Introduction to Heterogeneous Agents Models

Economic Analysis and the Computation of the Distribution

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(based on Violante's and first year's lecture notes)

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The Benchmark: A ”Bewely-Aiyagari” Economy

What is a ”Bewley-Aiyagari” Model:

- Heterogeneous agents model + incomplete markets + only isiosyncratic risk.
- Class of models with non-trivial endogenous distribution of wealth.
- Growing literature in a wide variety of fields:
    - Kaplan, Moll and Violante (2017) [macro]
    - Gomes [2016] [finance]
    - Nuno and Thomas (2016) [monetary and international]

What types of questions?:

- How does uninsurable earnings variation across agents explain wealth inequality?
- What’s the link between inequality and precautionary savings?
- What are the redistributional implications of various policies?
- How do we measure social welfare?

DESCRIPTION OF THE ECONOMY

Households:

- Preferences: time-separable over streams of consumption
- Endowments: stochastic endowment of efficiency units of labor (ϵ) [Markov]
- Savings: productive capital
- Borrowing constraint: $a_{t+1} \geq -b$
- Budget constraint: $c_t + a_{t+1} = (1 + r_t)a_t + w_t\epsilon_t$

Technology:

- Representative firm with CRS production function
- Combines capital and labor (no aggregate shocks)

Competitive Markets: Final goods, labor and capital (savings)

Resource Constraint: $F(K_t, H_t) = C_t + I_t = C_t + K_{t+1} - (1 - \delta)K_t$

Some Notation:

- $g_x(a, \epsilon) \rightarrow$ policy rule for variable x
- $\lambda_t \rightarrow$ time-t measure of households on the state space.
- $A = [-b, \bar{a}] \rightarrow$ compact set of possible assets holdings
- $E = \{\epsilon_l, \epsilon_h\} \rightarrow$ countable set of labor states
- $S \rightarrow$ Cartesian product $A \times E$ with σ-algebra Σ_s
- $\mathcal{S} = (\mathcal{A} \times \mathcal{E}) \rightarrow$ subset of Σ_s, with measure $\lambda(\mathcal{S})$
- $Q((a, \epsilon), \mathcal{A} \times \mathcal{E}) \rightarrow$ probability that a household with (a, ϵ) transits to $\mathcal{A} \times \mathcal{E}$ next period

Law of Motion of λ_t:

$$Q((a,\epsilon),\mathcal{A}\times\mathcal{E}) = \mathbb{I}_{g_a(a,\epsilon)\in\mathcal{A}} \sum_{\epsilon',\in\mathcal{E}} \Pi(\epsilon',\epsilon)$$

$$\lambda_{n+1}(\mathcal{A}\times\mathcal{E}) = \int_{A\times E} Q((a,\epsilon),\mathcal{A}\times\mathcal{E})d\lambda_n$$

Problem of the Household:

$$v(a,\epsilon;\lambda) = \max_{c,a'} \{u(c) + \beta\mathbb{E}v(a',\epsilon';\lambda')\}$$

$$\text{s.t. } c + a' = (1 + r(\lambda))a + w(\lambda)\epsilon$$

$$a' \geq -b$$

A stationary RCE is $v : S \to \mathcal{R}$, HH's policy functions $g_a : S \to \mathcal{R}$ and $g_c : S \to \mathcal{R}_+$, firm's choices H^* and K^*, prices r^* and w^* and a stationary measure λ^* s.t.

(i) given prices, HH's policy functions solve their problem

(ii) given prices, the firm optimally chooses (K, H) (i.e. $r^* + \delta =$ MPK and $w^* =$ MPL)

(iii) markets clear

- Labor: $H^* = \int_{A\times E} \epsilon d\lambda^*$
- Asset: $K^* = \int_{A\times E} g_a(a, \epsilon) d\lambda^*$
- Goods: $\int_{A\times E} g_c(a, \epsilon) d\lambda^* + \delta K^* = F(K^*, H^*)$

(iv) for all $\mathcal{A} \times \mathcal{E} \in \Sigma_s$, the stationary measure λ^* is a fixed point the above LoM

THE STATIONARY EQUILIBRIUM: EXISTENCE AND UNIQUENESS

- Existence and uniqueness is shown by proving that the excess demand function is continuos, strictly monotone and intersects "zero".
- By Walras Law, it is sufficient to show existence and uniqueness in the assets market.
- Aggregate Demand and Supply for Capital:

$$K(r) = F_k^{-1}(r+\delta)$$
$$A(r) = \int_{A\times E} g_a(a,\epsilon;r)d\lambda_r^*$$

- Demand is continuos, strictly decreasing, $\lim_{r\to-\delta}K(r) = +\infty$ and $\lim_{r\to+\infty}K(r) = 0$.

