NUMERICAL SIMULATION OF SDES WITH DISTRIBUTIONAL DRIFT

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1 INTRODUCTION

We would like to simulate numerically sample paths of the solution of the one-dimensional stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\, \mathrm{d}t + \mathrm{d}W_t \tag{1}$$

where $b \in H_q^{-\beta}(\mathbb{R})$, $\beta \in \left(0, \frac{1}{2}\right)$, $q \in \left(\frac{1}{1-\beta}, \frac{1}{\beta}\right)$, $t \in [0,T]$, and W_t is a standard Brownian motion. Equation (1) is studied by F. Flandoli, E. Issoglio, and F. Russo in [1] in which they define a concept of virtual solution. The authors prove then existence and unicity in law of this solution.

Example 1. *An example of such drift b on a bounded interval is given by the derivative of a sample path of a fractional Brownian motion B_x^H with Hurst index $1/2 < H < 1$. These stochastic processes are centered gaussian processes verifying*

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} + |t-s|^{2H}\right). \tag{2}$$

For $H = 1/2$, we retrieve the usual brownian motion. We note $-\beta = H - 1$. Given $B_x^H(\omega) \in H_q^{1-\beta}([0, 2K])$, we can take in a distributional sense $b = \frac{\mathrm{d}}{\mathrm{d}u} B_u^H(\omega) \in H_q^{-\beta}([0, 2K])$. We will use a translated and cut to zero version of this type of drift in our numerical simulations.

2 PRINCIPLE OF THE APPROXIMATION ALGORITHM

As far as the drift b is not a function but a distribution, it must be approximated if we want to evaluate it at points. In order to do so, we will use a series representation of b and truncate it. That is why we will consider three steps in our algorithm:

1. approximate b by a function b^N meant to converge to b in $H_q^{-\beta}(\mathbb{R})$ as $N \to \infty$. In practise we will choose $b^N \in L^q(\mathbb{R}) \cap H_q^{-\beta}(\mathbb{R})$ with Supp $b^N \subset [-N, N]$.

2. approximate the stopped solution $X_{t \wedge \tau_N}^N$ of the approximated SDE

$$\mathrm{d}X_t^N = b^N\left(X_t^N\right)\, \mathrm{d}t + \mathrm{d}W_t \tag{3}$$

by $X_{t \wedge \tau_N}^{N,n}$ defined with the Euler-Maruyama scheme by

$$X_t^{N,n} = X_0 + \int_0^t b^N\left(X_{\eta_n(t)}^{N,n}\right) \mathrm{d}t + W_{\eta_n(t)} \tag{4}$$

where $\eta_n(t) = t_k$ if $t \in [t_k, t_k + 1[$, for $t_k = \frac{k}{n}$ with $k \in [\![0, \lceil nT \rceil]\!]$ and $\tau_N = \inf\{t \in [0,T] \mid X_t^N \notin [-N, N]\}$.

3 Virtual solutions of the original SDE and its approximation

In order to approximate the solution of the SDE (1), we must go back to the definition of its virtual solution given in [1]. Let $(\delta, p) \in K(\beta, q) := \{(\delta, p) \mid \beta < \delta < 1 - \beta, \ \frac{1}{\delta} < p < q\}$. The authors of [1] define the virtual solution of SDE (1) by X_t such that:

$$\begin{cases} Y_t = y + (\lambda + 1) \int_0^t u(s, \Psi(s, Y_s)) \ \mathrm{d}s + \int_0^t (\nabla u(s, \Psi(s, Y_s)) + 1) \ \mathrm{d}W_s \\ X_t = \Psi(t, Y_t) = \varphi^{-1}(t, Y_t) \end{cases} \tag{5}$$

where u is the mild solution in $H_p^{1+\delta}$ of the following parabolic PDE:

$$\begin{cases} \partial_t u + \frac{1}{2} \Delta u + b \nabla u - (\lambda + 1) u = -b & \text{on } [0, T] \times \mathbb{R} \\ u(T) = 0 & \text{on } \mathbb{R} \end{cases} \tag{6}$$

with $\varphi(t, x) = x + u(t, x)$, and $y = \varphi(0, x)$.

We also define another similar PDE by replacing b by $b^N \in L^q(\mathbb{R}) \cap H_q^{-\beta}(\mathbb{R})$. We call u^N its mild solution in $H_p^{1+\delta}$:

$$\begin{cases} \partial_t u^N + \frac{1}{2} \Delta u^N + b^N \nabla u^N - (\lambda + 1) u^N = -b^N & \text{on } [0, T] \times \mathbb{R} \\ u^N(T) = 0 & \text{on } \mathbb{R} \end{cases} \tag{7}$$

Then we consider an approximated version of (5):

$$\begin{cases} Y_t^N = y^N + (\lambda + 1) \int_0^t u^N \left(s, \Psi^N \left(s, Y_s^N \right) \right) \ \mathrm{d}s + \int_0^t \left(\nabla u^N \left(s, \Psi^N \left(s, Y_s^N \right) \right) + 1 \right) \ \mathrm{d}W_s \\ X_t^N = \Psi^N(t, Y_t^N) = (\varphi^N)^{-1}(t, Y_t^N) \end{cases} \tag{8}$$

with $\varphi^N(t, x) = x + u^N(t, x)$, and $y^N = \varphi^N(0, x)$.

The following result from Flandoli, Issoglio and Russo expresses the link between classical and virtual solutions and justifies our approximation algorithm.

Proposition 1 (Proposition 26 in [1]). *If $b^N \in L^q(\mathbb{R})$, then the classical solution $(X, \mathbb{F})$ to the SDE (3) is also a virtual solution.*

Remark 1. *Proposition 1 assures us that the virtual solution of (8), X_t^N, defined above in (8) is in fact the classical solution of (3), as far as $b^N \in L^q(\mathbb{R})$. That is why for each fixed N our Euler scheme is meant to converge to the virtual solution X_t^N. This convergence will be made explicit later in this report.*

Remark 2. *For clarity purpose, we will note $\tilde{u}(s, x) = u(s, \Psi(s, x))$, $\tilde{u}^N(s, x) = u^N(s, \Psi(s, x))$, $\overline{u}^N(s, x) = u^N \left(s, \Psi^N(s, x) \right)$ and use the same notations for the gradient and the approximated mild solution.*

Our project concerns a large class of very irregular drifts. But in order to do numerical studies, we must select examples of such drifts. The easiest example we can think of is the derivative of a sample path of a fractional brownian motion. That's why we will explain here how we can apply our approximation algorithm in this special case.
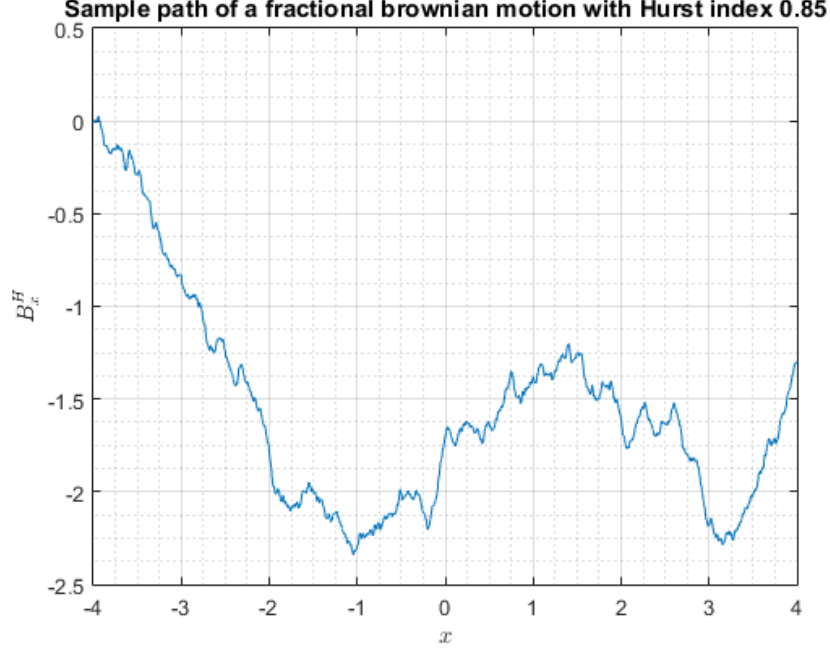
Figure 1: Translated sample path of a fractional brownian motion

4   NUMERICAL SIMULATION OF FRACTIONAL BROWNIAN MOTION

To simulate a sample path of a fractional brownian motion B_x^H on a finite grid $(x_k)_{k\in[\![1,N]\!]}$, we simulate n independent standard gaussian random variables $(X_k)_{k\in[\![1,N]\!]}$ and then correlate them with the definite positive correlation matrix

$$C_{k,s} = \mathbb{E}\left[B_{x_k}^H B_{x_s}^H\right] = \frac{1}{2}\left(x_k^{2H} + x_s^{2H} + |x_k - x_s|^{2H}\right).$$

To do so, we use the Cholesky decomposition method and calculate the lower triangular matrix M such that $C = MM^\top$. Therefore, defining the multidimensional random values

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ and } B^H = MX,$$

B^H is a fractional brownian motion evaluated on the grid $(x_k)_{k\in[\![1,n]\!]}$. Indeed, numerically we will only consider some realizations of these random values. Figure 1 shows the result of such a simulation. We will always translate the path in order to have a symmetric domain. In fact, what does matter in our case is the regularity of a fixed path and not the domain of the underlying process.

