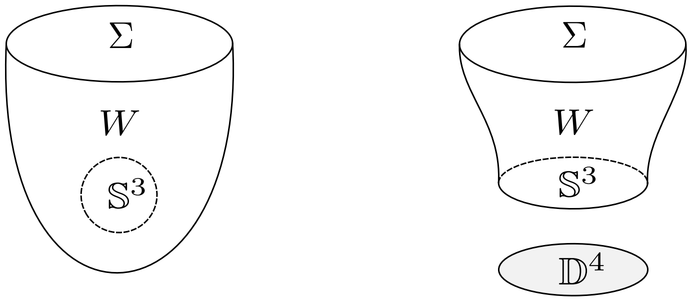


Figure 1: Schematic picture of the construction above.

**Proposition 1.5.** *Let $\Sigma_1, \Sigma_2$ be integer homology spheres. Then $\Sigma_1 \# \Sigma_2$ is an integer homology sphere as well.*

*Proof.* First of all, $\Sigma_1 \# \Sigma_2$ is again a closed connected and oriented 3-manifold. We consider the long exact sequence of the pair $(\Sigma_1 \# \Sigma_2, \mathbb{S}^2)$, with $\mathbb{S}^2$ obtained from the construction of the connected sum. Note, that we have the following relation between homology of good pairs and reduced homology.

$$\begin{aligned} H_n(\Sigma_1 \# \Sigma_2, \mathbb{S}^2; \mathbb{Z}) &\cong \tilde{H}_n((\Sigma_1 \# \Sigma_2)/\mathbb{S}^2; \mathbb{Z}) \\ &\cong \tilde{H}_n(\Sigma_1 \vee \Sigma_2; \mathbb{Z}) \cong \tilde{H}_n(\Sigma_1; \mathbb{Z}) \oplus \tilde{H}_n(\Sigma_2; \mathbb{Z}) \end{aligned}$$

The long exact sequence of pairs in homology now yields the following.

$$\begin{aligned} &0 \longrightarrow 0 \longrightarrow H_3(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \\ &\hookrightarrow \mathbb{Z} \longrightarrow H_2(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \longrightarrow 0 \\ &\hookrightarrow 0 \longrightarrow H_1(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \longrightarrow 0 \\ &\hookrightarrow \mathbb{Z} \longrightarrow H_0(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \longrightarrow 0 \longrightarrow 0 \end{aligned}$$

We can directly see that $H_0(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \cong \mathbb{Z}$ and $H_1(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \cong 0$. Using Poincaré Duality and the Universal Coefficient Theorem for cohomology yields $H_2(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \cong 0$. As $\Sigma_1 \# \Sigma_2$ is oriented, we already know that $H_3(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \cong \mathbb{Z}$. One can also see it from the long exact sequence above, as the short exact sequence splits.

$$0 \to H_3(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

□

**Proposition 1.6.** *$\Theta^3_{\mathbb{Z}}$ is an abelian group under the operation of connected sum.*

*Proof.* The neutral element is given by the sphere $\mathbb{S}^3$. This follows from the fact that $\mathbb{S}^3\#\mathbb{S}^3 \cong \mathbb{S}^3$, hence $[\![\mathbb{S}^3]\!]_H = 0$. Inverses are given by reversing orientation. We need to show that for $\Sigma$, an integer homology sphere, $\Sigma\# - \Sigma$ is homology cobordant to $\mathbb{S}^3$. Let $W = \Sigma \times [0,1]$. This defines an $H$-cobordism $W : \Sigma \to \Sigma$. Let $\gamma : [0,1] \to W$ be a path connecting $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$. Consider the thickening of $\gamma$, $U(\gamma) \cong \mathbb{D}^3 \times \gamma \subseteq W$. Note, that $\partial U(\gamma) \cong \mathbb{S}^2 \times \gamma$. This is closed, and $W \setminus \dot{U}(\gamma)$ is closed as well. Here, $\dot{U}(\gamma)$ denotes $U(\gamma) \setminus \partial U(\gamma)$. The boundary of $W \setminus \dot{U}(\gamma)$ now consists of the connected sum of three components, namely $\Sigma\#(\mathbb{S}^2 \times \gamma)\# - \Sigma$. A quick calculation yields the following result.

