

NUMERICAL SIMULATION OF SDES WITH DISTRIBUTIONAL DRIFT

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1 Introduction

We would like to simulate numerically sample paths of the solution of the stochastic differential equation

$$dX_t = b(X_t) dt + dW_t \tag{1}$$

where $b \in H_q^{-\beta}(\mathbb{R})$, $\beta \in (0, \frac{1}{2})$, $q \in (\frac{1}{1-\beta}, \frac{1}{\beta})$, $t \in [0, T]$, and W_t is a standard Brownian motion. Equation (1) is studied by F. Flandoli, E. Issoglio, and F. Russo in [1] in which they define a concept of virtual solution. The authors prove then existence and unicity in law of this solution.

Example 1. *An example of such drift b is given by the derivative of a sample path of a fractional Brownian motion B_x^H with Hurst index $1/2 < H < 1$. These stochastic processes are gaussian processes verifying*

$$\mathbb{E} [B_t^H B_s^H] = \frac{1}{2} (t^{2H} + s^{2H} + |t - s|^{2H}).$$

We note $-\beta = H - 1$. Given $B_x^H(\omega) \in H_q^{1-\beta}(\mathbb{R})$, we can take $b(x) = \frac{\partial}{\partial x} B_x^H(\omega) \in H_q^{-\beta}(\mathbb{R})$. We will use this in our numerical simulations.

As far as the drift b is not a function but a distribution, it must be approximated if we want to evaluate it at points. In order to do so, we will use a series representation of b and truncate it. That is why we will consider two steps in our algorithm:

1. approximate the drift b by a function b^N meant to converge to b as $N \rightarrow \infty$.
2. approximate the solution X_t^N of the approximated SDE

$$dX_t^N = b^N(X_t^N) dt + dW_t \quad (2)$$

by $X_t^{N,n}$ defined with the Euler-Maruyama scheme

$$X_t^{N,n} = X_0 + \int_0^t b^N(X_{\eta_n(t)}) dt + W_{\eta_n(t)}$$

where $\eta_n(t) = t_k$ if $t \in [t_k, t_k + 1]$, for $t_k = \frac{k}{n}$ with $k \in \llbracket 0, \lceil nT \rceil \rrbracket$.

2 Numerical simulation of fractional Brownian motion

To simulate a sample path of a fractional brownian motion B_x^H on a finite grid $(x_k)_{k \in \llbracket 1, n \rrbracket}$, we simulate n independent standard gaussian random variables $(X_k)_{k \in \llbracket 1, n \rrbracket}$ and then correlate them with the definite positive correlation matrix

$$C_{k,s} = \mathbb{E}[B_{x_k}^H B_{x_s}^H] = \frac{1}{2} (x_k^{2H} + x_s^{2H} + |x_k - x_s|^{2H}).$$

To do so, we use the Cholesky decomposition method and calculate the triangular matrix M such that $C = MM^\top$. Therefore, defining

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ and } B^H = MX,$$

B^H contains the values of a fractional brownian motion evaluated on the grid $(x_k)_{k \in \llbracket 1, n \rrbracket}$.

3 Approximation of the drift

3.1 Series representation

We use Haar wavelets to give a series representation of b . By doing so, we will be able to approximate it numerically by truncating the series.

Definition 1 (Haar wavelets). *We define the Haar wavelets $h_{j,m}$ on \mathbb{R} with $j \in \mathbb{N} \cup \{-1\}$ and $m \in \mathbb{Z}$ by:*

$$\begin{cases} h_M & : x \mapsto \left(\mathbf{1}_{[0, \frac{1}{2}[} - \mathbf{1}_{[\frac{1}{2}, 1[} \right) (x) \\ h_{-1,m} & : x \mapsto \sqrt{2} |h_M(x - m)| \\ h_{j,m} & : x \mapsto h_M(2^j x - m) \end{cases}$$

Theorem 1 (See [2]). *Let $b \in H_q^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left] -\frac{1}{2}, \frac{1}{q} \right[$. Therefore,*

$$b = \sum_{j=-1}^{+\infty} \sum_{m \in \mathbb{Z}} \mu_{j,m} h_{j,m} \quad (3)$$

where $\mu_{j,m} = 2^j \int_{\mathbb{R}} b(x) h_{j,m}(x) \, dx$ in the sense of dual pairing.

