NUMERICAL SIMULATION OF SDES WITH DISTRIBUTIONAL DRIFT

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1 Introduction

We would like to simulate numerically sample paths of the solution of the one-dimensional stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\ \mathrm{d}t + \mathrm{d}W_t \tag{1}$$

where $b \in H_q^{-\beta}(\mathbb{R})$, $\beta \in \left(0, \frac{1}{2}\right)$, $q \in \left(\frac{1}{1-\beta}, \frac{1}{\beta}\right)$, $t \in [0,T]$, and W_t is a standard Brownian motion. Equation (1) is studied by F. Flandoli, E. Issoglio, and F. Russo in [1] in which they define a concept of virtual solution. The authors prove then existence and unicity in law of this solution.

Example 1. *An example of such drift b is given by the derivative of a sample path of a fractional Brownian motion B_x^H with Hurst index $1/2 < H < 1$. These stochastic processes are gaussian processes verifying*

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} + |t-s|^{2H}\right).$$

We note $-\beta = H - 1$. Given $B_x^H(\omega) \in H_q^{1-\beta}(\mathbb{R})$, we can take $b(x) = \frac{\mathrm{d}}{\mathrm{d}x} B_x^H(\omega) \in H_q^{-\beta}(\mathbb{R})$. We will use this in our numerical simulations.

As far as the drift b is not a function but a distribution, it must be approximated if we want to evaluate it at points. In order to do so, we will use a series representation of b and truncate it. That is why we will consider two steps in our algorithm:

1. approximate the drift b by a function b^N meant to converge to b as $N \to \infty$.
2. approximate the solution X_t^N of the approximated SDE

$$\mathrm{d}X_t^N = b^N\left(X_t^N\right)\ \mathrm{d}t + \mathrm{d}W_t \tag{2}$$

by $X_t^{N,n}$ defined with the Euler-Maruyama scheme

$$X_t^{N,n} = X_0 + \int_0^t b^N\left(X_{\eta_n(t)}^{N,n}\right)\mathrm{d}t + W_{\eta_n(t)}$$

where $\eta_n(t) = t_k$ if $t \in [t_k, t_k + 1[$, for $t_k = \frac{k}{n}$ with $k \in [\![0, \lceil nT \rceil]\!]$.

2 Numerical simulation of fractional Brownian motion

To simulate a sample path of a fractional brownian motion B_x^H on a finite grid $(x_k)_{k\in[\![1,n]\!]}$, we simulate n independent standard gaussian random variables $(X_k)_{k\in[\![1,n]\!]}$ and then correlate them with the definite positive correlation matrix

$$C_{k,s} = \mathbb{E}\left[B_{x_k}^H B_{x_s}^H\right] = \frac{1}{2}\left(x_k^{2H} + x_s^{2H} + |x_k - x_s|^{2H}\right).$$

To do so, we use the Cholesky decomposition method and calculate the triangular matrix M such that $C = MM^\top$. Therefore, defining

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ and } B^H = MX,$$

B^H contains the values of a fractional brownian motion evaluated on the grid $(x_k)_{k\in[\![1,n]\!]}$.

3 Approximation of the drift

3.1 Series representation

We use Haar wavelets to give a series representation of b. By doing so, we will be able to approximate it numerically by truncating the series.

Definition 1 (Haar wavelets)**.** *We define the Haar wavelets $h_{j,m}$ on $\mathbb{R}$ with $j \in \mathbb{N} \cup \{-1\}$ and $m \in \mathbb{Z}$ by:*

$$\begin{cases} h_M & : x \longmapsto \left(\mathbb{1}_{[0,\frac{1}{2}[} - \mathbb{1}_{[\frac{1}{2},1[}\right)(x) \\ h_{-1,m} & : x \longmapsto \sqrt{2}|h_M(x-m)| \\ h_{j,m} & : x \longmapsto h_M(2^j x - m) \end{cases}$$

Theorem 1 (See [2])**.** *Let $b \in H_q^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left]-\frac{1}{2}, \frac{1}{q}\right[$. Therefore,*

$$b = \sum_{j=-1}^{+\infty} \sum_{m\in\mathbb{Z}} \mu_{j,m} h_{j,m} \tag{3}$$

where $\mu_{j,m} = 2^j \int_{\mathbb{R}} b(x) h_{j,m}(x)\,\mathrm{d}x$ in the sense of dual pairing.

Definition 2. *With the same notation $\mu_{j,m}$, let $b \in H_q^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left]-\frac{1}{2}, \frac{1}{q}\right[$. Given $N \in \mathbb{N}^*$ we define b^N by:*

$$b^N = \sum_{m=N}^{N-1} \mu_{-1,m} h_{-1,m} + \sum_{j=0}^{N} \sum_{m=-N2^j}^{N2^j-1} \mu_{j,m} h_{j,m}. \tag{4}$$

Remark 1. *We can note that* Supp $b^N \subset [-N, N]$. *Moreover, we have:*

$$\left\|b - b^N\right\|_{H_q^s(\mathbb{R})} \underset{N\to+\infty}{\longrightarrow} 0.$$

4 Convergence

4.1 Weak convergence of $X_t^{N,n}$ to X_t^N

Recently, Leobacher and Szölgyenyi proved in [3] the convergence of the Euler-Maruyama scheme for SDE with discontinuous but piecewise Lipschitz drift and with a degenerate diffusion coefficient. This framework applies to the scheme we use with piecewise constant drift b^N and a constant diffusion coefficient.

Theorem 2 (Theorem 3.1. in [3])**.** *$\exists C_N > 0$ independent of n such that it holds $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$:*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t^{N,n} - X_t^N\right|^2\right]^{1/2} \leq C_N \delta^{1/4-\varepsilon} \tag{5}$$

with $\delta = \frac{1}{n}$ the step size and C_N depending on $\left\|b^N\right\|_\infty$.

Theorem 3. *Let f be μ-Hölder with constant $C_f > 0$, $\mu \in (0,1]$ and $t \in [0,T]$. Then, exists $C'_N > 0$ independent of n such that it holds $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$:*

$$\left|\mathbb{E}\left[f\left(X_t^{N,n}\right) - f\left(X_t^N\right)\right]\right| \leq C'_N \delta^{\mu/4-\varepsilon} \tag{6}$$

with $\delta = \frac{1}{n}$ the step size.

