

# NUMERICAL SIMULATION OF SDES WITH DISTRIBUTIONAL DRIFT

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## 1 Introduction

We would like to simulate numerically sample paths of the solution of the stochastic differential equation

$$dX_t = b(X_t) dt + dW_t \tag{1}$$

where  $b \in H_q^{-\beta}(\mathbb{R})$ ,  $\beta \in (0, \frac{1}{2})$ ,  $q \in (\frac{1}{1-\beta}, \frac{1}{\beta})$ ,  $t \in [0, T]$ , and  $W_t$  is a standard Brownian motion. Equation (1) is studied by F. Flandoli, E. Issoglio, and F. Russo in [1] in which they define a concept of virtual solution. The authors prove then existence and unicity in law of this solution.

**Example 1.** *An example of such drift  $b$  is given by the derivative of a sample path of a fractional Brownian motion  $B_x^H$  with Hurst index  $1/2 < H < 1$ . These stochastic processes are gaussian processes verifying*

$$\mathbb{E} [B_t^H B_s^H] = \frac{1}{2} (t^{2H} + s^{2H} + |t - s|^{2H}).$$

*We note  $-\beta = H - 1$ . Given  $B_x^H(\omega) \in H_q^{1-\beta}(\mathbb{R})$ , we can take  $b(x) = \frac{\partial}{\partial x} B_x^H(\omega) \in H_q^{-\beta}(\mathbb{R})$ . We will use this in our numerical simulations.*

As far as the drift  $b$  is not a function but a distribution, it must be approximated if we want to evaluate it at points. In order to do so, we will use a series representation of  $b$  and truncate it. That is why we will consider two steps in our algorithm:

1. approximate the drift  $b$  by a function  $b^N$  meant to converge to  $b$  as  $N \rightarrow \infty$ .
2. approximate the solution  $X_t^N$  of the approximated SDE

$$dX_t^N = b^N(X_t^N) dt + dW_t \quad (2)$$

by  $X_t^{N,n}$  defined with the Euler-Maruyama scheme

$$X_t^{N,n} = X_0 + \int_0^t b^N(X_{\eta_n(t)}^{N,n}) dt + W_{\eta_n(t)}$$

where  $\eta_n(t) = t_k$  if  $t \in [t_k, t_k + 1]$ , for  $t_k = \frac{k}{n}$  with  $k \in \llbracket 0, \lceil nT \rceil \rrbracket$ .

## 2 Numerical simulation of fractional Brownian motion

To simulate a sample path of a fractional brownian motion  $B_x^H$  on a finite grid  $(x_k)_{k \in \llbracket 1, n \rrbracket}$ , we simulate  $n$  independent standard gaussian random variables  $(X_k)_{k \in \llbracket 1, n \rrbracket}$  and then correlate them with the definite positive correlation matrix

$$C_{k,s} = \mathbb{E}[B_{x_k}^H B_{x_s}^H] = \frac{1}{2} (x_k^{2H} + x_s^{2H} + |x_k - x_s|^{2H}).$$

To do so, we use the Cholesky decomposition method and calculate the triangular matrix  $M$  such that  $C = MM^\top$ . Therefore, defining

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ and } B^H = MX,$$

$B^H$  contains the values of a fractional brownian motion evaluated on the grid  $(x_k)_{k \in \llbracket 1, n \rrbracket}$ .

## 3 Approximation of the drift

### 3.1 Series representation

We use Haar wavelets to give a series representation of  $b$ . By doing so, we will be able to approximate it numerically by truncating the series.

**Definition 1** (Haar wavelets). *We define the Haar wavelets  $h_{j,m}$  on  $\mathbb{R}$  with  $j \in \mathbb{N} \cup \{-1\}$  and  $m \in \mathbb{Z}$  by:*

$$\begin{cases} h_M & : x \mapsto \left( \mathbf{1}_{[0, \frac{1}{2}[} - \mathbf{1}_{[\frac{1}{2}, 1[} \right) (x) \\ h_{-1,m} & : x \mapsto \sqrt{2} |h_M(x - m)| \\ h_{j,m} & : x \mapsto h_M(2^j x - m) \end{cases}$$

**Theorem 1** (See [2]). *Let  $b \in H_q^s(\mathbb{R})$  for  $2 \leq q \leq \infty$ , and  $s \in \left] -\frac{1}{2}, \frac{1}{q} \right[$ . Therefore,*

$$b = \sum_{j=-1}^{+\infty} \sum_{m \in \mathbb{Z}} \mu_{j,m} h_{j,m} \quad (3)$$

where  $\mu_{j,m} = 2^j \int_{\mathbb{R}} b(x) h_{j,m}(x) \, dx$  in the sense of dual pairing.

