Accuracy of Solution and Models with Liquid/Illiquid Assets

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July 20, 2017

Accuracy Tests

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Goal:

- Measure accuracy of numerical solution
- Issue could be number/position of nodes, basis functions, ...

Tests based on

- Euler equation errors
- Den Haan-Marcet statistic

Apply to income fluctuation problem (IFP), but can be applied to other problems

Standard Euler-equation test

- Euler equation if unconstrained

$$u_c(c_t) - \beta R \mathbb{E}_{s_{t+1}}[u_c(c_{t+1})|s_t] = 0 \tag{EE}$$

Idea

EE ≈ 0 at nodes, but may be far from 0 outside of nodes

- Define relative approximation error η as value such that, at given point (a, s) in state space, EE holds with equality

$$u_c(c(a,s)(1-\eta)) = \beta R \mathbb{E}_{s'}[u_c(c(a'(a,s),s'))|s]$$

Then Euler equation error is

$$\eta = 1 - \frac{u_c^{-1}\left(\beta R \mathbb{E}_{s'}[u_c(c(a'(a,s),s'))|s]\right)}{c(a,s)}$$

Interpretation

- Error of 0.01 means agent makes a mistake equivalent to \$ 1 for each \$ 100 consumed in any period
- Error in value function (i.e. implied cost in terms of agent's welfare) is of order of η^2 (Santos 2000)

Reporting

- Long simulation, exclude states where agent is constrained, take maximum and average of errors in absolute value
- Build fine grid for assets $\{\alpha_i\}_i$ (different than grid used for solution approximation), compute $\eta(\alpha_i, s_j)$ for all i, j and plot
- Usually reported in base 10 log (-2 means \$ 1 for each \$ 100). Acceptable average error ≤ -4

Test only looks at one-period ahead inaccuracies: tiny errors can accumulate ...

Dynamic Euler-equation test

Excluding periods when constaint binds, compare series $\{\hat{c}_t, \hat{a}_t\}_{t=1}^{T}$ generated with numerical solution and $\{\tilde{c}_t, \tilde{a}_t\}_{t=1}^{T}$, where

- $\tilde{a}_1 = \hat{a}_1$
- from EE and budget constraint

$$\tilde{c}_t = u_c^{-1}\left(\beta R \mathbb{E}_{s'}[u_c(c(a'(\tilde{a}_t, s_t), s'))|s_t]\right)$$

$$\tilde{a}_{t+1} = R\tilde{a}_t - \tilde{c}_t + y(s_t)$$

Notes

- In $\{\tilde{c}_t, \tilde{a}_t\}_{t=1}^{T}$, numerical solution used only indirectly to compute conditional expectation
- Report average or maximum % absolute errors
- Demanding test: occasional large error may not mean solution is generally bad. If large difference between average and maximum, worth inspecting the path

Den Haan-Marcet Statistic

- EE implies that residual

$$\epsilon_{t+1} = u_c(c_t) - \beta R u_c(c_{t+1})$$

  is uncorrelated with any variable in the information set at time t
- Therefore for each t

$$\mathbb{E}_t[\epsilon_{t+1}\mathbf{z}_t] = 0 \tag{1}$$

  where $\mathbf{z}_t$ is vector of variables known at t, e.g. $[\{y_j, c_j, a_j\}_{j=0}^{t}]$

Idea

A poor approximate solution does not satisfy this property

- Compute LHS of 1 under a given approximate solution by simulating a path of length T and constructing

$$\mathbf{q}_T = \frac{\sum_{t=1}^{T} \hat{\epsilon}_{t+1}\hat{\mathbf{z}}_t}{T}$$

  where $\hat{\epsilon}_{t+1}$ and $\hat{\mathbf{z}}_t$ are simulated counterparts of ϵ_{t+1} and $\mathbf{z}_t$ in 1

- Under mild conditions, $\sqrt{T}\mathbf{q}_T \xrightarrow{d} N(0,V)$, and the quadratic form of the test statistic
$$T\mathbf{q}_T'\hat{V}_T^{-1}\mathbf{q}_T \xrightarrow{d} \chi^2_r$$
where $\hat{V}_T^{-1}$ is the inverse of the matrix
$$\hat{V}_T = \frac{\sum_{t=1}^{T}\hat{\epsilon}_{t+1}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\hat{\epsilon}_{t+1}}{T}$$

Notes

- Given a level of approximation of the solution, can always find T large enough that the null is rejected, i.e. test is failed by the approximate solution
- Not a problem if **comparing different approximation methods**: can compare them by computing - for each method - the smallest T such that the test is failed, and then compare these thresholds

Notes (continued)

- In order to **assess the accuracy of a given solution**, Den Haan and Marcet suggest this procedure:
  1. Draw a long simulation path from the approximate solution and compute the statistic (e.g. T=5000)
  2. Record whether the value of the statistic is above/below the 2.5% critical value from the χ^2_r distribution
  3. Repeat several times the previous steps (e.g. N=500) and calculate the fraction of simulations where the test fails (i.e. $>$ critical value). If this fraction is substantially different than the theoretical 5%, then the test indicates the solution is inaccurate