What about Supply? Use of Supermartingale Theorem (Doob 1995)

- Define $M_t \equiv [\beta(1+r)]^t u'(c_t) \rightarrow$ **Euler:** $M_t \geq \mathbb{E}_t M_{t+1}$ (supermartingale)
- $M_t \geq 0$ and if $\sup_t \mathbb{E}[M_t] < \infty$, then (SMT) M_t converges a.s. to positive and finite $\bar{M}$.

Case: $\beta R > 1 \implies (\beta R)^t \rightarrow \infty$. Since $M_t \xrightarrow{\text{a.s.}} \bar{M} < \infty$, $u'(c_t) \rightarrow 0 \implies c_t \rightarrow \infty \implies$ assets diverge.

Case: $\beta R = 1$.

- Chamberlain and Wilson (2000) $\rightarrow$ if the income process is sufficiently volatile, then assets will diverge.
- If we assume $u''' > 0$, by Jensen's inequality $u'(c_t) \geq \mathbb{E}_t[u'(c_{t+1})] > u'(\mathbb{E}_t(c_{t+1})] \implies \mathbb{E}_t(c_{t+1}) > c_t$, so it diverges and so do assets.

Finally, $A(-1) \to -b$. Continuity?$\to$ IVT guarantees existence.
Monotonicity? $\to$ uniqueness.

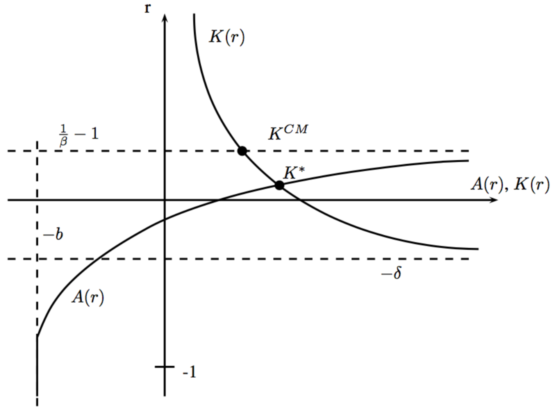


Figure 1: Existence of an equilibrium interest rate

Computation and Implications

Fixed point algorithm over the interest rate:

(i) Initial guess $r^0 \in \left(-\delta, \frac{1}{\beta} - 1\right)$, and obtain $w(r^0)$ by CRS property of production function.

(ii) Given prices, obtain HH's optimal policies $g_a(a, \epsilon; r^0)$ and $g_c(a, \epsilon; r^0)$.

(iii) Using the optimal policies, solve for the invariant distribution $\lambda(r^0)$. More on this later. Key step for speed.

(iv) Compute aggregate demand (firm's FOC) and aggregate supply (integrate policies) for capital.

(v) If the capital market does not clear, update according to a bisection method:

$$r^1 = \frac{1}{2}\{r^0 + [F_K(A(r^0), H) - \delta]\}$$

(vi) Iterate from step 1 until interest rates have converged.

COMMENT: TRICKS FOR STABILITY

There are 2 tricks for stability, but make the algorithm slower:

1. Use bisection method + dampening to solve for r

$$r^1 = \omega r^0 + (1-\omega)\left[F_K(A(r^0)) - \delta\right]$$

2. When solving for the HH problem using a new r^{n+1}, don't initialize the algorithm using the solution associated to r^n. Instead, always use the same initial guess to avoid propagation of errors.

Precautionary Savings:

- K^{CM} corresponds to the SS equilibrium in a neoclassical growth model.
- $K^* - K^{CM}$ denotes the addiotional savings for self-insurance due to precautionary savings.

Comparative Statics:

- Borrowing Limit: if $\uparrow b$, $A(r)$ shifts leftwards so that $\uparrow r$ and there is less precautionary savings.
- Risk-aversion: By increasing γ, HH are more concerced about consumption smoothing so $A(r)$ shifts downwards, and there is higher precautionary savings and a lower r.
- Higer income risk: $A(r)$ shifts downwards, higher precautioanry savings and lower interest rate.

