NUMERICAL SIMULATION OF SDES WITH DISTRIBUTIONAL DRIFT

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1 Introduction

We would like to simulate numerically sample paths of the solution of the stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\ \mathrm{d}t + \mathrm{d}W_t \tag{1}$$

where $b \in H_q^s(\mathbb{R}),\ s \in \left]-\frac{1}{2}, 0\right[,\ t \in [0,T]$, and W_t is a standard Brownian motion. This equation is studied by F. Flandoli, E. Issoglio, and F. Russo in [1] in which they define a virtual solution for equation (1). The authors prove then existence and unicity in law of this solution.

Example 1. *An example of such drift b is given by the derivative of a sample path of a fractional Brownian motion B_x^H with Hurst index $1/2 < H < 1$. These stochastic processes are gaussian processes verifying*

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} + |t-s|^{2H}\right).$$

We note $s = H - 1$. Given $B_x^H(\omega) \in H_q^{s+1}(\mathbb{R})$, we can take $b(x) = \frac{\partial}{\partial x} B_x^H(\omega) \in H_q^s(\mathbb{R})$. We will use this in our numerical simulations.

As far as the drift b is not a function but a distribution, it must be approximated if we want to evaluate it at points. In order to do so, we will use a series representation of b and truncate it. That is why we will consider two steps in our algorithm:

1. approximate the drift b by a function b^N meant to converge to b as $N \to \infty$.
2. approximate the solution X_t^N of the approximated SDE

   $$\mathrm{d}X_t^N = b^N\left(X_t^N\right)\ \mathrm{d}t + \mathrm{d}W_t \tag{2}$$

   by $X_t^{N,n}$ defined with the Euler-Maruyama scheme

   $$X_t^{N,n} = X_0 + \int_0^t b^N\left(X_{\eta_n(t)}^{N,n}\right)\mathrm{d}t + W_{\eta_n(t)}$$

   where $\eta_n(t) = t_k$ if $t \in [t_k, t_k+1]$, for $t_k = \frac{k}{n}$ with $k \in [\![0, \lceil 2^n T\rceil]\!]$.

2 Numerical simulation of fractional Brownian motion

To simulate a sample path of a fractional brownian motion B_x^H on a finite grid $(x_k)_{k\in[\![1,n]\!]}$, we simulate n independent standard gaussian random variables $(X_k)_{k\in[\![1,n]\!]}$ and then correlate them with the definite positive correlation matrix

$$C_{k,s} = \mathbb{E}\left[B_{x_k}^H B_{x_s}^H\right] = \frac{1}{2}\left(x_k^{2H} + x_s^{2H} + |x_k - x_s|^{2H}\right).$$

To do so, we use the Cholesky decomposition method and calculate the triangular matrix M such that $C = MM^\top$. Therefore, defining

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ and } B^H = MX,$$

B^H contains the values of a fractional brownian motion evaluated on the grid $(x_k)_{k\in[\![1,n]\!]}$.

3 Approximation of the drift

3.1 Series representation

We use Haar wavelets to give a series representation of b. By doing so, we will be able to approximate it numerically by truncating the series.

Definition 1 (Haar wavelets). *We define the Haar wavelets $h_{j,m}$ on $\mathbb{R}$ with $j \in \mathbb{N} \cup \{-1\}$ and $m \in \mathbb{Z}$ by:*

$$\begin{cases} h_M & : x \longmapsto \left(\mathbb{1}_{[0,\frac{1}{2}[} - \mathbb{1}_{[\frac{1}{2},1[}\right)(x) \\ h_{-1,m} & : x \longmapsto \sqrt{2}|h_M(x-m)| \\ h_{j,m} & : x \longmapsto h_M(2^j x - m) \end{cases}$$

Theorem 1 (See [2]). *Let $b \in H_q^s(\mathbb{R})$ for $2 \le q \le \infty$, and $s \in \left]-\frac{1}{2}, \frac{1}{q}\right[$. Therefore,*

$$b = \sum_{j=-1}^{+\infty} \sum_{m\in\mathbb{Z}} \mu_{j,m} h_{j,m} \tag{3}$$

where $\mu_{j,m} = 2^j \int_{\mathbb{R}} b(x) h_{j,m}(x)\, \mathrm{d}x$ in the sense of dual pairing.