Proof. By Jensen's inequality and the μ-Hölder property of f, we obtain:

$$\begin{aligned}\left|\mathbb{E}\left[f\left(X_t^{N,n}\right) - f\left(X_t^N\right)\right]\right| &\leq C_f \mathbb{E}\left[\left|X_t^{N,n} - X_t^N\right|^\mu\right] \\ &\leq C_f \mathbb{E}\left[\left|Y_t - Y_t^N\right|^2\right]^{\mu/2} \\ &\leq C_f C_N^\mu \delta^{\mu/4-\varepsilon}.\end{aligned}$$

□

4.2 Weak convergence of X_t^N to X_t

The goal of this section is to estimate the weak error $\left|\mathbb{E}\left[f\left(X_t\right) - f\left(X_t^N\right)\right]\right|$ with suitable functions f. In order to do so, we must go back to the definition of the virtual solution of the SDE (1) given in [1]. Let $(\delta, p) \in K(\beta, q) := \{(\delta, p) \mid \beta < \delta < 1-\beta,\ \frac{1}{\delta} < p < q\}$. The authors of [1] define the virtual solution of SDE (1) by X_t such that:

$$\begin{cases} Y_t = y + (\lambda+1)\int_0^t u(s, \Psi\left(s, Y_s\right))\ \mathrm{d}s + \int_0^t (\nabla u(s, \Psi\left(s, Y_s\right)) + 1)\ \mathrm{d}W_s \\ X_t = \Psi(t, Y_t) = \varphi^{-1}(t, Y_t) \end{cases} \tag{7}$$

where u is the mild solution in $H_p^{1+\delta}$ of the following parabolic PDE:

$$\begin{cases} \partial_t u + \frac{1}{2}\Delta u + b\nabla u - (\lambda+1)u = -b & \text{on } [0,T]\times\mathbb{R} \\ u(T) = 0 & \text{on } \mathbb{R} \end{cases} \tag{8}$$

with $\varphi(t,x) = x + u(t,x)$, and $y = \varphi(0,x)$.

We also define another similar PDE by replacing b by b^N. We call u^N its mild solution in $H_p^{1+\delta}$:

$$\begin{cases} \partial_t u^N + \frac{1}{2}\Delta u^N + b^N \nabla u^N - (\lambda+1)u^N = -b^N & \text{on } [0,T]\times\mathbb{R} \\ u^N(T) = 0 & \text{on } \mathbb{R} \end{cases}. \tag{9}$$

Then we consider an approximated version of (7):

$$\begin{cases} Y_t^N = y^N + (\lambda+1)\int_0^t u^N\left(s, \Psi^N\left(s, Y_s^N\right)\right)\, \mathrm{d}s + \int_0^t \left(\nabla u^N\left(s, \Psi^N\left(s, Y_s^N\right)\right) + 1\right)\, \mathrm{d}W_s \\ X_t^N = \Psi^N(t, Y_t^N) = \left(\varphi^N\right)^{-1}(t, Y_t^N) \end{cases}. \tag{10}$$

with $\varphi^N(t,x) = x + u^N(t,x)$, and $y^N = \varphi^N(0,x)$.

Remark 2. *Proposition 26 in [1] assures us that X_t^N defined above in (10) is in fact the classical solution of (2), as far as $b^N \in L^p$. That is why for each fixed N our Euler scheme converges to the virtual solution X_t^N.*

We also recall a useful lemma concerning the solutions of (8) and (9).

Lemma 4 (Lemma 20 in [1])**.** *Let $(\delta,p)\in K(\beta,q)$ and let $u,\ u^N$ be the mild solutions to (8), (9) in $H_p^{1+\delta}$. Fix ρ such that the integral operator is a contraction and let $\lambda > \lambda^*$. Then $u(t),\ u^N(t) \in \mathcal{C}^{1,\alpha}$ with $\alpha = \delta - 1/p$ for each fixed t and $\forall \varepsilon > 0,\ \exists \lambda_0 > 0$ such that*

$$\begin{cases} \forall \lambda \geq \lambda_0, & \sup\limits_{(t,x)\in[0,T]\times\mathbb{R}} |\nabla u(t,x)| \leq \varepsilon \\ \forall \lambda \geq \lambda_0, & \sup\limits_{(t,x)\in[0,T]\times\mathbb{R}} |\nabla u^N(t,x)| \leq \varepsilon \end{cases}$$

where the choice of λ_0 depends only on $\delta, \beta, \|b\|_{H_p^{-\beta}}$, and $\left\|b^N\right\|_{H_q^{-\beta}}$.

Lemma 5. *Let $(\delta,p)\in K(\beta,q)$ and let $u,\ u^N$ be the mild solutions to (8),(9) in $H_p^{1+\delta}$, $\alpha = \delta - 1/p$. Exists $c, K > 0$ such that for both $N\in\mathbb{N}$ and $\rho > 1$ big enough, $\forall t\in[0,T]$,*

$$\begin{cases} \left\|u^N(t) - u(t)\right\|_{L^\infty} \leq \kappa \left\|b - b^N\right\|_{H_q^{-\beta}} \\ \left\|\nabla u^N(t) - \nabla u(t)\right\|_{L^\infty} \leq \kappa \left\|b - b^N\right\|_{H_q^{-\beta}}. \end{cases}. \tag{11}$$

with $\kappa = cKe^{\rho T}$.

