NUMERICAL SIMULATION OF SDES WITH DISTRIBUTIONAL DRIFT

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1 Introduction

We would like to simulate numerically sample paths of the solution of the stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\ \mathrm{d}t + \mathrm{d}W_t \tag{1}$$

where $b \in H_q^s(\mathbb{R})$, $s \in \left]-\frac{1}{2}, 0\right[$, $t \in [0,T]$, and W_t is a standard Brownian motion. This equation is studied in [1] in which the authors prove existence and unicity in law of a virtual solution for equation (1).

Example 1.1. *An example of such drift b is given by the derivative of a sample path of a fractional Brownian motion B_x^H with Hurst index $1/2 < H < 1$. These stochastic processes are gaussian processes verifying*

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} + |t-s|^{2H}\right).$$

We note $s = H - 1$. Given $B_x^H(\omega) \in H_q^{s+1}(\mathbb{R})$, we can take $b(x) = \frac{\partial}{\partial x} B_x^H(\omega) \in H_q^s(\mathbb{R})$. We will use this in our numerical simulations.

As far as the drift b is not a function but a distribution, it must be approximated if we want to evaluate it at points. In order to do so, we will use a series representation of b and truncate it. That is why we will consider two steps in our algorithm:

1. approximate the drift b by b^N.
2. approximate the solution X_t^N of the approximated SDE:

$$\mathrm{d}X_t^N = b^N\left(X_t^N\right)\ \mathrm{d}t + \mathrm{d}W_t \tag{2}$$

   by $X_t^{N,n}$ with a Euler-Maruyama scheme.

2 Numerical simulation of fractional Brownian motion

To simulate a sample path of a fractional brownian motion B^H_x on a finite grid $(x_k)_{k\in[\![1,n]\!]}$, we simulate n independent standard gaussian random variables $(X_k)_{k\in[\![1,n]\!]}$ and then correlate them with the definite positive correlation matrix

$$C_{k,s} = \mathbb{E}\left[B^H_{x_k} B^H_{x_s}\right] = \frac{1}{2}\left(x_k^{2H} + x_s^{2H} + |x_k - x_s|^{2H}\right).$$

To do so, we use the Cholesky decomposition method and calculate the triangular matrix M such that $C = MM^\top$. Therefore, defining

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ and } B^H = MX,$$

B^H contains the values of a fractional brownian motion evaluated on the grid $(x_k)_{k\in[\![1,n]\!]}$.

3 Approximation of the drift

3.1 Series representation

We use Haar wavelets to give a series representation of b. By doing so, we will be able to approximate it numerically by truncating the series.

Definition 3.1 (Haar wavelets). *We define the Haar wavelets $h_{j,m}$ on $\mathbb{R}$ with $j \in \mathbb{N} \cup \{-1\}$ and $m \in \mathbb{Z}$ by:*

$$\begin{cases} h_M & : x \longmapsto \left(\mathbb{1}_{[0,\frac{1}{2}[} - \mathbb{1}_{[\frac{1}{2},1[}\right)(x) \\ h_{-1,m} & : x \longmapsto \sqrt{2}|h_M(x-m)| \\ h_{j,m} & : x \longmapsto h_M(2^j x - m) \end{cases}$$

Theorem 3.1 (See [2]). *Let $b \in H^s_q(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left]-\frac{1}{2}, \frac{1}{q}\right[$. Therefore,*

$$b = \sum_{j=-1}^{+\infty} \sum_{m\in\mathbb{Z}} \mu_{j,m} h_{j,m} \tag{3}$$

where $\mu_{j,m} = 2^j \int_{\mathbb{R}} b(x) h_{j,m}(x)\, \mathrm{d}x$ in the sense of dual pairing.

Definition 3.2. *Let $b \in H^s_q(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left]-\frac{1}{2}, 0\right[$. For $N \in \mathbb{N}$ we define $b^N \in H^s_q(\mathbb{R})$ by:*

$$b^N = \sum_{j=-1}^{N} \sum_{m=-N2^j}^{N2^j-1} \mu_{j,m} h_{j,m}. \tag{4}$$

Remark 3.1. *We can note that* Supp $b^N \subset [-N, N]$. *Moreover, we have:*

$$\left\| b - b^N \right\|_{H_q^s(\mathbb{R})} \underset{N \to +\infty}{\longrightarrow} 0.$$

3.2 Computation of the coefficients $\mu_{j,m}$ when b is the derivative of a fractional brownian motion

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4 Numerical results

5 Convergence

5.1 Convergence of the Euler-Maruyama scheme

Ngo and Taguchi proved in [3] the convergence of the Euler-Maruyama scheme for SDE (2) in the following case which applies to our problem.

Theorem 5.1 (Corollary 2.9. in [3])**.** *Assume that $b \in L^1(\mathbb{R}) \cap H^\beta$ for some $\beta \in (0, 1]$ and the diffusion coefficient σ is Lipschitz continuous and uniformly elliptic. Then for any $p \geq 1$, there exists positive constant C which depends on $K_\sigma, \|b\|_\beta, \|b\|_{L^1(\mathbb{R})}, T, x_0, \alpha, \beta$ and p such that*

$$\mathbb{E}\left[\sup_{0 \leq s \leq T} \left| X_s^N - X_s^{N,n} \right|^p\right] \leq \frac{C}{n^{p\beta/2}} \tag{5}$$

TO DO: make explicit the dependance of C in N.

5.2 Convergence of X_s^N to X_s

References

[1] F. Flandoli, E. Issoglio, and F. Russo. Multidimensional stochastic differential equations with distributional drift. *Transactions of the American Mathematical Society*, 369 (3):1655–1688, 3 2017.

[2] E. Issoglio and F. Russo. On a class of markov bsdes with generalized driver. soumis.

[3] H.-L. Ngo and D. Taguchi. On the Euler-Maruyama approximation for one-dimensional stochastic differential equations with irregular coefficients. *IMA Journal of Numerical Analysis*, 2017.