$$\begin{aligned} H_n(W, W \setminus \dot{U}(\gamma); \mathbb{Z}) &\cong H_n(W/(W \setminus \dot{U}(\gamma)), *; \mathbb{Z}) \\ &\cong \tilde{H}_n(W/(W \setminus \dot{U}(\gamma)); \mathbb{Z}) \cong \tilde{H}_n(\mathbb{S}^3 \times [0,1]; \mathbb{Z}) \end{aligned}$$

The long exact sequence of pairs in homology now yields

$$\cdots \to H_n(W \setminus \dot{U}(\gamma); \mathbb{Z}) \xrightarrow{i_*} H_n(W; \mathbb{Z}) \to \tilde{H}_n(\mathbb{S}^3 \times [0,1]; \mathbb{Z}) \to \dots$$

and as $W$ is homotopy equivalent to $\Sigma$ which has the same homology groups as $\mathbb{S}^3$, $\mathbb{S}^3 \times [0,1]$ is homotopy equivalent to $\mathbb{S}^3$, we obtain the following exact sequence.

$$\begin{tikzcd} 0 \arrow[r] & H_3(W \setminus \dot{U}(\gamma); \mathbb{Z}) \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[dll, rounded corners] & \\ & H_2(W \setminus \dot{U}(\gamma); \mathbb{Z}) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, rounded corners] & \\ & H_1(W \setminus \dot{U}(\gamma); \mathbb{Z}) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, rounded corners] & \\ & H_0(W \setminus \dot{U}(\gamma); \mathbb{Z})) \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

Going through the sequence, we can deduce $H_n(W \setminus \dot{U}(\gamma); \mathbb{Z}) \cong H_n(\mathbb{D}^4; \mathbb{Z})$. Hence by Lemma 1.4, $\Sigma\# - \Sigma$ is $H$-cobordant to $\mathbb{S}^3$, and $-[\![\Sigma]\!]_H = [\![-\Sigma]\!]_H$. □

**Proposition 1.7.** *The map $\Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z}$ given by the Rokhlin invariant is a surjective group homomorphism.*

*Proof.* Let $\Sigma_1$ and $\Sigma_2$ be two integer homology spheres. Consider $\Sigma_1\#\Sigma_2$. By Proposition 1.5, this is again an integer homology sphere. Let $W_1$ be a compact connected smooth spin 4-manifold with boundary $\Sigma_1$, $W_2$ one which $\Sigma_2$ bounds. Then $W_1 \natural W_2$ has boundary $\Sigma_1\#\Sigma_2$ and its intersection form is given by $Q_{W_1} \oplus Q_{W_2}$. The signature is additive. Therefore, $\mu(\Sigma_1\#\Sigma_2) = \mu(\Sigma_1) + \mu(\Sigma_2) \mod 2$. We already know that $\mu(\mathbb{S}^3) = 0$. Suppose $\Sigma$ is an integer homology sphere that is $H$-cobordant to $\mathbb{S}^3$ via a compact smooth 4-manifold $W$ with $H_\bullet(W; \mathbb{Z}) \cong H_\bullet(\mathbb{D}^4; \mathbb{Z})$. Its first homology group is trivial and the intersection form $Q_W$ is empty, hence has even signature, and a spin structure. Attaching $\mathbb{D}^4$ to the boundary component $\mathbb{S}^3$ yields the following.

$$\mu(\Sigma) = \frac{1}{8}\sigma(W) = 0 \mod 2$$

Therefore, the Rokhlin invariant truly defines a group homomorphism $\Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z}$. As $\mu(\Sigma(2,3,5)) = 1$, it is a surjective mapping. □

We will now give an argument on why the Poincaré homology sphere $\Sigma(2,3,5)$ has infinite order in $\Theta^3_{\mathbb{Z}}$. For that, we need the following classical result by Donaldson.