4.1   Refining a sample path of a fractional Brownian motion

A first numerical concern was about the way to refine a path of brownian motion, it is to say to add new points in between the simulated path. For instance, if you have simulated a

fractional brownian motion on a regular grid $x_k = \frac{k}{N}$, you may wish to add new points to work on the grid 2 times more precise $\widetilde{x_k} = \frac{k}{2N}$ without changing the points already defined. Let X, Y be two vectors of N independent gaussian random variables and C the correlation matrix associated with the grid $(x_k)_{k \in [\![1,n]\!]}$. We note $\widetilde{X} = \begin{pmatrix} X \\ Y \end{pmatrix}$.

If you have already simulated B^H on the grid $(x_k)_{k \in [\![1,n]\!]}$ with the random variable X, it verifies $B^H = MX$ with $C = MM^\top$. Then, let's consider the new correlation matrix $\widetilde{C} = \begin{pmatrix} C & A^\top \\ A & B \end{pmatrix}$ with

$$A_{k,s} = \mathbb{E}\left[B^H_{\frac{x_k + x_{k-1}}{2}} B^H_{x_s}\right] = \frac{1}{2}\left(\left(\frac{x_k + x_{k-1}}{2}\right)^{2H} + x_s^{2H} + \left|\frac{x_k + x_{k-1}}{2} - x_s\right|^{2H}\right) \qquad (9)$$

$$B_{k,s} = \mathbb{E}\left[B^H_{\frac{x_k + x_{k-1}}{2}} B^H_{\frac{x_s + x_{s-1}}{2}}\right]$$
$$= \frac{1}{2}\left(\left(\frac{x_k + x_{k-1}}{2}\right)^{2H} + \left(\frac{x_s + x_{s-1}}{2}\right)^{2H} + \left|\frac{x_k + x_{k-1}}{2} - \frac{x_s + x_{s-1}}{2}\right|^{2H}\right) \qquad (10)$$

where we chose by convention $x_0 = 0$. Looking at $\widetilde{M}$, the Cholesky root of $\widetilde{C}$, we obtain:

$$\widetilde{M} = \begin{pmatrix} D & 0 \\ E & F \end{pmatrix} \qquad (11)$$

with D lower triangular matrix verifying $DD^\top = C$ so $D = M$. Therefore, we will have

$$\widetilde{B}^H = \widetilde{M}\widetilde{X} = \begin{pmatrix} M & 0 \\ E & F \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} MX \\ EX + FY \end{pmatrix} = \begin{pmatrix} B^H \\ EX + FY \end{pmatrix} \qquad (12)$$

and we obtain $\widetilde{B}^H$ on the grid $(x_k, \frac{x_k + x_{k-1}}{2})$ without changing the previous values. Therefore, one just have to reorder $\widetilde{B}^H$ to obtain a realization of a fractional brownian motion on the grid $\widetilde{x_k} = \left(\frac{x_1}{2}, x_1, \cdots, x_{k-1}, \frac{x_k + x_{k-1}}{2}, x_k, \cdots, x_N\right)$. Numerical results are shown in figures 2 and 3.

5   Approximation of the drift

5.1   Series representation

We use Haar wavelets to give a series representation of b. By doing so, we will be able to approximate it numerically by truncating the series.

Definition 1 (Haar wavelets). *We define the Haar wavelets $h^k_{j,m}$ on $\mathbb{R}$ with $j \in \mathbb{N} \cup \{-1\}$, $m \in \mathbb{Z}$ and $k \in \mathbb{Z}$ by:*

$$\begin{cases} h_M & : x \longmapsto \left(\mathbb{1}_{[0,\frac{1}{2})} - \mathbb{1}_{[\frac{1}{2},1)}\right)(x) \\ h^k_{-1,0} & : x \longmapsto \sqrt{2}\left|h_M(x - k)\right| \\ h_{j,m} & : x \longmapsto h_M(2^j x - m) \\ h^k_{j,m} & : x \longmapsto h_{j,m}(x - k) \end{cases} \qquad (13)$$
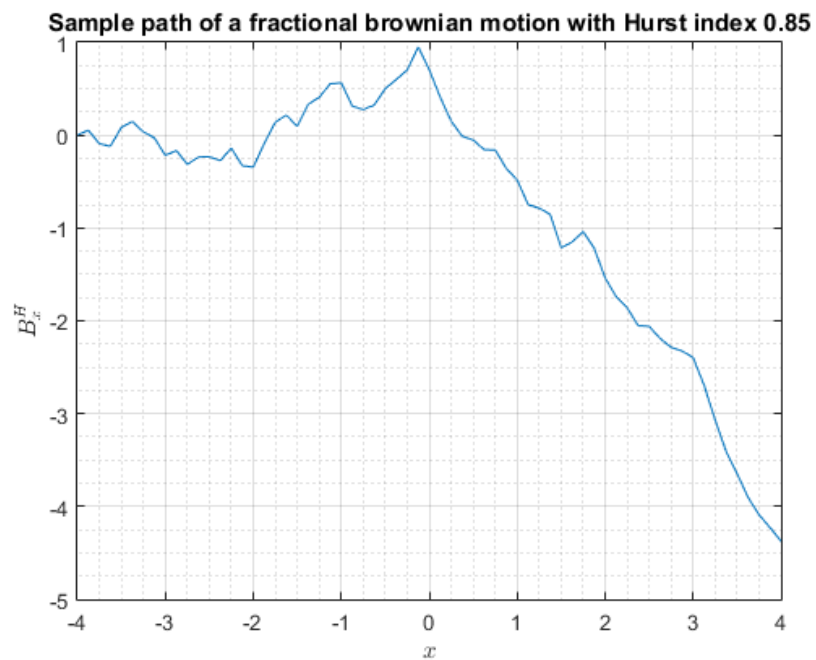
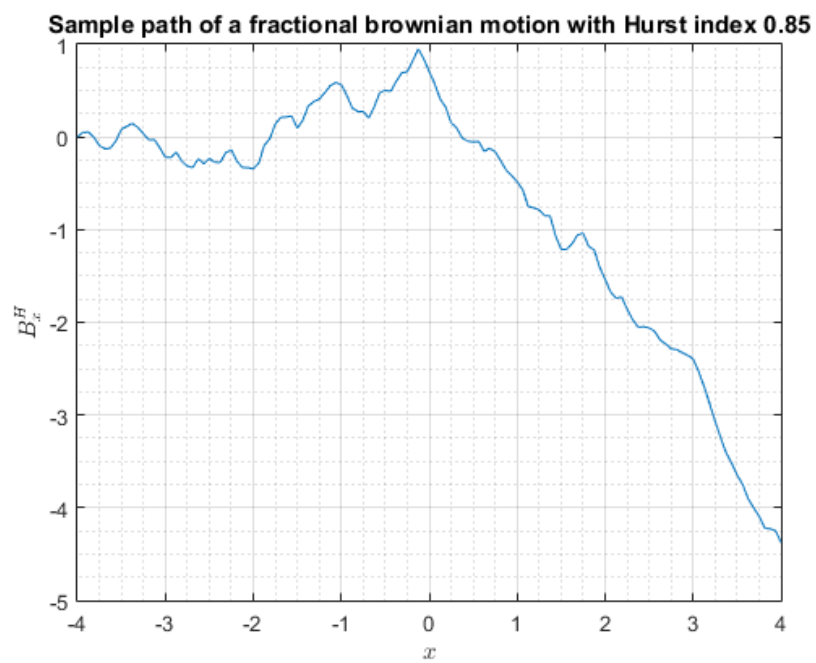Figure 2: Fractional brownian path with 64 points

Figure 3: Refined version of the previous fractional brownian path with 128 points

Theorem 1 (See Theorem 2.9 in [4] and Remark 3.4 in [5]). *Let $f \in H_q^s(\mathbb{R}) = F_{q,2}^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left(-\frac{1}{2}, \frac{1}{q}\right)$. Therefore, we have the representation*

$$f = \sum_{j=-1}^{+\infty} \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^j-1} \mu_{j,m}^k 2^{-j\left(s-\frac{1}{q}\right)} h_{j,m}^k \tag{14}$$

where $\mu_{j,m}^k = 2^{j\left(s-\frac{1}{q}+1\right)} \int_{\mathbb{R}} f(x) h_{j,m}^k(x) \, \mathrm{d}x$ in the sense of dual pairing and $\sum_{m=0}^{2^j-1}$ means $m = 0$ when $j = -1$.