Definition 2. *With the same notation $\mu_{j,m}$, let $b \in H_q^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left] -\frac{1}{2}, \frac{1}{q} \right[$. Given $N \in \mathbb{N}^*$ we define b^N by:*

$$b^N = \sum_{m=N}^{N-1} \mu_{-1,m} h_{-1,m} + \sum_{j=0}^N \sum_{m=-N2^j}^{N2^j-1} \mu_{j,m} h_{j,m}. \quad (4)$$

Remark 1. *We can note that $\text{Supp } b^N \subset [-N, N]$. Moreover, we have:*

$$\|b - b^N\|_{H_q^s(\mathbb{R})} \xrightarrow{N \rightarrow +\infty} 0.$$

4 Convergence

4.1 Weak convergence of $X_T^{N,n}$ to X_T^N

Recently, Leobacher and Szölgyenyi proved in [3] the convergence of the Euler-Maruyama scheme for SDE with discontinuous but piecewise Lipschitz drift and with a degenerate diffusion coefficient. This framework applies to the scheme we use with piecewise constant drift b^N and a constant diffusion coefficient.

Theorem 2 (Theorem 3.1. in [3]). $\exists C_N > 0$ independent of n such that it holds $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0$:

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t^{N,n} - X_t^N|^2 \right]^{1/2} \leq C_N \delta^{1/4-\varepsilon} \quad (5)$$

with $\delta = \frac{1}{n}$ the step size and C_N depending on $\|b^N\|_\infty$.

Theorem 3. Let f be μ -Hölder with constant $C_f > 0$ and $\mu \in (0, 1]$. Then, exists $C'_N > 0$ independent of n such that it holds $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0$:

$$\left| \mathbb{E} \left[f \left(X_T^{N,n} \right) - f \left(X_T^N \right) \right] \right| \leq C'_N \delta^{\mu/4-\varepsilon} \quad (6)$$

with $\delta = \frac{1}{n}$ the step size.

Proof. By Jensen's inequality and the μ -Hölder property of f , we obtain:

$$\begin{aligned} \left| \mathbb{E} \left[f \left(X_T^{N,n} \right) - f \left(X_T^N \right) \right] \right| &\leq C_f \mathbb{E} \left[|X_T^{N,n} - X_T^N|^\mu \right] \\ &\leq C_f \mathbb{E} \left[|Y_T - Y_T^N|^2 \right]^{\mu/2} \\ &\leq C_f C_N^\mu \delta^{\mu/4-\varepsilon}. \end{aligned}$$

□

4.2 Weak convergence of X_T^N to X_T

The goal of this section is to estimate the weak error $|\mathbb{E} [f(X_T) - f(X_T^N)]|$ with suitable functions f . In order to do so, we must go back to the definition of the virtual solution of the SDE (1) given in [1]. Let $(\delta, p) \in K(\beta, q) :=$

$\{(\delta, p) \mid \beta < \delta < 1 - \beta, \frac{1}{\delta} < p < q\}$. The authors define the virtual solution of SDE (1) by X_t such that:

$$\begin{cases} Y_t = y + (\lambda + 1) \int_0^t u(s, Y_s) \, ds + \int_0^t (\nabla u(s, Y_s) + 1) \, dW_s \\ X_t = \Psi(t, Y_t) = \varphi^{-1}(t, Y_t) \end{cases} \quad (7)$$

where u is the mild solution in $H_p^{1+\delta}$ of the following parabolic PDE:

$$\begin{cases} \partial_t u + \frac{1}{2} \Delta u + b \nabla u - (\lambda + 1)u = -b & \text{on } [0, T] \times \mathbb{R} \\ u(T) = 0 & \text{on } \mathbb{R} \end{cases} \quad (8)$$

and $\varphi(t, x) = x + u(t, x)$.

