OPROIU MATEI

311

PROIECT STATISTICĂ

I 1) a) $X \sim Cauchy(x_0, \gamma)$

$$f(x; x_0, \gamma) = \frac{1}{\pi\gamma\left(1+\left(\frac{x-x_0}{\gamma}\right)^2\right)}$$

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi\gamma\left(1+\left(\frac{t-x_0}{\gamma}\right)^2\right)} dt =$$

SV: $\frac{t-x_0}{\gamma} = u$ $\quad t=-\infty \Rightarrow u=-\infty$, $\quad t=x \Rightarrow u = \frac{x-x_0}{\gamma}$

$+\frac{1}{\gamma} dt = du$

$$= \frac{1}{\pi\gamma}\int_{-\infty}^{\frac{x-x_0}{\gamma}} \frac{1}{(1+u)^2}(+\gamma)\,du = \frac{1}{\pi} arctg(u)\Big|_{-\infty}^{\frac{x-x_0}{\gamma}} = \frac{1}{2} + \frac{1}{\pi} arctg\left(\frac{x-x_0}{\gamma}\right)$$

Aplic met inversei

Fie $U \sim U[0,1]$

$$F(X) = U \Leftrightarrow \frac{1}{\pi} arctg\left(\frac{X-x_0}{\gamma}\right) + \frac{1}{2} = U \Leftrightarrow \frac{X-x_0}{\gamma} = tg\left(\pi\left(U-\tfrac{1}{2}\right)\right)$$

$$\Leftrightarrow X = x_0 + \gamma\, tg\left(\pi\left(U - \tfrac{1}{2}\right)\right)$$

b) $X \sim \begin{pmatrix} 1 & 2 & 3 \\ 1/2 & 1/3 & 1/6 \end{pmatrix}$ (în R putem face sample direct din X)

$$F(x) = \begin{cases} 0 & , x<1 \\ \frac{1}{2} & , x\in[1,2) \\ \frac{1}{2}+\frac{1}{3} = \frac{5}{6} & , x\in[2,3) \\ 1 & , x \geq 3 \end{cases}$$

Fie $U \sim U[0,1]$

$F(X) = U$

$$\left.\begin{aligned} 0 \leq U < \tfrac{1}{2} &\Rightarrow X=1 \\ \tfrac{1}{2} \leq U < \tfrac{5}{6} &\Rightarrow X=2 \\ \tfrac{5}{6} \leq U \leq 1 &\Rightarrow X=3 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow X = \begin{cases} 1, & 0 \leq U < \frac{1}{2} \\ 2, & \frac{1}{2} \leq U < \frac{5}{6} \\ 3, & \frac{5}{6} \leq U < 1 \end{cases}$$

c) $X \sim \begin{pmatrix} x_1 & x_2 \\ 1-p & p \end{pmatrix}$

$F(x) = \begin{cases} 0 & , x < x_1 \\ 1-p & , x \in [x_1, x_2) \\ 1 & , x \geq x_2 \end{cases}$ $\quad F(x) = U \Leftrightarrow X = \begin{cases} x_1 & , U \leq 1-p \\ x_2 & , \text{altfel} \end{cases}$

II - Def: $p(x|\theta) = h(x) \cdot \exp\left(\sum_{i=1}^{n} \eta_i(\theta) T_i(x) - A(\theta)\right)$

1) a) $X \sim B(3, p)$, $n = 3$

$f(x; p) = C_3^x p^x (1-p)^{3-x} = C_3^x \exp(x \ln(p) + (3-x)\ln(1-p))$

$h(x) = C_3^x$, $\eta_1(p) = \ln(p) - \ln(1-p)$ ✓

$T_1(x) = x$, $A(p) = -3\ln(1-p)$

b) $X \sim B(n, p)$

$f(x; n, p) = \underbrace{C_n^x}_{\downarrow} p^x (1-p)^{n-x}$

funcție de x și n => Nu face parte din fam exponențială

c) $X \sim Geom(p)$

$f(x; p) = p(1-p)^{x-1}$, $x \in \mathbb{N}^*$

$= \exp((x-1)\ln(1-p) + \ln(p))$

$h(x) = 1$, $\eta_1(p) = \ln(1-p)$, $A(p) = -\ln(p)$ ✓

$T_1(x) = x - 1$

d) $X \sim Poisson(\lambda)$

$f(x; \lambda) = \frac{\lambda^x}{x!} \exp(-\lambda) = \frac{1}{x!} \exp(x \ln(\lambda) - \lambda)$

$h(x) = \frac{1}{x!}$, $\eta_1(\lambda) = \ln(\lambda)$, $A(\lambda) = \lambda$ ✓

$T_1(x) = x$

e) $X \sim Gamma(\alpha, \beta)$

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right)$$

$$= \exp\left(-\frac{x}{\beta} + (\alpha-1)\ln(x) - \ln(\Gamma(\alpha)) - \alpha\ln(\beta)\right)$$

$h(x) = 1$, $\eta_1(\alpha,\beta) = \alpha - 1$, $\eta_2(\alpha,\beta) = -\frac{1}{\beta}$

$T_1(x) = \ln(x)$ $T_2(x) = x$ ✓

$A(\alpha,\beta) = \ln(\Gamma(\alpha)) + \alpha\ln(\beta)$

f) $X \sim Beta(\alpha,\beta)$

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} =$$

$$= \exp\left((\alpha-1)\ln(x) + (\beta-1)\ln(1-x) + \ln(\Gamma(\alpha+\beta)) - \ln(\Gamma(\alpha)) - \ln(\Gamma(\beta))\right)$$

$h(x) = 1$, $\eta_1(\alpha,\beta) = \alpha - 1$, $\eta_2(\alpha,\beta) = \beta - 1$

$T_1(x) = \ln(x)$ $T_2(x) = \ln(1-x)$ ✓

$A(\alpha,\beta) = -\ln(\Gamma(\alpha+\beta)) + \ln(\Gamma(\alpha)) + \ln(\Gamma(\beta))$

g) $X \sim \chi^2(k)$

$$f(x;k) = \frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} \exp\left(-\frac{x}{2}\right) =$$

$$= \exp\left(-\frac{1}{2}x + \left(\frac{k}{2}-1\right)\ln(x) - \frac{k}{2}\ln(2) - \ln\left(\Gamma\left(\frac{k}{2}\right)\right)\right)$$