Remark 3. *Theorem 3.3 in [5] gives the same Haar representation in $B_{pq}^s(\mathbb{R})$ for $\frac{1}{p} - 1 < s < \min\left(1, \frac{1}{2}\right)$.*

Definition 2. *With the same notation $\mu_{j,m}^k$, let $f \in H_q^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left(-\frac{1}{2}, \frac{1}{q}\right)$. Given $N \in \mathbb{N}^*$ we define f^N by:*

$$f^N = 2^{\left(s-\frac{1}{q}\right)} \sum_{k=-N}^{N-1} \mu_{-1,0}^k h_{-1,0}^k + \sum_{j=0}^{N} \sum_{k=-N}^{N-1} \sum_{m=0}^{2^j-1} \mu_{j,m}^k 2^{-j\left(s-\frac{1}{q}\right)} h_{j,m}^k. \tag{15}$$

Remark 4. *We can note that $\mathrm{Supp}\ f^N \subset [-N, N]$. Moreover, we have:*

$$\left\| f - f^N \right\|_{H_q^s(\mathbb{R})} \xrightarrow[N \to +\infty]{} 0.$$

5.2 Computation of the coefficients $\mu_{j,m}^k$ in the special case of the derivative of a fractional brownian motion

For our numerical studies, we now fix the realization of a fractional brownian motion B_x^H and cut its path to zero continuously. We recall that there exists a modification of B_x^H which is Hölder-continuous and so belongs to a fractional Sobolev space. We note $\mathcal{C}_0(\mathbb{R}) = \{f \in \mathcal{C}(\mathbb{R}) \mid |f(x)| \xrightarrow[|x| \to \infty]{} 0\}$.

Proposition 2 (Elena). $\forall s \in \left(\frac{1}{2}, H\right),$

$$B_x^H \in H_2^s(\mathbb{R}). \tag{16}$$

Therefore we will consider a compactly supported continuous modification of B_x^H and will still note it B_x^H. By Proposition 2, we fix $s = H - \varepsilon$ for $\varepsilon > 0$ small such that $B_s^H \in H_2^s(\mathbb{R})$. Then, $b = \frac{\mathrm{d}}{\mathrm{d}u} B_u^H$ belongs to $H_2^{s-1}(\mathbb{R})$ so we have by Theorem 1 its Haar decomposition:

$$b(x) = \sum_{j=-1}^{+\infty} \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^j-1} \mu_{j,m}^k 2^{-j\left(s-\frac{1}{q}\right)} h_{j,m}^k(x). \tag{17}$$

with $\mu_{j,m}^k = 2^{j\left(s-\frac{1}{q}+1\right)} \int_{\mathbb{R}} f(x) h_{j,m}^k(x) \, \mathrm{d}x$. The problem is the fact that we cannot directly compute these coefficients $\mu_{j,m}^k$. We will need a representation of B_s^H as another series in order to do that.

In [5], H. Triebel defines the Faber system for the weighted Besov spaces $B_{pq}^s(\mathbb{R}, \xi) = \{f \in S'(\mathbb{R}) \mid \|f\|_{B_{pq}^s(\mathbb{R},\xi)} = \|(1+x^2)^{\xi/2} f\|_{B_{pq}^s(\mathbb{R})} < \infty\}$ for $\xi \in \mathbb{R}$. We will use his results in the unweighted case ($\xi = 0$) to find the Faber representation of B_x^H, and then the Haar coefficients $\mu_{j,m}^k$ of its derivative. In fact, Triebel expresses in [3] the links between fractional Sobolev spaces and unweighted Besov spaces $B_{pq}^s(\mathbb{R}) = B_{pq}^s(\mathbb{R}, 0)$.

Lemma 2 (Theorem 1.3.2 in [3]). *Let $s > 0$, then*

$$H_2^s(\mathbb{R}) = B_{22}^s(\mathbb{R}). \tag{18}$$

Lemma 3 (Proposition 3.5 in [5]). *Let $s > \frac{1}{2}$, then*

$$B_{22}^s(\mathbb{R}) \hookrightarrow \mathcal{C}(\mathbb{R}) \tag{19}$$

and

$$\|f\|_{B_{22}^s(\mathbb{R})} \sim \left(\sum_{k\in\mathbb{Z}} |f(k)|^2\right)^{1/2} + \|f'\|_{B_{22}^{s-1}(\mathbb{R})}. \tag{20}$$

Definition 3 (Faber basis). *We define the Faber system $\{v^k, \ v_{j,m}^k\}$ on $\mathbb{R}$ for $j \in \mathbb{N}$, $k \in \mathbb{Z}$, $m \in \mathbb{N}$ by:*

$$\begin{cases} v_j(x) & = (1 - 2^{j+1}|x|)_+ \\ v^k(x) & = v_{-1}(x - k) \\ v_{jm}(x) & = v_j(x - 2^{-j-1} - 2^{-j}m) \\ v_{jm}^k(x) & = v_{jm}(x - k) \end{cases} \tag{21}$$

Then, by Theorem 3.1 in [5], the Faber system is a conditional basis of $\mathcal{C}_0(\mathbb{R})$.

We adapt Theorem 3.6 in [5] to the case $\alpha = 0$ and $B^H \in \mathcal{C}_0(\mathbb{R})$ to obtain:

Theorem 4. *(Faber representation of B_x^H) In $H_2^s(\mathbb{R}) = B_{22}^s(\mathbb{R})$ we have the unique Faber representation:*

$$B_x^H = \sum_{j=-1}^{+\infty} \sum_{k\in\mathbb{Z}} \sum_{m=0}^{2^j-1} \lambda_{j,m}^k 2^{-j\left(s-\frac{1}{2}\right)} v_{j,m}^k(x) \tag{22}$$

with

$$\begin{cases} \lambda_{-1,0}^k & = 2^{-s+\frac{1}{2}} B_k^H \\ \lambda_{j,m}^k & = -2^{j(s-\frac{1}{2})-1} (\Delta_{2^{-j-1}}^2 B^H)(k + 2^{-j}m) \end{cases} \tag{23}$$

and $-\frac{1}{2}(\Delta_{2^{-j-1}}^2 B^H)(2^{-j}m) = B_{2^{-j}m+2^{-j-1}}^H - \frac{1}{2} B_{2^{-j}m}^H - \frac{1}{2} B_{2^{-j}m+2^{-j}}^H$.

Proof. Remark 3 gives a Haar decomposition of $b = \frac{\mathrm{d}}{\mathrm{d}u} B_u^H$ in the unweighted Besov space $B_{22}^{s-1}(\mathbb{R}) = B_{22}^{s-1}(\mathbb{R}, 0)$:

$$b = \sum_{k\in\mathbb{Z}} \left(B_{k+1}^H - B_k^H\right) \chi_k + \sum_{j=0}^{+\infty} \sum_{k\in\mathbb{Z}} \sum_{m=0}^{2^j-1} \mu_{j,m}^k(b) 2^{-j\left(s-\frac{1}{2}\right)} h_{j,m}^k. \tag{24}$$

with $\mu_{j,m}^k(b) = 2^{j\left(s-\frac{1}{2}+1\right)} \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}u} B_u^H(x) h_{j,m}^k(x) \, \mathrm{d}x = -2^{j\left(s-\frac{1}{2}\right)} \left(\Delta_{2^{-j-1}}^2 B^H\right)(k+2^{-j}m) = 2\lambda_{j,m}^k(B^H)$
and:

$$\|b\|_{B_{22}^{s-1}(\mathbb{R})} \sim \left(\sum_{k \in \mathbb{Z}} \left|B_{k+1}^H - B_k^H\right|^2\right)^{1/2} + \left(\sum_{j=0}^{\infty} \left(\sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^j-1} \left|\mu_{j,m}^k(b)\right|^2\right)\right)^{1/2}. \tag{25}$$

The representation can also be written as:

$$b = \sum_{k \in \mathbb{Z}} B_k^H (\chi_{k-1} - \chi_k) + 2 \sum_{j=-0}^{+\infty} \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^j-1} \lambda_{j,m}^k \left(B^H\right) 2^{-j\left(s-1-\frac{1}{2}\right)} h_{j,m}^k.$$

Suppose that $B^H \in \mathcal{C}_0^\infty(\mathbb{R})$. Then by integration it follows:

$$B^H = \sum_{k \in \mathbb{Z}} B_k^H v_{-1,0}^k + \sum_{j=-0}^{+\infty} \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^j-1} \lambda_{j,m}^k \left(B^H\right) 2^{-j\left(s-\frac{1}{2}\right)} h_{j,m}^k$$

$$= \sum_{j=-1}^{+\infty} \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^j-1} \lambda_{j,m}^k \left(B^H\right) 2^{-j\left(s-\frac{1}{2}\right)} h_{j,m}^k. \tag{26}$$

By 25 and Lemma 3 we obtain:

$$\left\|\lambda(B^H)\right\|_{b_{22}(\mathbb{R})} \leq c \left\|B^H\right\|_{B_{22}^s(\mathbb{R})}. \tag{27}$$

Then, by density of $\mathcal{C}_0^\infty(\mathbb{R})$ in $\mathcal{C}_0(\mathbb{R})$, 27 can be extended to $B^H \in \mathcal{C}_0(\mathbb{R})$.

