Lecture 3: DSGE and perturbation methods

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Intro

Roadmap

Purpose:

- You can find equations, or exact references here

Content:

- linear DSGEs
- perturbation methods

Tools and Algorithms

Why Japanese lost Pacific war? Comparing to Americans they have

- more experienced pilots
- better technique
- better local strategy

... But “Mitsubishi A6M Zero” is slower than “Grumman F6F”!

It was just like an old-time turkey shoot down home!

Choose a not-bad programming language / library to avoid being turkey...
... while using your brain to think about algorithms.

General Concerns

No one can solve a problem satisfying three conditions

- Fully rational solution
- Continuous shock value choices
- Global solution

Relaxing first condition - guess/verify limited functional forms
Relaxing second condition - projection methods (what Laszlo taught)
Relaxing third condition - DSGE, perturbation methods

We talk about the third today. LOCAL METHODS!

Notations

- x: state variables ("exogenous" variables)
- y: control variables ("endogenous" variables)
- x', y': variables in next period
- x_{-1}, y_{-1}: variables in last period
- $y = g(x)$: policy function
- $\mathbb{E}x' = h(x)$: law of motion of states
- ϵ: innovations, zero-mean, i.i.d., (bounded support in perturbation)

x, y, ϵ are all (column) vectors.

Representations

Any model can be summarized into equations $\mathcal{H}$ in the following forms,

"Canonical" representation (Blanchard and Kahn (1980)):

$$\mathbb{E}_t \mathcal{H}\left(x', x, x_{-1}; \epsilon\right) = 0$$

Schmitt-Grohé and Uribe (2004) representation:

$$\mathbb{E}_t \mathcal{H}\left(y', y, x', x; \epsilon\right) = 0$$

For the sake of consistency and generality, I use Schmitt-Grohé and Uribe (2004) way.

DSGE

Literature

We start with linear DSGE case. Higher order DSGE is easier to describe with perturbation method so we describe it there.
With the original equation system

$$\mathbb{E}_t \mathcal{H}\left(y', y, x', x; \epsilon\right) = 0$$

linearization / log-linearization on steady state, we get

$$A \left(\begin{array}{c} x' \\ \mathbb{E}y' \end{array} \right) + B \left(\begin{array}{c} x \\ y \end{array} \right) + C\epsilon = 0$$

and we propose linear solutions for $y = h(x)$ and $\mathbb{E}x' = h\left(x\right)$.

Blanchard and Kahn (1980)

$$A\begin{pmatrix} x' \\ \mathbb{E}y' \end{pmatrix} + B\begin{pmatrix} x \\ y \end{pmatrix} + C\epsilon = 0$$

Blanchard and Kahn (1980) says "if A is invertible then we only need to check Blanchard-Kahn condition".
If A is invertible, the system can be written as

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}\begin{pmatrix} x' \\ \mathbb{E}y' \end{pmatrix} + \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}\epsilon = 0$$

Iterating "forward", the unique stable solution for y is

$$y = S_{22}^{-1}\sum_{k=0}^{\infty}\left(S_{22}^{-1}T_{22}\right)^k D_2\mathbb{E}\left(\epsilon_{t+k}\right)$$

B-K condition: S_{22} should be explosive (all eigenvalue > 1), and they should be the only explosive eigenvalues to ensure uniqueness.

Klein (2000)

$$A\begin{pmatrix} x' \\ \mathbb{E}y' \end{pmatrix} + B\begin{pmatrix} x \\ y \end{pmatrix} + C\epsilon = 0$$

What if A is in general not invertible?
Klein (2000) gives answer - Schur decomposition.
General Schur decomposition has another name as QZ decomposition.

$$QTZ'\begin{pmatrix} x' \\ \mathbb{E}y' \end{pmatrix} + QSZ'\begin{pmatrix} x \\ y \end{pmatrix} + G\epsilon = 0$$

- Q and Z are unitary squares
- T and S are upper-triangular

Klein (2000)

Multiply Q out, use Blanchard-Kahn explosive trick,

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} Z'_{11} & Z'_{12} \\ Z'_{21} & Z'_{22} \end{pmatrix} \begin{pmatrix} x' \\ \mathbb{E}y' \end{pmatrix} + \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} Z'_{11} & Z'_{12} \\ Z'_{21} & Z'_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Solution:

$$y = g\left(x\right) = -\left(Z'_{22}\right)^{-1} Z'_{21} x$$

$$\mathbb{E}x' = h(x) = -\left(B_{11} - B_{12}\left(Z'_{22}\right)^{-1} Z'_{21}\right)^{-1} \left(A_{11} - A_{12}\left(Z'_{22}\right)^{-1} Z'_{21}\right) x$$

Sims (2002)

What if we don't know which variables are x or y?
Do a singular value decomposition, then use Klein (2000).
So-called *gensys* method.