$h(x) = 1$, $\eta_1(k) = -\frac{1}{2}$, $\eta_2(k) = \frac{k}{2} - 1$

$T_1(x) = x$ $T_2(x) = \ln(x)$ ✓

$A(k) = \frac{k}{2}\ln(2) + \ln\left(\Gamma\left(\frac{k}{2}\right)\right)$

3) $p(x;\theta) = h(x)\exp\left(\sum_{i=1}^{?} \eta_i(\theta) T_i(x) - A(\theta)\right)$

Fie $X_1, \dots, X_n$ iid cu val de selecție $x_1, \dots, x_n$

$$\log L(\theta) = \sum_{j=1}^{n} \log p(x_j;\theta) =$$

$$= \sum_{j=1}^{n} \left(\ln(h(x_j)) + \sum_{i=1}^{?} \eta_i(\theta) T_i(x_j) - A(\theta)\right)$$

$$= \boxed{\sum_{j=1}^{n} \ln(h(x_j)) + \sum_{j=1}^{n}\sum_{i=1}^{?} \eta_i(\theta) T_i(x_j) - nA(\theta)}$$

Vrem să arătăm că $\log L(\theta)$ este concavă:

Obs că: $A(\theta) = \log \int_{\mathbb{R}} h(x)\exp\left(\sum_{i=1}^{?} \eta_i(\theta) T_i(x)\right)dx$ | $A(\theta)$ depinde de celelalte componente (log normalizare)

Pentru a face următoarele calcule mai ușoare vom aduce familia exponențială la parametrizarea canonică.

Înlocuim $T_i(x)$ cu $c \cdot T_i(x)$ și θ_i cu $\frac{\theta_i}{c}$ și obținem aceeași distribuție deci parametrizarea nu este unică.

$$\boxed{p(x;\theta) = h(x)\exp\left(\sum_{i=1}^{?} \theta_i T_i(x) - A(\theta)\right)}$$

$$\frac{\partial A(\theta)}{\partial \theta_i} = \frac{1}{\int_{\mathbb{R}} \exp\left(\sum_{i=1}^{?} \theta_i T_i(x)\right)h(x)dx} \frac{\partial}{\partial \theta_i}\left(\int_{\mathbb{R}} h(x)\exp\left(\sum_{i=1}^{?} \theta_i T_i(x)\right)\right)$$

$$= \frac{\int_{\mathbb{R}} T_i(x)\exp\left(\sum_{i=1}^{?} \theta_i T_i(x)\right)h(x)\,dx}{\underbrace{\int_{\mathbb{R}} \exp\left(\sum_{i=1}^{?} \theta_i T_i(x)\right)h(x)\,dx}_{\overset{(*)}{=} \exp(A(\theta))}} = \mathbb{E}[T_i(X)]$$

$(*)$ $\int p(x,\theta)dx = e^{-A(\theta)} \int h(x)\exp\left(\sum_{i=1}^{?} \theta_i T_i(x)\right)dx \overset{\text{pt că } p \text{ este densitate}}{=} 1$

$$\frac{\partial^2 A(\theta)}{\partial\theta_i\partial\theta_j} = \frac{\partial}{\partial\theta_j}\mathbb{E}_\theta[T_i(X)] = \frac{\partial}{\partial\theta_j}\int_{\mathbb{R}} T_i(x)\,p(x;\theta)\,dx =$$

$$= \frac{\partial}{\partial\theta_j}\int_{\mathbb{R}} T_i(x)\exp(-A(\theta))\,h(x)\exp\Big(\sum_{i=1}\theta_i T_i(x)\Big)dx$$

$$= \int_{\mathbb{R}} T_i(x)\,p(x;\theta)\big(T_j(x) - \mathbb{E}_\theta[T_j(\cancel{x}X)]\big)$$

$$= \underbrace{\int_{\mathbb{R}} T_i(x)T_j(x)\,p(x;\theta)\,dx}_{=E[T_i(X)T_j(X)]} - \underbrace{\mathbb{E}_\theta[T_j(X)]\int_{\mathbb{R}} T_i(x)\,p(x;\theta)\,dx}_{=\mathbb{E}[T_j(X)]\mathbb{E}[T_i(X)]}$$

$$= E_\theta[T_i(X)T_j(X)] - E_\theta[T_j(X)]E_\theta[T_i(X)] = cov_\theta(T_i(X),T_j(X))$$

$$\frac{\partial}{\partial\theta_i}\log L(\theta) = \sum_{j=1}^{n} T_i(x_j) - n\underbrace{\frac{\partial A(\theta)}{\partial\theta_i}}_{E_\theta[T_i(X)]}$$

$$\frac{\partial^2}{\partial\theta_i\theta_k}\log L(\theta) = -n\frac{\partial^2 A(\theta)}{\partial\theta_i\partial\theta_k} = -n\,cov_\theta(T_i(X),T_k(X))$$

Cum $n \geq 0$ nr de observații => Hessiana funcț. $\log L(\theta)$ este negativ definită.
(Putem observa acest lucru și în R când facem graficul funcției de log verosimilitate)

4) c) $X \sim Geom(p)$

$$f(x;p) = \exp\big((x-1)\ln(1-p) + \ln(p)\big)$$

$$\boxed{\log L(p) = \sum_{j=1}^{n}(x_j-1)\ln(1-p) + n\ln(p)}$$

d) $X \sim \text{Poisson}(\lambda)$

$$f(x;\lambda) = \frac{1}{x!} \exp(x \ln(\lambda) - \lambda)$$

$$\boxed{\log L(\lambda) = \sum_{j=1}^{n} \ln\left(\frac{1}{x_j!}\right) + \ln(\lambda) \sum_{j=1}^{n} x_j - n\lambda}$$

g) $X \sim \chi^2(k)$

$$f(x;k) = \exp\left(-\frac{1}{2}x + \left(\frac{k}{2}-1\right)\ln(x) - \frac{k}{2}\ln(2) - \ln\left(\Gamma\left(\frac{k}{2}\right)\right)\right)$$

$$\boxed{\log L(k) = -\frac{1}{2}\sum_{j=1}^{n} x_j + \left(\frac{k}{2}-1\right)\sum_{j=1}^{n} \ln(x_j) - \frac{kn}{2}\ln(2) - n\ln\left(\Gamma\left(\frac{k}{2}\right)\right)}$$

