Projection-Perturbation Method (Reiter Method)

Pierre Mabille

NYU Stern

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Overview

Reiter method (JEDC, 2009)

Objective: solve heterogeneous agents model with aggregate risk

Combine building blocks (taught in this course)

- Projection methods
- Perturbation methods
- Income Fluctuation Problem, Bewley-Huggett-Aiyagari and Krusell-Smith

Idea: compute a solution that is fully nonlinear in the idiosyncratic shocks, but only linear in the aggregate shocks

Motivations

- Alternative to KS algorithm
- Continuous aggregate shocks
- Linear filtering
- Estimation

Main steps

1. Provide a finite representation of the economy at date t
   (a) represent policy functions by arrays containing the policy values at grid points (if approximated by splines) or the polynomial coefficients (if approximated by orthogonal polynomials)
   (b) represent the distribution as a vector of probability mass of agents of each type within the regions of the state space
2. Compute the stationary steady state of the economy (no aggregate shocks) $\rightarrow$ finite representation of the stationary policy functions and the invariant distribution
3. Compute a first-order perturbation of all variables (policy functions and distribution) w.r.t. aggregate shocks, around the stationary steady state

Example economy

Household problem in recursive form (inelastically supplies ϵ units of efficient labor):

$$\begin{aligned} v\left(a,\epsilon;z,\lambda\right) &= \max_{c,a'}\left\{\frac{c^{1-\gamma}}{1-\gamma}+\beta\mathbb{E}\left[v\left(a',\epsilon';z',\lambda'\right)|z,\epsilon\right]\right\} \\ \text{s.t.}\quad c+a' &= w\left(z,\lambda\right)\epsilon+R\left(z,\lambda\right)a \\ a' &\geq -\underline{a} \\ \lambda' &= \Psi\left(\lambda,z,z'\right) \end{aligned}$$

Firm problem static:

$$\max_{K,L} e^{z}K^{1-\alpha}L^{\alpha}-w\left(z,\lambda\right)L-\left(R\left(z,\lambda\right)-1+\delta\right)K$$

$$z'=\rho z+\sigma\eta'$$

Recursive Competitive Equilibrium

Policy and price functions, distribution law of motion

$$\left\{c\left(a,\epsilon;z,\lambda\right),a'\left(a,\epsilon;z,\lambda\right);R\left(z,\lambda\right),w\left(z,\lambda\right);\Psi\left(\lambda,z,z'\right)\right\} \quad \text{s.t.}$$

- HH optimization: given price functions $r(.), w(.)$ and the law of motion $\Psi(.)$, policy functions satisfy

$$\begin{aligned} c\left(a,\epsilon;z,\lambda\right)^{-\gamma} &\geq \beta\mathbb{E}_{z,\epsilon}\Big[R\left(z',\lambda'\right) \\ &\times \; c\left(a'\left(a,\epsilon;z,\lambda\right),\epsilon';z',\Psi\left(z,\lambda\right)\right)^{-\gamma}\Big] \\ c\left(a,\epsilon;z,\lambda\right)+a'\left(a,\epsilon;z,\lambda\right) &= R\left(z,\lambda\right)a+w\left(z,\lambda\right) \end{aligned}$$

- Firm optimization: given price functions $r(.), w(.)$, firms K and N satisfy

$$\begin{aligned} R\left(z,\lambda\right) &= 1+(1-\alpha)e^{z}K^{-\alpha}L^{\alpha}-\delta \\ w\left(z,\lambda\right) &= \alpha e^{z}K^{1-\alpha}L^{\alpha-1} \end{aligned}$$

Recursive Competitive Equilibrium (cont'd)

- Market clearing

$$
\begin{aligned}
L &= \int \epsilon d\lambda\left(a,\epsilon;z\right) \quad \text{(labor)} \\
K &= \int a'\left(a,\epsilon;z\right) d\lambda\left(a,\epsilon;z\right) \quad \text{(capital)} \\
e^{z}K^{1-\alpha}L^{\alpha} + (1-\delta)K &= \int c\left(a,\epsilon;z\right) d\lambda\left(a,\epsilon;z\right) \\
&+ \int a'\left(a,\epsilon;z\right) d\lambda\left(a,\epsilon;z\right) \quad \text{(goods)}
\end{aligned}
$$

- Distribution's law of motion: $\forall \epsilon' \in E$ and measurable set $\mathcal{A} \subseteq A$, next distribution consistent with policy functions

$$
\Psi\left(z,\lambda\right)\left(\mathcal{A}\times\left\{\epsilon'\right\}\right) = \int \mathbb{1}\left\{a'\left(a,\epsilon;z,\lambda\right)\in\mathcal{A}\right\}\pi\left(\epsilon'|\epsilon\right) d\lambda\left(a,\epsilon\right)
$$

Step 1: finite representation of the economy

Discretize the state space (idiosyncratic states)