$\square$

6   Weak convergence rate of $X_t^{N,n}$ to X_t^N

Recently, Leobacher and Szölgyenyi proved in [2] the convergence of the Euler-Maruyama scheme for SDE with discontinuous but piecewise Lipschitz drift and with a degenerate diffusion coefficient. This framework applies to the scheme we use with piecewise constant drift b^N and a constant diffusion coefficient.

Theorem 5 (Theorem 3.1. in [2]). $\exists C_N > 0$ independent of n such that it holds $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$:

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \left|X_t^{N,n} - X_t^N\right|^2\right]^{1/2} \leq C_N \delta^{1/4-\varepsilon} \tag{28}$$

with $\delta = \frac{1}{n}$ the step size and C_N depending on $\left\|b^N\right\|_\infty$.

This strong error obviously gives us a control of the weak error for μ-Hölder functions with $\mu \in (0,1]$.

Corollary 6. Let f be μ-Hölder with constant $C_f > 0$, $\mu \in (0,1]$ and $t \in [0,T]$. Then, exists $C_N' > 0$ independent of n such that it holds $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$:

$$\sup_{0 \leq t \leq T} \left|\mathbb{E}\left[f\left(X_t^{N,n}\right) - f\left(X_t^N\right)\right]\right| \leq C_N' \delta^{\mu/4-\varepsilon} \tag{29}$$

with $\delta = \frac{1}{n}$ the step size.

Proof. By Jensen's inequality and the μ-Hölder property of f, we obtain:

$$\sup_{0 \le t \le T} \left| \mathbb{E} \left[f\left(X_t^{N,n} \right) - f\left(X_t^N \right) \right] \right| \le C_f \sup_{0 \le t \le T} \mathbb{E} \left[\left| X_t^{N,n} - X_t^N \right|^{\mu} \right]$$

$$\le C_f \mathbb{E} \left[\left| Y_t - Y_t^N \right|^2 \right]^{\mu/2} \le C_f \sup_{0 \le t \le T} \mathbb{E} \left[\sup_{0 \le t \le T} \left| Y_t - Y_t^N \right|^2 \right]^{\mu/2}$$

$$\le C_f C_N^{\mu} \delta^{\mu/4 - \varepsilon}.$$

$\square$

7 WEAK CONVERGENCE RATE OF X_t^N TO X_t

The goal of this section, which is the core of this research project, is to estimate the weak error $\left| \mathbb{E} \left[f\left(X_t \right) - f\left(X_t^N \right) \right] \right|$ with suitable functions f. The main result relies on several lemmas. We will recall and use results from [1] where the stochastic differential equation (1) was first studied. Some of our technical proofs are developed in the appendix.

7.1 Useful lemmas

Most of these results allow us to study the behaviour of the solutions of PDEs (6) and (7) and then of the transformations φ and Ψ. We will use these lemmas to prove our main result, Theorem 14.

We first recall a useful lemma concerning the solutions of (6) and (7). It will allow us to choose λ big enough such that u and u^N are lipschitz.

Lemma 7 (Lemma 20 in [1]). *Let $(\delta, p) \in K(\beta, q)$ and let u, u^N be the mild solutions to (6), (7) in $H_p^{1+\delta}$. Fix ρ such that the integral operator is a contraction and let $\lambda > \lambda^*$. Then $u(t)$, $u^N(t) \in \mathcal{C}^{1,\alpha}$ with $\alpha = \delta - 1/p$ for each fixed t and $\forall \varepsilon > 0$, $\exists \lambda_0 > 0$ such that*

$$\begin{cases} \forall \lambda \ge \lambda_0, & \displaystyle\sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla u(t,x)| \le \varepsilon \\ \forall \lambda \ge \lambda_0, & \displaystyle\sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla u^N(t,x)| \le \varepsilon \end{cases} \tag{30}$$

where the choice of λ_0 depends only on $\delta, \beta, \|b\|_{H_p^{-\beta}}$, and $\|b^N\|_{H_q^{-\beta}}$.

The next lemma uses the embedding between $H_p^{1+\delta}$ and $\mathcal{C}^{1,\alpha}$ to control the error of approximation of u by u^N.

Lemma 8. *Let $(\delta, p) \in K(\beta, q)$ and let u, u^N be the mild solutions to (6),(7) in $H_p^{1+\delta}$, $\alpha = \delta - 1/p$. Exists $c, K > 0$ such that for both $N \in \mathbb{N}$ and $\rho > 1$ big enough, $\forall t \in [0,T]$,*

$$\begin{cases} \left\| u^N(t) - u(t) \right\|_{L^{\infty}} \le \kappa \left\| b - b^N \right\|_{H_q^{-\beta}} \\ \left\| \nabla u^N(t) - \nabla u(t) \right\|_{L^{\infty}} \le \kappa \left\| b - b^N \right\|_{H_q^{-\beta}}. \end{cases} \tag{31}$$

with $\kappa = cKe^{\rho T}$.

9

Proof. Applying fractional Morrey inequality, $\exists c > 0$, $\forall t \in [0, T]$:

$$
\begin{cases}
\left\| u^N(t) - u(t) \right\|_{L^\infty} \leq \left\| u^N(t) - u(t) \right\|_{\mathcal{C}^{1,\alpha}} \leq c \left\| u^N(t) - u(t) \right\|_{H_p^{1+\delta}} \\
\left\| \nabla u^N(t) - \nabla u(t) \right\|_{L^\infty} \leq \left\| u^N(t) - u(t) \right\|_{\mathcal{C}^{1,\alpha}} \leq c \left\| u^N(t) - u(t) \right\|_{H_p^{1+\delta}}.
\end{cases}
$$

Now, we can conclude with

$$
\left\| u^N - u \right\|_{\infty, H_p^{1+\delta}} \leq e^{\rho T} \left\| u^N - u \right\|_{\infty, H_p^{1+\delta}}^{(\rho)} \leq K e^{\rho T} \left\| b - b^N \right\|_{H_q^{-\beta}}
$$

from Lemma 23 in [1], for both $N \in \mathbb{N}$ and $\rho > 1$ big enough, and where $\|f(t)\|_{\infty, X}^{(\rho)} :=$
$\sup\limits_{0 \leq t \leq T} e^{-\rho t} \|f(t)\|_X$. $\square$

Lemma 9. *For λ big enough,*

$$
\left| \Psi^N\left(t, Y_t^N\right) - \Psi\left(t, Y_t^N\right) \right| \leq 2\kappa \left\| b - b^N \right\|_{H_q^{-\beta}} \tag{32}
$$

with $\kappa = cK e^{\rho T}$.

Proof. For λ big enough, by Lemma 7, $\forall t \in [0, T]$, $\sup\limits_{x \in \mathbb{R}} |\nabla u(t, x)| \leq 1/2$, so we obtain with
$\varphi(t, x) = x + u(t, x)$:

$$
\left| \varphi\left(t, \Psi^N\left(t, Y_t^N\right)\right) - \varphi\left(t, \Psi\left(t, Y_t^N\right)\right) \right|
$$

$$
\geq \inf\limits_{x \in \mathbb{R}} |\nabla \varphi(t, x)| \left| \Psi^N\left(t, Y_t^N\right) - \Psi\left(t, Y_t^N\right) \right|
$$

$$
\geq \frac{1}{2} \left| \Psi^N\left(t, Y_t^N\right) - \Psi\left(t, Y_t^N\right) \right|
$$

then

$$
\left| \Psi^N\left(t, Y_t^N\right) - \Psi\left(t, Y_t^N\right) \right| \leq 2 \left| \varphi\left(t, \Psi^N\left(t, Y_t^N\right)\right) - \varphi\left(t, \Psi\left(t, Y_t^N\right)\right) \right|
$$

$$
= 2 \left| \varphi\left(t, \Psi^N\left(t, Y_t^N\right)\right) - \varphi^N\left(t, \Psi^N\left(t, Y_t^N\right)\right) \right|
$$

$$
\leq 2 \left\| u(t) - u^N(t) \right\|_\infty
$$

$$
\leq 2\kappa \left\| b - b^N \right\|_{H_q^{-\beta}} \tag{33}
$$

where we have used Lemma 8 and the fact that

$$
\varphi^N\left(t, \Psi^N\left(t, Y_t^N\right)\right) = \varphi\left(t, \Psi\left(t, Y_t^N\right)\right) = Y_t^N. \tag{34}
$$

$\square$

We will need an adapted version of a local time inequality (Lemma 4.2 in [6]) from Liqing Yan. The method he introduced will allow us to estimate the error $\mathbb{E}[|Y - Y^N|]$ and therefore calculate the weak error of approximation our algorithm commit. In his paper he studies the case where $X_0 = 0$ whereas in our case we don't assume anything about the initial value of the semimartingale X_t. That's why the proof is mainly the same. The difference is that σ_1 is now a stopping time when its value was 0 in the proof of Yan.