Excercise: apply one of these 3 tests (one-period/dynamic Euler equation tests, DHM statistic) to one of the solutions to the IFP you computed in the past weeks and discuss the results

IFP with Liquid/Illiquid Assets

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IFP with 2 assets:

- one adjusted at no cost ("liquid asset", a)
- to adjust the other, must pay fixed cost κ ("illiquid asset", d)
- illiquid asset has higher return, $R^d > R^a$ (or in terms of prices: $q^a > q^d$)

When adjusting illiquid asset d,

$$V^A(a, d, s) = \max_{c, a', d'} \left\{ u\left(c\right) + \beta \mathbb{E}_{s'}[\mathbf{V}(a', d', s')|s] \right\}$$

subject to

$$c + q^a a' + q^d d' = a + d + y(s) - \kappa;\ a' \geq 0;\ d' \geq 0$$

When not adjusting,

$$V^N(a, d, s) = \max_{c, a'} \left\{ u\left(c\right) + \beta \mathbb{E}_{s'}[\mathbf{V}(a', d', s')|s] \right\}$$

subject to

$$c + q^a a' = a + y(s);\ d' = \frac{d}{q^d};\ a' \geq 0$$

where

$$\mathbf{V}(a, d, s) = \max \left\{ V^A(a, d, s), V^N(a, d, s) \right\}$$

When adjusting illiquid asset d,

$$V^A(x, s) = \max_{c,a',d'} \{u(c) + \beta \mathbb{E}_{s'}[\mathbf{V}(a', d', s')|s]\}$$

subject to

$$c + q^a a' + q^d d' = x - \kappa;\ a' \geq 0;\ d' \geq 0$$

When not adjusting,

$$V^N(x, d, s) = \max_{c,a'} \{u(c) + \beta \mathbb{E}_{s'}[\mathbf{V}(a', d', s')|s]\}$$

subject to

$$c + q^a a' = x;\ d' = \frac{d}{q^d};\ a' \geq 0$$

where

$$\mathbf{V}(a, d, s) = \max\left\{V^A(a + d + y(s), s), V^N(a + y(s), d, s)\right\}$$

FOCs

- Due to non-convexity, necessary but not sufficient

Define operator

$$\tilde{\max}\left\{f,g\right\}(a,d,s) = \begin{cases} f(\cdot) & \text{if } V^A(a+d+y(s),s) > V^N(a+y(s),d,s) \\ g(\cdot) & \text{otherwise} \end{cases}$$

and functions

$$FV_a(a,d,s) = \mathbb{E}_{s'}\left[\tilde{\max}\left\{V_x^A, V_x^N\right\}(a,d,s')|s\right]$$

$$FV_d(a,d,s) = \mathbb{E}_{s'}\left[\tilde{\max}\left\{V_x^A, V_d^N\right\}(a,d,s')|s\right]$$

No-adjust case:

$$u_c(c) \geq \frac{\beta}{q^a} F V_a(a', d', s) \tag{EE-N}$$

$$V_x^N(x, d, s) = u_c(c) \tag{Env-N 1}$$

$$V_d^N(x, d, s) = \frac{\beta}{q^d} F V_d(a', d', s) \tag{Env-N 2}$$

Adjust case:

$$u_c(c) \geq \frac{\beta}{q^a} F V_a(a', d', s) \tag{EE-A}$$

$$u_c(c) \geq \frac{\beta}{q^d} F V_d(a', d', s) \tag{Portfolio}$$

$$V_x^A(x, s) = u_c(c) \tag{Env-A}$$

"PFI" Algorithm

Essentially, Kaplan, Violante (2014)

- Choose grids for continuous variables x, a, d
- **Iteration 0:**
  - Guess $c^A(x_i, s_j)$, $c^N(x_i, d_k, s_j)$, $V^A(x_i, s_j)$, $V^N(x_i, d_k, s_j)$, $V_d^N(x_i, d_k, s_j)$. For instance,

$$c^A(x_i, s_j) = \max\{x_i - \kappa, \underline{c}\}$$
$$c^N(x_i, d_k, s_j) = x_i$$
$$V^A(x_i, s_j) = u(c^A(x_i, s_j))$$
$$V^N(x_i, d_k, s_j) = u(c^N(x_i, d_k, s_j))$$
$$V_d^N(x_i, d_k, s_j) = 0$$