Equilibrium

Algorithms to Compute the Invariant Distribution

There are 3 main approaches to compute the distribution:

1. Monte-Carlo simulation (easiest but inefficient)
2. Piecewise linear approximation (requires non-linear solver)
3. Discretization of the density function (exploits sparsity)

- It mainly involves generating a large panel of households ($I = 10,000, T \geq 1,000$) and keeping track of them over time.
- This makes it memory and time consuming.
- It may be useful if the dimension of the state space is high, since it does not have the curse of dimensionality.

The algorithm is as follows:

1. Initialize the sample with some (a_0, ϵ_0). Use the policy rule $g_a(a, \epsilon)$ and $\Pi(\epsilon', \epsilon)$ to simulate next period's cross-sectional distribution.
2. Drop the first (100?) observations.
3. At each period t compute a vector of moments μ_t^I.
4. Stop when $\|\mu_t^I - \mu_{t-1}^I\| \approx 0$.

- Approximate distribution Λ by piecewise-linear functions and iterate on the LoM for the distribution.
- Should use piecewise linear basis due to it's shape-preserving property. We need Λ to be increasing on a.

The algorithm is as follows:

1. Make a finer grid for assets with size N. It doesn't need to be evenly spaced.
2. Choose an initial Λ^0 over $A \times E$, for example

$$\Lambda^0(a_k, \epsilon_j) = \frac{a_k - \underline{a}}{\bar{a} - \underline{a}} \Pi_j^*$$

3. Update the distribution. For each (a_k, ϵ_j)

$$\Lambda^1(a_k, \epsilon_j) = \sum_{\epsilon_i \in E} \Pi_{ij} \Lambda^0(g_a^{-1}(a_k, \epsilon_i), \epsilon_i)$$

   Note that

   - we need a non-linear solver to compute $a = g_a^{-1}(a_k, \epsilon_i)$ using the budget constraint $c(a, \epsilon_i) + a_k = Ra + \epsilon_i$.
   - $g_a^{-1}(a_k, \epsilon_i)$ is not on the grid so we need to (linearly) interpolate.
4. Iterate until $\|\Lambda^n - \Lambda^{n-1}\| \approx 0$

To compute the aggregate supply of capital we need to take a stance on how to integrate between gridpoints (a_n, a_{n+1}). We assume the distribution of assets is uniform within this range:

Assume the distribution of assets is <u>uniform</u> within $[a_n, a_{n+1}]$:

$$\begin{aligned}\int_{a_n}^{a_{n+1}} a d\Lambda(a, y_i) &= \int_{a_n}^{a_{n+1}} a \left[\frac{\Lambda(a_{n+1}, \epsilon_i) - \Lambda(a_n, \epsilon_i)}{a_{n+1} - a_n}\right] da \\ &= \left[\frac{\Lambda(a_{n+1}, \epsilon_i) - \Lambda(a_n, \epsilon_i)}{2}(a_{n+1} + a_n)\right]\end{aligned}$$

so that

$$A = \sum_{\epsilon \in E} \sum_{n=1}^{N-1} \frac{\Lambda(a_{n+1}, \epsilon_i) - \Lambda(a_n, \epsilon_i)}{2}(a_{n+1} + a_n) + \Lambda(\underline{a}, \epsilon_i)\underline{a}$$

- The idea is to approximate the invariant density function $\lambda(a, \epsilon)$.
- We need to build a finer grid for assets.
- We will construct the sparse transition matrix Q and exploit the sparsity of this matrix (saves memory!).