- Grid over ϵ, call it E, with n_ϵ points
- Grid for the consumption and savings policy rules over a, call it A^p, with n_a^p points for each ϵ
  - For HH of each type ϵ, the borrowing constraint starts binding when $a \leq \chi_\epsilon$ (EE w/ inequality, solve for $a' = \underline{a}$ from constraint), and is slack when $a > \chi_\epsilon$ (solve for a' from EE w/ equality). χ_ϵ are unknown and solved for in step 2
  - Define A^p grid, for each ϵ, as $a_{i,\epsilon} = \chi_\epsilon + x_i$, where $0 = x_1 < x_2 < ... < x_{n_a^p}$
  - Policy functions are represented by $n_\epsilon \times n_a^p$ coefficients: for each ϵ: χ_ϵ and coefficients at $a_{2,\epsilon}, ..., a_{n_a^p,\epsilon}$ (projection)
- Denser grid over a for the discretized density (histogram), call it A^d, with $n_a^d > n_a^p$ points
- The histogram is a series of vector of weights $\{\lambda_{\epsilon,t}\}$ (one for each value of ϵ), each of dimension $n_a^d - 1$ (sum to 1)

Step 1 (cont'd)

Define a system of equations representing the economy at date t:

- Law of motion for exogenous aggregate state (1 eqn):

$$z_{t+1} = \rho z_t + \sigma \eta_{t+1} \tag{1}$$

- Euler equations and budget constraints: one eqn for each point $(\epsilon, a_{j,\epsilon}) \in E \times A^p$ ($2 \times (n_\epsilon \times n_a^p)$ eqns)
  - collect policy function coefficients in vector $\Theta(z_t, \lambda_t)$, of dimension $n_\epsilon \times n_a^p$
  - define policy functions on grids, $\hat{g}(a_{i,\epsilon}, \epsilon; \Theta(z_t, \lambda_t))$ ($g = c, a'$): dependence on aggregate states *via* $\Theta(z_t, \lambda_t)$

$$\begin{aligned} \hat{c}(a_{i,\epsilon}, \epsilon; \Theta(z_t, \lambda_t))^{-\gamma} &= \beta \sum_{z_{t+1}} R(z_{t+1}, \lambda_{t+1}) \\ \times & \sum_{\epsilon' \in E} \hat{c}(a'(a_{i,\epsilon}, \epsilon; \Theta(z_t, \lambda_t)), \epsilon'; \Theta(z_{t+1}, \lambda_{t+1}))^{-\gamma} \\ \times & \pi(\epsilon' | \epsilon) \pi(z_{t+1} | z_t) \end{aligned} \tag{2}$$

$$\begin{aligned} & \hat{c}(a_{i,\epsilon}, \epsilon; \Theta(z_t, \lambda_t))^{-\gamma} + \hat{a}'(a_{i,\epsilon}, \epsilon; \Theta(z_t, \lambda_t))^{-\gamma} \\ = & R(z_t, \lambda_t) a_{i,\epsilon} + w(z_t', \lambda_t) \epsilon \end{aligned} \tag{3}$$

Step 1 (cont'd)

- Equilibrium prices (2 eqns):

$$\begin{aligned} R(z_t, \lambda_t) &= 1 + (1-\alpha) e^{z_t} K_t^{-\alpha} L_t^{\alpha} - \delta \\ &= 1 + (1-\alpha) e^{z_t} \left(\sum_{\epsilon \in E} \sum_{i=1}^{n_a^d} a_i \lambda_{\epsilon,t}(a_i) \right)^{-\alpha} \left(\sum_{\epsilon \in E} \sum_{i=1}^{n_a^d} \epsilon \lambda_{\epsilon,t}(a_i) \right)^{\alpha} \\ &- \delta \qquad (4) \\ w(z_t, \lambda_t) &= \alpha e^{z_t} K_t^{1-\alpha} L_t^{\alpha-1} \\ &= \alpha e^{z_t} \left(\sum_{\epsilon \in E} \sum_{i=1}^{n_a^d} a_{i,\epsilon} \lambda_{\epsilon,t}(a_i) \right)^{1-\alpha} \left(\sum_{\epsilon \in E} \sum_{i=1}^{n_a^d} \epsilon \lambda_{\epsilon,t}(a_i) \right)^{\alpha-1} \qquad (5) \end{aligned}$$

Step 1 (cont'd)

- Law of motion for the density (histogram): for each next period point $(\epsilon', a_{i'}) \in E \times A^d$, next period weights are $((n_\epsilon \times n_a^d)$ eqns):

$$\lambda_{\epsilon',t+1}(a_{i'}) = \sum_{\epsilon} \pi(\epsilon'|\epsilon)\, \omega_{i,\epsilon,i'} \lambda_{\epsilon,t}(i) \tag{6}$$

$$\text{where} \quad \omega_{i,\epsilon,i'} = \begin{cases} \frac{a_{i'+1} - \hat{a}'(a_{i,\epsilon}, \epsilon; \Theta(z_t, \lambda_t))}{a_{i'+1} - a_{i'}} & \text{if } \hat{a}'(.) \in [a_{i'}, a_{i'+1}] \\ \frac{\hat{a}'(a_{i,\epsilon}, \epsilon; \Theta(z_t, \lambda_t)) - a_{i'-1}}{a_{i'+1} - a_{i'}} & \text{if } \hat{a}'(.) \in [a_{i'}, a_{i'-1}] \end{cases}$$