Proof. Applying fractional Morrey inequality, $\exists c > 0,\ \forall t\in[0,T]$:

$$\begin{cases} \left\|u^N(t) - u(t)\right\|_{L^\infty} \leq \left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1,\alpha}} \leq c\left\|u^N(t) - u(t)\right\|_{H_p^{1+\delta}} \\ \left\|\nabla u^N(t) - \nabla u(t)\right\|_{L^\infty} \leq \left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1,\alpha}} \leq c\left\|u^N(t) - u(t)\right\|_{H_p^{1+\delta}}. \end{cases}$$

Now, we can conclude with

$$\left\|u^N - u\right\|_{\infty, H_p^{1+\delta}} \leq e^{\rho T}\left\|u^N - u\right\|_{\infty, H_p^{1+\delta}}^{(\rho)} \leq Ke^{\rho T}\left\|b - b^N\right\|_{H_q^{-\beta}}$$

from Lemma 23 in [1], for both $N\in\mathbb{N}$ and $\rho > 1$ big enough, and where $\|f(t)\|_{\infty,X}^{(\rho)} := \sup\limits_{0\leq t\leq T} e^{-\rho t}\|f(t)\|_X$. ∎

Lemma 6. *For λ big enough,*

$$\left|\Psi^N\left(t, Y_t^N\right) - \Psi\left(t, Y_t^N\right)\right| \leq 2\kappa \left\|b - b^N\right\|_{H_q^{-\beta}}$$

with $\kappa = cKe^{\rho T}$.

Proof. For λ big enough, by Lemma 4, $\forall t \in [0,T]$, $\sup_{x\in\mathbb{R}} |\nabla u(t,x)| \leq 1/2$, so we obtain with $\varphi(t,x) = x + u(t,x)$:

$$\begin{aligned}
&\left|\varphi\left(t, \Psi^N\left(t, Y_t^N\right)\right) - \varphi\left(t, \Psi\left(t, Y_t^N\right)\right)\right| \\
&\geq \inf_{x\in\mathbb{R}} |\nabla\varphi(t,x)| \left|\Psi^N\left(t, Y_t^N\right) - \Psi\left(t, Y_t^N\right)\right| \\
&\geq \frac{1}{2}\left|\Psi^N\left(t, Y_t^N\right) - \Psi\left(t, Y_t^N\right)\right|
\end{aligned}$$

and

$$\begin{aligned}
\left|\Psi^N\left(t, Y_t^N\right) - \Psi\left(t, Y_t^N\right)\right| &\leq 2\ \left|\varphi\left(t, \Psi^N\left(t, Y_t^N\right)\right) - \varphi\left(t, \Psi\left(t, Y_t^N\right)\right)\right| \\
&\leq 2\ \left|\varphi\left(t, \Psi^N\left(t, Y_t^N\right)\right) - \varphi^N\left(t, \Psi^N\left(t, Y_t^N\right)\right)\right| \\
&\leq 2\ \left\|u(t) - u^N(t)\right\|_\infty \\
&\leq 2cKe^{\rho T}\ \left\|b - b^N\right\|_{H_q^{-\beta}} \qquad (12)
\end{aligned}$$

where we have used Lemma 5 and the fact that

$$\varphi^N\left(t, \Psi^N\left(t, Y_t^N\right)\right) = \varphi\left(t, \Psi\left(t, Y_t^N\right)\right) = Y_t^N.$$

□

We will need an adapted version of a local time inequality (Lemma 4.2 in [4]) from Liqing Yan:

Lemma 7. *Let X be a continuous semimartingale. For $\varepsilon > 0$ we define a double sequence of stopping times by $\sigma_1 = \inf\{t \geq 0 | X_t = 0\}$, $\tau_1 = \inf\{t > \sigma_1 | X_t = \varepsilon\}$, $\sigma_n = \inf\{t > \tau_{n-1} | X_t = 0\}$, $\tau_n = \inf\{t > \sigma_n | X_t = \varepsilon\}$. For any real function $F(\cdot) \in \mathcal{C}^2(\mathbb{R})$ with $F(0) = 0$, $F'(0) = 0$, $F(\cdot) > 0$ on $(0, \varepsilon_0)$ with some $\varepsilon_0 > 0$, and for any $0 < \varepsilon < \varepsilon_0$ we have*

$$0 \leq L_t^0(X) \leq 2\varepsilon - \frac{2}{F(\varepsilon)} \int_{\sigma_1\wedge t}^t \theta_s(X)\left(F(\varepsilon) - \varepsilon F'\left(X_s^+\right)\right)\ \mathrm{d}X_s + \frac{\varepsilon}{F(\varepsilon)} \int_{\sigma_1\wedge t}^t \theta_s(X) F''(X_s^+)\ \mathrm{d}[X]_s$$

with $\theta_s(X) = \sum_{n=1}^\infty \mathbb{1}_{\{\sigma_n < s \leq \tau_n,\ 0 < X_s \leq \varepsilon\}}(X)$.

Applying lemma 7 with $F : x \in \mathbb{R} \mapsto x^2$, it follows:

Corollary 8. *Let X be a continuous semimartingale. With the same notations as in lemma 7, for any $\varepsilon > 0$ we have*

$$0 \leq L_t^0(X) \leq 2\varepsilon - \frac{2}{\varepsilon} \int_{\sigma_1\wedge t}^t \theta_s(X)\left(\varepsilon - 2X_s^+\right)\ \mathrm{d}X_s + \frac{2}{\varepsilon} \int_{\sigma_1\wedge t}^t \theta_s(X)\ \mathrm{d}[X]_s \qquad (13)$$

Proof of Lemma 7. We note $U_t(X) = \sup\{n \in \mathbb{N} | \tau_n < t\}$ and $n(t) = t \wedge \sigma_{U_t(X)+1}$. By Meyer-Tanaka's formula, $\forall i \in \mathbb{N}^*$:

$$X^+_{\tau_i \wedge t} - X^+_{\sigma_i \wedge t} = \int_{\sigma_i \wedge t}^{\tau_i \wedge t} \mathbb{1}_{\{X_s > 0\}} \, \mathrm{d}X_s + \frac{1}{2} \left\{ L^0_{\tau_i \wedge t}(X) - L^0_{\sigma_i \wedge t}(X) \right\}. \tag{14}$$

Because $\forall i \in \mathbb{N}^*$, $L^0_{\tau_i \wedge t}(X) = L^0_{\sigma_{i+1} \wedge t}(X)$ and $L^0_{\sigma_1 \wedge t}(X) = 0$, we have

$$\sum_{i=1}^{U_t(X)+1} \left(X^+_{\tau_i \wedge t} - X^+_{\sigma_i \wedge t} \right) = \int_{\sigma_1 \wedge t}^{t} \theta_s(X) \, \mathrm{d}X_s + \frac{1}{2} \, L^0_t(X).$$