Review

- Blanchard-Kahn: B-K condition
- Klein: relax inversibility of A
- Sims: relax definition of states/controls

My comment:

- Although all system can be written in B-K way, some are not very convenient
- Klein method is the clearest
- *Dynare* uses Sims method to identify state/control, sometimes hard to interpret

Anyway, now we know how to solve linear-quadratic system!

Perturbation Method

A Simple Way to Understand Perturbation

I don't know function $f(x)$ in general, but I know value of $f(\bar{x})$ at $\bar{x}$. If f is smooth enough, a nth-order approximation would be Taylor expansion

$$f\left(x\right)=f\left(\bar{x}\right)+\sum\frac{1}{n!}\frac{\partial^{n}f}{\partial x^{n}}\left(\bar{x}\right)\left(x-\bar{x}\right)^{n}$$

Perturbation is thinking about deviation to a system.
In the example above, we perturb a *function*.

A Simple Way to Understand Perturbation

What if we perturb an *equation*?
Suppose we know function $f(x,y)$. We are trying to find a solution $y=g(x)$ such that $f(x,g(x))=0$.
We don't know the solution in general; however, we do have some $\bar{x}$ and $\bar{y}$ such that $f(\bar{x},\bar{y})=0$.
Knowing $f(x,g(x))$ should always be 0 and $f(\bar{x},\bar{y})=0$, a first order perturbation gives

$$f(x,g(x)) = f(\bar{x},\bar{y}) + \frac{\partial f}{\partial x}(\bar{x})(x-\bar{x}) + \frac{\partial f}{\partial y}(\bar{y})\, g'(\bar{x})(x-\bar{x}) + o\left((x-\bar{x})^2\right)$$

This equation should hold for all possible x, which means coefficient should match

$$g'(\bar{x}) = -\left(\frac{\partial f}{\partial y}(\bar{y})\right)^{-1} \frac{\partial f}{\partial x}(\bar{x})$$

A Simple Way to Understand Perturbation

We can also do second, third... order perturbation and match coefficients. We don't know g in general; we know $g(\bar{x})$ from its derivatives.

Perturbation method is a LOCAL method as we approximate neighborhood of $\bar{x}$ by its behavior at $\bar{x}$.

Reference

Fernandez-Villaverde's perturbation notes
Schmitt-Grohé and Uribe (2004)

Two Key Points

The general question is still

$$\mathbb{E}_t \mathcal{H}\left(y', y, x', x\right) = 0$$

Two key points:

- Perturbation parameter is σ, the variance scalar
- There can be multiple aggregate shocks, but ONE perturbation parameter

Proposed Solution

$$y = g\left(x;\sigma\right)$$
$$x' = h\left(x;\sigma\right) + \sigma\eta\epsilon$$

We wish to find an approximate solution of g and h around $x = \bar{x}, \sigma = 0$.

Perturbation

Plug in the proposed solution

$$F(x;\sigma) \equiv \mathbb{E}_t \mathcal{H}\left(g\left(h\left(x;\sigma\right)\right), g\left(x;\sigma\right), h\left(x;\sigma\right), x;\sigma\right) = 0$$

Since $F = 0$ for any values of x and σ, the derivatives of F at any direction should be 0

$$F_{x^i \sigma^j} = 0, \forall x, \sigma, i, j$$

We consider the special case around $x = \bar{x}, \sigma = 0$,

$$F_{x^i \sigma^j}\left(\bar{x};0\right) = 0, \forall i, j$$

General solution

A n-th order perturbation solution result is

- n-th order Taylor expansion of functions g, h
- solves n-th order partial derivatives of F

We only need to solve / evaluate all the derivatives around non-stochastic steady state!