5) a) $X \sim B(n, p)$ $n = \text{fixat} \overset{\text{aleg}}{=} 3$ (pt că n este luat notăm m nr de observații)

$$f(x;p) = C_3^{x} \exp(x \ln(p) + (3-x)\ln(1-p))$$

$$\boxed{\log L(p) = \sum_{j=1}^{m} \log(C_3^{x_j}) + (\ln(p) - \ln(1-p))\sum_{j=1}^{m} x_j + n \cdot m \ln(1-p)}$$

(n fixat)

e) $X \sim \text{Gamma}(\alpha, \beta)$

$$f(x;\alpha,\beta) = \exp\left(-\frac{x}{\beta} + (\alpha-1)\ln(x) - \ln(\Gamma(\alpha)) - \alpha\ln(\beta)\right)$$

$$\boxed{\log L(\alpha,\beta) = -\frac{1}{\beta}\sum_{j=1}^{n} x_j + (\alpha-1)\sum_{j=1}^{n}\ln(x_j) - n\ln(\Gamma(\alpha)) - n\alpha\ln(\beta)}$$

f) $X \sim \text{Beta}(\alpha, \beta)$

$$f(x;\alpha,\beta) = \exp((\alpha-1)\ln(x) + (\beta-1)\ln(1-x) + \ln(\Gamma(\alpha+\beta)) - \ln(\Gamma(\alpha)) - \ln(\Gamma(\beta)))$$

$$\boxed{\log L(\alpha,\beta) = (\alpha-1)\sum_{j=1}^{n}\ln(x) + (\beta-1)\sum_{j=1}^{n}\ln(1-x) + n\ln(\Gamma(\alpha+\beta)) - n\ln(\Gamma(\alpha)) - n\ln(\Gamma(\beta))}$$

b) $MIRC = \frac{1}{I_n(\theta)}$ unde $I_n(\theta) = E_\theta\left(\left(\frac{\partial}{\partial\theta}\log f_\theta(x)\right)^2\right)$

$$= -E_\theta\left(\frac{\partial^2}{\partial\theta^2}\log f_\theta(x)\right)$$

Știm că funcția de informație fisher este aditivă i.e.

$I_n(\theta) = n \cdot I(\theta)$. În continuare consider $n=1$ și calc $I(\theta)$

Suntem tot în ipoteza că $p(x,\theta) = h(x)\exp\left(\sum_{i=1}^{?}\theta_i T_i(x) - A(\theta)\right)$

Am demonstrat la ex 3) că:

$$\frac{\partial}{\partial\theta}\log L(\theta) = T(X) - \frac{\partial A(\theta)}{\partial\theta} = T(X) - E_\theta[T(X)]$$

$$\frac{\partial^2}{\partial\theta^2}\log L(\theta) = -Var_\theta(T(X))$$

$$I(\theta) = E_\theta\left(\frac{\partial}{\partial\theta}\log L(\theta)\right)^2 = E_\theta\left[\left(T(X) - E_\theta[T(X)]\right)^2\right] = Var_\theta(T(X))$$

$$I(\theta) = -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log L(\theta)\right] = -(-Var_\theta(T(X)) = Var_\theta(T(X))$$

$$\boxed{MIRC = \frac{1}{n \cdot Var_\theta(T(X))}}$$ pentru familia exponențială cu 1 parametru (parametrizată canonic)

Dacă am făcut schimbarea $\eta(p) = \theta$ atunci ca să aflăm $I(p)$ calculăm $I(p) = I(\theta)\left(\frac{\partial\theta}{\partial p}\right)^2$ și în loc de θ punem $\eta(p)$

c) $X \sim Geom(p)$

$$\log L(p) = \sum_{j=1}^{n}(x_j - 1)\ln(1-p) + n\ln(p)$$

Vrem $I_n(p)$. Calculăm $I(p)$. Prima dată aducem la forma canonică:

$\theta = \ln(1-p) \Rightarrow p = 1 - e^\theta$, aleg $n=1$ pentru simplitate

$$\log L(\theta) = \theta(x-1) + \ln(1-e^\theta)$$

$$I(\theta) = Var_\theta(X-1) = \frac{1-(1-e^\theta)}{(1-e^\theta)^2} = \frac{e^\theta}{(1-e^\theta)^2}$$

$$I(p) = I(\theta)\cdot\left(\frac{\partial\theta}{\partial p}\right)^2 = \frac{e^\theta}{(1-e^\theta)^2}\cdot\frac{1}{(1-p)^2} \overset{e^\theta = 1-p}{=\!=} \frac{(1-p)}{p^2}\cdot\frac{1}{(1-p)^2} = \frac{1}{p^2(1-p)}$$

$$i_n(r)=\frac{n}{r^2(1-r)} \Rightarrow \boxed{MIRC=\frac{r^2(1-r)}{n}}$$

Alternativ în caz că nu știm Var(X) putem calcula informația fisher folosind formulele

$$i_n(\theta)=E_\theta\left(\left(\frac{\partial}{\partial\theta}\log L(\theta)\right)^2\right)=-E_\theta\left(\frac{\partial^2}{\partial\theta^2}\log L(\theta)\right)$$, luând $n=1$

$$\frac{\partial}{\partial r}\log L(r)=\frac{1}{r}-\frac{1}{1-r}(x-1)$$

$$\frac{\partial^2}{\partial r^2}\log L(r)=-\frac{1}{r^2}-\frac{x-1}{(1-r)^2}$$

$$i(r)=-E_r\left(-\frac{1}{r^2}-\frac{x-1}{(1-r)^2}\right)=E_r\left(\frac{1}{r^2}\right)+E_r\left(\frac{x-1}{(1-r)^2}\right)=\frac{1}{r^2}+\frac{1}{(1-r)^2}E_r(x-1)=$$

$$=\frac{1}{r^2}+\frac{1}{(1-r)^2}\left(\frac{1}{r}-1\right)=\frac{1}{r^2}+\frac{1}{(1-r)r}=\frac{1}{r^2(1-r)}$$

$$i_n(r)=\frac{n}{r^2(1-r)} \Rightarrow \boxed{MIRC=\frac{r^2(1-r)}{n}}$$

Observăm că rezultatele sunt asemenea. Alegem metoda care ne este cea mai la îndemână.