**Definition 2.** *With the same notation  $\mu_{j,m}$ , let  $b \in H_q^s(\mathbb{R})$  for  $2 \leq q \leq \infty$ , and  $s \in \left] -\frac{1}{2}, \frac{1}{q} \right[$ . Given  $N \in \mathbb{N}^*$  we define  $b^N$  by:*

$$b^N = \sum_{m=N}^{N-1} \mu_{-1,m} h_{-1,m} + \sum_{j=0}^N \sum_{m=-N2^j}^{N2^j-1} \mu_{j,m} h_{j,m}. \quad (4)$$

**Remark 1.** *We can note that  $\text{Supp } b^N \subset [-N, N]$ . Moreover, we have:*

$$\|b - b^N\|_{H_q^s(\mathbb{R})} \xrightarrow{N \rightarrow +\infty} 0.$$

## 4 Convergence

### 4.1 Weak convergence of $X_T^{N,n}$ to $X_T^N$

Recently, Leobacher and Szölgyenyi proved in [3] the convergence of the Euler-Maruyama scheme for SDE with discontinuous but piecewise Lipschitz drift and with a degenerate diffusion coefficient. This framework applies to the scheme we use with piecewise constant drift  $b^N$  and a constant diffusion coefficient.

**Theorem 2** (Theorem 3.1. in [3]).  $\exists C_N > 0$  independent of  $n$  such that it holds  $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0$ :

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^{N,n} - X_t^N|^2 \right]^{1/2} \leq C_N \delta^{1/4-\varepsilon} \quad (5)$$

with  $\delta = \frac{1}{n}$  the step size and  $C_N$  depending on  $\|b^N\|_\infty$ .

**Theorem 3.** Let  $f$  be  $\mu$ -Hölder with constant  $C_f > 0$  and  $\mu \in (0, 1]$ . Then, exists  $C'_N > 0$  independent of  $n$  such that it holds  $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0$ :

$$\left| \mathbb{E} \left[ f(X_T^{N,n}) - f(X_T^N) \right] \right| \leq C'_N \delta^{\mu/4-\varepsilon} \quad (6)$$

with  $\delta = \frac{1}{n}$  the step size.

*Proof.* By Jensen's inequality and the  $\mu$ -Hölder property of  $f$ , we obtain:

$$\begin{aligned} \left| \mathbb{E} \left[ f(X_T^{N,n}) - f(X_T^N) \right] \right| &\leq C_f \mathbb{E} \left[ |X_T^{N,n} - X_T^N|^\mu \right] \\ &\leq C_f \mathbb{E} \left[ |Y_T - Y_T^N|^2 \right]^{\mu/2} \\ &\leq C_f C_N^\mu \delta^{\mu/4-\varepsilon}. \end{aligned}$$

□

### 4.2 Weak convergence of $X_T^N$ to $X_T$

The goal of this section is to estimate the weak error  $|\mathbb{E} [f(X_T) - f(X_T^N)]|$  with suitable functions  $f$ . In order to do so, we must go back to the definition of the virtual solution of the SDE (1) given in [1]. Let  $(\delta, p) \in K(\beta, q) :=$

$\{(\delta, p) \mid \beta < \delta < 1 - \beta, \frac{1}{\delta} < p < q\}$ . The authors define the virtual solution of SDE (1) by  $X_t$  such that:

$$\begin{cases} Y_t = y + (\lambda + 1) \int_0^t u(s, Y_s) \, ds + \int_0^t (\nabla u(s, Y_s) + 1) \, dW_s \\ X_t = \Psi(t, Y_t) = \varphi^{-1}(t, Y_t) \end{cases} \quad (7)$$

where  $u$  is the mild solution in  $H_p^{1+\delta}$  of the following parabolic PDE:

$$\begin{cases} \partial_t u + \frac{1}{2} \Delta u + b \nabla u - (\lambda + 1)u = -b & \text{on } [0, T] \times \mathbb{R} \\ u(T) = 0 & \text{on } \mathbb{R} \end{cases} \quad (8)$$

and  $\varphi(t, x) = x + u(t, x)$ .