- **Iteration t - preliminary step:**

- Define interpolants $\Phi c^A(x,s)$, $\Phi c^N(x,d,s)$, $\Phi V^A(x,s)$, $\Phi V^N(x,d,s)$, $\Phi V_d^N(x,d,s)$ for $c^A(x,s)$, $c^N(x,d,s)$, $V^A(x,s)$, $V^N(x,d,s)$, $V_d^N(x,d,s)$, respectively
- Compute
$$\tilde{V}(x_i,d_k,s_j) = \Phi V^A(x_i+d_k,s_j) - V_x^N(x_i,d_k,s_j)$$
and define associated interpolant $\Phi\tilde{V}(x,d,s)$ to solve $m\tilde{a}x$ decision
- Compute
$$FV_a(a_m,d_k,s_j) = \mathbb{E}_{s'}[\tilde{\max}\{u_c(\Phi c^A(a_m+d_k+y(s'),s')), u_c(\Phi c^N(a_m+y(s'),d_k,s'))\}|s_j]$$
$$FV_d(a_m,d_k,s_j) = \mathbb{E}_{s'}[\tilde{\max}\{u_c(\Phi c^A(a_m+d_k+y(s'),s')), \Phi V_d^N(a_m+y(s'),d_k,s')\}|s_j]$$
$$CV(a_m,d_k,s_j) = \mathbb{E}_{s'}[\max\{\Phi V^A(a_m+d_k+y(s'),s')), \Phi V^N(a_m+y(s'),d_k,s')\}|s_j]$$
and define associated interpolants $\Phi FV_a(a,d,s)$, $\Phi FV_d(a,d,s)$ and $CV(a,d,s)$

- **Iteration t - Adjust case:**

- Define

$$G^A_{EE}(a', d', x, s) = u_c(x - \kappa - q^a a' - q^d d') - \frac{\beta}{q^a}\Phi FV_a(a', d', s)$$

$$G^A_{Port}(a', d', x, s) = u_c(x - \kappa - q^a a' - q^d d') - \frac{\beta}{q^d}\Phi FV_d(a', d', s)$$

- For each x_i and s_j ...
  1. if $x_i - \kappa < 0$, then $c^A(x_i, s_j) = \underline{c}$ and $V^A(x_i, s_j) = -\infty$
- ...otherwise consider the following cases:
  1. if

$$G^A_{EE}(0, 0, x_i, s_j) > 0,\ G^A_{Port}(0, 0, x_i, s_j) > 0$$

then store

$$\hat{a}' = 0,\ \hat{d}' = 0,\ \hat{c} = x_i - \kappa,\ \hat{V}^A = u(\hat{c}) + \beta\Phi CV(0, 0, s_j)$$

as a local solution

- **Iteration t - Adjust case (continued):**

2. if
$$G^A_{EE}(0, \hat{d}', x_i, s_j) > 0 \text{ where } \hat{d}' : G^A_{Port}(0, \hat{d}', x_i, s_j) = 0$$
then store
$$\hat{a}' = 0,\ \hat{d}',\ \hat{c} = x_i - \kappa - q^d \hat{d}',\ \hat{V}^A = u(\hat{c}) + \beta \Phi CV(0, \hat{d}', s_j)$$
as a local solution
3. if
$$G^A_{Port}(\hat{a}', 0, x_i, s_j) > 0 \text{ where } \hat{a}' : G^A_{EE}(\hat{a}', 0, x_i, s_j) = 0$$
then store
$$\hat{a}',\ \hat{d}' = 0,\ \hat{c} = x_i - \kappa - q^a \hat{a}',\ \hat{V}^A = u(\hat{c}) + \beta \Phi CV(\hat{a}', 0, s_j)$$
as a local solution
4. look for local solution(s) of
$$\begin{cases} G^A_{EE}(\hat{a}', \hat{d}', x_i, s_j) = 0 \\ G^A_{Port}(\hat{a}', \hat{d}', x_i, s_j) = 0 \end{cases}$$
and store them

- **Iteration t - Adjust case (continued):**
  - Set $c^A(x_i, s_j) = c^*$ and $V^A(x_i, s_j) = V^{A*}$ where (c^*, V^{A*}) is the local solution with $V^{A*} > \hat{V}^A$ for all other $\hat{V}^A$
- **Iteration t - No-Adjust case:**
  - Analogous, but remember to set

$$V_d^N(x_i, d_k, s_j) = \frac{\beta}{q^d} \Phi F V_d(a^*, d^*, s_j)$$

  after determining V^{N*} among the local solutions
- Repeat all parts of **Iteration t** until convergence

EGM Algorithm

Described in Druedahl, Jørgensen (2017)

- Faster than previous algorithm because it avoids root-finding
- Main challenges (in addition to non-sufficiency of FOCs):
  1. no simple algorithm for finding neighboring points in the multi-dimensional irregular grid (1d EGM: easy, bisection search)
  2. no prior knowledge of where the constraints bind (1d EGM: all a lower than endogenous asset level such that $a' = -\underline{a}$)
- In addition to extensive accuracy and speed comparisons, Druedahl and Jørgensen define a broad class of models, in terms of sufficient and necessary conditions on model fundaments, where their method can be applied

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