d) $X\sim Poisson(\lambda)$

$$\log L(\lambda)=\sum_{j=1}^{n}\ln\left(\frac{1}{x_j!}\right)+\ln(\lambda)\sum_{j=1}^{n}x_j-n\lambda$$

$$\frac{\partial^2}{\partial\lambda^2}\log L(\lambda)=-\frac{1}{\lambda^2}x$$ (este ușor de calculat)

$$i(\lambda)=-E_\lambda\left(-\frac{1}{\lambda^2}x\right)=\frac{1}{\lambda^2}\cdot\lambda=\frac{1}{\lambda}\Rightarrow i_n(\lambda)=\frac{n}{\lambda}\Rightarrow\boxed{MIRC=\frac{\lambda}{n}}$$

SAU: $\ln(\lambda)=\theta\Rightarrow\lambda=e^\theta$

$$i(\theta)=Var_\theta(x)=e^\theta$$

$$i(\lambda)=i(\theta)\left(\frac{\partial\theta}{\partial\lambda}\right)^2=e^\theta\left(-\frac{1}{\lambda}\right)^2=\frac{\lambda}{\lambda^2}=\frac{1}{\lambda}\Rightarrow i_n(\lambda)=\frac{n}{\lambda}\Rightarrow\boxed{MIRC=\frac{\lambda}{n}}$$

$g) X \sim \chi^2(k) \quad n=1$

$$\log L(k) = -\frac{1}{2}x + \left(\frac{k}{2}-1\right)\ln(x) - \frac{k}{2}\ln(2) - \ln\left(\Gamma\left(\frac{k}{2}\right)\right)$$

$$\frac{\partial^2}{\partial k^2}\log L(k) = -\frac{1}{4}\psi'\left(\frac{k}{2}\right),\ \psi' = \text{trigamma}$$

$$i(k) = \frac{1}{4}\psi'\left(\frac{k}{2}\right) \Rightarrow i_n(k) = \frac{n}{4}\psi'\left(\frac{k}{2}\right) \Rightarrow \boxed{MIRC = \frac{4}{n\,\psi'\left(\frac{k}{2}\right)}}$$

In cazul acesta folosind formula am avea:

$i(k) = Var_k(\ln(X)) = E_k((\log X)^2) - E(\log X)^2$ lucru care este evident mai laborios decât metoda cu derivata.

$$E(\log X) = \int_0^\infty \log(x)\frac{x^{\frac{k}{2}-1}e^{-\frac{x}{2}}}{2^{k/2}\Gamma(k/2)}dx$$

SV: $y = \frac{x}{2} \Rightarrow x = 2y$
$dx = 2dy$
$x=0 \Rightarrow y=0$
$x=\infty \Rightarrow y=\infty$

$$= \int_0^\infty \frac{2}{2^{k/2}\Gamma\left(\frac{k}{2}\right)}\left(\log(2)+\log(y)\right)\cdot 2^{\frac{k}{2}-1}y^{\frac{k}{2}-1}e^{-y}dy$$

$$= \log(2) + \frac{1}{\Gamma\left(\frac{k}{2}\right)}\int_0^\infty \log(y)\,y^{\frac{k}{2}-1}e^{-y}dy$$

$$= \log(2) + \frac{1}{\Gamma\left(\frac{k}{2}\right)}\Gamma\left(\frac{k}{2}\right)\psi\left(\frac{k}{2}\right),\ \psi = \text{digamma}$$

$$= \log(2) + \psi\left(\frac{k}{2}\right)$$

$$E(\log(x)^2) = \int_0^\infty \log(x)^2\frac{x^{\frac{k}{2}-1}e^{-\frac{x}{2}}}{2^{k/2}\Gamma\left(\frac{k}{2}\right)}dx =$$

SV: $y = \frac{x}{2} \Rightarrow x = 2y$
$dx = 2dy$
$x=0 \Rightarrow y=0$
$x=\infty \Rightarrow y=\infty$

$$= \frac{1}{\Gamma(k/2)}\int_0^\infty\left(\log(2)^2 + 2\log(2)\log(y) + \log(y)^2\right)y^{\frac{k}{2}-1}e^{-y}dy$$

$$= \frac{\log(2)^2}{\Gamma(k/2)}\underbrace{\int_0^\infty y^{\frac{k}{2}-1}e^{-y}dy}_{\Gamma(k/2)} + \frac{2\log(2)}{\Gamma(k/2)}\underbrace{\int_0^\infty \log(y)y^{\frac{k}{2}-1}e^{-y}dy}_{\Gamma\left(\frac{k}{2}\right)\psi(k/2)} + \frac{1}{\Gamma(k/2)}\underbrace{\int_0^\infty \log(y)^2 y^{\frac{k}{2}-1}e^{-y}dy}_{\Gamma(k/2)\left(\psi^2(k/2)+\psi'(k/2)\right)}$$

$$E(\log(x)^2) = \log(2)^2 + 2\log(2)\,\psi(\tfrac{k}{2}) + \psi^2(k/2) + \psi'(k/2)$$

$$Var_{\mathbb{R}}(\log(X)) = \log(2)^2 + 2\log(2)\,\psi(k/2) + \psi'(\tfrac{k}{2}) + \psi^2(\tfrac{k}{2}) -$$

$$-(\log(2) + \psi(k/2))^2 = \psi'(k/2)$$

Dacă facem $\theta = \frac{k}{2}$

$$i(\theta) = \psi'(\theta) \Rightarrow i(k) = \psi'(\tfrac{k}{2}) \cdot (\tfrac{1}{2})^2 = \tfrac{1}{4}\psi'(\tfrac{k}{2}) \Rightarrow i_n(\theta) = \tfrac{n}{4}\psi'(\tfrac{k}{2})$$

$$\Rightarrow \boxed{MIRC = \frac{4}{n\,\psi'(k)}}$$

7) Calculăm $i_n(\theta)$ pentru următoarele distribuții:

i) $X \sim U[a,b]$, $f(x;a,b) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & altfel \end{cases}$