The left term is equal to $\varepsilon U_t(X) + X^+_t - X^+_{n(t)}$ so

$$\varepsilon U_t(X) = \int_{\sigma_1 \wedge t}^{t} \theta_s(X) \, \mathrm{d}X_s + \frac{1}{2} \, L^0_t(X) - X^+_t + X^+_{n(t)}. \tag{15}$$

Now we express differently $U_t(X)$. $F \in \mathcal{C}^2(\mathbb{R})$ so by Itô's formula:

$$F\left(X^+_{\tau_i \wedge t}\right) - F\left(X^+_{\sigma_i \wedge t}\right) = \int_{\sigma_i \wedge t}^{\tau_i \wedge t} F'\left(X^+_s\right) \mathrm{d}X^+_s + \frac{1}{2} \int_{\sigma_i \wedge t}^{\tau_i \wedge t} F''\left(X^+_s\right) \mathrm{d}[X^+]_s.$$

By (14), $\mathrm{d}X^+_s = \mathbb{1}_{\{X_s > 0\}} \mathrm{d}X_s + \frac{1}{2} \, \mathrm{d}L^0_t(X)$ and $\mathrm{d}[X^+]_s = \mathbb{1}_{\{X_s > 0\}} \mathrm{d}[X]_s$. It follows

$$\begin{aligned} F\left(X^+_{\tau_i \wedge t}\right) - F\left(X^+_{\sigma_i \wedge t}\right) = \int_{\sigma_i \wedge t}^{\tau_i \wedge t} F'\left(X^+_s\right) \mathbb{1}_{\{X_s > 0\}} \, \mathrm{d}X_s + \frac{1}{2} \int_{\sigma_i \wedge t}^{\tau_i \wedge t} F'(X^+_s) \, \mathrm{d}L^0_t(X) \\ + \frac{1}{2} \int_{\sigma_i \wedge t}^{\tau_i \wedge t} F''\left(X^+_s\right) \mathbb{1}_{\{X_s > 0\}} \, \mathrm{d}[X]_s. \end{aligned}$$

Adding up for i, with $F(0) = 0$ we obtain

$$\begin{aligned} F(\varepsilon) U_t(X) + F\left(X^+_t\right) - F\left(X^+_{n(t)}\right) &= \sum_{i=1}^{U_t(X)+1} \left(F\left(X^+_{\tau_i \wedge t}\right) - F\left(X^+_{\sigma_i \wedge t}\right) \right) \\ = \int_{\sigma_1 \wedge t}^{t} F'(X^+_s) \, \theta_s(X) \, \mathrm{d}X_s + \frac{1}{2} \int_{\sigma_1 \wedge t}^{t} F'\left(X^+_s\right) \Xi_s \, \mathrm{d}L^0_t(X) &+ \frac{1}{2} \int_{\sigma_1 \wedge t}^{t} F''\left(X^+_s\right) \theta_s(X) \, \mathrm{d}[X]_s \end{aligned}$$

with $\Xi_s = \sum_{n=1}^{\infty} \mathbb{1}_{\{\sigma_n < s \leq \tau_n\}}$ The measure $\mathrm{d}L^0_t(X)$ is almost surely carried by $\{t | X_t = 0\}$ so we can simplify $\int_{\sigma_1 \wedge t}^{t} F'\left(X^+_s\right) \Xi_s \, \mathrm{d}L^0_t(X) = F'(0) \int_{\sigma_1 \wedge t}^{t} \Xi_s \, \mathrm{d}L^0_t(X)$ in order to have, with $F'(0) = 0$:

$$F(\varepsilon) U_t(X) = -F\left(X^+_t\right) + F\left(X^+_{n(t)}\right) + \int_{\sigma_1 \wedge t}^{t} F'\left(X^+_s\right) \theta_s(X) \, \mathrm{d}X_s + \frac{1}{2} \int_{\sigma_1 \wedge t}^{t} F''\left(X^+_s\right) \theta_s(X) \, \mathrm{d}[X]_s. \tag{16}$$

Combining (15) and (16), it follows

$$\begin{aligned} L^0_t(X) F(\varepsilon) = 2F(\varepsilon)(X^+_t - X^+_{n(t)}) - 2\varepsilon(F\left(X^+_t\right) - F\left(X^+_{n(t)}\right) \\ - 2 \int_{\sigma_1 \wedge t}^{t} \left(F(\varepsilon) - \varepsilon F'\left(X^+_s\right) \right) \theta_s(X) \, \mathrm{d}X_s + \varepsilon \int_{\sigma_1 \wedge t}^{t} F''\left(X^+_s\right) \theta_s(X) \, \mathrm{d}[X]_s. \end{aligned}$$

Then, if $n(t) = t$, the two first right terms of the equality are equal to zero. Else, if $n(t) = \sigma_{U_t(X)+1}$, $0 \leq F(\varepsilon)(X_t^+ - X_{n(t)}^+) = F(\varepsilon)X_t^+ \leq F(\varepsilon)\varepsilon$ and $-2\varepsilon(F\left(X_t^+\right) - F\left(X_{n(t)}^+\right) = -2\varepsilon F\left(X_t^+\right) \leq 0$ because of the positivity of F. Finally we obtain:

$$L_t^0(X) \leq 2\varepsilon - \frac{2}{F(\varepsilon)} \int_{\sigma_1 \wedge t}^{t} \left(F(\varepsilon) - \varepsilon F'\left(X_s^+\right)\right) \theta_s(X)\, \mathrm{d}X_s + \frac{\varepsilon}{F(\varepsilon)} \int_{\sigma_1 \wedge t}^{t} F''\left(X_s^+\right) \theta_s(X)\, \mathrm{d}[X]_s.$$

$\square$

Lemma 9. *Let $(\delta, p) \in K(\beta, q)$, $\alpha = \delta - 1/p < 1$, u, u^N be the mild solutions to (8), (9) in $H_p^{1+\delta}$, and Y, Y^N solutions of the SDEs (7), (10). Then, if $\alpha > 1/2$, for λ big enough we have $\forall \varepsilon \in (0,1]$,*

$$0 \leq \mathbb{E}\left[L_T^0(Y - Y^N)\right] \leq g(\varepsilon).$$

where

$$g(\varepsilon) \leq 4(\lambda+1)T\kappa \left\|b - b^N\right\|_{H_q^{-\beta}} + \left(2 + 2(\lambda+1)\, T + 6\, \|u\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right) \varepsilon^{2\alpha-1}$$
$$+ 6T\left(\Omega^2 4^\alpha \kappa^{2\alpha} \left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2 \left\|b - b^N\right\|_{H_q^{-\beta}}^2\right) \varepsilon^{-1}.$$