We also define another similar PDE by replacing  $b$  by  $b^N$ . We call  $u^N$  its mild solution in  $H_p^{1+\delta}$ :

$$\begin{cases} \partial_t u^N + \frac{1}{2} \Delta u^N + b^N \nabla u^N - (\lambda + 1)u^N = -b^N & \text{on } [0, T] \times \mathbb{R} \\ u^N(T) = 0 & \text{on } \mathbb{R} \end{cases}. \quad (9)$$

Then we consider an approximated version of (7):

$$\begin{cases} Y_t^N = y + (\lambda + 1) \int_0^t u^N(s, Y_s^N) \, ds + \int_0^t (\nabla u^N(s, Y_s^N) + 1) \, dW_s \\ X_t^N = \Psi^N(t, Y_t^N) = (\varphi^N)^{-1}(t, Y_t^N) \end{cases}. \quad (10)$$

**Remark 2.** Proposition 26 in [1] assures us that  $X_t^N$  defined above in (10) is in fact the classical solution of (2), as far as  $b^N \in L^p$ . That is why for each fixed  $N$  our Euler scheme converges to the virtual solution  $X_t^N$ .

We also recall a useful lemma concerning the solutions of (8) and (9).

**Lemma 4** (Lemma 20 in [1]). *Let  $(\delta, p) \in K(\beta, q)$  and let  $u, u^N$  be the mild solution to (8) in  $H_p^{1+\delta}$ . Fix  $\rho$  such that the integral operator is a contraction and let  $\lambda > \rho$ . Then  $u(t), u^N(t) \in \mathcal{C}^{1,\alpha}$  with  $\alpha = \delta - 1/p$  for each fixed  $t$  and*

$$\begin{cases} \sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla u(t, x)| \rightarrow 0, \text{ as } \lambda \rightarrow \infty \\ \sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla u^N(t, x)| \rightarrow 0, \text{ as } \lambda \rightarrow \infty \end{cases}$$

where the choice of  $\lambda$  depends only on  $\delta, \beta, \|b\|_{H_p^{-\beta}}$ , and  $\|b\|_{H_q^{-\beta}}$ .

**Lemma 5.** *Let  $(\delta, p) \in K(\beta, q)$  and let  $u, u^N$  be the mild solutions to (8), (9) in  $H_p^{1+\delta}$ ,  $\alpha = \delta - 1/p$ . Exists  $c, K > 0$  such that for both  $N \in \mathbb{N}$  and  $\rho > 1$  big enough,  $\forall t \in [0, T]$ ,*

$$\begin{cases} \|u^N(t) - u(t)\|_{L^\infty} \leq cKe^{\rho T} \|b^N - b\|_{H_q^{-\beta}} \\ \|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq cKe^{\rho T} \|b^N - b\|_{H_q^{-\beta}}. \end{cases} \quad (11)$$

*Proof.* Applying fractional Morrey inequality,  $\exists c > 0, \forall t \in [0, T]$ :

$$\begin{cases} \|u^N(t) - u(t)\|_{L^\infty} \leq \|u^N(t) - u(t)\|_{C^{1,\alpha}} \leq c \|u^N(t) - u(t)\|_{H_p^{1+\delta}} \\ \|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \|u^N(t) - u(t)\|_{C^{1,\alpha}} \leq c \|u^N(t) - u(t)\|_{H_p^{1+\delta}}. \end{cases}$$

Now, we can conclude with

$$\|u^N - u\|_{\infty, H_p^{1+\delta}} \leq e^{\rho T} \|u^N - u\|_{\infty, H_p^{1+\delta}}^{(\rho)} \leq Ke^{\rho T} \|b^N - b\|_{H_q^{-\beta}}$$

from Lemma 23 in [1], for both  $N \in \mathbb{N}$  and  $\rho > 1$  big enough, and where  $\|f(t)\|_{\infty, X}^{(\rho)} := \sup_{0 \leq t \leq T} e^{-\rho t} \|f(t)\|_X$ .  $\square$

We will need the following local time inequality from Liqing Yan:

**Lemma 6** (Lemma 4.2 in [4]). *Let  $X$  be a continuous semimartingale with  $X_0 = 0$ . For  $\varepsilon > 0$  we define a double sequence of stopping times by  $\sigma_1 = 0$ ,  $\tau_1 = \inf\{t > 0 | X_t = \varepsilon\}$ ,  $\sigma_n = \inf\{t > \tau_{n-1} | X_t = 0\}$ ,  $\tau_n = \inf\{t > \sigma_n | X_t = \varepsilon\}$ . For any real function  $F(\cdot) \in \mathcal{C}^2$  with  $F(0) = 0$ ,  $F'(0) = 0$  and  $F(\cdot) > 0$  on  $(0, \varepsilon_0)$  for some  $\varepsilon_0 > 0$ , then for any  $0 < \varepsilon < \varepsilon_0$  we have*

$$\begin{aligned} 0 \leq L_t^0(X) &\leq 2\varepsilon - \frac{2}{F(\varepsilon)} \int_0^t \theta_s(X) (F(\varepsilon) - \varepsilon F'(X_s^+)) dX_s \\ &\quad + \frac{1}{F(\varepsilon)} \int_0^t \theta_s(X) \varepsilon F''(X_s^+) d[X]_s \end{aligned}$$

with  $\theta_s(X) = \sum_{n=1}^{\infty} \mathbb{1}_{\{\sigma_n < s \leq \tau_n, 0 < X_s \leq \varepsilon\}}(X)$ .

Applying lemma 6 with  $F(x) = x^2$ , it follows:

**Corollary 7.** *Let  $X$  be a continuous semimartingale with  $X_0 = 0$ . With the same notations as in lemma 6, for any  $\varepsilon > 0$  we have*

$$0 \leq L_t^0(X) \leq 2\varepsilon - \frac{2}{\varepsilon} \int_0^t \theta_s(X) (\varepsilon - 2X_s^+) dX_s + \frac{2}{\varepsilon} \int_0^t \theta_s(X) d[X]_s \quad (12)$$

**Lemma 8.** *Let  $(\delta, p) \in K(\beta, q)$ ,  $\alpha = \delta - 1/p < 1$ ,  $u, u^N$  be the mild solutions to (8), (9) in  $H_p^{1+\delta}$ , and  $Y, Y^N$  solutions of the SDEs (7), (10). Then, if  $\alpha > 1/2$ , for  $\lambda$  big enough we have  $\forall \varepsilon \in (0, 1]$ ,*

$$0 \leq \mathbb{E} [L_T^0(Y - Y^N)] \leq g(\varepsilon).$$

where

$$\begin{aligned} g(\varepsilon) = & 2(\lambda+1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + (16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2(1 + (\lambda + 1)T)) \varepsilon^{2\alpha-1} \\ & + 4c^2 T K^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-1} \end{aligned}$$

*Proof.* By Lemma 4, we choose  $\lambda$  big enough for  $\nabla u$  and  $\nabla u^N$  to be bounded by  $\frac{1}{2}$ .  $\lambda$  can be chosen independently of  $N$  as far as  $\|b - b^N\|_{H_q^s(\mathbb{R})} \xrightarrow{N \rightarrow \infty} 0$ . Therefore  $u^N(t, \cdot)$  and  $u(t, \cdot)$  are  $\frac{1}{2}$ -lipschitz. We recall that in this case, by Lemma 22 in [1],  $\Psi(t, \cdot)$  is 2-lipschitz. Let  $\varepsilon \in (0, 1]$ . Corollary 7 gives us:

$$\begin{aligned} 0 \leq L_T^0(Y - Y^N) \leq & 2\varepsilon - \frac{2}{\varepsilon} \int_0^T \theta_s(Y - Y^N) (\varepsilon - 2(Y_s - Y_s^N)^+) d(Y_s - Y_s^N) \\ & + \frac{2}{\varepsilon} \int_0^T \theta_s(Y - Y^N) d[(Y - Y^N)]_s. \end{aligned}$$

**Remark 3.** *Note that  $\theta_s(Y - Y^N) |\varepsilon - 2(Y_s - Y_s^N)^+| \leq \varepsilon \theta_s(Y - Y^N)$ .*

Let  $t \in [0, T]$ .

$$\begin{aligned} Y_t - Y_t^N = & (\lambda + 1) \int_0^t \{u(s, \Psi(s, Y_s)) - u^N(s, \Psi(s, Y_s^N))\} ds \\ & + \int_0^t \{\nabla u(s, \Psi(s, Y_s)) - \nabla u^N(t, \Psi(s, Y_s^N))\} dW_s. \end{aligned}$$

**Remark 4.** *For clarity purpose, we note  $\tilde{u}(s, x) = u(s, \Psi(s, x))$  and use the same notation for the gradient and the approximated mild solution. We can notice that  $\tilde{u}$  is 1-lipschitz in space and  $\widetilde{\nabla u}$  is  $\alpha$ -Hölder with constant  $2 \|u\|_{\mathcal{C}^{1,\alpha}}$ .*

