Incomplete markets and aggregate uncertainty
The Krusell-Smith algorithm

James Graham

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Overview

Households

- Continuum of households, i
- Households consume and accumulate assets/capital
- Idiosyncratic stochastic process determines household employment status
- A household's problem is

$$\max_{\{c_t^i, k_{t+1}^i\}_{t=0}^{\infty}} E \sum_{t=0}^{\infty} \beta^t \frac{(c_t^i)^{1-\gamma} - 1}{1-\gamma}$$

$$\text{s.t} \quad c_t^i + k_{t+1}^i = r_t k_t^i + w_t [(1-\tau_t)\bar{l}\varepsilon_t^i + \mu(1-\varepsilon_t^i)] + (1-\delta)k_t^i$$

$$k_{t+1}^i \geq 0$$

- Where τ_t is the tax rate, μ is unemployment benefits, $\bar{l}$ is the labor time endowment, and ε_t^i is employment status

Firms

- Markets are competitive $\Rightarrow$ representative firm
- Production function:

$$Y_t = a_t K_t^{\alpha} (\bar{l} L_t)^{1-\alpha}$$

- Where L_t is the measure of households employed
- Firm optimization yields factor prices:

$$w_t = (1-\alpha) a_t \left(\frac{K_t}{\bar{l} L_t} \right)^{\alpha}, \quad r_t = \alpha a_t \left(\frac{K_t}{\bar{l} L_t} \right)^{\alpha - 1}$$

Government

- Unemployment insurance fully funded via taxes on employed households' wages
- Government budget constraint:

$$\tau_t = \frac{\mu u_t}{\bar{l} L_t}$$

- Where the measure of unemployed households is $u_t = 1 - L_t$

Exogenous processes

- Employment status reflects *idiosyncratic uncertainty*
  - Transitions between employment and unemployment directly affect household's budget constraint
  - Transition probabilities depend on aggregate productivity
- Aggregate productivity reflects *aggregate uncertainty*
  - Directly affects production through productivity a_t
  - Two-state process:

$$a_g = 1 + \Delta, \quad a_b = 1 - \Delta$$

  - Also affects production through aggregate employment/unemployment:

$$u(a_g) = 0.04, \quad u(a_b) = 0.1$$

Parameterization

Parameters

β	γ	α	δ	$\bar{l}$	μ	Δ
0.99	1	0.36	0.025	1/0.9	0.15	0.01

Transition probabilities

$(a, \varepsilon)/(a', \varepsilon')$	$(1-\Delta, 0)$	$(1-\Delta, 1)$	$(1+\Delta, 0)$	$(1+\Delta, 1)$
$(1-\Delta, 0)$	0.525	0.35	0.03125	0.09375
$(1-\Delta, 1)$	0.038889	0.836111	0.002083	0.122917
$(1+\Delta, 0)$	0.09375	0.03125	0.291667	0.583333
$(1+\Delta, 1)$	0.009115	0.115885	0.024306	0.850694

The recursive problem

- Let Γ be a measure (distribution) over individual employment and wealth
- Then (Γ, a) is the aggregate state
- Transitions from a to a' are given by the exogenous transition matrix
- Transitions from Γ to Γ' are endogenous, and given by a law of motion:

$$\Gamma' = H(\Gamma, a, a')$$

- The household's problem written recursively:

$$\begin{aligned} v(k, \varepsilon; \Gamma, a) &= \max_{c,k'} \left\{ U(c) + \beta E[v(k', \varepsilon'; \Gamma', a')|a, \varepsilon] \right\} \\ \text{s.t} \quad c + k' &= (1 + r(K, L, a) - \delta)k \\ &\quad + w(K, L, a)[(1 - \tau(L))\bar{l}\varepsilon + \mu(1 - \varepsilon)] \\ k' &\geq 0 \\ \Gamma' &= H(\Gamma, a, a') \end{aligned}$$