First Order Perturbation

We propose solution of g, h

$$g(x;\sigma) = g(\bar{x};0) + g_x(x - \bar{x}) + g_\sigma \sigma$$
$$h(x;\sigma) = h(\bar{x};0) + h_x(x - \bar{x}) + h_\sigma \sigma$$

$g_x, h_x, g_\sigma, h_\sigma$ should solve $F_x = 0, F_\sigma = 0$.

$$F_x = \mathcal{H}_{y'} g_x h_x + \mathcal{H}_y g_x + \mathcal{H}_{x'} h_x + \mathcal{H}_x = 0$$
$$F_\sigma = \mathcal{H}_{y'} g_x h_\sigma + \mathcal{H}_y g_\sigma + \mathcal{H}_y g_\sigma + \mathcal{H}_{x'} h_\sigma = 0$$

First Order Perturbation

$$F_x = \mathcal{H}_{y'} g_x h_x + \mathcal{H}_y g_x + \mathcal{H}_{x'} h_x + \mathcal{H}_x = 0$$
$$F_\sigma = \mathcal{H}_{y'} g_x h_\sigma + \mathcal{H}_y g_\sigma + \mathcal{H}_y g_\sigma + \mathcal{H}_{x'} h_\sigma = 0$$

Observations:

- First equation is a linear-quadratic system. We can use B-K, Klein, Sims method to solve.
- Second equation is homogeneous in g_σ and h_σ. Unique solution gives 0 - certainty equivalence.

Corollaries:

- A first order perturbation gives the same solution as linear DSGE.
- Linearization cannot capture risk regardless of state choice.

Second Order Perturbation

In addition to first order solutions,
$g_{xx}, h_{xx}, g_{x\sigma}, h_{x\sigma}, g_{\sigma\sigma}, h_{\sigma\sigma}$ should solve $F_{xx} = 0, F_{x\sigma} = 0, F_{\sigma\sigma} = 0$.
Observations: $F_{x\sigma} = 0$ is homogeneous in $g_{x\sigma}, h_{x\sigma}$, therefore 0.

And the rest is brute-force. (I'll put equations here after talk)
It is a linear equation system
What we get from this?

- non-linear state-dependent behavior
- risk adjustment (σ^2 kicks in)
- welfare measure

Example: A Simple RBC

Model

The simplest model,

$$\max \sum \beta^t u\left(c\right)$$

subject to

$$\begin{aligned} c_t + k_{t+1} &= \left(1-\delta\right) k_t + \exp\left(z_t\right) k_t^{\alpha} \\ z_{t+1} &= \rho z_t + \sigma \epsilon_t \end{aligned}$$

Euler equation

$$u'\left(c_t\right) = \beta \mathbb{E}_t u'\left(c_{t+1}\right) \left(1 + \alpha \exp\left(z_{t+1}\right) k_{t+1}^{\alpha-1} - \delta\right)$$

We can solve this model

- globally with projection method (Chase teaches next time)
- locally with first/second/whatever-order perturbation

Equations

$$\begin{aligned} u'\left(c_t\right) &= \beta \mathbb{E}_t u'\left(c_{t+1}\right)\left(1+\alpha \exp\left(z_{t+1}\right) k_{t+1}^{\alpha-1}-\delta\right) \\ c_t + k_{t+1} &= \left(1-\delta\right) k_t + \exp\left(z_t\right) k_t^{\alpha} \\ z_{t+1} &= \rho z_t + \sigma \epsilon_t \end{aligned}$$

form the $\mathcal{H}$ system we have.
There are two states (z, k) and one control c.

Given the perturbation routine we have, we only need to take Jacobian and Hessian of $\mathcal{H}$

- pen and pencil
- numerical differentiation
- symbolic differentiation
- automatic differentiation

Pen and Pencil

Jacobians

$$\left[\mathcal{H}_{y'}\right] = \left(\begin{array}{c} \beta\alpha\bar{k}^{\alpha-1}\bar{c}^{-2} \\ 0 \\ 0 \end{array}\right)$$

$$\left[\mathcal{H}_{y}\right] = \left(\begin{array}{c} -\bar{c}^{-2} \\ 1 \\ 0 \end{array}\right)$$

$$\left[\mathcal{H}_{x'}\right] = \left(\begin{array}{cc} -\frac{\beta}{\bar{c}}\alpha\left(\alpha-1\right)\bar{k}^{\alpha-2} & -\frac{\beta}{\bar{c}}\alpha\bar{k}^{\alpha-1} \\ 1 & 0 \\ 0 & 1 \end{array}\right)$$