Lemma 10. *Let X_t be a continuous semimartingale. For $\varepsilon > 0$ we define a double sequence of stopping times by $\sigma_1 = \inf\{t \geq 0 | X_t = 0\}$, $\tau_1 = \inf\{t > \sigma_1 | X_t = \varepsilon\}$, $\sigma_n = \inf\{t > \tau_{n-1} | X_t = 0\}$, $\tau_n = \inf\{t > \sigma_n | X_t = \varepsilon\}$. For any real function $F(\cdot) \in \mathcal{C}^2(\mathbb{R})$ with $F(0) = 0$, $F'(0) = 0$, $F(\cdot) > 0$ on $(0, \varepsilon_0)$ with some $\varepsilon_0 > 0$, and for any $0 < \varepsilon < \varepsilon_0$ we have*

$$0 \leq L_t^0(X) \leq 2\varepsilon - \frac{2}{F(\varepsilon)} \int_{\sigma_1 \wedge t}^t \theta_s(X) \left(F(\varepsilon) - \varepsilon F'\left(X_s^+\right)\right) \, \mathrm{d}X_s + \frac{\varepsilon}{F(\varepsilon)} \int_{\sigma_1 \wedge t}^t \theta_s(X) F''(X_s^+) \, \mathrm{d}[X]_s \tag{35}$$

with $\theta_s(X) = \sum_{n=1}^\infty \mathbb{1}_{\{\sigma_n < s \leq \tau_n, \ 0 < X_s \leq \varepsilon\}}(X)$.

Applying lemma 10 with $F : x \in \mathbb{R} \mapsto x^2$, it follows:

Corollary 11. *Let X be a continuous semimartingale. With the same notations as in lemma 10, for any $\varepsilon > 0$ we have*

$$0 \leq L_t^0(X) \leq 2\varepsilon - \frac{2}{\varepsilon} \int_{\sigma_1 \wedge t}^t \theta_s(X) \left(\varepsilon - 2X_s^+\right) \, \mathrm{d}X_s + \frac{2}{\varepsilon} \int_{\sigma_1 \wedge t}^t \theta_s(X) \, \mathrm{d}[X]_s \tag{36}$$

Proof of Lemma 10. We note $U_t(X) = \sup\{n \in \mathbb{N} | \tau_n < t\}$ and $n(t) = t \wedge \sigma_{U_t(X)+1}$. By Meyer-Tanaka's formula, $\forall i \in \mathbb{N}^*$:

$$X_{\tau_i \wedge t}^+ - X_{\sigma_i \wedge t}^+ = \int_{\sigma_i \wedge t}^{\tau_i \wedge t} \mathbb{1}_{\{X_s > 0\}} \, \mathrm{d}X_s + \frac{1}{2} \left\{L_{\tau_i \wedge t}^0(X) - L_{\sigma_i \wedge t}^0(X)\right\}. \tag{37}$$

Because $\forall i \in \mathbb{N}^*$, $L_{\tau_i \wedge t}^0(X) = L_{\sigma_{i+1} \wedge t}^0(X)$ and $L_{\sigma_1 \wedge t}^0(X) = 0$, we have

$$\sum_{i=1}^{U_t(X)+1} \left(X_{\tau_i \wedge t}^+ - X_{\sigma_i \wedge t}^+\right) = \int_{\sigma_1 \wedge t}^t \theta_s(X) \, \mathrm{d}X_s + \frac{1}{2} L_t^0(X).$$

The left term is equal to $\varepsilon U_t(X) + X_t^+ - X_{n(t)}^+$ so

$$\varepsilon U_t(X) = \int_{\sigma_1 \wedge t}^t \theta_s(X) \, \mathrm{d}X_s + \frac{1}{2} L_t^0(X) - X_t^+ + X_{n(t)}^+. \tag{38}$$

Now we express differently $U_t(X)$. $F \in \mathcal{C}^2(\mathbb{R})$ so by Itô's formula:

$$F\left(X_{\tau_i \wedge t}^+\right) - F\left(X_{\sigma_i \wedge t}^+\right) = \int_{\sigma_i \wedge t}^{\tau_i \wedge t} F'\left(X_s^+\right) \mathrm{d}X_s^+ + \frac{1}{2} \int_{\sigma_i \wedge t}^{\tau_i \wedge t} F''\left(X_s^+\right) \mathrm{d}[X^+]_s.$$

By (37), $\mathrm{d}X_s^+ = \mathbb{1}_{\{X_s > 0\}} \mathrm{d}X_s + \frac{1}{2} \, \mathrm{d}L_t^0(X)$ and $\mathrm{d}[X^+]_s = \mathbb{1}_{\{X_s > 0\}} \mathrm{d}[X]_s$. It follows

$$F\left(X_{\tau_i \wedge t}^+\right) - F\left(X_{\sigma_i \wedge t}^+\right) = \int_{\sigma_i \wedge t}^{\tau_i \wedge t} F'\left(X_s^+\right) \mathbb{1}_{\{X_s > 0\}} \, \mathrm{d}X_s + \frac{1}{2} \int_{\sigma_i \wedge t}^{\tau_i \wedge t} F'(X_s^+) \, \mathrm{d}L_t^0(X)$$

$$+ \frac{1}{2} \int_{\sigma_i \wedge t}^{\tau_i \wedge t} F''\left(X_s^+\right) \mathbb{1}_{\{X_s > 0\}} \, \mathrm{d}[X]_s.$$

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Adding up for i, with $F(0) = 0$ we obtain

$$F(\varepsilon)U_t(X) + F\left(X_t^+\right) - F\left(X_{n(t)}^+\right) = \sum_{i=1}^{U_t(X)+1} \left(F\left(X_{\tau_i \wedge t}^+\right) - F\left(X_{\sigma_i \wedge t}^+\right)\right)$$

$$= \int_{\sigma_1 \wedge t}^t F'(X_s^+)\,\theta_s(X)\,\mathrm{d}X_s + \frac{1}{2}\int_{\sigma_1 \wedge t}^t F'\left(X_s^+\right)\Xi_s\,\mathrm{d}L_t^0(X) + \frac{1}{2}\int_{\sigma_1 \wedge t}^t F''\left(X_s^+\right)\theta_s(X)\,\mathrm{d}[X]_s$$

with $\Xi_s = \sum_{n=1}^\infty \mathbb{1}_{\{\sigma_n < s \le \tau_n\}}$. The measure $\mathrm{d}L_t^0(X)$ is almost surely carried by $\{t|X_t = 0\}$ so we can simplify $\int_{\sigma_1 \wedge t}^t F'(X_s^+)\Xi_s\,\mathrm{d}L_t^0(X) = F'(0)\int_{\sigma_1 \wedge t}^t \Xi_s\,\mathrm{d}L_t^0(X)$ in order to have, with $F'(0) = 0$:

$$F(\varepsilon)U_t(X) = -F\left(X_t^+\right) + F\left(X_{n(t)}^+\right) + \int_{\sigma_1 \wedge t}^t F'\left(X_s^+\right)\theta_s(X)\,\mathrm{d}X_s + \frac{1}{2}\int_{\sigma_1 \wedge t}^t F''\left(X_s^+\right)\theta_s(X)\,\mathrm{d}[X]_s. \tag{39}$$

Combining (38) and (39), it follows

$$L_t^0(X)F(\varepsilon) = 2F(\varepsilon)(X_t^+ - X_{n(t)}^+) - 2\varepsilon(F\left(X_t^+\right) - F\left(X_{n(t)}^+\right)$$

$$- 2\int_{\sigma_1 \wedge t}^t \left(F(\varepsilon) - \varepsilon F'\left(X_s^+\right)\right)\theta_s(X)\,\mathrm{d}X_s + \varepsilon\int_{\sigma_1 \wedge t}^t F''\left(X_s^+\right)\theta_s(X)\,\mathrm{d}[X]_s.$$

