Symplectic Representation of the Mapping Class Group

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4th October 2022

Abstract

We study the action of $Mod(\Sigma_g)$ on $H_1(\Sigma_g; \mathbb{Z})$, following [FM11]. This is given by the map $\Psi : Mod(\Sigma_g) \to Aut(H_1(\Sigma_g; \mathbb{Z}))$ sending $[f]$ to f_*. It is well defined, as, by homotopy invariance, isotopic maps induce the same map in homology. Functoriality of homology results in homeomorphisms inducing isomorphisms in homology. This yields a representation into the symplectic group $Sp(2g, \mathbb{Z})$.

1 Symplectic Representation

Definition 1.1 (Symplectic structure over $\mathbb{R}^{2g}$)**.** A symplectic structure on a finite-dimensional vector space V is a skew-symmetric alternating bilinear form ω. We now consider $V = \mathbb{R}^{2g}$. Given a basis of $\mathbb{R}^{2g}$, one can represent ω as a matrix J which is skew-symmetric, invertible and has only zero entries on the diagonal. We now set $Sp(2g, \mathbb{R}) = \{M \in GL(2g, \mathbb{R}) : MJM^T = J\}$ and similarly $Sp(2g, \mathbb{Z}) = Sp(2g, \mathbb{R}) \cap GL(2g, \mathbb{Z})$.

Remark 1.2. For our use, we mostly consider the matrix

$$J = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}$$

with I_g, the $g \times g$ identity matrix.

Remark 1.3. One can easily check that J satisfies the needed properties. In the upcoming pages, it will get clear why we have chosen J in such a way.

Remark 1.4. For $g = 1$ we have $Sp(2, \mathbb{Z}) \cong SL(2, \mathbb{Z})$. In this case, Ψ is an isomorphism.

Definition 1.5 (Symplectic structure on $H_1(\Sigma_g; \mathbb{Z})$)**.** We obtain a symplectic structure on $H_1(\Sigma_g; \mathbb{Z})$ by the Poincaré Duality isomorphism

$$PD : H^1(\Sigma_g; \mathbb{Z}) \xrightarrow{\cong} H_1(\Sigma_g; \mathbb{Z})$$

and the intersection form:

$$Q_{\Sigma_g} : H^1(\Sigma_g;\mathbb{Z}) \otimes H^1(\Sigma_g;\mathbb{Z}) \to \mathbb{Z}$$
$$\alpha \otimes \beta \mapsto \langle \alpha \smile \beta, [\Sigma_g] \rangle$$

Let $(a_1, b_1, \ldots, a_g, b_g)$ be the standard basis of $H_1(\Sigma;\mathbb{Z})$. By Poincaré Duality, there is a corresponding dual basis $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ of $H_1(\Sigma_g;\mathbb{Z})$ satisfying

(i) $\langle \alpha_i \smile \beta_j, [\Sigma_g] \rangle = \delta_{ij}$

(ii) $\langle \alpha_i \smile \alpha_j, [\Sigma_g] \rangle = 0$

(iii) $\langle \beta_i \smile \beta_j, [\Sigma_g] \rangle = 0$

as the cohomology ring of Σ_g is known to be the following.

$$H^\bullet(\Sigma_g;\mathbb{Z}) \cong \frac{\mathbb{Z}[\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g]}{\langle \alpha_i\alpha_j, \beta_i\beta_j, \alpha_i\beta_j (i \neq j), \alpha_i\beta_i + \beta_i\alpha_i \rangle}$$

Remark 1.6. The condition $MJM^T = J$ in the definition of $Sp(2g,\mathbb{Z})$ is sufficient in order that M represents an element in $Aut(H_1(\Sigma_g;\mathbb{Z}))$. It yields the condition that an element in $Mod(\Sigma_g)$ must preserve the Poincaré dual of the intersection form.