We also define another similar PDE by replacing b by b^N . We call u^N its mild solution in $H_p^{1+\delta}$:

$$\begin{cases} \partial_t u^N + \frac{1}{2} \Delta u^N + b^N \nabla u^N - (\lambda + 1)u^N = -b^N & \text{on } [0, T] \times \mathbb{R} \\ u^N(T) = 0 & \text{on } \mathbb{R} \end{cases}. \quad (9)$$

Then we consider an approximated version of (7):

$$\begin{cases} Y_t^N = y + (\lambda + 1) \int_0^t u^N(s, Y_s^N) \, ds + \int_0^t (\nabla u^N(s, Y_s^N) + 1) \, dW_s \\ X_t^N = \Psi(t, Y_t^N) = \varphi^{-1}(t, Y_t^N) \end{cases}. \quad (10)$$

Remark 2. Proposition 26 in [1] assures us that X_t^N defined above in (10) is in fact the classical solution of (2), as far as $b^N \in L^p$. That is why for each fixed N our Euler scheme converges to the virtual solution X_t^N .

We also recall a useful lemma concerning the solutions of (8) and (9).

Lemma 4 (Lemma 20 in [1]). *Let $(\delta, p) \in K(\beta, q)$ and let u, u^N be the mild solution to (8) in $H_p^{1+\delta}$. Fix ρ such that the integral operator is a contraction and let $\lambda > \rho$. Then $u(t), u^N(t) \in \mathcal{C}^{1,\alpha}$ with $\alpha = \delta - 1/p$ for each fixed t and*

$$\begin{cases} \sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla u(t, x)| \rightarrow 0, \text{ as } \lambda \rightarrow \infty \\ \sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla u^N(t, x)| \rightarrow 0, \text{ as } \lambda \rightarrow \infty \end{cases}$$

where the choice of λ depends only on $\delta, \beta, \|b\|_{H_p^{-\beta}}$, and $\|b\|_{H_q^{-\beta}}$.

Lemma 5. *Let $(\delta, p) \in K(\beta, q)$ and let u, u^N be the mild solutions to (8),(9) in $H_p^{1+\delta}$, $\alpha = \delta - 1/p$. Exists $c, K > 0$ such that for both $N \in \mathbb{N}$ and $\rho > 1$ big enough, $\forall t \in [0, T]$,*

$$\begin{cases} \|u^N(t) - u(t)\|_{L^\infty} \leq cKe^{\rho T} \|b^N - b\|_{H_q^{-\beta}} \\ \|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq cKe^{\rho T} \|b^N - b\|_{H_q^{-\beta}}. \end{cases} \quad (11)$$

Proof. Applying fractional Morrey inequality, $\exists c > 0, \forall t \in [0, T]$:

$$\begin{cases} \|u^N(t) - u(t)\|_{L^\infty} \leq \|u^N(t) - u(t)\|_{C^{1,\alpha}} \leq c \|u^N(t) - u(t)\|_{H_p^{1+\delta}} \\ \|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \|u^N(t) - u(t)\|_{C^{1,\alpha}} \leq c \|u^N(t) - u(t)\|_{H_p^{1+\delta}}. \end{cases}$$

Now, we can conclude with

$$\|u^N - u\|_{\infty, H_p^{1+\delta}} \leq e^{\rho T} \|u^N - u\|_{\infty, H_p^{1+\delta}}^{(\rho)} \leq Ke^{\rho T} \|b^N - b\|_{H_q^{-\beta}}$$

from Lemma 23 in [1], for both $N \in \mathbb{N}$ and $\rho > 1$ big enough, and where $\|f(t)\|_{\infty, X}^{(\rho)} := \sup_{0 \leq t \leq T} e^{-\rho t} \|f(t)\|_X$. \square

We will need the following local time inequality from Liqing Yan:

Lemma 6 (Lemma 4.2 in [4]). *Let X be a continuous semimartingale with $X_0 = 0$. For $\varepsilon > 0$ we define a double sequence of stopping times by $\sigma_1 = 0$, $\tau_1 = \inf\{t > 0 | X_t = \varepsilon\}$, $\sigma_n = \inf\{t > \tau_{n-1} | X_t = 0\}$, $\tau_n = \inf\{t > \sigma_n | X_t = \varepsilon\}$. For any real function $F(\cdot) \in \mathcal{C}^2$ with $F(0) = 0$, $F'(0) = 0$ and $F(\cdot) > 0$ on $(0, \varepsilon_0)$ for some $\varepsilon_0 > 0$, then for any $0 < \varepsilon < \varepsilon_0$ we have*

$$\begin{aligned} 0 \leq L_t^0(X) &\leq 2\varepsilon - \frac{2}{F(\varepsilon)} \int_0^t \theta_s(X) (F(\varepsilon) - \varepsilon F'(X_s^+)) dX_s \\ &\quad + \frac{1}{F(\varepsilon)} \int_0^t \theta_s(X) \varepsilon F''(X_s^+) d[X]_s \end{aligned}$$

with $\theta_s(X) = \sum_{n=1}^{\infty} \mathbb{1}_{\{\sigma_n < s \leq \tau_n, 0 < X_s \leq \varepsilon\}}(X)$.