The Krusell-Smith algorithm

- Central problem:
  - Household needs to forecast prices $r'(K', L', a')$, $w'(K', L', a')$
  - Prices depend on the aggregate state (Γ, a):
    - Exogenously through a and L
    - Endogenously through K
  - Need to forecast the entire (infinite dimensional?) distribution Γ
- Krusell-Smith solution:
  - Search for a *boundedly rational* solution in a finite, aggregate state space
  - Assume that households perceive that prices depend only on the first I moments $\boldsymbol{m}$ of the distribution of capital
  - Let H_I be the Perceived Law of Motion (PLM) for $\boldsymbol{m}$:

$$\boldsymbol{m}' = H_I(\boldsymbol{m}', a, a')$$

The Krusell-Smith algorithm

- An example PLM:
  - Set $I = 1$, i.e. $\boldsymbol{m} \equiv \{K\}$
  - Assume functional form of H_I is log-linear:

$$a = a_g : \quad \log K' = \alpha_0 + \alpha_1 \log K, \tag{1}$$

$$a = a_b : \quad \log K' = \beta_0 + \beta_1 \log K \tag{2}$$

- The aggregate state is now (K, a), and the household solves its problem subject to the PLM for K.

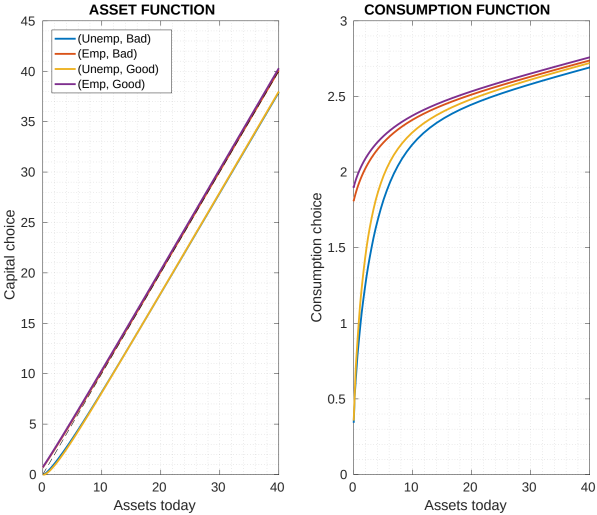
The Krusell-Smith algorithm

- Goal: find a PLM for the aggregate state which is close to the ALM in the model
- Algorithm:
  1. Guess PLM coefficients $\{\alpha_0, \alpha_1, \beta_0, \beta_1\}$, solve model, and simulate data for K
  2. Run the regressions (1) and (2) on simulated K
  3. Check the fit of the regressions: high R^2 indicates PLM is close to ALM
  4. If low R^2, update PLM using estimated regression coefficients $\{\hat{\alpha}_0, \hat{\alpha}_1, \hat{\beta}_0, \hat{\beta}_1\}$
  5. Return to step 1, and repeat until convergence of estimated coefficients
  6. If final coefficients return low R^2, increase number of moments (e.g. set $I = 2$, $\boldsymbol{m} \equiv \{K, \sigma_k\}$) and repeat

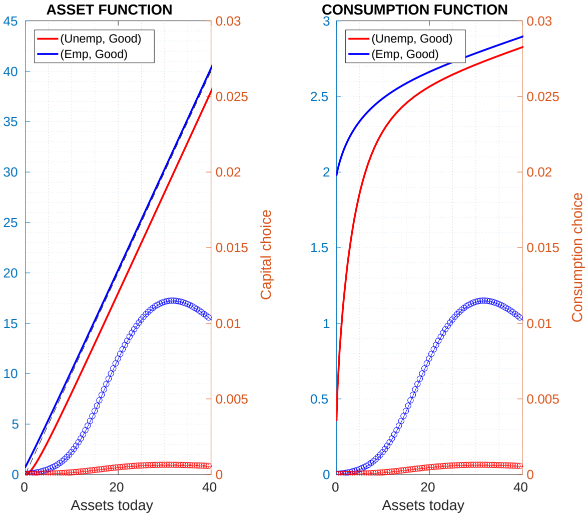
The Krusell-Smith intuition

- The standard Krusell-Smith model works with $I = 1$, m, and a log-linear functional form for the PLM
- This is due to “Near-aggregation”:
  - The wealth-rich have close-to-linear saving rules
  - Since the wealth-rich hold most of the capital stock, they matter most for determining aggregate capital
  - Aggregate shocks do not induce significant wealth redistribution across agents (e.g. little transfer from high-MPC to low-MPC agents)

Decision rules



Decision rules and distributions



Decision rules and distributions

"Approximate aggregation in dynamic economics", Walker et al. (2017)