$\widetilde{\nabla u}$  and  $\widetilde{\nabla u}^N$  are bounded so the Itô integral is a martingale. We take

the expectation:

$$\begin{aligned}
 \mathbb{E} [L_T^0(Y - Y^N)] &\leq 2\varepsilon + 2(\lambda + 1) \mathbb{E} \left[ \int_0^T \theta_s(Y - Y^N) \{ \tilde{u}(s, Y_s) - \tilde{u}^N(s, Y_s^N) \} ds \right] \\
 &\quad + \frac{2}{\varepsilon} \mathbb{E} \left[ \int_0^T \theta_s(Y - Y^N) \left\{ \widetilde{\nabla} u(s, Y_s) - \widetilde{\nabla} u^N(s, Y_s^N) \right\}^2 ds \right] \\
 &\leq 2\varepsilon + 2(\lambda + 1) \mathbb{E} \left[ \int_0^T \theta_s(Y - Y^N) \{ \tilde{u}(s, Y_s) - \tilde{u}(s, Y_s^N) \} ds \right] \\
 &\quad + 2(\lambda + 1) \mathbb{E} \left[ \int_0^T \theta_s(Y - Y^N) \{ \tilde{u}(s, Y_s^N) - \tilde{u}^N(s, Y_s^N) \} ds \right] \\
 &\quad + \frac{4}{\varepsilon} \mathbb{E} \left[ \int_0^T \theta_s(Y - Y^N) \left\{ \widetilde{\nabla} u(s, Y_s) - \widetilde{\nabla} u(s, Y_s^N) \right\}^2 ds \right] \\
 &\quad + \frac{4}{\varepsilon} \mathbb{E} \left[ \int_0^T \theta_s(Y - Y^N) \left\{ \widetilde{\nabla} u(s, Y_s^N) - \widetilde{\nabla} u^N(s, Y_s^N) \right\}^2 ds \right] \\
 &\leq 2\varepsilon + 2(\lambda + 1) \mathbb{E} \left[ \int_0^T \theta_s(Y - Y^N) |Y_s - Y_s^N| ds \right] \\
 &\quad + \frac{16 \|u\|_{\mathcal{C}^{1,\alpha}}^2}{\varepsilon} \mathbb{E} \left[ \int_0^T \theta_s(Y - Y^N) |Y_s - Y_s^N|^{2\alpha} ds \right] \\
 &\quad + 2(\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + 4c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-1}
 \end{aligned}$$

where we have used Lemma 5, the 1-lipschitz property of  $\tilde{u}$  and the  $\alpha$ -Hölder property of  $\widetilde{\nabla} u$  (with constant  $2 \|u\|_{\mathcal{C}^{1,\alpha}}$ ). As  $\theta_s(Y - Y^N) |Y_s - Y_s^N| \leq \varepsilon$ , we have

$$\begin{aligned}
 \mathbb{E} [L_T^0(Y - Y^N)] &\leq 2\varepsilon + 2(\lambda + 1) T\varepsilon + 16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T\varepsilon^{2\alpha-1} \\
 &\quad + 2(\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + 4c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-1}
 \end{aligned}$$

As  $1 > 2\alpha - 1 > 0$ , the result follows from  $\varepsilon \leq \varepsilon^{2\alpha-1}$  when  $0 < \varepsilon \leq 1$ .  $\square$

**Lemma 9.** *With assumptions and notations of Lemma 8, and  $1 > \alpha > 1/2$  we have  $\forall \varepsilon \in (0, 1]$ ,*

$$g(\varepsilon) \leq \sigma \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \quad (13)$$



for  $\|b^N - b\|_{H_q^{-\beta}}$  small enough (it is to say  $N$  big enough) where

$$\sigma = 2(\lambda+1) cTK e^{\rho T} + (16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2(1 + (\lambda + 1)T)) \omega^{2\alpha-1} + 4c^2TK^2 e^{2\rho T} \omega^{-1}$$

and

$$\omega = \left( \frac{4c^2TK^2 e^{2\rho T}}{(2\alpha - 1) (16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2(1 + (\lambda + 1)T))} \right)^{\frac{1}{2\alpha}}.$$