Applying lemma 6 with $F(x) = x^2$, it follows:

Corollary 7. *Let X be a continuous semimartingale with $X_0 = 0$. With the same notations as in lemma 6, for any $\varepsilon > 0$ we have*

$$0 \leq L_t^0(X) \leq 2\varepsilon - \frac{2}{\varepsilon} \int_0^t \theta_s(X) (\varepsilon - 2X_s^+) dX_s + \frac{2}{\varepsilon} \int_0^t \theta_s(X) d[X]_s \quad (12)$$

Lemma 8. *Let $(\delta, p) \in K(\beta, q)$, $\alpha = \delta - 1/p < 1$, u, u^N be the mild solutions to (8), (9) in $H_p^{1+\delta}$, and Y, Y^N solutions of the SDEs (7), (10). Then, if $\alpha > 1/2$, for λ big enough we have $\forall \varepsilon \in (0, 1]$,*

$$0 \leq \mathbb{E} [L_T^0(Y - Y^N)] \leq g(\varepsilon).$$

where

$$\begin{aligned} g(\varepsilon) = & 2(\lambda+1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + (16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2(1 + (\lambda + 1)T)) \varepsilon^{2\alpha-1} \\ & + 4c^2 T K^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-1} \end{aligned}$$

Proof. By Lemma 4, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\|b - b^N\|_{H_q^s(\mathbb{R})} \xrightarrow{N \rightarrow \infty} 0$. Therefore $u^N(t, \cdot)$ and $u(t, \cdot)$ are $\frac{1}{2}$ -lipschitz. We recall that in this case, by Lemma 22 in [1], $\Psi(t, \cdot)$ is 2-lipschitz. Let $\varepsilon \in (0, 1]$. Corollary 7 gives us:

$$\begin{aligned} 0 \leq L_T^0(Y - Y^N) \leq & 2\varepsilon - \frac{2}{\varepsilon} \int_0^T \theta_s(Y - Y^N) (\varepsilon - 2(Y_s - Y_s^N)^+) d(Y_s - Y_s^N) \\ & + \frac{2}{\varepsilon} \int_0^T \theta_s(Y - Y^N) d[(Y - Y^N)]_s. \end{aligned}$$

Remark 3. *Note that $\theta_s(Y - Y^N) |\varepsilon - 2(Y_s - Y_s^N)^+| \leq \varepsilon \theta_s(Y - Y^N)$.*

Let $t \in [0, T]$.

$$\begin{aligned} Y_t - Y_t^N = & (\lambda + 1) \int_0^t \{u(s, \Psi(s, Y_s)) - u^N(s, \Psi(s, Y_s^N))\} ds \\ & + \int_0^t \{\nabla u(s, \Psi(s, Y_s)) - \nabla u^N(t, \Psi(s, Y_s^N))\} dW_s. \end{aligned}$$

Remark 4. *For clarity purpose, we note $\tilde{u}(s, x) = u(s, \Psi(s, x))$ and use the same notation for the gradient and the approximated mild solution. We can notice that \tilde{u} is 1-lipschitz in space and $\widetilde{\nabla u}$ is α -Hölder with constant $2 \|u\|_{\mathcal{C}^{1,\alpha}}$.*

$\widetilde{\nabla u}$ and $\widetilde{\nabla u}^N$ are bounded so the Itô integral is a martingale. We take

the expectation:

