Homotopy Theory as a Cohomology Theory

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Abstract

We follow [Hat02] to establish a well-known, but surprising connection between homotopy theory and cohomology. The nth-homotopy group of a pointed topological space (X, x_0) is defined to be the set of all basepoint-preserving maps $f : (S^n, s_0) \to (X, x_0)$ up to basepoint-preserving homotopy, denoted as $[S^n, X]$. As we generally work in the category $\mathbf{CW_\bullet}$ of pointed CW-complexes, we will mostly omit any mention of the basepoint. Similarly, one could consider two spaces X, Y and study $[X, Y]$, though to define a general group structure could prove difficult. We will study Ω-spectra which will give rise to special families of CW-complexes $\{K_n\}$ such that $[X, K_n]$ with a certain group structure turns out to define a cohomology theory.

1 Establishing a group structure on $[X, K]$

The first step will consist of discussing which characteristics either X or K have to have, to be able to define an interesting group structure on $[X, K]$, similarly to the group structure of the homotopy groups.

Definition 1.1. Let X, K be spaces. We consider the reduced suspension

$$\Sigma X = \frac{X \times I}{X \times \{0\} \cup X \times \{1\} \cup x_0 \times I}$$

and define a group structure on $[\Sigma X, K]$. Let $f, g \in [\Sigma X, K]$. We define

$$(f + g) : \Sigma X \xrightarrow{p} \Sigma X \vee \Sigma X \xrightarrow{f \vee g} K$$

with p being the equatorial pinch map. This given structure turns $[\Sigma X, K]$ into a group.

Definition 1.2 (Loop space)**.** Let X be a pointed topological space. The loop space ΩX is defined to be the set of all pointed loops $f : (S^1, s_0) \to (X, x_0)$. Equivalently, we can think about elements in ΩX to be maps $f : I \to X$ such that $f(0) = x_0 = f(1)$. Considering $\Omega^2 X := \Omega(\Omega X)$ we can inductively define $\Omega^n X := \Omega^{n-1}(\Omega X)$.

Lemma 1.3. *The sets* $[\Sigma X, K]$ *and* $[X, \Omega K]$ *stand in bijective relation.*

Proof. We define $\phi : [\Sigma X, K] \to [X, \Omega K]$ by $\phi(f(x))(t) \coloneqq f(x,t)$. Note that by definition of ΣX we have $f(x,0) = f(x_0) = f(x,1)$. □

This is an example of Eckmann-Hilton duality, see [Hat02, 4.H]

Definition 1.4. We will now define a group structure on $[X, \Omega K]$ and compare it to the one defined above. Let $f, g \in [X, \Omega K]$. We define

$$(f+g)(x) \coloneqq (f \circ g)(x)$$

by composition of the loops in ΩK. Under the relation asserted in Lemma 1.3 this agrees with the previously introduced group structure.

For locally compact Hausdorff spaces X, Y, we have a homeomorphism $(K^X)^Y \cong K^{X \times Y}$, see for example [Hat02, A.16]. Here, K^X denotes the set of morphisms $f : X \to K$. Hence, we can view $\Omega^n K$ as the set of maps $f : (I^n, \partial I^n) \to (K, k_0)$.

Lemma 1.5. *$[X, \Omega^n K]$ is abelian for $n \geq 2$.*

Proof. This is the same proof as for $\pi_n(X, x_0)$, $n \geq 2$. □

Considering Lemma 1.3, we have that $[\Sigma^n X, K] = [X, \Omega^n K]$ is abelian for $n \geq 2$, thus a functor $[-, K] : \mathbf{CW}_\bullet^{\mathbf{op}} \to \mathbf{Ab}$.

2 Respresenting Homotopy Theory as a Cohomology Theory

From the previously established facts, to view $h^\bullet(X) \coloneqq [X, K_\bullet]$ as a cohomology theory, we would need each $K_\bullet$ to be a double loop space. There is a weaker condition that suffices. Let $K_\bullet \to \Omega L_\bullet$ be a weak homotopy equivalence for some spaces $L_\bullet$. Then $[X, K_\bullet] = [X, \Omega L_\bullet]$. This naturally leads to the following definition.

Definition 2.1 (Ω-spectrum)**.** An Ω-spectrum is a sequence of CW-complexes K_n together with weak homotopy equivalences $K_n \to \Omega K_{n+1}$ for all $n \in \mathbb{N}$.

Note that we thus have the following weak homotopy equivalence for each $n \in \mathbb{N}$.

$$K_n \to \Omega K_{n+1} \to \Omega^2 K_{n+2}$$

Considering each K_{n-1} as a CW-approximation of ΩK_n, we can extend the sequence for $n \in \mathbb{Z}$. Furthermore, since we work in $\mathbf{CW}_\bullet$, Whitehead's theorem gives us $[X, K_n] = [X, \Omega K_{n+1}]$.