Remark 1.7. Given the standard basis of $H_1(\Sigma_2;\mathbb{Z})$, see Figure 1, and using the corresponding basis $a_i \sim \alpha_i, b_i \sim \beta_i$ of $H^1(\Sigma_2;\mathbb{Z})$ we obtain the following matrices, depending on the ordering of the basis elements.

$$\begin{array}{cc} & \begin{array}{cccc} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \end{array} \\ \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cccc} \alpha_1 & \beta_1 & \alpha_2 & \beta_2 \end{array} \\ \begin{array}{c} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

If not otherwise stated, we consider the ordering of the left matrix. In the literature, one might find a different ordering.

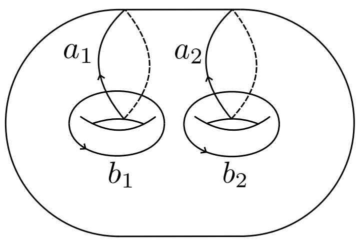

Figure 1: Geometric realisation of the standard generators of $H_1(\Sigma_2;\mathbb{Z})$.

There is the obvious representation given by the induced map on homology.

$$\Psi : Mod(\Sigma_g) \to Aut(H_1(\Sigma_g;\mathbb{Z})) \cong GL(2g,\mathbb{Z})$$

2 This Mapping is surjective

Now, we have equipped $H_1(\Sigma_g;\mathbb{Z})$ with a symplectic structure. Since this structure is preserved by the image of Ψ, we immediately get the refined mapping $\Psi : Mod(\Sigma_g) \to Sp(2g,\mathbb{Z})$. We want to prove that this is indeed surjective. To do so, we need the following results.

Lemma 2.1. *Let γ be an embedded oriented circle on Σ_g. Then for all $a \in H_1(\Sigma_g)$ we have the following. Let γ_* denote the induced class in $H_1(\Sigma_g;\mathbb{Z})$.*

$$\Psi(\tau_\gamma)(a) = (\tau_\gamma)_*(a) = a + \langle PD^{-1}(\gamma_*) \smile PD^{-1}(a), [\Sigma_g]\rangle \gamma_*$$

Proof. A geometric proof can be found in [FM11, Proposition 6.3]. □

Theorem 2.2 (Birman)**.** *$Sp(2g,\mathbb{Z})$ is generated by the following matrices.*

$$A_i = \begin{bmatrix} I_g & -E_{i,i} \\ 0 & I_g \end{bmatrix},\ B_i = \begin{bmatrix} I_g & 0 \\ E_{i,i} & I_g \end{bmatrix},\ C_j = \begin{bmatrix} I_g & X_j \\ 0 & I_g \end{bmatrix}$$

for $i \in \{1,\dots,g\}$ and $j \in \{1,\dots,g-1\}$. Here, we consider elementary matrices and

$$X_j = -E_{j,j} - E_{j+1,j+1} + E_{j,j+1} + E_{j+1,j}$$

given by sums of elementary matrices. Note, that those matrices are invertible.

Proof. This algebraic result is due to Birman. □

Theorem 2.3. *The map $\Psi : Mod(\Sigma_g) \to Sp(2g,\mathbb{Z})$ is surjective.*

Proof. Consider the family of $3g-1$ circles given by $\alpha_i, \beta_i, \gamma_j$ for $i \in \{1,\dots,g\}$ and $j \in \{1,\dots,g-1\}$ generating $Mod(\Sigma_g)$.

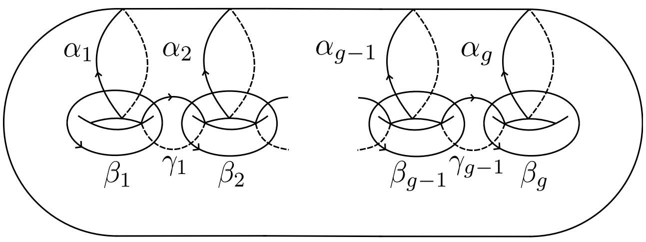

Figure 2: Generators of $Mod(\Sigma_g)$ as given by Lickorish in [Lic64].