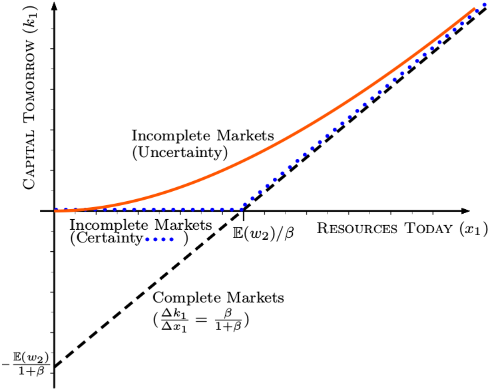


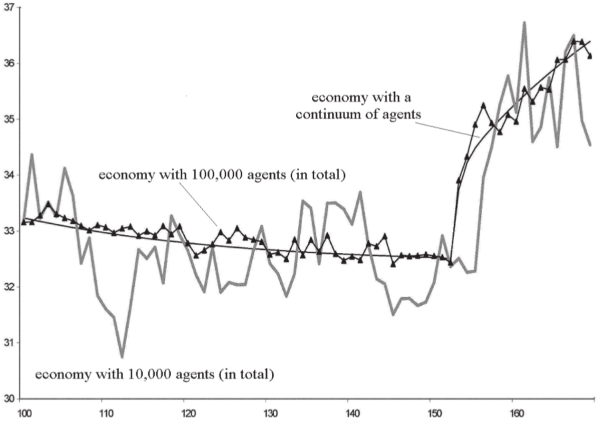
Figure 2: Savings Function ($k^{(1)}$) plotted against resources (x_1) for complete markets (dashed line), incomplete markets with certainty (dotted), and incomplete markets with uncertainty (solid). We define $w_2 = W_2\ell_2/(1-\delta+R_2)$.

Computational tips and tricks

Solving the household problem

- Value function iteration is very stable/robust
- To speed up value function iteration:
  - Use Golden Section Search to choose (a', c): robust and fast
  - Use Howard Improvement Algorithm to avoid slow maximization steps
  - During the KS simulation steps:
    - Start VFI from value function solution on previous iteration: solutions often don't change much; speed, possibly propagate errors
    - After solving VFI problem, approximate policy functions (i.e. $k' = f(k, \varepsilon; K, a)$) to avoid re-maximizing at each step of the simulation: speed, possible propagate errors

- A simulated panel of agents can generate the required distribution of wealth
- However, this is slow and there is a lot of room for simulation error:



Computational tips and tricks

Simulating aggregate state variables

- Run regressions on the same DGP each simulation by setting same 'seed' for aggregate shocks: important for stability
- Reasonable initial guess for PLM is a random walk, e.g. $\alpha_0 = \beta_0 = 0, \alpha_1 = \beta_1 = 1$
- To prevent instability from bad PLM coefficient guesses, force forecasts of K' under the PLM to remain on your state space grid:

$$\hat{K}_{t+1} = \max(\min(K_{t+1}, \overline{K}), \underline{K})$$

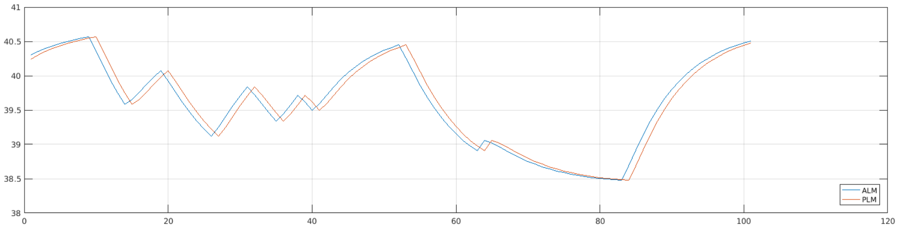
- In complicated models, may be simpler to run regressions of the form:

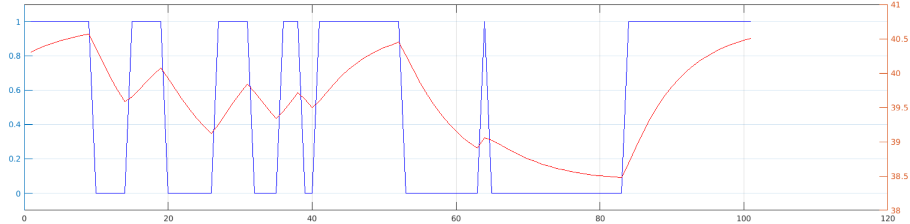
$$\log K_{t+1} = \beta_0 + \beta_1 \log a_t + \beta_2 \log K_t + \beta_3 (\log K_t)(\log A_t)$$