Pen and Pencil

Hessians

$$\left[\mathcal{H}_{y'y'}\right]^1 = -2\beta\alpha\bar{k}^{\alpha-1}\bar{c}^{-3}$$
$$\left[\mathcal{H}_{y'y'}\right]^2 = 0$$

$$\left[\mathcal{H}_{y'y}\right]^1 = 0$$
$$\left[\mathcal{H}_{y'y}\right]^2 = 0$$

$$\left[\mathcal{H}_{y'x'}\right]^1 = \begin{pmatrix} \beta\alpha\left(\alpha-1\right)\bar{k}^{\alpha-2}\bar{c}^{-2} & \beta\alpha\bar{k}^{\alpha-1}\bar{c}^{-2} \end{pmatrix}$$
$$\left[\mathcal{H}_{y'x'}\right]^2 = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$$\left[\mathcal{H}_{y'x}\right]^1 = \begin{pmatrix} 0 & 0 \end{pmatrix}$$
$$\left[\mathcal{H}_{y'x}\right]^2 = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

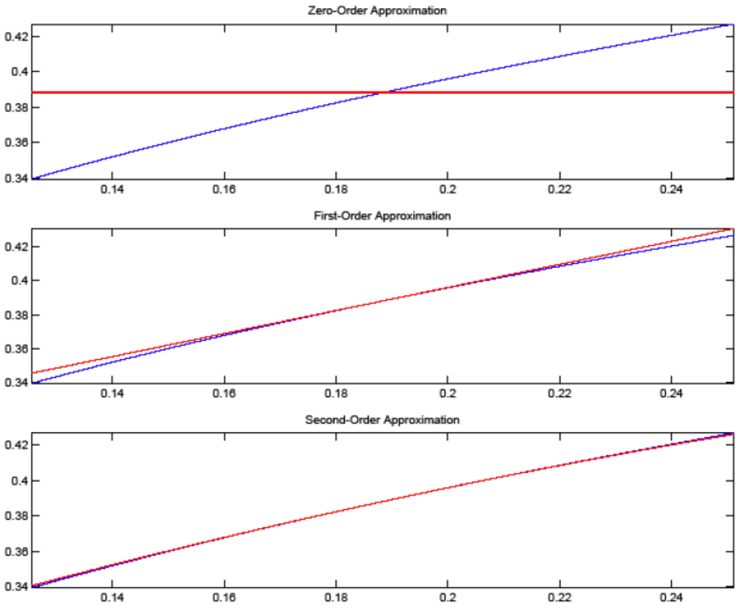
Pen and Pencil

Hessians

$$[\mathcal{H}_{yy}]^1 = 2\bar{c}^{-3}$$
$$[\mathcal{H}_{yy}]^2 = 0$$

$$[\mathcal{H}_{x'x'}]^1 = \begin{pmatrix} -\frac{\beta}{\bar{c}}\alpha(\alpha-1)(\alpha-2)\bar{k}^{\alpha-3} & -\frac{\beta}{\bar{c}}\alpha(\alpha-1)\bar{k}^{\alpha-2} \\ -\frac{\beta}{\bar{c}}\alpha(\alpha-1)\bar{k}^{\alpha-2} & -\frac{\beta}{\bar{c}}\alpha\bar{k}^{\alpha-1} \end{pmatrix} [\mathcal{H}_{x'x'}]^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$[\mathcal{H}_{xx}]^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$[\mathcal{H}_{xx}]^2 = \begin{pmatrix} -\alpha(\alpha-1)\bar{k}^{\alpha-2} & -\alpha\bar{k}^{\alpha-1} \\ -\alpha\bar{k}^{\alpha-1} & -\bar{k}^{\alpha} \end{pmatrix}$$


Zero-Order Approximation
First-Order Approximation
Second-Order Approximation
0.42
0.4
0.38
0.36
0.34
0.14
0.16
0.18
0.2
0.22
0.24

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