Lemma 2.1 now yields the following result.

$$\Psi(\tau_{\alpha_i}) = A_i^{-1},\ \Psi(\tau_{\beta_i}) = B_i^{-1}, \Psi(\tau_{\gamma_j}) = C_j$$

Thus, $\Psi : Mod(\Sigma_g) \to Sp(2g, \mathbb{Z})$ is surjective. The calculation itself is rather tedious. We give some examples. For that, we focus on Σ_2 with the standard basis of $H_1(\Sigma_2; \mathbb{Z})$ with the ordering chosen in Remark 1.7. Note, that the circles α_i and β_i in the figure above correspond to the geometric realisation of the standard basis.

Consider the Dehn-Twist τ_{a_1}. It acts non-trivially on b_1 only.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\Psi(\tau_{a_1})} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Consider the Dehn-Twist τ_{b_1}. It acts non-trivially on a_1 only.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\Psi(\tau_{b_1})} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In both cases, one can easily observe that the matrices are truly the suggested ones.

Now, onto γ_1. Firstly, we observe that γ_1 can be represented by the homology class given by $a_1 - a_2$. This can be seen in Figure 3.

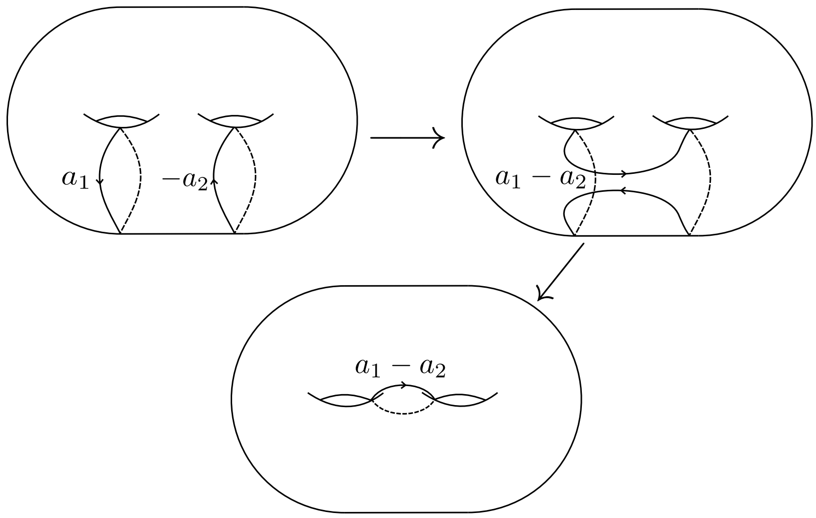

Figure 3: The circle γ_1 represented by the homology class $a_1 - a_2$.

Now we consider the Dehn-Twist τ_{γ_1} and its action on the homology generators. Firstly, it is clear that τ_{γ_1} acts trivially on both a_1 and a_2.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\Psi(\tau_{\gamma_1})} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This agrees with C_1 and hence concludes the exemplary proof.

□

Remark 2.4. A similar proof involving Burkhardt generators of $Sp(2g, \mathbb{Z})$ was given by Burkhardt. This representation can be used to prove that a generating set of $Mod(\Sigma_g)$ consists of at least $2g+1$ elements. This was originally proven by Humphries, see [Hum06], and is well explained by Farb and Margalit, see [FM11, Section 6.3.3].

References

[Lic64] W. B. R. Lickorish. "A finite set of generators for the homeotopy group of a 2-manifold". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 60.4 (1964), pp. 769–778.

[Hum06] Stephen Humphries. "Generators for the mapping class group". In: 2006, pp. 44–47.

[FM11] Benson Farb and Dan Margalit. *A Primer on Mapping Class Groups.* Princeton University Press, 2011.