- Keep a running, iteration-by-iteration plot of the PLM and ALM in order to 'eyeball' convergence (see next slides)

Computational tips and tricks

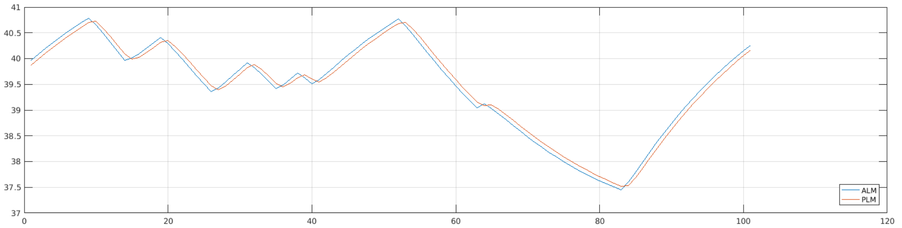
Iteration 1

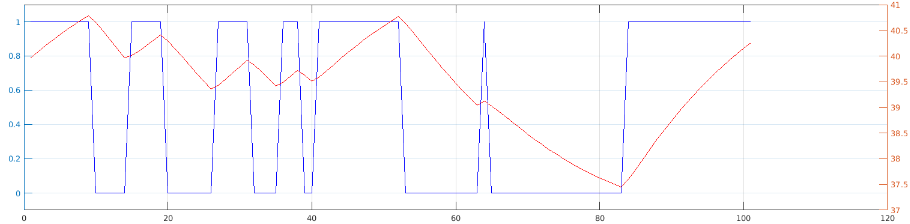




Computational tips and tricks

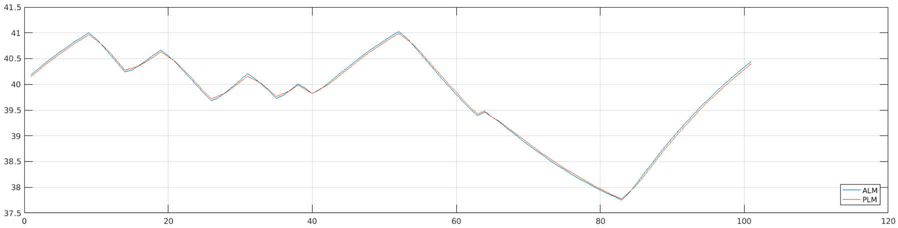
Iteration 2

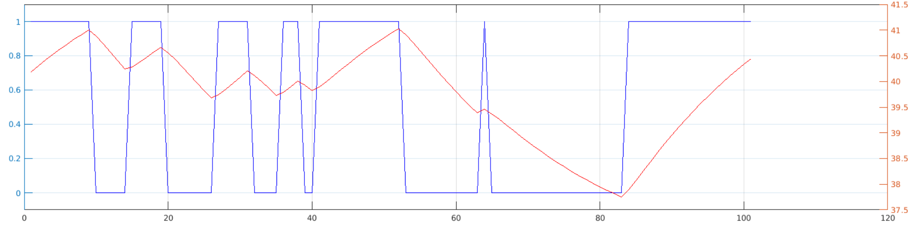




Computational tips and tricks

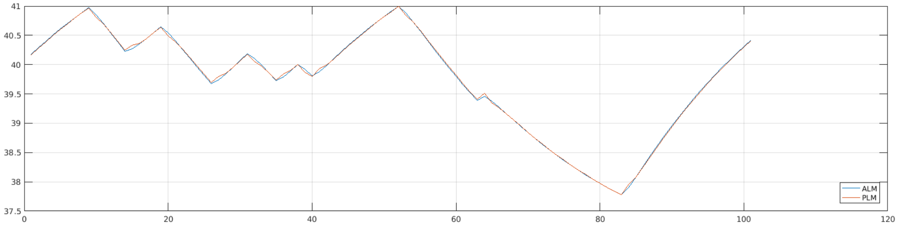
Iteration 5

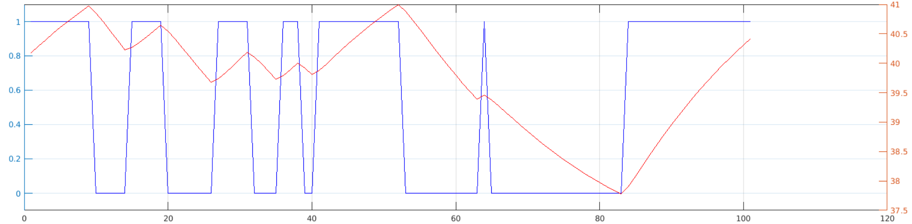




Computational tips and tricks

Iteration 11





How to think about near-aggregation

- Walker et al. (2017)
  - Give a more formal treatment of the KS near-aggregation result
  - Departure from near-aggregation depends on proportion of households with non-linear policies
  - In KS, proportion of households with large non-linearities is small
  - ‘Bin’ households across wealth distribution by size of non-linear errors, take representative households within each bin: significantly reduce the state space
- Prohl (2017)
  - Solves for the fully *rational* expectations equilibrium
  - Hugely complicated mathematical problem (polynomial chaos theory?!)
  - However, shows that the REE does *not* exhibit near aggregation: agents need to know more than K to generate accurate approximations to optimal policies
  - Thus, near aggregation is only a feature of the KS bounded rationality assumption

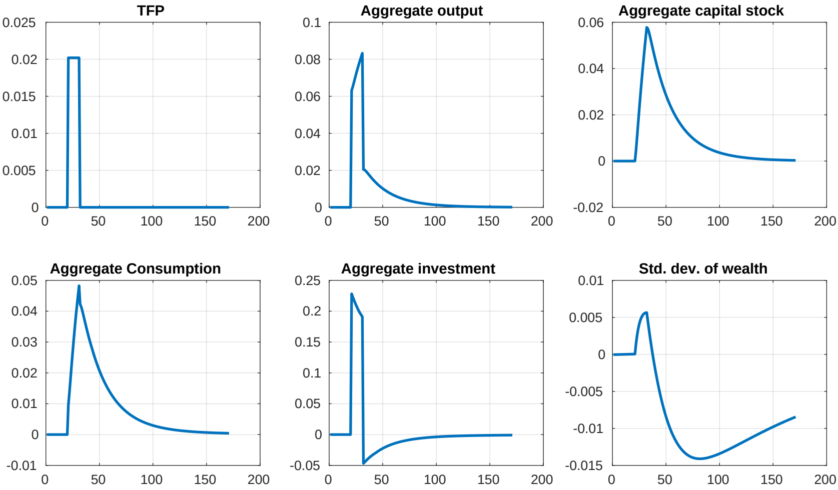
Whither the REE?

Favilukis, Ludvigson, Van Nieuwerburgh (2017)

We close this section by noting an important caution about these numerical checks. The model we are solving has a bounded rationality equilibrium, while the numerical checks are aimed at evaluating whether the model solution is consistent with the fully rational one. This incongruity between the numerical checks and the model environment is compounded by the complexity of the framework: when a