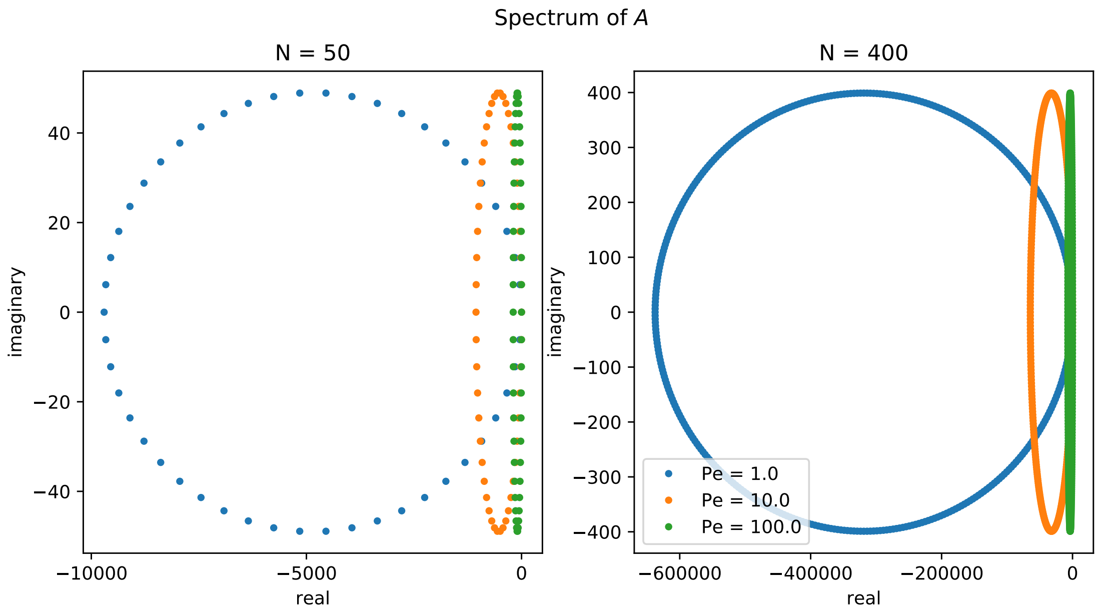

Figure 1: The spectrum of A. For this visualization we assume periodic boundary conditions. For Dirichlet boundary conditions all eigenvalues are negative real numbers.

1 Analysis of the power iteration

We analyse the rate of convergence of the power method for the linear advection-diffusion equation. Let

$$F(u) = a\partial_x u + b\partial_{xx} u$$

be the right hand side of the differential equation. Let $u \in L^2([0,1]/1)$

The eigenvalues of discretized one-dimensional Laplace operator $A_{Dif} \in \mathbb{R}^{N\times N}$ on the interval $[0,1]$ with periodic boundary conditions are given by

$$\lambda_j = \begin{cases} -\frac{4}{h^2}\sin^2\left(\frac{\pi(j-1)}{2(N+1)}\right), & \text{if j is odd} \\ -\frac{4}{h^2}\sin^2\left(\frac{\pi j}{2(N+1)}\right), & \text{if j is even} \end{cases}, \quad j = 1,\dots,N.$$

We investigate the rate of convergence for the power method given A_{Dif} and an initial vector v. Consider $v = \frac{1}{N}\sum_{j=1}^N v_j$, where each v_j is the normalized eigenvector corresponding to the eigenvalue λ_j. After n power iteration we underestimate the absolutely largest eigenvalue λ_N by a factor of

$$\frac{\|A_{Dif}^{n-1}v\|_2}{\|A_{Dif}^{n}v\|_2}|\lambda_N| = \sqrt{\frac{\sum_{j=1}^N \lambda_j^{2(n-1)}}{\sum_{j=1}^N \lambda_j^{2n}}}|\lambda_N| = \sqrt{\frac{\sum_{j=1}^N \sin^{4(n-1)}\left(\frac{\pi j}{2(N+1)}\right)}{\sum_{j=1}^N \sin^{4n}\left(\frac{\pi j}{2(N+1)}\right)}}\sin^2\left(\frac{\pi N}{2(N+1)}\right).$$

The first equality holds since A_{Dif} is symmetric. Therefore all eigenvectors are orthogonal. In order to continue our analysis and get some asymptotic bounds we interpret the sum of sine functions as an integral approximated by the trapezoidal rule. We use the nodes $j/(N+1)$ for $j = 0, \dots, N+1$.

$$\int_0^1 \sin^{4n}\left(\frac{\pi x}{2}\right) = \frac{1}{(N+1)}\left(2\sum_{j=1}^{N} \sin^{4n}\left(\frac{\pi j}{(N+1)}\right) + \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{12(N+1)^2}\right)$$

Note that the error of the approximation is strictly positive since the second derivative math.stackexchange[1] we can blissfully accept the identity

$$I_n := \int_0^1 \sin^{4n}\left(\frac{\pi x}{2}\right) = \frac{\Gamma(2n+0.5)}{\sqrt{\pi}\,\Gamma(2n+1)}.$$

In order to simplify our calculations we take the limit of N

$$\frac{\|A_{Dif}^{n-1}v\|}{\|A_{Dif}^{n}v\|}|\lambda_N| \xrightarrow{N\to\infty} \sqrt{\frac{I_n}{I_{n-1}}},$$

where

$$\begin{aligned}
\frac{I_n}{I_{n-1}} &= \frac{\Gamma(2n-1)\Gamma(2n+0.5)}{\Gamma(2n+1)\Gamma(2n-1.5)} \\
&= \frac{(2n-2)!}{(2n)!}\frac{\Gamma(2n+0.5)}{\Gamma(2n-1.5)}\frac{\Gamma(2n)}{\Gamma(2n)}\frac{\Gamma(2n-2)}{\Gamma(2n-2)} \\
&= \frac{1}{2n(2n-1)}\frac{2^{1-4n}\sqrt{\pi}}{2^{5-4n}\sqrt{\pi}}\frac{\Gamma(4n)}{\Gamma(4n-4)}\frac{\Gamma(2n-2)}{\Gamma(2n)} \\
&= \frac{1}{32n(2n-1)}\frac{(4n-1)!}{(4n-5)!}\frac{(2n-3)!}{(2n-1)!} \\
&= \frac{(4n-1)(4n-2)(4n-3)(4n-4)}{32n(2n-1)^2(2n-2)} \\
&= \frac{(4n-1)(4n-3)}{8n(2n-1)} \\
&= \frac{4n-1}{4n}\frac{4n-3}{4n-2} \\
&= \left(1-\frac{1}{4n}\right)\left(1-\frac{1}{4n-2}\right)
\end{aligned}$$

For the third equality we applied the duplication formula for the gamma function. All in all we underestimate the absolutely largest eigenvalue λ_N by a factor of

$$\lim_{N\to\infty}\frac{\|A_{Dif}^{n-1}v\|}{\|A_{Dif}^{n}v\|}|\lambda_N| = \sqrt{\left(1-\frac{1}{4n}\right)\left(1-\frac{1}{4n-2}\right)} \approx 1 - \frac{1}{4n-1}$$

at the limit $N \to \infty$.

[1] `https://math.stackexchange.com/questions/50447/integration-of-powers-of-the-sin-x`