Proof. By Lemma 4, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\left\|b - b^N\right\|_{H_q^s(\mathbb{R})} \underset{N\to\infty}{\longrightarrow} 0$. Therefore $u^N(t,\cdot)$ and $u(t,\cdot)$ are $\frac{1}{2}$-lipschitz. Let $\varepsilon \in (0,1]$. Corollary 8 gives us:

$$0 \leq L_T^0(Y - Y^N) \leq 2\varepsilon - \frac{2}{\varepsilon} \int_{\sigma_1 \wedge T}^{T} \theta_s(Y - Y^N)\left(\varepsilon - 2(Y_s - Y_s^N)^+\right)\, \mathrm{d}\left(Y_s - Y_s^N\right)$$
$$+ \frac{2}{\varepsilon} \int_{\sigma_1 \wedge T}^{T} \theta_s(Y - Y^N)\, \mathrm{d}\left[Y - Y^N\right]_s$$

with

$$Y_T - Y_T^N = y - y^N + (\lambda+1) \int_0^T \left\{u\left(s, \Psi\left(s, Y_s\right)\right) - u^N\left(s, \Psi^N\left(s, Y_s^N\right)\right)\right\}\, \mathrm{d}s$$
$$+ \int_0^T \left\{\nabla u\left(s, \Psi\left(s, Y_s\right)\right) - \nabla u^N\left(t, \Psi^N\left(s, Y_s^N\right)\right)\right\}\, \mathrm{d}W_s.$$

Remark 3. *Note that $\theta_s(Y - Y^N)\left|\varepsilon - 2(Y_s - Y_s^N)^+\right| \leq \varepsilon\theta_s(Y - Y^N)$.*

Remark 4. *For clarity purpose, we note $\tilde{u}\left(s,x\right) = u\left(s, \Psi\left(s,x\right)\right)$, $\tilde{u}^N\left(s,x\right) = u^N\left(s, \Psi\left(s,x\right)\right)$, $\overline{u}^N\left(s,x\right) = u^N\left(s, \Psi^N\left(s,x\right)\right)$ and use the same notations for the gradient and the approximated mild solution. We recall that in this case, by Lemma 22 in [1], $\Psi(t,\cdot)$ and $\Psi^N(t,\cdot)$ are 2-lipschitz. We can notice that $\tilde{u}$ is 1-lipschitz in space and $\widetilde{\nabla u}$ is α-Hölder with constant $2^\alpha \|u\|_{\mathcal{C}^{1,\alpha}}$. The same properties hold for $\overline{u}$ and $\overline{\nabla u}$ except that the Hölder constant for $\overline{\nabla u}$ can be bounded by a constant Ω for N big enough (see Lemma 24 in [1]).*

∇u and ∇u^N are bounded so the Itô integral is a martingale. We take the expectation:

$$\begin{aligned}\mathbb{E}\left[L_T^0(Y-Y^N)\right] \leq 2\varepsilon + 2(\lambda+1)\ \mathbb{E}\left[\int_{\sigma_1\wedge T}^T \theta_s\left(Y-Y^N\right)\left|\widetilde{u}\left(s,Y_s\right)-\overline{u}^N\left(s,Y_s^N\right)\right|\mathrm{d}s\right] \\ +\frac{2}{\varepsilon}\ \mathbb{E}\left[\int_{\sigma_1\wedge T}^T \theta_s\left(Y-Y^N\right)\left\{\widetilde{\nabla u}\left(s,Y_s\right)-\overline{\nabla u}\left(s,Y_s^N\right)\right\}^2\ \mathrm{d}s\right]\end{aligned}$$

$$\begin{aligned}&\leq 2\varepsilon + 2(\lambda+1)\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|\widetilde{u}\left(s,Y_s\right)-\widetilde{u}\left(s,Y_s^N\right)\right|\ \mathrm{d}s\right] \\ &\quad + 2(\lambda+1)\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|\widetilde{u}\left(s,Y_s^N\right)-\widetilde{u}^N\left(s,Y_s^N\right)\right|\ \mathrm{d}s\right] \\ &\quad + 2(\lambda+1)\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|\widetilde{u}^N\left(s,Y_s^N\right)-\overline{u}^N\left(s,Y_s^N\right)\right|\mathrm{d}s\right] \\ &\quad + \frac{6}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left\{\widetilde{\nabla u}\left(s,Y_s\right)-\widetilde{\nabla u}\left(s,Y_s^N\right)\right\}^2\ \mathrm{d}s\right] \\ &\quad + \frac{6}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left\{\widetilde{\nabla u}\left(s,Y_s^N\right)-\widetilde{\nabla u}^N\left(s,Y_s^N\right)\right\}^2\ \mathrm{d}s\right] \\ &\quad + \frac{6}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left\{\widetilde{\nabla u}^N\left(s,Y_s^N\right)-\overline{\nabla u}^N\left(s,Y_s^N\right)\right\}^2\ \mathrm{d}s\right]\end{aligned}$$

$$\begin{aligned}&\leq 2\varepsilon + 2(\lambda+1)\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|Y_s-Y_s^N\right|\ \mathrm{d}s\right] + 2(\lambda+1)T\kappa\left\|b^N-b\right\|_{H_q^{-\beta}} \\ &\quad + (\lambda+1)\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|\Psi\left(s,Y_s^N\right)-\Psi^N\left(s,Y_s^N\right)\right|\mathrm{d}s\right] \\ &\quad + \frac{6\times 4^\alpha\|u\|_{\mathcal{C}^{1,\alpha}}^2}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|Y_s-Y_s^N\right|^{2\alpha}\ \mathrm{d}s\right] + 6T\kappa^2\left\|b^N-b\right\|_{H_q^{-\beta}}^2\varepsilon^{-1} \\ &\quad + \frac{6\Omega^2}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|\Psi\left(s,Y_s^N\right)-\Psi^N\left(s,Y_s^N\right)\right|^{2\alpha}\ \mathrm{d}s\right]\end{aligned}$$