Definition 2. *With the same notation $\mu_{j,m}$, let $b \in H_q^s(\mathbb{R})$ for $2 \le q \le \infty$, and $s \in \left]-\frac{1}{2}, \frac{1}{q}\right[$. Given $N \in \mathbb{N}^*$ we define b^N by:*

$$b^N = \sum_{m=N}^{N-1} \mu_{-1,m} h_{-1,m} + \sum_{j=0}^{N} \sum_{m=-N2^j}^{N2^j-1} \mu_{j,m} h_{j,m}. \tag{4}$$

Remark 1. *We can note that* Supp $b^N \subset [-N, N]$. *Moreover, we have:*

$$\left\|b - b^N\right\|_{H_q^s(\mathbb{R})} \underset{N \to +\infty}{\longrightarrow} 0.$$

4 Convergence

4.1 Weak convergence of X^N to X

We want to estimate the weak error $\mathbb{E}\left[f\left(X_T\right) - f\left(X_T^N\right)\right]$ with suitable functions f. In order to do so, we must go back to the definition of the virtual solution of the SDE (1) given in [1]. The authors define the virtual solution of SDE (1) by X_t such that:

$$\begin{cases} X_t = \Psi(t, Y_t) = \varphi^{-1}(t, Y_t) \\ Y_t = y + (\lambda + 1)\int_0^t u(s, Y_s)\ \mathrm{d}s + \int_0^t (\nabla u(s, Y_s) + 1)\ \mathrm{d}W_s \end{cases} \tag{5}$$

where u is the mild solution in $H_p^{1+\delta}$ of the following parabolic PDE:

$$\begin{cases} \partial_t u + \frac{1}{2}\Delta u + b\nabla u - (\lambda + 1)u = -b & \text{on } [0, T] \times \mathbb{R} \\ u(T) = 0 & \text{on } \mathbb{R} \end{cases} \tag{6}$$

and $\varphi(t, x) = x + u(t, x)$.

We also define another similar PDE by replacing b by b^N. We call u^N its mild solution in $H_p^{1+\delta}$.

$$\begin{cases} \partial_t u^N + \frac{1}{2}\Delta u^N + b^N\nabla u^N - (\lambda + 1)u^N = -b^N & \text{on } [0, T] \times \mathbb{R} \\ u^N(T) = 0 & \text{on } \mathbb{R} \end{cases}. \tag{7}$$

We will need to use the following local time inequality from Liqing Yan:

Lemma 2 (Lemma 4.2 in [4])**.** *Let X be a continuous semimartingale with $X_0 = 0$. For $\varepsilon > 0$ we define a double sequence of stopping times by $\sigma_1 = 0$, $\tau_1 = \inf\{t > 0 | X_t = \varepsilon\}$, $\sigma_n = \inf\{t > \tau_{n-1} | X_t = 0\}$, $\tau_n = \inf\{t > \sigma_n | X_t = \varepsilon\}$. For any real function $F(\cdot) \in \mathcal{C}^2$ with $F(0) = 0$, $F'(0) = 0$ and $F(\cdot) > 0$ on $(0, \varepsilon_0)$ for some $\varepsilon_0 > 0$, then for any $0 < \varepsilon < \varepsilon_0$ we have*

$$0 \leq L_t^0(X) \leq 2\varepsilon - \frac{2}{F(\varepsilon)}\int_0^t \theta_s(X)\left(F\left(\varepsilon\right) - \varepsilon F'\left(X_s^+\right)\right)\ \mathrm{d}X_s + \frac{1}{F(\varepsilon)}\int_0^t \theta_s(X)\varepsilon F''(X_s^+)\ \mathrm{d}[X]_s$$

with $\theta_s(X) = \sum_{n=1}^\infty \mathbb{1}_{\sigma_n < s \leq \tau_n,\ 0 < X_s \leq \varepsilon}(X)$.

Applying lemma 2 with $F(x) = x^2$, it follows:

Corollary 3. *Let X be a continuous martingale with $X_0 = 0$. With the same notations as in lemma 2, for any $\varepsilon > 0$ we have*

$$0 \;\leq\; L_t^0(X) \;\leq\; 2\varepsilon - \frac{2}{\varepsilon}\int_0^t \theta_s(X)\left(\varepsilon - 2X_s^+\right)\ \mathrm{d}X_s + \frac{2}{\varepsilon}\int_0^t \theta_s(X)\ \mathrm{d}[X]_s \quad (8)$$

We also recall a useful lemma concerning the solutions of (6) and (7).