Algorithm to construct Q:

1. Since ϵ' is independent of a,

$$Q(a, \epsilon; a', \epsilon') = \Pi(\epsilon, \epsilon') \text{ rowkron } Q_a(a, \epsilon; a')$$

2. Since $a' = g_a(a, \epsilon)$ is not on the grid, we need to find the two adjacent points $a_{k+1} \leq a' <, a_k$ and define

$$Q_a(a, \epsilon; a_k) = \frac{a_{k+1} - g_a(a, \epsilon)}{a_{k+1} - a_k}$$
$$Q_a(a, \epsilon; a_{k+1}) = \frac{g_a(a, \epsilon) - a_k}{a_{k+1} - a_k}$$

3. Choose an initial $\lambda^0(a_k, \epsilon_j) = \frac{1}{N_e N_a}$
4. Use the linearity of the LoM

$$\lambda_{n+1} = Q' \lambda_n$$

   It is important to define Q as a sparse matrix in your coding to save memory.
5. Iterate until $\|\lambda^n - \lambda^{n-1}\| \approx 0$

The aggregate supply of capital is

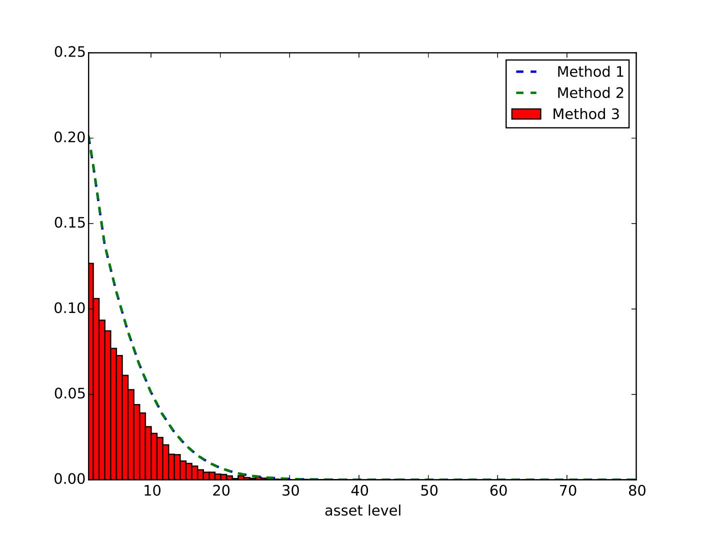
$$A = \sum_{\epsilon_j \in E} \sum_{a_k \in A} a_k \lambda(a_k, \epsilon_j)$$

Instead of iterating on λ_n, one could use an eigenvector approach:

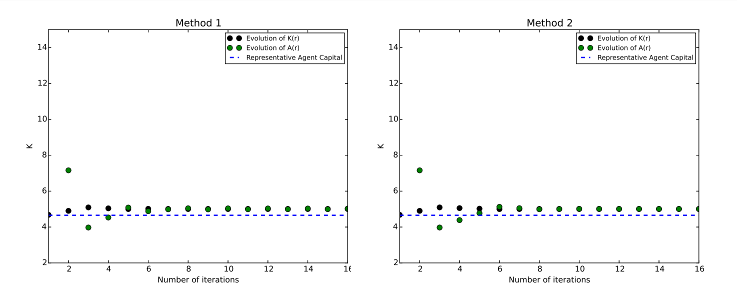
- An invariant distribution is the (normalized) eigenvector of the matrix Q associated to eigenvalue 1.
- We need to guarantee this eigenvector is unique $\rightarrow$ Perron-Frobenius Theorem.
- If possible, try to use ”eigs” (Julia) to exploit the sparsity of Q. Note that you can ask for the eigenvector associated to the largest eigenvalue only.
- But sometimes Q can be very large and very sparse, having many eigenvalues very close to one, some with eigenvectors with negative components.
- Then one has to perturb the zero entries of Q by a constant $\eta \leq \frac{\text{min}Q}{2N}$.
- Then find the unique non-negative eigenvector.

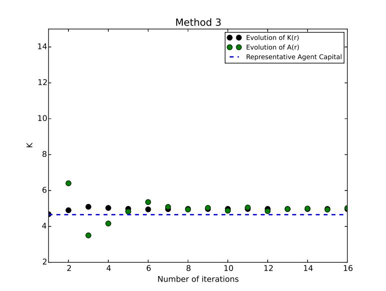
Solving the Aiyagari Model by the 3 Methods (From Felipe Alves' Solution to PSET)

method1: discretization(iteration). method2: discretization(eigenvector); method3 : simulation.



EXAMPLE: CONVERGENCE





Example: Government Bonds

EXAMPLE: GOVERNMENT BONDS

- In this example we allow for government bonds as only savings vehicle.
- The government collects lump-sum taxes to finance expenditures and debt repayment.
- We model model a symmetric (left) and asymmetric (right) endowment of labor.