Then, if $n(t) = t$, the two first right terms of the equality are equal to zero. Else, if $n(t) = \sigma_{U_t(X)+1}$, $0 \le F(\varepsilon)(X_t^+ - X_{n(t)}^+) = F(\varepsilon)X_t^+ \le F(\varepsilon)\varepsilon$ and $-2\varepsilon(F\left(X_t^+\right) - F\left(X_{n(t)}^+\right)) = -2\varepsilon F\left(X_t^+\right) \le 0$ because of the positivity of F. Finally we obtain:

$$L_t^0(X) \le 2\varepsilon - \frac{2}{F(\varepsilon)}\int_{\sigma_1 \wedge t}^t \left(F(\varepsilon) - \varepsilon F'\left(X_s^+\right)\right)\theta_s(X)\,\mathrm{d}X_s + \frac{\varepsilon}{F(\varepsilon)}\int_{\sigma_1 \wedge t}^t F''\left(X_s^+\right)\theta_s(X)\,\mathrm{d}[X]_s.$$

$$\square$$

Lemma 12. *Let $(\delta, p) \in K(\beta, q)$, $\alpha = \delta - 1/p < 1$, u, u^N be the mild solutions to (6), (7) in $H_p^{1+\delta}$, and Y, Y^N solutions of the SDEs (5), (8). Then, if $\alpha > 1/2$, for λ big enough we have $\forall \varepsilon \in (0, 1]$,*

$$0 \le \mathbb{E}\left[L_T^0(Y - Y^N)\right] \le g(\varepsilon). \tag{40}$$

where

$$g(\varepsilon) = 4(\lambda + 1)T\kappa \left\|b - b^N\right\|_{H_q^{-\beta}} + \left(2 + 2(\lambda + 1)\,T + 6\|u\|_{\mathcal{C}^{1,\alpha}}^2\,4^\alpha T\right)\varepsilon^{2\alpha - 1}$$

$$+ 6T\left(\Omega^2 4^\alpha \kappa^{2\alpha}\left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2\left\|b - b^N\right\|_{H_q^{-\beta}}^2\right)\varepsilon^{-1}. \tag{41}$$

Proof of Lemma 12. By Lemma 7, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\left\|b - b^N\right\|_{H_q^s(\mathbb{R})} \underset{N \to \infty}{\longrightarrow} 0$. Therefore $u^N(t, \cdot)$ and $u(t, \cdot)$ are $\frac{1}{2}$-lipschitz. We recall that in this case, by Lemma 22 in [1], $\Psi(t, \cdot)$ and $\Psi^N(t, \cdot)$ are 2-lipschitz. Therefore $\widetilde{u}^N(t, \cdot)$ and $\widetilde{u}(t, \cdot)$ are 1-lipschitz. We can notice that $\widetilde{\nabla u}$ is α-Hölder

with constant $2^\alpha \|u\|_{\mathcal{C}^{1,\alpha}}$ and ∇u^N is α-Hölder with a constant which can be bounded by Ω for N big enough (see Lemma 24 in [1]). Let $\varepsilon \in (0,1]$. Corollary 11 gives us:

$$0 \le L_T^0(Y - Y^N) \le 2\varepsilon - \frac{2}{\varepsilon} \int_{\sigma_1 \wedge T}^T \theta_s(Y - Y^N)\left(\varepsilon - 2(Y_s - Y_s^N)^+\right) \, \mathrm{d}\left(Y_s - Y_s^N\right)$$

$$+ \frac{2}{\varepsilon} \int_{\sigma_1 \wedge T}^T \theta_s(Y - Y^N) \, \mathrm{d}\left[Y - Y^N\right]_s$$

with

$$Y_T - Y_T^N = y - y^N + (\lambda + 1) \int_0^T \left\{ u\left(s, \Psi\left(s, Y_s\right)\right) - u^N\left(s, \Psi^N\left(s, Y_s^N\right)\right) \right\} \, \mathrm{d}s$$

$$+ \int_0^T \left\{ \nabla u\left(s, \Psi\left(s, Y_s\right)\right) - \nabla u^N\left(t, \Psi^N\left(s, Y_s^N\right)\right) \right\} \, \mathrm{d}W_s.$$

Remark 5. *Note that* $\theta_s(Y - Y^N)\left|\varepsilon - 2(Y_s - Y_s^N)^+\right| \le \varepsilon\theta_s(Y - Y^N)$.

∇u and ∇u^N are bounded so the Itô integral is a true martingale. We take the expectation:

$$\mathbb{E}\left[L_T^0(Y - Y^N)\right] \le 2\varepsilon + 2(\lambda + 1) \, \mathbb{E}\left[\int_{\sigma_1 \wedge T}^T \theta_s\left(Y - Y^N\right)\left|\widetilde{u}\left(s, Y_s\right) - \overline{u}^N\left(s, Y_s^N\right)\right| \mathrm{d}s\right]$$

$$+ \frac{2}{\varepsilon} \, \mathbb{E}\left[\int_{\sigma_1 \wedge T}^T \theta_s\left(Y - Y^N\right)\left\{\widetilde{\nabla u}\left(s, Y_s\right) - \overline{\nabla u}\left(s, Y_s^N\right)\right\}^2 \, \mathrm{d}s\right]$$

$$\le 2\varepsilon + 2(\lambda + 1) \, \mathbb{E}\left[\int_0^T \theta_s\left(Y - Y^N\right)\left|\widetilde{u}\left(s, Y_s\right) - \widetilde{u}\left(s, Y_s^N\right)\right| \, \mathrm{d}s\right]$$

$$+ 2(\lambda + 1) \, \mathbb{E}\left[\int_0^T \theta_s\left(Y - Y^N\right)\left|\widetilde{u}\left(s, Y_s^N\right) - \widetilde{u}^N\left(s, Y_s^N\right)\right| \, \mathrm{d}s\right]$$

$$+ 2(\lambda + 1) \, \mathbb{E}\left[\int_0^T \theta_s\left(Y - Y^N\right)\left|\widetilde{u}^N\left(s, Y_s^N\right) - \overline{u}^N\left(s, Y_s^N\right)\right| \mathrm{d}s\right]$$

$$+ \frac{6}{\varepsilon} \, \mathbb{E}\left[\int_0^T \theta_s\left(Y - Y^N\right)\left\{\widetilde{\nabla u}\left(s, Y_s\right) - \widetilde{\nabla u}\left(s, Y_s^N\right)\right\}^2 \, \mathrm{d}s\right]$$

$$+ \frac{6}{\varepsilon} \, \mathbb{E}\left[\int_0^T \theta_s\left(Y - Y^N\right)\left\{\widetilde{\nabla u}\left(s, Y_s^N\right) - \widetilde{\nabla u}^N\left(s, Y_s^N\right)\right\}^2 \, \mathrm{d}s\right]$$

$$+ \frac{6}{\varepsilon} \, \mathbb{E}\left[\int_0^T \theta_s\left(Y - Y^N\right)\left\{\widetilde{\nabla u}^N\left(s, Y_s^N\right) - \overline{\nabla u}^N\left(s, Y_s^N\right)\right\}^2 \, \mathrm{d}s\right]$$

$$\leq 2\varepsilon + 2(\lambda + 1) \, \mathbb{E}\left[\int_0^T \theta_s \left(Y - Y^N\right) \left|Y_s - Y_s^N\right| \, \mathrm{d}s\right] + 2(\lambda + 1)T\kappa \left\|b^N - b\right\|_{H_q^{-\beta}}$$

$$+ (\lambda + 1) \, \mathbb{E}\left[\int_0^T \theta_s \left(Y - Y^N\right) \left|\Psi\left(s, Y_s^N\right) - \Psi^N\left(s, Y_s^N\right)\right| \, \mathrm{d}s\right]$$

$$+ \frac{6 \times 4^\alpha \left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2}{\varepsilon} \, \mathbb{E}\left[\int_0^T \theta_s \left(Y - Y^N\right) \left|Y_s - Y_s^N\right|^{2\alpha} \, \mathrm{d}s\right] + 6T\kappa^2 \left\|b^N - b\right\|_{H_q^{-\beta}}^2 \varepsilon^{-1}$$

$$+ \frac{6\Omega^2}{\varepsilon} \, \mathbb{E}\left[\int_0^T \theta_s \left(Y - Y^N\right) \left|\Psi\left(s, Y_s^N\right) - \Psi^N\left(s, Y_s^N\right)\right|^{2\alpha} \, \mathrm{d}s\right]$$

where we have used Lemma 8, the 1-lipschitz property of $\widetilde{u}$, the 1/2-lipschitz property of u^N, the α-Hölder property of $\widetilde{\nabla u}$ (with constant $2^\alpha \left\|u\right\|_{\mathcal{C}^{1,\alpha}}$), and the α-Hölder property of ∇u^N (with constant Ω). Lemma 9 gives us:

$$\leq 2\varepsilon + 2(\lambda + 1) \, \mathbb{E}\left[\int_0^T \theta_s \left(Y - Y^N\right) \left|Y_s - Y_s^N\right| \, \mathrm{d}s\right] + 4(\lambda + 1)T\kappa \left\|b^N - b\right\|_{H_q^{-\beta}}$$

$$+ \frac{6 \times 4^\alpha \left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2}{\varepsilon} \, \mathbb{E}\left[\int_0^T \theta_s \left(Y - Y^N\right) \left|Y_s - Y_s^N\right|^{2\alpha} \, \mathrm{d}s\right] + 6T\kappa^2 \left\|b - b^N\right\|_{H_q^{-\beta}}^2 \varepsilon^{-1}$$

$$+ 6\Omega^2 T 4^\alpha \kappa^{2\alpha} \left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha} \varepsilon^{-1}.$$

As $\theta_s(Y - Y^N) \left|Y_s - Y_s^N\right| \leq \varepsilon$, we have

$$\mathbb{E}\left[L_T^0(Y - Y^N)\right] \leq 2\varepsilon + 2(\lambda + 1) \, T\varepsilon + 4(\lambda + 1)T\kappa \left\|b - b^N\right\|_{H_q^{-\beta}} + 6 \left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T \varepsilon^{2\alpha - 1}$$

$$+ 6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} \left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2 \left\|b - b^N\right\|_{H_q^{-\beta}}^2\right) \varepsilon^{-1}.$$

As $1 > 2\alpha - 1 > 0$, the result follows from $\varepsilon \leq \varepsilon^{2\alpha - 1}$ when $0 < \varepsilon \leq 1$.