where we have used Lemma 5, the 1-lipschitz property of $\widetilde{u}$, the 1/2-lipschitz property of u^N, the α-Hölder property of $\widetilde{\nabla u}$ (with constant $2^\alpha\|u\|_{\mathcal{C}^{1,\alpha}}$), and the α-Hölder property of ∇u^N (with constant Ω). Lemma 6 gives us:

$$\begin{aligned}&\leq 2\varepsilon + 2(\lambda+1)\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|Y_s-Y_s^N\right|\ \mathrm{d}s\right] + 4(\lambda+1)T\kappa\left\|b^N-b\right\|_{H_q^{-\beta}} \\ &\quad + \frac{6\times 4^\alpha\|u\|_{\mathcal{C}^{1,\alpha}}^2}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|Y_s-Y_s^N\right|^{2\alpha}\ \mathrm{d}s\right] + 6T\kappa^2\left\|b-b^N\right\|_{H_q^{-\beta}}^2\varepsilon^{-1} \\ &\quad + 6\Omega^2 T 4^\alpha\kappa^{2\alpha}\left\|b-b^N\right\|_{H_q^{-\beta}}^{2\alpha}\varepsilon^{-1}.\end{aligned}$$

As $\theta_s(Y-Y^N)\left|Y_s-Y_s^N\right|\leq\varepsilon$, we have

$$\mathbb{E}\left[L_T^0(Y-Y^N)\right]\leq 2\varepsilon+2(\lambda+1)\,T\varepsilon+4(\lambda+1)T\kappa\left\|b-b^N\right\|_{H_q^{-\beta}}+6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}}4^\alpha T\varepsilon^{2\alpha-1}$$
$$+6T\left(\Omega^2 4^\alpha\kappa^{2\alpha}\left\|b-b^N\right\|^{2\alpha}_{H_q^{-\beta}}+\kappa^2\left\|b-b^N\right\|^2_{H_q^{-\beta}}\right)\varepsilon^{-1}.$$

As $1>2\alpha-1>0$, the result follows from $\varepsilon\leq\varepsilon^{2\alpha-1}$ when $0<\varepsilon\leq 1$.

□

Lemma 10. *With assumptions and notations of Lemma 9, and $1>\alpha>1/2$ we have*

$$\mathbb{E}\left[L_T^0(Y-Y^N)\right]\leq g(\varepsilon_N)=\sigma\left\|b^N-b\right\|^{2-1/\alpha}_{H_q^{-\beta}} \tag{17}$$

for $\left\|b^N-b\right\|_{H_q^{-\beta}}$ small enough (it is to say N big enough) where

$$\sigma=4(\lambda+1)T\kappa+\left(2+2(\lambda+1)\,T+6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}}4^\alpha T\right)2\omega_\infty^{2\alpha-1}+6T\left(\Omega^2 4^\alpha\kappa^{2\alpha}+\kappa^2\right)\omega_\infty^{-1}$$

and

$$\omega_\infty=\left(\frac{6T\Omega^2 4^\alpha\kappa^{2\alpha}}{(2\alpha-1)\left(2+2(\lambda+1)\,T+6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}}4^\alpha T\right)}\right)^{\frac{1}{2\alpha}}.$$

Proof. Let $\varepsilon\in(0,1]$, by Lemma 9,

$$0\leq\mathbb{E}\left[L_T^0(Y-Y^N)\right]\leq g(\varepsilon)$$

where

$$g(\varepsilon)=4(\lambda+1)T\kappa\left\|b-b^N\right\|_{H_q^{-\beta}}+\left(2+2(\lambda+1)\,T+6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}}4^\alpha T\right)\varepsilon^{2\alpha-1}$$
$$+6T\left(\Omega^2 4^\alpha\kappa^{2\alpha}\left\|b-b^N\right\|^{2\alpha}_{H_q^{-\beta}}+\kappa^2\left\|b-b^N\right\|^2_{H_q^{-\beta}}\right)\varepsilon^{-1}.$$

With

$$g'(\varepsilon)=(2\alpha-1)\left(2+2(\lambda+1)\,T+6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}}4^\alpha T\right)\varepsilon^{2\alpha-2}$$
$$-6T\left(\Omega^2 4^\alpha\kappa^{2\alpha}\left\|b-b^N\right\|^{2\alpha}_{H_q^{-\beta}}+\kappa^2\left\|b-b^N\right\|^2_{H_q^{-\beta}}\right)\varepsilon^{-2},$$

and

$$g''(\varepsilon)=(2\alpha-2)(2\alpha-1)\left(2+2(\lambda+1)\,T+6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}}4^\alpha T\right)\varepsilon^{2\alpha-3}$$
$$+12T\left(\Omega^2 4^\alpha\kappa^{2\alpha}\left\|b-b^N\right\|^{2\alpha}_{H_q^{-\beta}}+\kappa^2\left\|b-b^N\right\|^2_{H_q^{-\beta}}\right)\varepsilon^{-3},$$

the minimum of g on $(0,1]$ is reached when N is big enough in

$$\varepsilon_N=\left(\frac{6T\left(\Omega^2 4^\alpha\kappa^{2\alpha}\left\|b-b^N\right\|^{2\alpha}_{H_q^{-\beta}}+\kappa^2\left\|b-b^N\right\|^2_{H_q^{-\beta}}\right)}{(2\alpha-1)\left(2+2(\lambda+1)\,T+6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}}4^\alpha T\right)}\right)^{\frac{1}{2\alpha}}=\omega_N\left\|b^N-b\right\|_{H_q^{-\beta}}.$$

where

$$g''(\varepsilon_N) = 12T\left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2 \left\|b - b^N\right\|_{H_q^{-\beta}}^{2(1-\alpha)}\right)\varepsilon_N^{-3}\alpha > 0.$$

and

$$\omega_N = \left(\frac{6T\left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2 \left\|b - b^N\right\|_{H_q^{-\beta}}^{2(1-\alpha)}\right)}{(2\alpha-1)\left(2 + 2(\lambda+1)\ T + 6\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right)}\right)^{\frac{1}{2\alpha}}$$

$$\geq \omega_\infty = \left(\frac{6T\Omega^2 4^\alpha \kappa^{2\alpha}}{(2\alpha-1)\left(2 + 2(\lambda+1)\ T + 6\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right)}\right)^{\frac{1}{2\alpha}}.$$