Theorem 2.2. *Let $\{K_n\}$ be an Ω-spectrum. Then the functors $h^n(-) : \mathbf{CW}_\bullet^{\mathbf{op}} \to \mathbf{Ab}$ defined by $h^n(-) \coloneqq [-, K_n]$, $n \in \mathbb{Z}$ define a reduced cohomology theory.*

The converse is true as well, this is Brown's representability theorem. Before we give a proof of Theorem 2.2, one more tool is needed. For a pointed space (X, x_0), the reduced cone CX is defined to be the following.

$$CX := \frac{X \times I}{X \times \{0\} \cup x_0 \times I}$$

For a CW-pair (X, A), consider the following diagram.

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "="] & X \arrow[r, hook] \arrow[d, "="] & X \cup CA \arrow[r, hook] \arrow[d, "\simeq"] & (X \cup CA) \cup CX \arrow[r, hook] \arrow[d, "\simeq"] & ((X \cup CA) \cup CX) \cup C(X \cup CA) \arrow[d, "\simeq"'] \\
A \arrow[r, hook] & X \arrow[r] & X/A \arrow[r] & \Sigma A \arrow[r, hook] & \Sigma X
\end{tikzcd}$$

The homotopy equivalences are gives by the fact that the cone is contractible, and there exists a homotopy from $i : A \hookrightarrow CA$ to $p : * \hookrightarrow CA$. The only map that has to be discussed is $X/A \to \Sigma A$, we define it to be the composition of choosing a homotopy inverse $X/A \to X \cup CA$ and then following the arrows of the third square. Thus, the diagram commutes up to homotopy. Repeating this process, and identifying $\Sigma(X/A) \cong \Sigma X/\Sigma A$ holds the following definition.

Definition 2.3 (Puppe sequence)**.** Let (X, A) be a CW-pair. The natural sequence

$$\begin{tikzcd}
A \arrow[r] & X \arrow[r] & X/A \arrow[dll, out=-10, in=170] \\
\Sigma A \arrow[r] & \Sigma X \arrow[r] & \Sigma X/\Sigma A \arrow[dll, out=-10, in=170] \\
\Sigma^2 A \arrow[r] & \Sigma^2 X \arrow[r] & \dots
\end{tikzcd}$$

is called the Puppe sequence of the pair (X, A).

Proof of Theorem 2.2. Let us start with homotopy invariance. A morphism $f : X \to Y$ induces $f^* : [Y, K_n] \to [X, K_n]$ given by pre-composition, $Y \to K_n \mapsto X \xrightarrow{f} Y \to K_n$. The fact that f^* is a homomorphism is evident if we replace K_n by ΩK_{n+1} and consider the group structure established in Definition 1.4. As we are dealing with homotopy classes, and the induced map is given by pre-composition, $f, g \in [X, Y]$ implies that $f^* = g^*$.

The wedge sum axiom holds for obvious reasons. A map $\bigvee_{i \in I} X_i \to K_n$ is a collection of maps $X_i \to K_n$ for every $i \in I$.

Given a CW-pair (X, A), we now want to define a long exact sequence. The Puppe sequence gives rise to the following sequence.

$$\begin{tikzcd}
{[A, K_n]} & {[X, K_n]} \arrow[l] & {[X/A, K_n]} \arrow[l] \\
{[\Sigma A, K_n]} \arrow[urr] & {[\Sigma X, K_n]} \arrow[l] & {[\Sigma X/\Sigma A, K_n]} \arrow[l] \\
{[\Sigma^2 A, K_n]} \arrow[urr] & {[\Sigma^2 X, K_n]} \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

Considering $[\Sigma X, K_n] = [X, \Omega K_n]$, and K_{n-1} as a CW-approximation of ΩK_n, we extend the sequence to a long sequence.

$$\begin{tikzcd}
\dots & {[A, K_n]} \arrow[l] & {[X, K_n]} \arrow[l] & {[X/A, K_n]} \arrow[l] \\
& {[A, K_{n-1}]} \arrow[urr] & {[X, K_{n-1}]} \arrow[l] & {[X/A, K_{n-1}]} \arrow[l] \\
& {[A, K_{n-2}]} \arrow[urr] & {[X, K_{n-2}]} \arrow[l] & {[X/A, K_{n-2}]} \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

It is left to show that this sequence is indeed exact. As each element in the Puppe sequence is obtained by its two predecessors, thus, it suffices to show that the sequence

$$[A, K_n] \leftarrow [X, K_n] \leftarrow [X \cup CA, K_n]$$

is exact. Note that $X/A \simeq X \cup CA$, as CA is homotopy equivalent to a point. Let $f \in [X, K_n]$. This is a zero element in $[A, K_n]$ if and only its restriction to A is nullhomotopic, equivalently, the map $f : X \to K_n$ extends to a map $f : X \cup CA \to K_n$. As we consider homotopy classes, this can be written as $f : X/A \to K_n$. Hence, the long sequence is exact. $\square$

References

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, 2002.