$\square$

We have been able with Lemma 12 to bound the expectation of the local time of $Y - Y^N$ by a function of $\varepsilon > 0$. Here we choose an optimal ε such that the bound is minimal.

Lemma 13. *With assumptions and notations of Lemma 12, and $1 > \alpha > 1/2$ we have*

$$\mathbb{E}\left[L_T^0(Y - Y^N)\right] \leq g(\varepsilon_N) = \sigma \left\|b^N - b\right\|_{H_q^{-\beta}}^{2\alpha - 1} \tag{42}$$

for $\left\|b^N - b\right\|_{H_q^{-\beta}}$ small enough (it is to say N big enough) where

$$\sigma = 4(\lambda + 1)T\kappa + \left(2 + 2(\lambda + 1) \, T + 6 \left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right) 2\omega_\infty^{2\alpha - 1} + 6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2\right) \omega_\infty^{-1}$$

and

$$\omega_\infty = \left(\frac{6T\Omega^2 4^\alpha \kappa^{2\alpha}}{(2\alpha - 1) \left(2 + 2(\lambda + 1) \, T + 6 \left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right)}\right)^{\frac{1}{2\alpha}}.$$

Proof of Lemma 13. Let $\varepsilon \in (0, 1]$, by Lemma 12,

$$0 \le \mathbb{E}\left[L_T^0(Y - Y^N)\right] \le g(\varepsilon)$$

where

$$g(\varepsilon) = 4(\lambda + 1)T\kappa \left\|b - b^N\right\|_{H_q^{-\beta}} + \left(2 + 2(\lambda + 1)\, T + 6 \left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right) \varepsilon^{2\alpha - 1}$$
$$+ 6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} \left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2 \left\|b - b^N\right\|_{H_q^{-\beta}}^2\right) \varepsilon^{-1}.$$

With

$$g'(\varepsilon) = (2\alpha - 1) \left(2 + 2(\lambda + 1)\, T + 6 \left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right) \varepsilon^{2\alpha - 2}$$
$$- 6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} \left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2 \left\|b - b^N\right\|_{H_q^{-\beta}}^2\right) \varepsilon^{-2},$$

and

$$g''(\varepsilon) = (2\alpha - 2)(2\alpha - 1) \left(2 + 2(\lambda + 1)\, T + 6 \left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right) \varepsilon^{2\alpha - 3}$$
$$+ 12T \left(\Omega^2 4^\alpha \kappa^{2\alpha} \left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2 \left\|b - b^N\right\|_{H_q^{-\beta}}^2\right) \varepsilon^{-3},$$

the minimum of g on $(0, 1]$ is reached when N is big enough in

$$\varepsilon_N = \left(\frac{6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} \left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2 \left\|b - b^N\right\|_{H_q^{-\beta}}^2\right)}{(2\alpha - 1) \left(2 + 2(\lambda + 1)\, T + 6 \left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right)}\right)^{\frac{1}{2\alpha}} = \omega_N \left\|b^N - b\right\|_{H_q^{-\beta}}.$$

where

$$g''(\varepsilon_N) = 12T \left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2 \left\|b - b^N\right\|_{H_q^{-\beta}}^{2(1-\alpha)}\right) \varepsilon_N^{-3} \alpha > 0.$$

and

$$\omega_N = \left(\frac{6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2 \left\|b - b^N\right\|_{H_q^{-\beta}}^{2(1-\alpha)}\right)}{(2\alpha - 1) \left(2 + 2(\lambda + 1)\, T + 6 \left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right)}\right)^{\frac{1}{2\alpha}}$$

$$\ge \omega_\infty = \left(\frac{6T\Omega^2 4^\alpha \kappa^{2\alpha}}{(2\alpha - 1) \left(2 + 2(\lambda + 1)\, T + 6 \left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right)}\right)^{\frac{1}{2\alpha}}.$$

Therefore $\mathbb{E}\left[L_T^0(Y - Y^N)\right] \le g(\varepsilon_N)$

$$\le 4(\lambda + 1)T\kappa \left\|b - b^N\right\|_{H_q^{-\beta}} + \left(2 + 2(\lambda + 1)\, T + 6 \left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right) \omega_N^{2\alpha - 1} \left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha - 1}$$
$$+ 6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} \left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2 \left\|b - b^N\right\|_{H_q^{-\beta}}^2\right) \omega_N^{-1} \left\|b - b^N\right\|_{H_q^{-\beta}}^{-1}$$

$$\le 4(\lambda + 1)T\kappa \left\|b - b^N\right\|_{H_q^{-\beta}} + \left(2 + 2(\lambda + 1)\, T + 6 \left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right) \omega_N^{2\alpha - 1} \left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha - 1}$$
$$+ 6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2 \left\|b - b^N\right\|_{H_q^{-\beta}}^{2(1-\alpha)}\right) \omega_\infty^{-1} \left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha - 1}$$

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$$\leq 4(\lambda+1)T\kappa \left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha-1} + \left(2 + 2(\lambda+1)\ T + 6\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right) 2\omega_\infty^{2\alpha-1} \left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha-1}$$

$$+ 6T\left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2\right)\omega_\infty^{-1}\left\|b - b^N\right\|_{H_q^{-\beta}}^{2\alpha-1}$$

for N big enough. The result follows. $\qquad\square$

7.2 Main result

Theorem 14. *Let f be μ-Hölder with constant $C_f > 0$ and $\mu \in (0,1]$. If $0 < \beta < 1/4$, $q \in \left(\frac{1}{1-\beta}, \frac{1}{\beta}\right)$, $\forall \varepsilon \in (0, 1-4\beta)$, with $(\delta,p) \in K(\beta,q)$ such that $\delta - 1/p = 1 - 2\beta - \varepsilon/2$, exists ξ independent of f such that for $N \in \mathbb{N}$, $\rho > 1$, λ big enough it holds:*

$$\sup_{0 \leq t \leq T} \mathbb{E}\left[\left|f\left(X_t\right) - f\left(X_t^N\right)\right|\right] \leq \xi C_f \left\|b^N - b\right\|_{H_q^{-\beta}}^{\mu(1-4\beta-\varepsilon)} \tag{43}$$

Proof of Theorem 14. We note as usual $\alpha = \delta - 1/p$. By Lemma 7, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\left\|b - b^N\right\|_{H_q^s(\mathbb{R})} \xrightarrow[N\to\infty]{} 0$. Therefore $u^N(t,\cdot)$ and $u(t,\cdot)$ are $\frac{1}{2}$-lipschitz. We recall that in this case, by Lemma 22 in [1], $\Psi(t,\cdot)$ and $\Psi^N(t,\cdot)$ are 2-lipschitz. Therefore $\widetilde{u}^N(t,\cdot)$ and $\widetilde{u}(t,\cdot)$ are 1-lipschitz. Let $t \in [0,T]$.

$$\mathbb{E}\left[\left|f\left(X_t\right) - f\left(X_t^N\right)\right|\right] = \mathbb{E}\left[\left|f\left(\Psi\left(t, Y_t\right)\right) - f\left(\Psi^N\left(t, Y_t^N\right)\right)\right|\right]$$

$$\leq \mathbb{E}\left[\left|f\left(\Psi\left(t, Y_t\right)\right) - f\left(\Psi\left(t, Y_t^N\right)\right)\right|\right] + \mathbb{E}\left[\left|f\left(\Psi\left(t, Y_t^N\right)\right) - f\left(\Psi^N\left(t, Y_t^N\right)\right)\right|\right]$$

$$\leq C_f\left(2^\mu \mathbb{E}\left[\left|Y_t - Y_t^N\right|^\mu\right] + \mathbb{E}\left[\left|\Psi\left(t, Y_t^N\right) - \Psi^N\left(t, Y_t^N\right)\right|^\mu\right]\right)$$

$$\leq C_f\left(2^\mu \mathbb{E}\left[\left|Y_t - Y_t^N\right|\right]^\mu + \mathbb{E}\left[\left|\Psi\left(t, Y_t^N\right) - \Psi^N\left(t, Y_t^N\right)\right|\right]^\mu\right) \tag{44}$$

by Jensen's inequality.