Lemma 4 (Lemma 20 in [1])**.** *Let $(\delta, p) \in K(\beta, q)$ and let v_λ be the mild solution to (6) in $H_p^{1+\delta}$. Fix ρ such that the integral operator is a contraction and let $\lambda > \rho$. Then $v_\lambda(t) \in \mathcal{C}^{1,\alpha}$ with $\alpha = \delta - 1/p$ for each fixed t and*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |\nabla v_\lambda(t,x)| \to 0, \text{ as } \lambda \to \infty$$

where the choice of λ depends only on $\delta, \beta, \|b\|_{H_p^{-\beta}}$, and $\|b\|_{H_q^{-\beta}}$.

Lemma 5. *Exists $c > 0$ such that for both $N \in \mathbb{N}$ and $\rho > 1$ big enough,*

$$\begin{cases} \left\|u^N(t) - u(t)\right\|_{L^\infty} \leq cKe^{\rho T}\left\|b^N - b\right\|_{H_q^{-\beta}} \\ \left\|\nabla u^N(t) - \nabla u(t)\right\|_{L^\infty} \leq cKe^{\rho T}\left\|b^N - b\right\|_{H_q^{-\beta}}. \end{cases} \quad . \quad (9)$$

Proof. Applying fractional Morrey inequality, $\exists c > 0,\ \forall t \in [0,T]$:

$$\begin{cases} \left\|u^N(t) - u(t)\right\|_{L^\infty} \leq \left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1,\alpha}} \leq c\left\|u^N(t) - u(t)\right\|_{H_p^{1+\delta}} \\ \left\|\nabla u^N(t) - \nabla u(t)\right\|_{L^\infty} \leq \left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1,\alpha}} \leq c\left\|u^N(t) - u(t)\right\|_{H_p^{1+\delta}}. \end{cases}$$

Now, with can conclude with

$$\left\|u^N(t) - u(t)\right\|_{\infty, H_p^{1+\delta}} \leq e^{\rho T}\left\|u^N(t) - u(t)\right\|_{\infty, H_p^{1+\delta}}^{(\rho)} \leq Ke^{\rho T}\left\|b^N - b\right\|_{H_q^{-\beta}}$$

from Lemma 23 in [1], for both $N \in \mathbb{N}$ and $\rho > 1$ big enough, and where $\|f(t)\|_{\infty,X}^{(\rho)} := \sup_{0\leq t\leq T} e^{-\rho t}\|f(t)\|_X$. $\square$

Theorem 6. *Let f be μ-Hölder with constant $C_f > 0$ and $\mu \in (0,1]$. Then for $N \in \mathbb{N}$, $\rho > 1$, λ big enough, exists ξ_f independent of N such that:*

$$\mathbb{E}\left[f\left(X_T\right) - f\left(X_T^N\right)\right] \leq \xi_f\left\|b^N - b\right\|_{H_q^{-\beta}}^\gamma$$

Proof. By Lemma 4, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\left\|b - b^N\right\|_{H_q^s(\mathbb{R})} \underset{N\to\infty}{\longrightarrow} 0$ (See Step 2 of the proof of Proposition 29 in [1]). Therefore u^N and u are $\frac{1}{2}$-lipschitz. We recall that in this case, by Lemma 22 in [1], $\Psi(t,\cdot)$ is 2-lipschitz.

$$\mathbb{E}\left[f\left(X_T\right) - f\left(X_T^N\right)\right] = \mathbb{E}\left[f\left(\Psi\left(T, Y_T\right)\right) - f\left(\Psi\left(T, Y_T^N\right)\right)\right]$$

$$\leq 2^{\mu} C_f \mathbb{E}\left[\left|Y_T - Y_T^N\right|^{\mu}\right] \leq 2^{\mu} C_f \mathbb{E}\left[\left|Y_T - Y_T^N\right|\right]^{\mu}$$

by Jensen's inequality. Let $t \in [0,T]$.