$$\begin{aligned}
 \mathbb{E} [L_T^0(Y - Y^N)] &\leq 2\varepsilon + 2(\lambda + 1) \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) \{ \tilde{u}(s, Y_s) - \tilde{u}^N(s, Y_s^N) \} ds \right] \\
 &\quad + \frac{2}{\varepsilon} \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) \left\{ \widetilde{\nabla} u(s, Y_s) - \widetilde{\nabla} u^N(s, Y_s^N) \right\}^2 ds \right] \\
 &\leq 2\varepsilon + 2(\lambda + 1) \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) \{ \tilde{u}(s, Y_s) - \tilde{u}(s, Y_s^N) \} ds \right] \\
 &\quad + 2(\lambda + 1) \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) \{ \tilde{u}(s, Y_s^N) - \tilde{u}^N(s, Y_s^N) \} ds \right] \\
 &\quad + \frac{4}{\varepsilon} \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) \left\{ \widetilde{\nabla} u(s, Y_s) - \widetilde{\nabla} u(s, Y_s^N) \right\}^2 ds \right] \\
 &\quad + \frac{4}{\varepsilon} \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) \left\{ \widetilde{\nabla} u(s, Y_s^N) - \widetilde{\nabla} u^N(s, Y_s^N) \right\}^2 ds \right] \\
 &\leq 2\varepsilon + 2(\lambda + 1) \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) |Y_s - Y_s^N| ds \right] \\
 &\quad + \frac{16 \|u\|_{\mathcal{C}^{1,\alpha}}^2}{\varepsilon} \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) |Y_s - Y_s^N|^{2\alpha} ds \right] \\
 &\quad + 2(\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + 4c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-1}
 \end{aligned}$$

where we have used Lemma 5, the 1-lipschitz property of \tilde{u} and the α -Hölder property of $\widetilde{\nabla} u$ (with constant $2 \|u\|_{\mathcal{C}^{1,\alpha}}$). As $\theta_s(Y - Y^N) |Y_s - Y_s^N| \leq \varepsilon$, we have

$$\begin{aligned}
 \mathbb{E} [L_T^0(Y - Y^N)] &\leq 2\varepsilon + 2(\lambda + 1) T\varepsilon + 16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T\varepsilon^{2\alpha-1} \\
 &\quad + 2(\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + 4c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-1}
 \end{aligned}$$

As $1 > 2\alpha - 1 > 0$, the result follows from $\varepsilon \leq \varepsilon^{2\alpha-1}$ when $0 < \varepsilon \leq 1$. \square

Lemma 9. *With assumptions and notations of Lemma 8, and $1 > \alpha > 1/2$ we have $\forall \varepsilon \in (0, 1]$,*

$$g(\varepsilon) \leq \sigma \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \quad (13)$$

for $\|b^N - b\|_{H_q^{-\beta}}$ small enough (it is to say N big enough) where

$$\sigma = 2(\lambda+1) cTK e^{\rho T} + (16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2(1 + (\lambda + 1)T)) \omega^{2\alpha-1} + 4c^2TK^2 e^{2\rho T} \omega^{-1}$$

and

$$\omega = \left(\frac{4c^2TK^2 e^{2\rho T}}{(2\alpha - 1) (16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2(1 + (\lambda + 1)T))} \right)^{\frac{1}{2\alpha}}.$$

Proof. By Lemma 8,

$$0 \leq \mathbb{E} [L_T^0(Y - Y^N)] \leq g(\varepsilon)$$

where

$$\begin{aligned} g(\varepsilon) = 2(\lambda+1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + (16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2(1 + (\lambda + 1)T)) \varepsilon^{2\alpha-1} \\ + 4c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-1}. \end{aligned}$$

With

$$\begin{aligned} g'(\varepsilon) = (2\alpha - 1) (16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2(1 + (\lambda + 1)T)) \varepsilon^{2\alpha-2} \\ - 4c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-2}, \end{aligned}$$

and

$$\begin{aligned} g''(\varepsilon) = (2\alpha - 2)(2\alpha - 1) (16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2(1 + (\lambda + 1)T)) \varepsilon^{2\alpha-3} \\ + 8c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-3}, \end{aligned}$$

the minimum of g on $(0, 1]$ is reached when N is big enough in

$$\varepsilon_0 = \left(\frac{4c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2}{(2\alpha - 1) (16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2(1 + (\lambda + 1)T))} \right)^{\frac{1}{2\alpha}} = \omega \|b^N - b\|_{H_q^{-\beta}}^{1/\alpha}.$$

where

$$g''(\varepsilon_0) = 8c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon_0^{-3} (1 - (1 - \alpha)) > 0.$$

and

$$\omega = \left(\frac{4c^2TK^2e^{2\rho T}}{(2\alpha - 1)(16\|u\|_{\mathcal{C}^{1,\alpha}}^2T + 2(1 + (\lambda + 1)T))} \right)^{\frac{1}{2\alpha}}.$$