Step 1 (end)

We get a dynamic nonlinear system of $1 + 2 \times (n_\epsilon \times n_a^p) + 2 + (n_\epsilon \times n_a^d)$ equations, which can be written as:

$$\mathbb{E}_t \left[\mathcal{F} \left(\mathbf{y_{t+1}}, \mathbf{y_t}, \mathbf{x_{t+1}}, \mathbf{x_t} \right) \right] = \mathbf{0}$$

- **y**: vector of control (jump) variables: policy functions and prices
- **x**: vector of state (predetermined) variables: aggregate shock (exogenous) and histogram weights (endogenous)

Step 2: computing the stationary steady state

The stationary steady state without aggregate risk ($z = 0$) is defined by

$$\mathcal{F}\left(\mathbf{y}^*, \mathbf{y}^*, \mathbf{x}^*, \mathbf{x}^*\right) = \mathbf{0},$$

a nonlinear system of $1 + 2 \times (n_\epsilon \times n_a^p) + 2 + (n_\epsilon \times n_a^d)$ equations with as many unknowns $\rightarrow$ very large, especially if n_a^p and n_a^d are large $\rightarrow$ don't solve directly

- solution algorithm for heterogeneous agents model without aggregate risk (Bewley-Huggett-Aiyagari)
- "smooth density approximation" (not today)

Step 2 (cont'd)

Use a solution algorithm for heterogeneous households model without aggregate risk (Bewley-Huggett-Aiyagari)

Guess capital $K^{(0)}$. For $n \geq 0$:

1. Back out prices $R^{(n)}, w^{(n)}$
2. Given prices, solve for HH policy functions $\Theta^{(n),*}$ (use an algorithm for the income fluctuation problem, e.g. collocation with numerical equation solver using initial guess $\Theta^{(n)}$)
3. Using implied policy functions, compute invariant distribution $\lambda^{(n),*}$
4. Compute aggregate supply of capital using the invariant distribution: stop if it is close enough to $K^{(n)}$, otherwise update $K^{(n+1)}$ and go back to sub-step 1

Step 2 (end)

Obtain nonlinear solution of the model without aggregate risk

- Jump variables $\mathbf{y}^*$: policy function coefficients $\Theta\left(0, \lambda^*\right)$, prices $R\left(0, \lambda^*\right), w\left(0, \lambda^*\right)$
- Predetermined variables $\mathbf{x}^*$: exogenous shock $z = 0$, histogram weights λ^*

For instance, consumption is a nonlinear function of the idiosyncratic states (a, ϵ) (resp. endogenous and exogenous) and of the aggregate state λ (endogenous)

Step 3: first-order perturbation around the steady state

Compute a first-order perturbation of the variables $\{\mathbf{y_t}, \mathbf{x_t}\}$, implicitly defined by

$$\mathbb{E}_t \left[\mathcal{F}\left(\mathbf{y_{t+1}}, \mathbf{y_t}, \mathbf{x_{t+1}}, \mathbf{x_t}\right)\right] = \mathbf{0},$$

around the stationary steady state $\{\mathbf{y}^*, \mathbf{x}^*\}$

Use standard perturbation methods

- Sims' `gensys` (2002)
- Klein (2000)

Step 3 (cont'd and end)

Obtain a linear (VAR) representation of the economy, the solution of the model with aggregate risk:

$$\begin{pmatrix} \mathbf{y}^*_{\mathbf{t+1}} \\ \mathbf{x}^*_{\mathbf{t+1}} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathbf{y}_{\mathbf{t}^*} \\ \mathbf{x}^*_{\mathbf{t}} \end{pmatrix} + \mathbf{B}\eta_{t+1}$$

Intuition: each policy function coefficient in $\Theta(z_{t+1}, \lambda_{t+1})$ and each histogram weight in λ_{t+1} varies linearly with η_{t+1} (innovation to z_{t+1}), and they are related nonlinearly to the other coefficients and weights

Applications: compute moments, IRFs, linear filtering and estimation (Kalman filter)

Application examples

- Reiter (2009): Krusell-Smith "near-aggregation" result (the mean of the asset distribution – aggregate capital – suffices to accurately forecast future prices) holds with only technology shocks, but not with large redistributive tax shocks (need ≥ 4 moments of the HH distribution)
- McKay&Reis (2016): assess the role of automatic stabilizers in the US, in a quantitatively realistic NK business cycle model with heterogeneous agents

Caveats

- Two sources of numerical error: errors in decision rules due to projection (even in stationary equilibrium), and errors due to first-order perturbation when nonlinear responses to aggregate shocks $\rightarrow$ compute EE errors
- Local method, not for large shocks
- Linear dynamics w.r.t. aggregate shocks, not for highly nonlinear problems

Homework

Visit Felipe Alves' repository at `https://github.com/FelipeAAlves/Reiter`, and run the code for the Krusell-Smith model (in Julia)

References

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