*Proof.* By Lemma 8,

$$0 \leq \mathbb{E} [L_T^0(Y - Y^N)] \leq g(\varepsilon)$$

where

$$\begin{aligned} g(\varepsilon) = 2(\lambda+1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + (16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2(1 + (\lambda + 1)T)) \varepsilon^{2\alpha-1} \\ + 4c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-1}. \end{aligned}$$

With

$$\begin{aligned} g'(\varepsilon) = (2\alpha - 1) (16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2(1 + (\lambda + 1)T)) \varepsilon^{2\alpha-2} \\ - 4c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-2}, \end{aligned}$$

and

$$\begin{aligned} g''(\varepsilon) = (2\alpha - 2)(2\alpha - 1) (16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2(1 + (\lambda + 1)T)) \varepsilon^{2\alpha-3} \\ + 8c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-3}, \end{aligned}$$

the minimum of  $g$  on  $(0, 1]$  is reached when  $N$  is big enough in

$$\varepsilon_0 = \left( \frac{4c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2}{(2\alpha - 1) (16 \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2(1 + (\lambda + 1)T))} \right)^{\frac{1}{2\alpha}} = \omega \|b^N - b\|_{H_q^{-\beta}}^{1/\alpha}.$$

where

$$g''(\varepsilon_0) = 8c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon_0^{-3} (1 - (1 - \alpha)) > 0.$$

and

$$\omega = \left( \frac{4c^2TK^2e^{2\rho T}}{(2\alpha - 1)(16\|u\|_{\mathcal{C}^{1,\alpha}}^2T + 2(1 + (\lambda + 1)T))} \right)^{\frac{1}{2\alpha}}.$$

Therefore  $\mathbb{E} [L_T^0(Y - Y^N)] \leq g(\varepsilon_0)$

$$\begin{aligned} &\leq 2(\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + (16\|u\|_{\mathcal{C}^{1,\alpha}}^2T + 2(1 + (\lambda + 1)T)) \omega^{2\alpha-1} \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \\ &\quad + 4c^2TK^2e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \omega^{-1} \|b^N - b\|_{H_q^{-\beta}}^{-1/\alpha} \\ &\leq 2(\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} \\ &+ ((16\|u\|_{\mathcal{C}^{1,\alpha}}^2T + 2(1 + (\lambda + 1)T)) \omega^{2\alpha-1} + 4c^2TK^2e^{2\rho T} \omega^{-1}) \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \\ &\leq 2(\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \\ &+ ((16\|u\|_{\mathcal{C}^{1,\alpha}}^2T + 2(1 + (\lambda + 1)T)) \omega^{2\alpha-1} + 4c^2TK^2e^{2\rho T} \omega^{-1}) \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \end{aligned}$$

for  $N$  big enough. The result follows.  $\square$

**Theorem 10.** *Let  $f$  be  $\mu$ -Hölder with constant  $C_f > 0$  and  $\mu \in (0, 1]$ . If  $0 < \beta < 1/4$ ,  $q \in \left(\frac{1}{1-\beta}, \frac{1}{\beta}\right)$ ,  $\forall \varepsilon \in (0, \frac{1-4\beta}{2})$ , with  $(\delta, p) \in K(\beta, q)$  such that  $\delta - 1/p = 1 - 2\beta - \varepsilon$ , exists  $\xi$  independent of  $f$  such that for  $N \in \mathbb{N}$ ,  $\rho > 1$ ,  $\lambda$  big enough it holds:*

$$|\mathbb{E} [f(X_T) - f(X_T^N)]| \leq \xi C_f \|b^N - b\|_{H_q^{-\beta}}^{\mu(2 - \frac{1}{1-2\beta-\varepsilon})}$$

*Proof.* We note as usual  $\alpha = \delta - 1/p$ . By Lemma 4, we choose  $\lambda$  big enough for  $\nabla u$  and  $\nabla u^N$  to be bounded by  $\frac{1}{2}$ .  $\lambda$  can be chosen independently of  $N$  as far as  $\|b - b^N\|_{H_q^s(\mathbb{R})} \xrightarrow{N \rightarrow \infty} 0$  (See Step 2 of the proof of Proposition 29 in [1]). Therefore  $u^N(t, \cdot)$  and  $u(t, \cdot)$  are  $\frac{1}{2}$ -lipschitz. We recall that in this case, by Lemma 22 in [1],  $\Psi(t, \cdot)$  and  $\Psi^N(t, \cdot)$  are 2-lipschitz.