Therefore $\mathbb{E}\left[L_T^0(Y - Y^N)\right] \leq g(\varepsilon_N)$

$$\leq 4(\lambda+1)T\kappa\left\|b - b^N\right\|_{H_q^{-\beta}} + \left(2 + 2(\lambda+1)\ T + 6\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right)\omega_N^{2\alpha-1}\left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha-1} + 6T\left(\Omega^2 4^\alpha \kappa^{2\alpha}\left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2\left\|b - b^N\right\|_{H_q^{-\beta}}^{2}\right)\omega_N^{-1}\left\|b - b^N\right\|_{H_q^{-\beta}}^{-1}$$

$$\leq 4(\lambda+1)T\kappa\left\|b - b^N\right\|_{H_q^{-\beta}} + \left(2 + 2(\lambda+1)\ T + 6\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right)\omega_N^{2\alpha-1}\left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha-1} + 6T\left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2\left\|b - b^N\right\|_{H_q^{-\beta}}^{2(1-\alpha)}\right)\omega_\infty^{-1}\left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha-1}$$

$$\leq 4(\lambda+1)T\kappa\left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha-1} + \left(2 + 2(\lambda+1)\ T + 6\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right)2\omega_\infty^{2\alpha-1}\left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha-1} + 6T\left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2\right)\omega_\infty^{-1}\left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha-1}$$

for N big enough. The result follows. □

Theorem 11. *Let f be μ-Hölder with constant $C_f > 0$ and $\mu \in (0,1]$. If $0 < \beta < 1/4$, $q \in \left(\frac{1}{1-\beta}, \frac{1}{\beta}\right)$, $\forall \varepsilon \in (0, 1-4\beta)$, with $(\delta, p) \in K(\beta, q)$ such that $\delta - 1/p = 1 - 2\beta - \varepsilon/2$, exists ξ independent of f such that for $N \in \mathbb{N}$, $\rho > 1$, λ big enough it holds:*

$$\sup_{0\leq t\leq T}\mathbb{E}\left[\left|f\left(X_t\right) - f\left(X_t^N\right)\right|\right] \leq \xi C_f\left\|b^N - b\right\|_{H_q^{-\beta}}^{\mu(1-4\beta-\varepsilon)}$$

Proof. We note as usual $\alpha = \delta - 1/p$. By Lemma 4, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\left\|b - b^N\right\|_{H_q^s(\mathbb{R})} \underset{N\to\infty}{\longrightarrow} 0$. We recall that in this case, by Lemma 22 in [1], $\Psi(t,\cdot)$ and $\Psi^N(t,\cdot)$ are 2-lipschitz. Therefore $\widetilde{u}^N(t,\cdot)$ and $\widetilde{u}(t,\cdot)$ are 1-lipschitz. Let $t \in [0,T]$.

$$\mathbb{E}\left[\left|f\left(X_t\right) - f\left(X_t^N\right)\right|\right] = \mathbb{E}\left[\left|f\left(\Psi\left(t, Y_t\right)\right) - f\left(\Psi^N\left(t, Y_t^N\right)\right)\right|\right]$$

$$\leq \mathbb{E}\left[\left|f\left(\Psi\left(t, Y_t\right)\right) - f\left(\Psi\left(t, Y_t^N\right)\right)\right|\right] + \mathbb{E}\left[\left|f\left(\Psi\left(t, Y_t^N\right)\right) - f\left(\Psi^N\left(t, Y_t^N\right)\right)\right|\right]$$

$$\leq C_f\left(2^\mu\mathbb{E}\left[\left|Y_t - Y_t^N\right|^\mu\right] + \mathbb{E}\left[\left|\Psi\left(t, Y_t^N\right) - \Psi^N\left(t, Y_t^N\right)\right|^\mu\right]\right)$$

$$\leq C_f \left(2^\mu \mathbb{E}\left[\left|Y_t - Y_t^N\right|\right]^\mu + \mathbb{E}\left[\left|\Psi\left(t, Y_t^N\right) - \Psi^N\left(t, Y_t^N\right)\right|\right]^\mu \right) \tag{18}$$

by Jensen's inequality.

$$\begin{aligned} Y_T - Y_T^N = y - y^N + (\lambda+1) \int_0^T & \left\{ u\left(s, \Psi\left(s, Y_s\right)\right) - u^N\left(s, \Psi^N\left(s, Y_s^N\right)\right)\right\} \, \mathrm{d}s \\ & + \int_0^T \left\{ \nabla u\left(s, \Psi\left(s, Y_s\right)\right) - \nabla u^N\left(t, \Psi^N\left(s, Y_s^N\right)\right)\right\} \, \mathrm{d}W_s. \end{aligned}$$

We apply Meyer-Tanaka's formula to obtain:

$$\begin{aligned} \left|Y_t - Y_t^N\right| = \left|y - y^N\right| + (\lambda+1) \int_0^t & \operatorname{sign}(Y_s - Y_s^N) \left\{ \widetilde{u}\left(s, Y_s\right) - \overline{u}^N\left(s, Y_s^N\right)\right\} \, \mathrm{d}s \\ & + \int_0^t \operatorname{sign}(Y_s - Y_s^N) \left\{ \widetilde{\nabla u}\left(s, Y_s\right) - \overline{\nabla u}^N\left(s, Y_s^N\right)\right\} \, \mathrm{d}W_s + L_t^0(Y - Y^N). \end{aligned}$$

Taking the expectation leads to:

$$\begin{aligned} \mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] &= \left|u(0,x) - u^N(0,x)\right| \\ & + (\lambda+1) \, \mathbb{E}\left[\int_0^t \operatorname{sign}(Y_s - Y_s^N) \left\{ \widetilde{u}\left(s, Y_s\right) - \overline{u}^N\left(s, Y_s^N\right)\right\} \, \mathrm{d}s\right] + \mathbb{E}\left[L_t^0(Y - Y^N)\right] \end{aligned}$$

because ∇u and ∇u^N are bounded so the Itô integral is a martingale.