Ex știm a și vrem să îl estimăm pe b. ! <u>suport dep. de b!</u>

$$\log L(b) = \begin{cases} -n\ln(b-a), & a \le x \le b \\ -\infty, & altfel \end{cases}$$

$$\frac{\partial}{\partial b}\log L(b) = -\frac{n}{(b-a)}$$

$$i_n(b) = E\left(\frac{n^2}{(b-a)^2}\right) = \frac{n^2}{(b-a)^2} \Rightarrow i_n(b) = \frac{n^2}{(b-a)^2} \Rightarrow \boxed{MIRC = \frac{(b-a)^2}{n^2}}$$

ii) $X \sim N(\mu, \sigma^2)$ $f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Ex: că știm σ^2

$$\log L(\mu) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(x-\mu)^2}{2\sigma^2}$$

$$\frac{\partial^2}{\partial\mu^2}\log L(\mu) = -\frac{1}{\sigma^2} \Rightarrow i(\mu) = \frac{1}{\sigma^2} \Rightarrow i_n(\mu) = \frac{n}{\sigma^2} \Rightarrow \boxed{MIRC = \frac{\sigma^2}{n}}$$

iii) $X \sim Exp(\lambda)$, $f(x;\lambda) = \lambda e^{-\lambda x}$, $x \geq 0$

$\log L(\lambda) = \ln(\lambda) - \lambda x$

$\frac{\partial^2}{\partial \lambda^2} \log L(\lambda) = -\frac{1}{\lambda^2} \Rightarrow i_n(\lambda) = \frac{n}{\lambda^2} \Rightarrow \boxed{MIRC = \frac{\lambda^2}{n}}$

iv) $X \sim Bernoulli(p)$, $f(x,p) = p^x(1-p)^{1-x}$

$\log L(p) = x\ln(p) + (1-x)\ln(1-p)$

$\frac{\partial^2}{\partial p^2} \log L(p) = -\frac{x}{p^2} - \frac{1-x}{(1-p)^2}$

$i(p) = -E\left(-\frac{x}{p^2} - \frac{1-x}{(1-p)^2}\right) = \frac{1}{p^2}E(x) + \frac{1}{(1-p)^2}E(1-x) = \frac{1}{p^2}p +$

$+\frac{1}{(1-p)^2}(1-p) = \frac{1}{p(1-p)} \Rightarrow i_n(p) = \frac{n}{p(1-p)} \Rightarrow \boxed{MIRC = \frac{p(1-p)}{n}}$

v) $X \sim Gamma(\alpha, \beta)$, $f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}$, $x>0$

Pp α este cunoscut

$\log L(\beta) = -\frac{1}{\beta}x + (\alpha-1)\ln(x) - \ln(\Gamma(\alpha)) - \alpha\ln(\beta)$

$\frac{\partial^2}{\partial \beta^2} \log L(\beta) = -\frac{2}{\beta^3}x + \frac{\alpha}{\beta^2}$

$i(\beta) = -E\left(-\frac{2}{\beta^3}x + \frac{\alpha}{\beta^3}\right) = \frac{2\alpha}{\beta^2} - \frac{\alpha}{\beta^2} = \frac{\alpha}{\beta^2} \Rightarrow i_n(\beta) = \frac{\alpha n}{\beta^2} \Rightarrow$

$\Rightarrow \boxed{MIRC = \frac{\beta^2}{\alpha n}}$

8) $p(x;\theta) = h(x)\exp\left(\sum_{i=1}^{?}\theta_i T_i(x) - A(\theta)\right)$ fam exponentialor parametrizată canonic

• MVM (met verosimilității maxime) (1 parametru)

$$\log L(\theta) = \sum_{j=1}^{n}\log h(x_j) + \sum_{j=1}^{n}\theta T(x_j) - nA(\theta)$$

$$\frac{\partial}{\partial\theta}\log L(\theta) = \sum_{j=1}^{n} T(x_j) - n\frac{\partial A(\theta)}{\partial\theta} \overset{\text{Vrem}}{=\!=} 0$$

$$\sum_{j=1}^{n} T(x_j) = n\frac{\partial A(\theta)}{\partial\theta} = E(T(X)) \quad , \text{not } \overline{T} = \frac{1}{n}\sum_{j=1}^{n}T(x_j)$$

$$\boxed{\overline{T} = \frac{\partial A(\theta)}{\partial\theta} = E(T(X))}$$

• MVM (2 parametrii)

$$\log L(\underset{(\theta_1,\theta_2)}{\underset{\shortparallel}{\theta}}) = \sum_{j=1}^{n}\log h(x_j) + \theta_1\sum_{j=1}^{n}T_1(x_j) + \theta_2\sum_{j=1}^{n}T_2(x_j) - nA(\theta_1,\theta_2)$$

$$\frac{\partial}{\partial\theta_1}\log L(\theta_1,\theta_2) = \sum_{j=1}^{n}T_1(x_j) - n\frac{\partial A(\theta_1,\theta_2)}{\partial\theta_1} \overset{\text{Vrem}}{=\!=} 0$$

$$\frac{\partial}{\partial\theta_2}\log L(\theta_1,\theta_2) = \sum_{j=1}^{n}T_2(x_j) - n\frac{\partial A(\theta_1,\theta_2)}{\partial\theta_2} \overset{\text{Vrem}}{=\!=} 0$$

Avem: $\overline{T}_1 = \dfrac{\partial A(\theta_1,\theta_2)}{\partial\theta_1}$, $\overline{T}_2 = \dfrac{\partial A(\theta_1,\theta_2)}{\partial\theta_2}$

$$\boxed{\overline{T}_1 = \frac{\partial A(\theta_1,\theta_2)}{\partial\theta_1} = E(T_1(X))} \qquad \boxed{\overline{T}_2 = \frac{\partial A(\theta_1,\theta_2)}{\partial\theta_2} = E(T_2(X))}$$

• MM (metoda momentelor)