$$|\mathbb{E} [f(X_T) - f(X_T^N)]| = |\mathbb{E} [f(\Psi(T, Y_T)) - f(\Psi^N(T, Y_T^N))]|$$

$$\begin{aligned}
 &\leq \mathbb{E} [|f(\Psi(T, Y_T)) - f(\Psi(T, Y_T^N))|] + \mathbb{E} [|f(\Psi(T, Y_T^N)) - f(\Psi^N(T, Y_T^N))|] \\
 &\leq C_f (2^\mu \mathbb{E} [|Y_T - Y_T^N|^\mu] + \mathbb{E} [|\Psi(T, Y_T^N) - \Psi^N(T, Y_T^N)|^\mu]) \\
 &\leq C_f (2^\mu \mathbb{E} [|Y_T - Y_T^N|]^\mu + \mathbb{E} [|\Psi(T, Y_T^N) - \Psi^N(T, Y_T^N)|]^\mu) \quad (14)
 \end{aligned}$$

by Jensen's inequality. Let  $t \in [0, T]$ .

$$\begin{aligned}
 Y_t - Y_t^N &= (\lambda + 1) \int_0^t \{u(s, \Psi(s, Y_s)) - u^N(s, \Psi(s, Y_s^N))\} \, ds \\
 &\quad + \int_0^t \{\nabla u(s, \Psi(s, Y_s)) - \nabla u^N(t, \Psi(s, Y_s^N))\} \, dW_s.
 \end{aligned}$$

We apply Meyer-Tanaka's formula to obtain:

$$\begin{aligned}
 |Y_t - Y_t^N| &= (\lambda + 1) \int_0^t \text{sign}(Y_s - Y_s^N) \{\tilde{u}(s, Y_s) - \tilde{u}^N(s, Y_s^N)\} \, ds \\
 &\quad + \int_0^t \text{sign}(Y_s - Y_s^N) \{\widetilde{\nabla} u(s, Y_s) - \widetilde{\nabla} u^N(s, Y_s^N)\} \, dW_s + L_t^0(Y - Y^N).
 \end{aligned}$$

Taking the expectation leads to:

$$\begin{aligned}
 \mathbb{E} [|Y_t - Y_t^N|] &= (\lambda + 1) \mathbb{E} \left[ \int_0^t \text{sign}(Y_s - Y_s^N) \{\tilde{u}(s, Y_s) - \tilde{u}^N(s, Y_s^N)\} \, ds \right] \\
 &\quad + \mathbb{E} [L_t^0(Y - Y^N)]
 \end{aligned}$$

because  $\widetilde{\nabla} u$  and  $\widetilde{\nabla} u^N$  are bounded so the Itô integral is a martingale.

$$\begin{aligned}
 \mathbb{E} [|Y_t - Y_t^N|] &\leq (\lambda + 1) \mathbb{E} \left[ \int_0^t \{\tilde{u}(s, Y_s) - \tilde{u}(s, Y_s^N)\} \, ds \right] \\
 &\quad + (\lambda + 1) \mathbb{E} \left[ \int_0^t \{\tilde{u}(s, Y_s^N) - \tilde{u}^N(s, Y_s^N)\} \, ds \right] + \mathbb{E} [L_t^0(Y - Y^N)].
 \end{aligned}$$

We use Lemma 5 and the 1-lipschitz property of  $\tilde{u}$ :

$$\begin{aligned}
 \mathbb{E} [|Y_t - Y_t^N|] &\leq (\lambda + 1) \mathbb{E} \left[ \int_0^t |Y_s - Y_s^N| \, ds \right] + (\lambda + 1) ctKe^{\rho T} \|b^N - b\|_{H_q^{-\beta}} \\
 &\quad + \mathbb{E} [L_t^0(Y - Y^N)]
 \end{aligned}$$

$$\begin{aligned} \leq (\lambda + 1) \int_0^t \mathbb{E} [|Y_s - Y_s^N|] ds + (\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} \\ + \mathbb{E} [L_T^0(Y - Y^N)] . \end{aligned}$$

where we have used the fact that  $L_t^0(Y - Y^N)$  is an increasing process.

By Gronwall's Lemma, it follows:

$$\mathbb{E} [|Y_T - Y_T^N|] \leq C(N) \exp((\lambda + 1)T) \quad (15)$$

with  $C(N) = (\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + \mathbb{E} [L_T^0(Y - Y^N)]$ .