very large number of agents face an infinite-dimensional state space, the fully rational equilibrium is not computable and the degree of departure from the fully rational equilibrium is unknowable. The fully rational equilibrium may not be a reasonable one with which to compare a model. The cost in terms of the agent's objectives of computing the fully rational policy could in principle be infinite, so that no expenditure of resources on computing better policies would be economically optimal for an agent. Although the numerical checks conducted here suggest that—for the aspects of the model evaluated by the checks—our equilibrium is close to what would be implied by a fully rational one, the fundamental question of how closely our equilibrium policies and prices correspond to those of the fully rational one cannot ultimately be answered. We simply conclude with a caution. As we address issues of contemporary economic importance, we would do well to acknowledge the enormous complexity of real-world problems economic players face and the possibility that the fully rational outcome is an unattainable theoretical construct appropriate only in unrealistically simplistic environments.

Simulations

Impulse response functions



Homework questions

- Use the Krusell-Smith algorithm to:
  1. Solve the incomplete markets model with aggregate uncertainty using a model with an AR(1) process for aggregate productivity
  2. Solve the incomplete markets model with aggregate uncertainty using an AR(1) process for idiosyncratic labor productivity
- Note: in each problem, you will need to consider how aggregate labor L_t depends on the aggregate state
- E.g. 1: assume that the idiosyncratic labor process is independent of the aggregate state (easy)
- E.g. 2: assume that the standard deviation of idiosyncratic shocks is a function of the aggregate state, e.g. counter-cyclical income risk (hard)