Fie $\varphi(\theta) = E(T(X))$

Vrem $\boxed{\overline{T} = \varphi(\theta)} \to$ aflu $\hat{\theta}$

- Observații: MVM și MM produc același rezultat. Acest lucru se datorează veridicităţii faptului că T_i sunt statistici suficiente i.e. conțin toată informația despre parametrul θ. MVM și MM depind de $\overline{T}$ (media eșantionului) care este un estimator consistent și nedeplasat. (Cat timp $E(T_i(x))$ fct. liniare)

7) a) $X \sim Bin(3, p)$

MVM: $p(x;p) = C_3^x \exp\left(x \ln\left(\frac{p}{1-p}\right) + 3\ln(1-p)\right)$

SV: $\theta = \ln\left(\frac{p}{1-p}\right) \Rightarrow \frac{p}{1-p} = e^{\theta}$

$$\overline{T} = E(T(X)) = \frac{\partial A(\theta)}{\partial \theta}$$

$$A(p) = -3\ln(1-p)$$

$$= 3\ln\left(\frac{1}{1-p}\right) = 3\ln\left(1 + \frac{p}{1-p}\right)$$

$$\frac{\partial A(\theta)}{\partial \theta} = \frac{3e^{\theta}}{1+e^{\theta}}$$

$$A(\theta) = 3\ln(1+e^{\theta})$$

$$\overline{T} = \frac{3e^{\ln\left(\frac{p}{1-p}\right)}}{1+e^{\ln\left(\frac{p}{1-p}\right)}} =$$

$T(x) = x$

$$= \frac{3 \cdot \frac{p}{1-p}}{1+\frac{p}{1-p}} = 3p \Rightarrow \boxed{\hat{p} = \frac{\overline{T}}{3} = \frac{\overline{X}}{3}}$$

unde $\overline{X} = \frac{1}{n}\sum_{j=1}^{n} x_j$

MM: $E[X] = 3p \Rightarrow \overline{X} = 3\hat{p} \Rightarrow \boxed{\hat{p} = \frac{\overline{X}}{3}}$

b) $X \sim Geom(p)$

În continuare vom aplica metoda verosimilității maxime prin metoda standard $\frac{\partial}{\partial\theta}\log L(\theta) = 0$. Merge și cu metoda alternativă la punctul 8) doar că am derivatele deja calculate și pot ajunge la un rezultat mai repede.

MVM: $\log L(p) = \sum_{j=1}^{n}(x_j - 1)\ln(1-p) + n\ln(p)$

$$\frac{\partial}{\partial p}\log L(p) = \frac{1}{1-p}\sum_{j=1}^{n}(x_j-1) + \frac{n}{p} = 0$$

$$\sum_{j=1}^{n} x_j - n = -\frac{n}{p}(-(1-p)) \Rightarrow \overline{X} = \frac{n-np}{p} + n = \frac{n}{p} \Rightarrow$$

$$\Rightarrow \boxed{\hat{p} = \frac{1}{\overline{X}}}$$

MM: $E[X] = \frac{1}{p} \Rightarrow \boxed{\tilde{p} = \frac{1}{\overline{X}}}$

c) $X \sim \text{Poisson}(\lambda)$

MVM: $\log L(\lambda) = \sum_{j=1}^{n} \ln\left(\frac{1}{x_j!}\right) + \ln(\lambda) \sum_{j=1}^{n} x_j - n\lambda$

$$\frac{\partial}{\partial \lambda} \log L(\lambda) = \frac{1}{\lambda} \sum_{j=1}^{n} x_j - n = 0 \Rightarrow \boxed{\hat{\lambda} = \overline{X}}$$

MM: $E(X) = \lambda \Rightarrow \boxed{\tilde{\lambda} = \overline{X}}$

d) $X \sim \text{Gamma}(\alpha, \beta)$

MVM: $\log L(\alpha,\beta) = -\frac{1}{\beta} \sum_{j=1}^{n} x_j + (\alpha-1) \sum_{j=1}^{n} \ln(x_j) - n \ln(\Gamma(\alpha)) - n\alpha \ln(\beta)$

$$\frac{\partial}{\partial \beta} \log L(\alpha,\beta) = \frac{1}{\beta^2} \sum_{j=1}^{n} x_j - \frac{\alpha n}{\beta} = 0 \Rightarrow \boxed{\hat{\beta} = \frac{1}{\alpha} \overline{X}}$$

$$\frac{\partial}{\partial \alpha} \log L(\alpha,\beta) = \sum_{j=1}^{n} \ln(x_j) - n\,\psi(\alpha) - n \ln(\beta) = 0 \Rightarrow$$

$$\boxed{\psi(\hat{\alpha}) = -\log(\beta) + \frac{1}{n} \sum_{j=1}^{n} \log(x_i)}$$

MM:

$E(X) = \alpha\beta$, $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} x_i$

$Var(X) = \alpha\beta^2$, $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{X})^2$

$\overline{X} = \alpha\beta \Rightarrow \alpha = \frac{\overline{X}}{\beta}$

$S^2 = \alpha\beta^2 \Rightarrow S^2 = \beta\overline{X} \Rightarrow \boxed{\tilde{\beta} = \frac{S^2}{\overline{X}}} \Rightarrow \boxed{\tilde{\alpha} = \frac{\overline{X}^2}{S^2}}$

e) $X \sim Beta(\alpha, \beta)$.

MVM: $\exp\left((\alpha-1)\ln(x) + (\beta-1)\ln(1-x) + \ln(\Gamma(\alpha+\beta)) - \ln(\Gamma(\alpha)) - \right.$

$\left. - \ln(\Gamma(\beta)\right) = \log L(\alpha, \beta)$

$$\frac{\partial}{\partial \alpha} \log L(\alpha,\beta) = \sum_{j=1}^{n} \ln(x_j) + n\,\Psi(\alpha+\beta) - n\,\Psi(\alpha) = 0$$

$$\boxed{\frac{1}{n}\sum_{j=1}^{n} \ln(x_j) = \Psi(\hat{\alpha}) - \Psi(\hat{\alpha}+\beta)}$$

$$\frac{\partial}{\partial \beta} \log L(\alpha,\beta) = \sum_{j=1}^{n} \ln(x_j) + n\,\Psi(\alpha+\beta) - n\,\Psi(\beta) = 0$$

$$\boxed{\frac{1}{n}\sum_{j=1}^{n} \ln(x_j) = \Psi(\hat{\beta}) - \Psi(\alpha+\hat{\beta})}$$