Therefore $\mathbb{E}[L_T^0(Y - Y^N)] \leq g(\varepsilon_0)$

$$\begin{aligned} &\leq 2(\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + (16\|u\|_{\mathcal{C}^{1,\alpha}}^2T + 2(1 + (\lambda + 1)T)) \omega^{2\alpha-1} \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \\ &\quad + 4c^2TK^2e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \omega^{-1} \|b^N - b\|_{H_q^{-\beta}}^{-1/\alpha} \\ &\leq 2(\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} \\ &+ ((16\|u\|_{\mathcal{C}^{1,\alpha}}^2T + 2(1 + (\lambda + 1)T)) \omega^{2\alpha-1} + 4c^2TK^2e^{2\rho T} \omega^{-1}) \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \\ &\leq 2(\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \\ &+ ((16\|u\|_{\mathcal{C}^{1,\alpha}}^2T + 2(1 + (\lambda + 1)T)) \omega^{2\alpha-1} + 4c^2TK^2e^{2\rho T} \omega^{-1}) \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \end{aligned}$$

for N big enough. The result follows. \square

Theorem 10. *Let f be μ -Hölder with constant $C_f > 0$ and $\mu \in (0, 1]$. If $0 < \beta < 1/4$, $q \in \left(\frac{1}{1-\beta}, \frac{1}{\beta}\right)$, $\forall \varepsilon \in (0, \frac{1-4\beta}{2})$, with $(\delta, p) \in K(\beta, q)$ such that $\delta - 1/p = 1 - 2\beta - \varepsilon$, exists ξ independent of f such that for $N \in \mathbb{N}$, $\rho > 1$, λ big enough it holds:*

$$|\mathbb{E}[f(X_T) - f(X_T^N)]| \leq \xi C_f \|b^N - b\|_{H_q^{-\beta}}^{\mu(2 - \frac{1}{1-2\beta-\varepsilon})}$$

Proof. We note as usual $\alpha = \delta - 1/p$. By Lemma 4, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\|b - b^N\|_{H_q^s(\mathbb{R})} \xrightarrow{N \rightarrow \infty} 0$ (See Step 2 of the proof of Proposition 29 in [1]). Therefore $u^N(t, \cdot)$ and $u(t, \cdot)$ are $\frac{1}{2}$ -lipschitz. We recall that in this case, by Lemma 22 in [1], $\Psi(t, \cdot)$ is 2-lipschitz.

$$|\mathbb{E}[f(X_T) - f(X_T^N)]| = |\mathbb{E}[f(\Psi(T, Y_T)) - f(\Psi(T, Y_T^N))]|$$

$$\leq 2^\mu C_f \mathbb{E} [|Y_T - Y_T^N|^\mu] \leq 2^\mu C_f \mathbb{E} [|Y_T - Y_T^N|]^\mu \quad (14)$$

by Jensen's inequality. Let $t \in [0, T]$.

$$\begin{aligned} Y_t - Y_t^N &= (\lambda + 1) \int_0^t \{u(s, \Psi(s, Y_s)) - u^N(s, \Psi(s, Y_s^N))\} \, ds \\ &\quad + \int_0^t \{\nabla u(s, \Psi(s, Y_s)) - \nabla u^N(t, \Psi(s, Y_s^N))\} \, dW_s. \end{aligned}$$

We apply Meyer-Tanaka's formula to obtain:

$$\begin{aligned} |Y_t - Y_t^N| &= (\lambda + 1) \int_0^t \text{sign}(Y_s - Y_s^N) \{\tilde{u}(s, Y_s) - \tilde{u}^N(s, Y_s^N)\} \, ds \\ &\quad + \int_0^t \text{sign}(Y_s - Y_s^N) \{\widetilde{\nabla} u(s, Y_s) - \widetilde{\nabla} u^N(s, Y_s^N)\} \, dW_s + L_t^0(Y - Y^N). \end{aligned}$$