**Theorem 1.8.** *(Donaldson) Let $W$ be a closed smooth oriented 4-manifold. If the intersection form $Q_W$ is positive or negative definite, then it can be diagonalised over $\mathbb{Z}$ to the identity matrix or its additive inverse, respectively.*

*Proof.* See [Don87] for a proof. □

**Proposition 1.9.** *$\Sigma(2,3,5)$ has infinite order in $\Theta^3_{\mathbb{Z}}$.*

*Proof.* Recall that $\Sigma(2,3,5)$ bounds a smooth closed oriented 4-manifold $W$ with intersection form $E_8$. Consider $\#_{i=1}^n \Sigma(2,3,5)$ for $n \geq 1$. This bounds the smooth closed oriented manifold 4-manifold $\natural_{i=1}^n W$. Its intersection form is isomorphic to $\bigoplus_{i=1}^n E_8$. Let us now assume that there exists a certain $k \in \mathbb{N}$ such that $[\![\#_{i=1}^k \Sigma(2,3,5)]\!]_H$ is trivial, meaning homology cobordant to $\mathbb{S}^3$. Let $N$ denote this cobordism and consider the manifold $\natural_{i=1}^k W \cup_\Sigma -N$. This is a smooth closed oriented manifold of dimension 4 with intersection form $\bigoplus_{i=1}^k E_8$ which is definite but not diagonalisable over $\mathbb{Z}$. This contradicts Donaldson's Theorem, proving the claim. □

This also implies that $\mathbb{Z} \leq \Theta^3_{\mathbb{Z}}$. We now will state some interesting known properties of the homology cobordism group. This will be held informally, proofs will be omitted.

**Theorem 1.10** (Furuta, Fintushel and Stern)**.** $\mathbb{Z}^\infty \leq \Theta^3_{\mathbb{Z}}$.

*Proof.* The classes $[\![\Sigma(2,3,6n-1)]\!]_H$ for $n \geq 1$ given by Brieskorn homology spheres are linearly independent. For example, see [Fur90]. □

## 2 A brief word on the Triangulation conjecture

In 2016, Manolescu published a paper in which he disproved the triangulation conjecture.

**Conjecture 2.1** (Triangulation conjecture)**.** Every topological manifold is triangulable.

Up to dimension 3, this is indeed true. Casson and Freedman disproved the triangulation conjecture in dimension 4, the $E_8$-manifold for example is not triangulable. The following result is due to the work of Matumoto, Galewski, and Stern.

**Theorem 2.2** (Matumoto, Galewski and Stern)**.** *For dimension greater or equal than* $5$*, every topological manifold* $M$ *with empty boundary is triangulable if and only if the following short exact sequence splits.*

$$0 \to ker(\mu) \to \Theta^3_{\mathbb{Z}} \xrightarrow{\mu} \mathbb{Z}/2\mathbb{Z} \to 0$$

*The map* $\mu$ *is given by the Rokhlin invariant.*

*Proof.* See [GS80] and [Mat78]. □

This means that any such manifold is triangulable if and only if there exists an integer homology sphere $\Sigma$ such that $\mu(\Sigma) = 1$ and $\Sigma \# \Sigma$ is homology cobordant to $\mathbb{S}^3$.

**Theorem 2.3** (Manolescu)**.** *There is no* $2$*-torsion in* $\Theta^3_{\mathbb{Z}}$*.*

*Proof.* See [Man16]. □

Hence, the short exact sequence in Theorem 2.2 does not split, and there do exist manifolds of dimension greater or equal than 5 which are not triangulable, disproving the triangulation conjecture.

Regarding the group $\Theta^3_{\mathbb{Z}}$, an interesting question to think about would be if it is torsion-free. To this date, this is an open question.

## References


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