$$\begin{aligned} Y_t - Y_t^N = (\lambda+1) \int_0^t & \left\{ u\left(s, \Psi\left(s, Y_s\right)\right) - u^N\left(s, \Psi\left(s, Y_s^N\right)\right)\right\} \, \mathrm{d}s \\ &+ \int_0^t \left\{ \nabla u\left(s, \Psi\left(s, Y_s\right)\right) - \nabla u^N\left(t, \Psi\left(s, Y_s^N\right)\right)\right\} \, \mathrm{d}W_s. \end{aligned}$$

Remark 2. *For clarity purpose, we note $\tilde{u}(s,x) = u(s,\Psi(s,x))$ and use the same notation for the gradient and the approximated mild solution. We can notice that $\tilde{u}$ is 1-lipschitz in space and $\nabla\tilde{u}$ is α-Hölder with constant $2\left\|u\right\|_{\mathcal{C}^{1,\alpha}}$.*

We apply Meyer-Tanaka's formula to obtain:

$$\begin{aligned} \left|Y_t - Y_t^N\right| &= (\lambda+1) \int_0^t \operatorname{sign}(Y_s - Y_s^N) \left\{\tilde{u}\left(s, Y_s\right) - \tilde{u}^N\left(s, Y_s^N\right)\right\} \, \mathrm{d}s \\ &+ \int_0^t \operatorname{sign}(Y_s - Y_s^N) \left\{\nabla\tilde{u}\left(s, Y_s\right) - \nabla\tilde{u}^N\left(s, Y_s^N\right)\right\} \, \mathrm{d}W_s + L_t^0(Y - Y^N). \end{aligned}$$

Taking the expectation leads to:

$$\begin{aligned} \mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] = (\lambda+1)\, \mathbb{E}\left[\int_0^t \operatorname{sign}(Y_s - Y_s^N) \left\{\tilde{u}\left(s, Y_s\right) - \tilde{u}^N\left(s, Y_s^N\right)\right\} \, \mathrm{d}s\right] \\ + \mathbb{E}\left[L_t^0(Y - Y^N)\right] \end{aligned}$$

because $\nabla\tilde{u}$ and $\nabla\tilde{u}^N$ are bounded so the Itô integral is a martingale.

$$\begin{aligned} \mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] &\leq (\lambda+1)\, \mathbb{E}\left[\int_0^t \left\{\tilde{u}\left(s, Y_s\right) - \tilde{u}\left(s, Y_s^N\right)\right\} \mathrm{d}s\right] \\ &+ (\lambda+1)\, \mathbb{E}\left[\int_0^t \left\{\tilde{u}\left(s, Y_s^N\right) - \tilde{u}^N\left(s, Y_s^N\right)\right\} \mathrm{d}s\right] + \mathbb{E}\left[L_t^0(Y - Y^N)\right]. \end{aligned}$$

We use Lemma 5 and the 1-lipschitz property of $\tilde{u}$:

$$\begin{aligned} \mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] \leq (\lambda+1)\, \mathbb{E}\left[\int_0^t \left|Y_s - Y_s^N\right| \mathrm{d}s\right] + (\lambda+1)\; ctKe^{\rho T} \left\|b^N - b\right\|_{H_q^{-\beta}} \\ + \mathbb{E}\left[L_t^0(Y - Y^N)\right] \end{aligned}$$

$$\begin{aligned} \leq (\lambda+1) \int_0^t \mathbb{E}\left[\left|Y_s - Y_s^N\right|\right] \mathrm{d}s + (\lambda+1)\; cTKe^{\rho T} \left\|b^N - b\right\|_{H_q^{-\beta}} \\ + \mathbb{E}\left[L_T^0(Y - Y^N)\right]. \end{aligned}$$

where we have used the fact that $L_t^0(Y - Y^N)$ is an increasing process.

By Gronwall's Lemma, it follows:

$$\mathbb{E}\left[\left|Y_T - Y_T^N\right|\right] \leq C(N)\exp((\lambda+1)T) \tag{10}$$

with $C(N) = (\lambda+1)\ cTKe^{\rho T}\left\|b^N - b\right\|_{H_q^{-\beta}} + \mathbb{E}\left[L_T^0(Y - Y^N)\right]$.