Taking the expectation leads to:

$$\begin{aligned} \mathbb{E} [|Y_t - Y_t^N|] &= (\lambda + 1) \mathbb{E} \left[\int_0^t \text{sign}(Y_s - Y_s^N) \{\tilde{u}(s, Y_s) - \tilde{u}^N(s, Y_s^N)\} \, ds \right] \\ &\quad + \mathbb{E} [L_t^0(Y - Y^N)] \end{aligned}$$

because $\widetilde{\nabla} u$ and $\widetilde{\nabla} u^N$ are bounded so the Itô integral is a martingale.

$$\begin{aligned} \mathbb{E} [|Y_t - Y_t^N|] &\leq (\lambda + 1) \mathbb{E} \left[\int_0^t \{\tilde{u}(s, Y_s) - \tilde{u}(s, Y_s^N)\} \, ds \right] \\ &\quad + (\lambda + 1) \mathbb{E} \left[\int_0^t \{\tilde{u}(s, Y_s^N) - \tilde{u}^N(s, Y_s^N)\} \, ds \right] + \mathbb{E} [L_t^0(Y - Y^N)]. \end{aligned}$$

We use Lemma 5 and the 1-lipschitz property of \tilde{u} :

$$\begin{aligned} \mathbb{E} [|Y_t - Y_t^N|] &\leq (\lambda + 1) \mathbb{E} \left[\int_0^t |Y_s - Y_s^N| \, ds \right] + (\lambda + 1) c t K e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} \\ &\quad + \mathbb{E} [L_t^0(Y - Y^N)] \\ &\leq (\lambda + 1) \int_0^t \mathbb{E} [|Y_s - Y_s^N|] \, ds + (\lambda + 1) c T K e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} \\ &\quad + \mathbb{E} [L_T^0(Y - Y^N)]. \end{aligned}$$

where we have used the fact that $L_t^0(Y - Y^N)$ is an increasing process.

By Gronwall's Lemma, it follows:

$$\mathbb{E} [|Y_T - Y_T^N|] \leq C(N) \exp((\lambda + 1)T) \quad (15)$$

with $C(N) = (\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + \mathbb{E} [L_T^0(Y - Y^N)]$.

With Lemma 8 and Lemma 9 we obtain

$$C(N) \leq (\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + \sigma \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \leq \zeta \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha}.$$

for $\|b^N - b\|_{H_q^{-\beta}}$ small enough where $\zeta = (\lambda + 1) cTK e^{\rho T} + \sigma$. It follows:

$$\mathbb{E} [|Y_T - Y_T^N|] \leq \zeta \exp((\lambda + 1)T) \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha}. \quad (16)$$

Finally, combining (14) and (16) we obtain:

$$\begin{aligned} |\mathbb{E} [f(X_T) - f(X_T^N)]| &\leq 2^\mu C_f \mathbb{E} [|Y_T - Y_T^N|]^\mu \\ &\leq 2^\mu C_f \zeta^\mu \exp(\mu(\lambda + 1)T) \|b^N - b\|_{H_q^{-\beta}}^{\mu(2-1/\alpha)} \end{aligned}$$

which is the expected result. □

5 Numerical results

5.1 Strong convergence of the Euler scheme

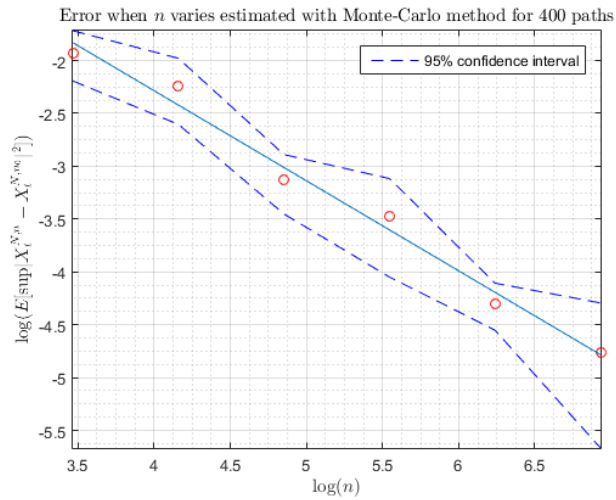


Figure 1: Estimation of the L^2 error of the Euler-Maruyama scheme with a Monte-Carlo method. 400 paths, $N = 5$, $n \in \{2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}\}$, reference solution with $n_0 = 2^{12}$ points.

We observe a numerical convergence rate of 0.85 when Theorem 2 shows a theoretical rate of $0.5 - \varepsilon$.

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