$$Y_T - Y_T^N = y - y^N + (\lambda+1)\int_0^T \left\{u\left(s, \Psi\left(s, Y_s\right)\right) - u^N\left(s, \Psi^N\left(s, Y_s^N\right)\right)\right\}\ \mathrm{d}s$$

$$+ \int_0^T \left\{\nabla u\left(s, \Psi\left(s, Y_s\right)\right) - \nabla u^N\left(t, \Psi^N\left(s, Y_s^N\right)\right)\right\}\ \mathrm{d}W_s.$$

We apply Meyer-Tanaka's formula to obtain:

$$\left|Y_t - Y_t^N\right| = \left|y - y^N\right| + (\lambda+1)\int_0^t \mathrm{sign}(Y_s - Y_s^N)\left\{\widetilde{u}\left(s, Y_s\right) - \overline{u}^N\left(s, Y_s^N\right)\right\}\ \mathrm{d}s$$

$$+ \int_0^t \mathrm{sign}(Y_s - Y_s^N)\left\{\widetilde{\nabla u}\left(s, Y_s\right) - \overline{\nabla u}^N\left(s, Y_s^N\right)\right\}\ \mathrm{d}W_s + L_t^0(Y - Y^N).$$

Taking the expectation leads to:

$$\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] = \left|u(0,x) - u^N(0,x)\right|$$

$$+ (\lambda+1)\mathbb{E}\left[\int_0^t \mathrm{sign}(Y_s - Y_s^N)\left\{\widetilde{u}\left(s, Y_s\right) - \overline{u}^N\left(s, Y_s^N\right)\right\}\ \mathrm{d}s\right] + \mathbb{E}\left[L_t^0(Y - Y^N)\right]$$

because ∇u and ∇u^N are bounded so the Itô integral is a martingale.

$$\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] \leq \left|u(0,x) - u^N(0,x)\right| + (\lambda + 1)\,\mathbb{E}\left[\int_0^t \left|\widetilde{u}\left(s,Y_s\right) - \widetilde{u}\left(s,Y_s^N\right)\right|\mathrm{d}s\right] +$$

$$+ (\lambda + 1)\,\mathbb{E}\left[\int_0^t \left|\widetilde{u}\left(s,Y_s^N\right) - \widetilde{u}^N\left(s,Y_s^N\right)\right|\mathrm{d}s\right] + (\lambda + 1)\,\mathbb{E}\left[\int_0^t \left|\widetilde{u}^N\left(s,Y_s^N\right) - \overline{u}^N\left(s,Y_s^N\right)\right|\mathrm{d}s\right]$$

$$+ \mathbb{E}\left[L_t^0(Y - Y^N)\right].$$

We use Lemma 8, the 1-lipschitz property of $\widetilde{u}$, and the $1/2$-lipschitz property of u:

$$\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] \leq \kappa\left\|b^N - b\right\|_{H_q^{-\beta}} + (\lambda + 1)\,\mathbb{E}\left[\int_0^t \left|Y_s - Y_s^N\right|\mathrm{d}s\right] + (\lambda + 1)t\kappa\left\|b^N - b\right\|_{H_q^{-\beta}}$$

$$+ \frac{\lambda + 1}{2}\,\mathbb{E}\left[\int_0^t \left|\Psi\left(s,Y_s^N\right) - \Psi^N\left(s,Y_s^N\right)\right|\mathrm{d}s\right] + \mathbb{E}\left[L_t^0(Y - Y^N)\right]$$

$$\leq (\lambda + 1)\int_0^t \mathbb{E}\left[\left|Y_s - Y_s^N\right|\right]\mathrm{d}s + (2(\lambda + 1)T + 1)\kappa\left\|b^N - b\right\|_{H_q^{-\beta}} + \mathbb{E}\left[L_T^0(Y - Y^N)\right].$$

where we have used the fact that $L_t^0(Y - Y^N)$ is an increasing process and Lemma 9.

By Gronwall's Lemma, it follows:

$$\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] \leq C(N)\,e^{(\lambda + 1)t} \leq C(N)\,e^{(\lambda + 1)T} \tag{45}$$

with $C(N) = (2(\lambda + 1)T + 1)\kappa\left\|b^N - b\right\|_{H_q^{-\beta}} + \mathbb{E}\left[L_T^0(Y - Y^N)\right].$

With Lemma 12 and Lemma 13 we obtain

$$C(N) \leq (2(\lambda + 1)T + 1)\kappa\left\|b^N - b\right\|_{H_q^{-\beta}} + \sigma\left\|b^N - b\right\|_{H_q^{-\beta}}^{2\alpha - 1} \leq \zeta\left\|b^N - b\right\|_{H_q^{-\beta}}^{2\alpha - 1}.$$

for $\left\|b^N - b\right\|_{H_q^{-\beta}}$ small enough where $\zeta = (2(\lambda + 1)T + 1)\kappa + \sigma$. It follows:

$$\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] \leq \zeta e^{(\lambda + 1)T}\left\|b^N - b\right\|_{H_q^{-\beta}}^{2\alpha - 1}. \tag{46}$$

Finally, combining (44), (46) and Lemma 9 we obtain:

$$\left|\mathbb{E}\left[f\left(X_t\right) - f\left(X_t^N\right)\right]\right|$$

$$\leq C_f\left(2^\mu\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right]^\mu + \mathbb{E}\left[\left|\Psi\left(t,Y_t^N\right) - \Psi^N\left(t,Y_t^N\right)\right|\right]^\mu\right)$$

$$\leq C_f\left(2^\mu\zeta^\mu e^{\mu(\lambda + 1)T}\left\|b^N - b\right\|_{H_q^{-\beta}}^{\mu(2\alpha - 1)} + 2^\mu\kappa^\mu\left\|b^N - b\right\|_{H_q^{-\beta}}^\mu\right)$$

$$\leq 2^\mu C_f\left(\zeta^\mu e^{\mu(\lambda + 1)T} + \kappa^\mu\right)\left\|b^N - b\right\|_{H_q^{-\beta}}^{\mu(2\alpha - 1)} \tag{47}$$

for N big enough, which is the expected result. $\qquad\square$

8   Convergence in law of X_t^N to X_t

To prove the convergence in law of the solution X_t^N of (3) to the virtual solution X_t of (1), we use the same arguments as in the Proposition 29 of [1].
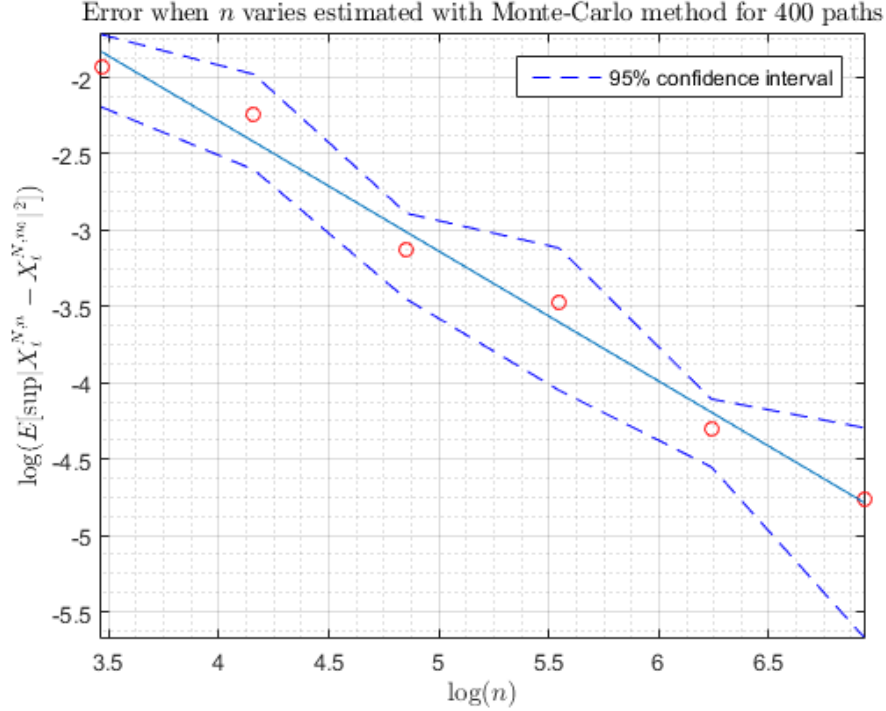
9   Strong convergence of the Euler scheme

Figure 4: Estimation of the L^2 error of the Euler-Marayuma scheme with a Monte-Carlo method. 400 paths, $N = 5$, $n \in \{2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}\}$, reference solution with $n_0 = 2^{12}$ points.

We observe a numerical strong convergence rate of 0.85 when Theorem 5 shows a theoretical rate of $0.5 - \varepsilon$.

10   Weak convergence of X_T^N to X_T

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