MM: $E(X) = \frac{\alpha}{\alpha+\beta}$, $Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$\overline{X} = \frac{\alpha}{\alpha+\beta}$, $S^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$\alpha = \underbrace{(\alpha+\beta)}_{g}\overline{X}$. *Natăm* $\alpha+\beta = g \Rightarrow \beta = (1-\overline{X})g$

$$\Rightarrow S^2 = \frac{(1-\overline{X})g \cdot g\overline{X}}{g^2 \cdot (g+1)} = \frac{\overline{X}(1-\overline{X})}{g+1} \Rightarrow g = \frac{\overline{X}(1-\overline{X})}{S^2} - 1$$

$$\boxed{\hat{\alpha} = \overline{X}\left(\frac{\overline{X}(1-\overline{X})}{S^2} - 1\right)} \quad \boxed{\hat{\beta} = (1-\overline{X})\left(\frac{\overline{X}(1-\overline{X})}{S^2} - 1\right)}$$

g) $X \sim \chi^2(k)$

MVM: $\log L(k) = -\frac{1}{2}\sum_{j=1}^{n} x_j + \left(\frac{k}{2}-1\right)\sum_{j=1}^{n} \ln(x_j) - \frac{kn}{2}\ln(2) - n\ln\left(\Gamma\left(\frac{k}{2}\right)\right)$

$$\frac{\partial}{\partial k} \log L(k) = \frac{1}{2}\sum_{j=1}^{n} \ln(x_j) - \frac{n}{2}\ln(2) - n\,\Psi\left(\frac{k}{2}\right) = 0$$

$$\boxed{\Psi\left(\frac{\hat{k}}{2}\right) = \ln(\overline{X}) - \ln(2)}$$

MM: $E(X)=k \Rightarrow \boxed{\hat{k}=\overline{X}}$

III $\sup_x |\mathbb{P}(Z_n \le x)-\phi(x)| \le \frac{33}{4} \frac{E|X_1-\mu|^3}{\sqrt{n}\,\sigma^3}$

$Z_n = \frac{\sqrt{n}}{\sigma}(\overline{X}_n-\mu)$, $\phi(x)=\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}dt$, $\mu=E(X_1)$, $\sigma=\sqrt{Var(X_1)}$

1) $\mathbb{P}(Z_n \le x) = \mathbb{P}\left(\sum_{i=1}^{n} X_i \le n\mu + x\sigma\sqrt{n}\right)$

a) Binomiala(r,p), $X \sim Bin(r,p)$, $X_1,\dots,X_n$ iid

$M_X(t)=E(e^{tX}) = \sum_{k=0}^{r} e^{tk}\mathbb{P}(X=k) = \sum_{k=0}^{r} e^{tk} C_r^k p^k (1-p)^{r-k} =$

$=(pe^t+(1-p))^r = (1-p+pe^t)^r$

Fie $S_n = X_1+\dots+X_n$

$M_{S_n}(t)=\prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n}(1-p+pe^t)^r = (1-p+pe^t)^{nr} \Rightarrow$

$\Rightarrow \boxed{S_n \sim Binomial(nr,p)}$

$E(S_n) = nrp = n\mu$ | $E(S_n)=nE(X)$

$\sigma_{S_n} = \sqrt{Var(S_n)} = \sqrt{nrp(1-p)} = \sigma\sqrt{n}$ | $Var(S_n)=nVar(X)$

$\boxed{\mathbb{P}(S_n \le nrp + x\sqrt{nrp(1-p)}) \sim \phi(x)}$

b) $X \sim Geom(p)$, $X_1,\dots,X_n$ iid

$S_n = X_1+\dots+X_n$

$E(S_n) = n\cdot\frac{1}{p}$, $Var(S_n) = n\frac{1-p}{p^2} \Rightarrow \sigma = \sqrt{n\frac{1-p}{p^2}}$

$\boxed{\mathbb{P}\left(S_n \le \frac{n}{p} + x\sqrt{\frac{n(1-p)}{p^2}}\right) \sim \phi(x)}$

c) $X \sim Poisson(\lambda)$, $X_1, \dots, X_n$ iid

$S_n = \sum_{i=1}^{n} X_i$

$M_X(t) = E(e^{tX}) =$

$= \sum_{k=0}^{\infty} e^{tk} \mathbb{P}(X=k) = \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}$

Știm că $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \forall z$. În cazul nostru $z = e^t \lambda$

$M_X(t) = e^{-\lambda} \cdot e^{e^t \lambda} = e^{\lambda(e^t - 1)}$

$M_{S_n}(t) = \prod_{i=1}^{n} e^{\lambda(e^t-1)} = e^{n\lambda(e^t-1)} \Rightarrow \boxed{S_n \sim Poisson(n\lambda)}$

$E(S_n) = n E(X) = n\lambda$

$Var(S_n) = n\lambda \Rightarrow \sigma = \sqrt{n\lambda}$

$\boxed{\mathbb{P}(S_n \leq n\lambda + x\sqrt{n\lambda}) \sim \phi(x)}$

d) Unif $\{a, \dots, b\}$, X_i iid

$E(X) = \frac{b+a}{2}$, $Var(X) = \frac{(b-a+1)^2 - 1}{12}$

$S_n = \sum_{i=1}^{n} X_i$

$\boxed{\mathbb{P}\left(S_n \leq n \frac{b+a}{2} + x\sqrt{n \frac{(b-a+1)^2-1}{12}}\right.}$

e) $X \sim Exp(\lambda)$, X_i iid

$S_n = \sum_{i=1}^{n} X_i$

$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{x(t-\lambda)} dx$ converge

dacă $t - \lambda < 0 \Rightarrow t < \lambda$

$M_X(t) = \frac{\lambda}{\lambda - t}, \; t < \lambda$

$M_{S_n}(t) = \left(\frac{\lambda}{\lambda - t}\right)^n$ ⊛

Fie $X \sim Gammă(n, \lambda)$, $f_X(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)}$, $x \geq 0$, $n > 0$, $\lambda > 0$

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)} dx =$$

$$= \frac{\lambda^n}{\Gamma(n)} \int_0^\infty x^{n-1} e^{-x(\lambda - t)} dx$$ converge dacă $\lambda - t > 0 \Rightarrow t < \lambda$