$$\begin{aligned} \mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] \leq \left|u(0,x) - u^N(0,x)\right| + (\lambda+1) \, \mathbb{E}\left[\int_0^t \left|\widetilde{u}\left(s, Y_s\right) - \widetilde{u}\left(s, Y_s^N\right)\right| \mathrm{d}s\right] + \\ + (\lambda+1) \, \mathbb{E}\left[\int_0^t \left|\widetilde{u}\left(s, Y_s^N\right) - \widetilde{u}^N\left(s, Y_s^N\right)\right| \mathrm{d}s\right] + (\lambda+1) \, \mathbb{E}\left[\int_0^t \left|\widetilde{u}^N\left(s, Y_s^N\right) - \overline{u}^N\left(s, Y_s^N\right)\right| \mathrm{d}s\right] \\ + \mathbb{E}\left[L_t^0(Y - Y^N)\right]. \end{aligned}$$

We use Lemma 5, the 1-lipschitz property of $\widetilde{u}$, and the 1/2-lipschitz property of u:

$$\begin{aligned} \mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] &\leq \kappa \left\|b^N - b\right\|_{H_q^{-\beta}} + (\lambda+1) \, \mathbb{E}\left[\int_0^t \left|Y_s - Y_s^N\right| \mathrm{d}s\right] + (\lambda+1) t \kappa \left\|b^N - b\right\|_{H_q^{-\beta}} \\ & \quad + \frac{\lambda+1}{2} \, \mathbb{E}\left[\int_0^t \left|\Psi\left(s, Y_s^N\right) - \Psi^N\left(s, Y_s^N\right)\right| \mathrm{d}s\right] + \mathbb{E}\left[L_t^0(Y - Y^N)\right] \\ & \leq (\lambda+1) \int_0^t \mathbb{E}\left[\left|Y_s - Y_s^N\right|\right] \mathrm{d}s + (2(\lambda+1)T + 1)\kappa \left\|b^N - b\right\|_{H_q^{-\beta}} + \mathbb{E}\left[L_T^0(Y - Y^N)\right]. \end{aligned}$$

where we have used the fact that $L_t^0(Y - Y^N)$ is an increasing process and Lemma 6.

By Gronwall's Lemma, it follows:

$$\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] \leq C(N) \, e^{(\lambda+1)t} \leq C(N) \, e^{(\lambda+1)T} \tag{19}$$

with $C(N) = (2(\lambda+1)T + 1)\kappa \left\|b^N - b\right\|_{H_q^{-\beta}} + \mathbb{E}\left[L_T^0(Y - Y^N)\right].$

With Lemma 9 and Lemma 10 we obtain

$$C(N) \leq (2(\lambda+1)T+1)\kappa \left\|b^N - b\right\|_{H_q^{-\beta}} + \sigma \left\|b^N - b\right\|_{H_q^{-\beta}}^{2\alpha-1} \leq \zeta \left\|b^N - b\right\|_{H_q^{-\beta}}^{2\alpha-1}.$$

for $\left\|b^N - b\right\|_{H_q^{-\beta}}$ small enough where $\zeta = (2(\lambda+1)T+1)\kappa + \sigma$. It follows:

$$\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] \leq \zeta e^{(\lambda+1)T} \left\|b^N - b\right\|_{H_q^{-\beta}}^{2\alpha-1}. \tag{20}$$

Finally, combining (18), (20) and Lemma 6 we obtain:

$$\begin{aligned}
&\left|\mathbb{E}\left[f\left(X_t\right) - f\left(X_t^N\right)\right]\right| \\
&\leq C_f\left(2^\mu \mathbb{E}\left[\left|Y_t - Y_t^N\right|\right]^\mu + \mathbb{E}\left[\left|\Psi\left(t, Y_t^N\right) - \Psi^N\left(t, Y_t^N\right)\right|\right]^\mu\right) \\
&\leq C_f\left(2^\mu \zeta^\mu e^{\mu(\lambda+1)T} \left\|b^N - b\right\|_{H_q^{-\beta}}^{\mu(2\alpha-1)} + 2^\mu \kappa^\mu \left\|b^N - b\right\|_{H_q^{-\beta}}^{\mu}\right) \\
&\leq 2^\mu C_f\left(\zeta^\mu e^{\mu(\lambda+1)T} + 2^\mu \kappa^\mu\right) \left\|b^N - b\right\|_{H_q^{-\beta}}^{\mu(2\alpha-1)}
\end{aligned}$$

for N big enough, which is the expected result.

□

5 Numerical results

5.1 Strong convergence of the Euler scheme

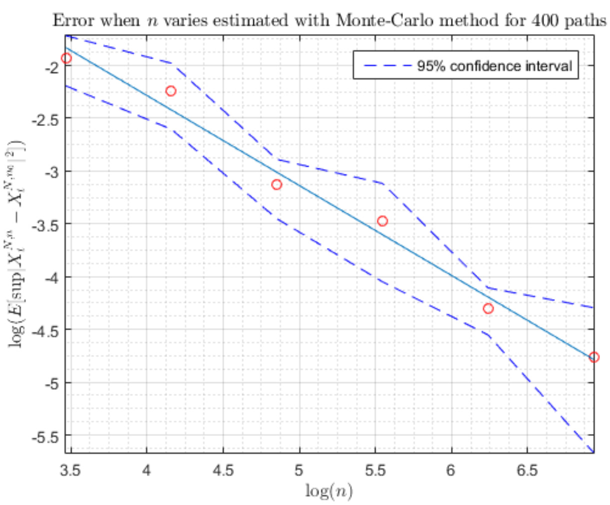

Figure 1: Estimation of the L^2 error of the Euler-Marayuma scheme with a Monte-Carlo method. 400 paths, $N = 5$, $n \in \{2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}\}$, reference solution with $n_0 = 2^{12}$ points.

We observe a numerical convergence rate of 0.85 when Theorem 2 shows a theoretical rate of $0.5 - \varepsilon$.

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