With Lemma 8 and Lemma 9 we obtain

$$C(N) \leq (\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + \sigma \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \leq \zeta \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} .$$

for  $\|b^N - b\|_{H_q^{-\beta}}$  small enough where  $\zeta = (\lambda + 1) cTK e^{\rho T} + \sigma$ . It follows:

$$\mathbb{E} [|Y_T - Y_T^N|] \leq \zeta \exp((\lambda + 1)T) \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} . \quad (16)$$

Using the  $\frac{1}{2}$  property of  $u$ , we obtain with  $\varphi = 1 + u$ :

$$\begin{aligned} |\varphi^N(t, \Psi^N(t, Y_T^N)) - \varphi^N(t, \Psi(t, Y_T^N))| &\geq \inf_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla \varphi| |\Psi(T, Y_T^N) - \Psi^N(T, Y_T^N)| \\ &\geq \frac{1}{2} |\Psi(T, Y_T^N) - \Psi^N(T, Y_T^N)| \end{aligned}$$

and

$$\begin{aligned} |\Psi^N(T, Y_T^N) - \Psi(T, Y_T^N)| &\leq 2 |\varphi^N(t, \Psi^N(t, Y_T^N)) - \varphi^N(t, \Psi(t, Y_T^N))| \\ &\leq 2 (|\varphi^N(t, \Psi^N(t, Y_T^N)) - \varphi(t, \Psi(t, Y_T^N))| + |\varphi(t, \Psi(t, Y_T^N)) - \varphi^N(t, \Psi(t, Y_T^N))|) \\ &\leq 2 |\varphi(t, \Psi(t, Y_T^N)) - \varphi^N(t, \Psi(t, Y_T^N))| \\ &\leq 2 \|u^N(t) - u(t)\|_\infty \\ &\leq 2cK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} \end{aligned} \quad (17)$$

where we have used Lemma 5 and the fact that  $\varphi^N(t, \Psi^N(t, Y_T^N)) = \varphi(t, \Psi(t, Y_T^N))$ .

Finally, combining (14), (16) and (17) we obtain:

$$\begin{aligned}
 & |\mathbb{E} [f(X_T) - f(X_T^N)]| \\
 & \leq C_f (2^\mu \mathbb{E} [|Y_T - Y_T^N|]^\mu + \mathbb{E} [|\Psi(T, Y_T^N) - \Psi^N(T, Y_T^N)|]^\mu) \\
 & \leq C_f \left( 2^\mu \zeta^\mu e^{\mu(\lambda+1)T} \|b^N - b\|_{H_q^{-\beta}}^{\mu(2-1/\alpha)} + 2^\mu c^\mu K^\mu e^{\mu\rho T} \|b^N - b\|_{H_q^{-\beta}}^\mu \right) \\
 & \leq 2^\mu C_f (\zeta^\mu e^{\mu(\lambda+1)T} + c^\mu K^\mu e^{\mu\rho T}) \|b^N - b\|_{H_q^{-\beta}}^{\mu(2-1/\alpha)}
 \end{aligned}$$

for  $N$  big enough, which is the expected result.  $\square$

## 5 Numerical results

### 5.1 Strong convergence of the Euler scheme

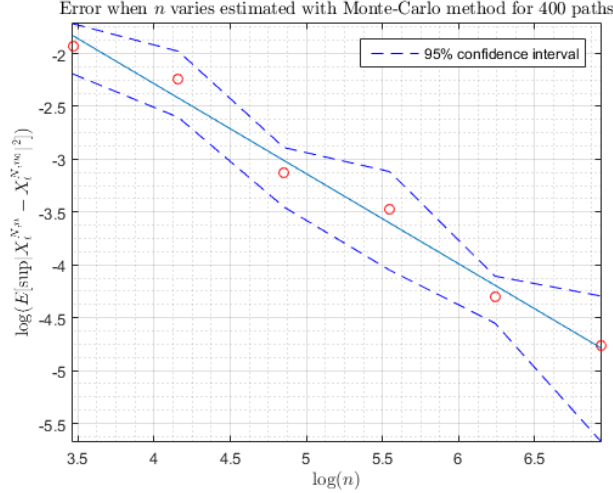


Figure 1: Estimation of the  $L^2$  error of the Euler-Maruyama scheme with a Monte-Carlo method. 400 paths,  $N = 5$ ,  $n \in \{2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}\}$ , reference solution with  $n_0 = 2^{12}$  points.

We observe a numerical convergence rate of 0.85 when Theorem 2 shows a theoretical rate of  $0.5 - \varepsilon$ .

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