We now have to study the term $\mathbb{E}\left[L_T^0(Y-Y^N)\right]$. Let $\varepsilon > 0$. Corollary 3 gives us:

$$\begin{aligned}0 \leq L_T^0(Y-Y^N) \leq 2\varepsilon - \frac{2}{\varepsilon}\int_0^T \theta_s(Y-Y^N)\left(\varepsilon - 2(Y_s - Y_s^N)^+\right)\ \mathrm{d}(Y_s - Y_s^N)\\ + \frac{2}{\varepsilon}\int_0^T \theta_s(Y-Y^N)\ \mathrm{d}[(Y-Y^N)]_s.\end{aligned}$$

Remark 3. *Note that* $\theta_s(Y-Y^N)\left|\varepsilon - 2(Y_s - Y_s^N)^+\right| \leq \varepsilon\theta_s(Y-Y^N)$.

We take the expectation to remove again the martingale part:

$$\begin{aligned}\mathbb{E}\left[L_T^0(Y-Y^N)\right] \leq 2\varepsilon + 2(\lambda+1)\,\mathbb{E}\left[\int_0^T \theta_s(Y-Y^N)\left\{\tilde{u}\left(s,Y_s\right) - \tilde{u}^N\left(s,Y_s^N\right)\right\}\mathrm{d}s\right]\\ + \frac{2}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s(Y-Y^N)\left\{\nabla\tilde{u}\left(s,Y_s\right) - \nabla\tilde{u}^N\left(s,Y_s^N\right)\right\}^2\ \mathrm{d}s\right]\end{aligned}$$

$$\begin{aligned}\leq 2\varepsilon + 2(\lambda+1)\ \mathbb{E}\left[\int_0^T \theta_s(Y-Y^N)\left\{\tilde{u}\left(s,Y_s\right) - \tilde{u}\left(s,Y_s^N\right)\right\}\ \mathrm{d}s\right]\\ + 2(\lambda+1)\ \mathbb{E}\left[\int_0^T \theta_s(Y-Y^N)\left\{\tilde{u}\left(s,Y_s^N\right) - \tilde{u}^N\left(s,Y_s^N\right)\right\}\ \mathrm{d}s\right]\\ + \frac{4}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s(Y-Y^N)\left\{\nabla\tilde{u}\left(s,Y_s\right) - \nabla\tilde{u}\left(s,Y_s^N\right)\right\}^2\ \mathrm{d}s\right]\\ + \frac{4}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s(Y-Y^N)\left\{\nabla\tilde{u}\left(s,Y_s^N\right) - \nabla\tilde{u}^N\left(s,Y_s^N\right)\right\}^2\ \mathrm{d}s\right]\end{aligned}$$

$$\begin{aligned}\leq 2\varepsilon + 2(\lambda+1)\ \mathbb{E}\left[\int_0^T \theta_s(Y-Y^N)\left|Y_s - Y_s^N\right|\ \mathrm{d}s\right]\\ + \frac{16\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s(Y-Y^N)\left|Y_s - Y_s^N\right|^{2\alpha}\ \mathrm{d}s\right]\\ + 2(\lambda+1)\ cTKe^{\rho T}\left\|b^N - b\right\|_{H_q^{-\beta}} + 4c^2TK^2e^{2\rho T}\left\|b^N - b\right\|_{H_q^{-\beta}}^2\varepsilon^{-1}\end{aligned}$$

by Lemma 5, the 1-lipschitz property of $\tilde{u}$ and the α-Hölder property of $\nabla\tilde{u}$ (with constant $2\|u\|_{\mathcal{C}^{1,\alpha}}$).

$$
\begin{aligned}
\mathbb{E}\left[L_T^0(Y-Y^N)\right] \leq 2(\lambda+1)\ cTKe^{\rho T}\left\|b^N-b\right\|_{H_q^{-\beta}} + 16\|u\|^2_{\mathcal{C}^{1,\alpha}} T\varepsilon^{2\alpha-1} \\
+ 2(1+(\lambda+1)T)\varepsilon + 4c^2TK^2e^{2\rho T}\left\|b^N-b\right\|^2_{H_q^{-\beta}}\varepsilon^{-1}
\end{aligned}
$$

We choose an optimal ε.

□

References

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