SV: $x(\lambda - t) = y \Rightarrow x = \frac{y}{\lambda - t}$

$(\lambda - t) dx = dy$

capetele rămân la fel

$$= \frac{\lambda^n}{\Gamma(n)} \int_0^\infty \left(\frac{y}{\lambda - t}\right)^{n-1} e^{-y} \frac{1}{\lambda - t} dy$$

$$= \frac{\lambda^n}{\Gamma(n)} \frac{\Gamma(n)}{(\lambda - t)^n} = \left(\frac{\lambda}{\lambda - t}\right)^n$$ ⊛

⊛ $\Rightarrow \boxed{S_n \sim Gamma(n, \lambda)}$

$E(S_n) = n \cdot \frac{1}{\lambda}$

$Var(S_n) = n \cdot \frac{1}{\lambda^2}$

$\boxed{P\left(S_n \leq \frac{n}{\lambda} + x\sqrt{\frac{n}{\lambda^2}}\right.}$

f) $X \sim Gamma(\alpha, \beta)$, X iid, $f_X(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$

$S_n = \sum_{i=1}^n X_i$

$M_X(t) = \left(\frac{\beta}{\beta - t}\right)^\alpha$

$M_{S_n}(t) = \left(\frac{\beta}{\beta - t}\right)^{n\alpha} \Rightarrow \boxed{S_n \sim Gamma(n\alpha, \beta)}$

$E(X) = \frac{\alpha}{\beta}$

$Var(X) = \frac{\alpha}{\beta^2}$

$$\boxed{\mathbb{P}\left(S_n \le \frac{n\alpha}{\beta} + * \sqrt{\frac{n\alpha}{\beta^2}}\right)}$$

g) $X \sim Beta(\alpha, \beta)$, X iid

$E(X) = \frac{\alpha}{\alpha+\beta}$, $Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$S_n = \sum_{i=1}^{n} X_i$

$$\boxed{\mathbb{P}\left(S_n \le \frac{n\alpha}{\alpha+\beta} + * \sqrt{\frac{n\alpha\beta}{(\alpha+\beta^2)(\alpha+\beta+1)}}\right.}$$

Observație: La ex: II g) pentru distribuțiile cu 2 parametrii în cazul metodei momentelor am folosit $Var(X)$ în loc de $E(X^2)$. $Var(X)$ este momentul de ordin 2 centrat deci în practică este același lucru. In plus cu $Var(X)$ este mai ușor pentru că mă pot lega de $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 \xrightarrow{a.s.} Var(X)$.
In caz că această metodă nu intră în contextul metodei momentelor reai scrie aici cu $E(X^2)$

d) $X \sim Gamma(\alpha, \beta)$

$E(X) = \alpha\beta$

$E(X^2) = Var(X) + E(X)^2 = \alpha\beta^2 + \alpha^2\beta^2 = \alpha(\alpha+1)\beta$

$\overline{X} \stackrel{not}{=} \frac{1}{n}\sum_{i=1}^{n} X_i$, $\overline{X}_2 \stackrel{not}{=} \frac{1}{n}\sum_{i=1}^{n} X_i^2$

$$\begin{cases}\overline{X}=\alpha\beta \Rightarrow \beta=\dfrac{\overline{X}}{\alpha}\\ \overline{X}_2=\alpha(\alpha+1)\beta^2 \Rightarrow \overline{X}_2=\alpha(\alpha+1)\dfrac{\overline{X}^2}{\alpha^2} \Rightarrow\end{cases}$$

$$\Rightarrow \alpha\overline{X}_2=(\alpha+1)\overline{X}^2 \Rightarrow \alpha\overline{X}_2-\alpha\overline{X}^2-\overline{X}^2=0 \Rightarrow$$

$$\Rightarrow \boxed{\hat{\alpha}=\frac{\overline{X}^2}{\overline{X}_2-\overline{X}^2}} \Rightarrow \boxed{\hat{\beta}=\overline{X}\cdot\frac{\overline{X}_2-\overline{X}^2}{\overline{X}^2}=\frac{\overline{X}_2-\overline{X}^2}{\overline{X}}}$$

e) $X \sim \text{Beta}(\alpha,\beta)$

$$E(X)=\frac{\alpha}{\alpha+\beta}$$

$$E(X^2)=Var(X)+E(X)^2=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}+\frac{\alpha^2}{(\alpha+\beta)^2}=$$

$$=\frac{\alpha\beta+\alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}=\frac{\alpha\beta+\alpha^3+\alpha^2\beta+\alpha^2}{(\alpha+\beta)^2(\alpha+\beta+1)}=\frac{\alpha(\beta+\alpha^2+\alpha\beta+\alpha)}{(\alpha+\beta)^2(\alpha+\beta+1)}=$$

$$=\frac{\alpha(\alpha+1)(\alpha+\beta)}{(\alpha+\beta)^2(\alpha+\beta+1)}=\frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$\begin{cases}\overline{X}=\dfrac{\alpha}{\alpha+\beta} \Rightarrow \alpha=\underbrace{(\alpha+\beta)}_{\text{not } g}\overline{X} \Rightarrow \alpha+\beta=g \Rightarrow \beta=(1-\overline{X})g\\ \overline{X}_2=\dfrac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}=\dfrac{g\overline{X}(g\overline{X}+1)}{g(g+1)}\end{cases}$$ nu mai merge așa

$$\beta=\frac{\alpha(1-\overline{X})}{\overline{X}}$$

$$\overline{X}_2=\frac{\alpha(\alpha+1)}{\left(\alpha+\frac{\alpha(1-\overline{X})}{\overline{X}}\right)\left(\alpha+\frac{\alpha(1-\overline{X})}{\overline{X}}+1\right)}=\frac{\overline{X}^2(\alpha+1)}{\alpha+\overline{X}} \Rightarrow \boxed{\hat{\alpha}=\frac{\overline{X}^2-\overline{X}_2\overline{X}}{\overline{X}_2-\overline{X}^2}}$$

$$\Rightarrow \boxed{\hat{\beta}=\frac{\overline{X}(1-\overline{X}+\overline{X}_2)-\overline{X}_2}{\overline{X}_2-\overline{X}^2}}$$