

NUMERICAL SIMULATION OF SDES WITH DISTRIBUTIONAL DRIFT

Maximilien GERMAIN

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1 Introduction

We would like to simulate numerically sample paths of the solution of the stochastic differential equation

$$dX_t = b(X_t) dt + dW_t \quad (1)$$

where $b \in H_q^s(\mathbb{R})$, $s \in]-\frac{1}{2}, 0[$, $t \in [0, T]$, and W_t is a standard Brownian motion. This equation is studied in [1] in which the authors prove existence and unicity in law of a virtual solution for equation (1).

Example 1.1. *An example of such drift b is given by the derivative of a sample path of a fractional Brownian motion B_x^H with Hurst index $1/2 < H < 1$. These stochastic processes are gaussian processes verifying*

$$\mathbb{E} [B_t^H B_s^H] = \frac{1}{2} (t^{2H} + s^{2H} + |t - s|^{2H}).$$

We note $s = H - 1$. Given $B_x^H(\omega) \in H_q^{s+1}(\mathbb{R})$, we can take $b(x) = \frac{\partial}{\partial x} B_x^H(\omega) \in H_q^s(\mathbb{R})$. We will use this in our numerical simulations.

As far as the drift b is not a function but a distribution, it must be approximated if we want to evaluate it at points. In order to do so, we will use a series representation of b and truncate it. That is why we will consider two steps in our algorithm:

1. approximate the drift b by a function b^N meant to converge to b as $N \rightarrow \infty$.
2. approximate the solution X_t^N of the approximated SDE

$$dX_t^N = b^N(X_t^N) dt + dW_t \quad (2)$$

by $X_t^{N,n}$ defined with the Euler-Maruyama scheme

$$X_t^{N,n} = X_0 + \int_0^t b^N(X_{\eta_n(t)}^{N,n}) dt + W_{\eta_n(t)}$$

where $\eta_n(t) = t_k$ if $t \in [t_k, t_k + 1]$, for $t_k = \frac{k}{2^n}$ with $k \in \llbracket 0, \lceil 2^n T \rceil \rrbracket$.

2 Numerical simulation of fractional Brownian motion

To simulate a sample path of a fractional brownian motion B_x^H on a finite grid $(x_k)_{k \in \llbracket 1, n \rrbracket}$, we simulate n independent standard gaussian random variables $(X_k)_{k \in \llbracket 1, n \rrbracket}$ and then correlate them with the definite positive correlation matrix

$$C_{k,s} = \mathbb{E} [B_{x_k}^H B_{x_s}^H] = \frac{1}{2} (x_k^{2H} + x_s^{2H} + |x_k - x_s|^{2H}).$$

To do so, we use the Cholesky decomposition method and calculate the triangular matrix M such that $C = MM^\top$. Therefore, defining

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ and } B^H = MX,$$

B^H contains the values of a fractional brownian motion evaluated on the grid $(x_k)_{k \in \llbracket 1, n \rrbracket}$.

3 Approximation of the drift

3.1 Series representation

We use Haar wavelets to give a series representation of b . By doing so, we will be able to approximate it numerically by truncating the series.

Definition 3.1 (Haar wavelets). *We define the Haar wavelets $h_{j,m}$ on \mathbb{R} with $j \in \mathbb{N} \cup \{-1\}$ and $m \in \mathbb{Z}$ by:*

$$\begin{cases} h_M & : x \mapsto \left(\mathbf{1}_{[0, \frac{1}{2}[} - \mathbf{1}_{[\frac{1}{2}, 1[} \right) (x) \\ h_{-1,m} & : x \mapsto \sqrt{2} |h_M(x - m)| \\ h_{j,m} & : x \mapsto h_M(2^j x - m) \end{cases}$$

Theorem 3.1 (See [2]). *Let $b \in H_q^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left] -\frac{1}{2}, \frac{1}{q} \right[$. Therefore,*

$$b = \sum_{j=-1}^{+\infty} \sum_{m \in \mathbb{Z}} \mu_{j,m} h_{j,m} \quad (3)$$

where $\mu_{j,m} = 2^j \int_{\mathbb{R}} b(x) h_{j,m}(x) dx$ in the sense of dual pairing.

Definition 3.2. *With the same notation $\mu_{j,m}$, let $b \in H_q^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left] -\frac{1}{2}, \frac{1}{q} \right[$. Given $N \in \mathbb{N}^*$ we define b^N by:*

$$b^N = \sum_{m=N}^{N-1} \mu_{-1,m} h_{-1,m} + \sum_{j=0}^N \sum_{m=-N2^j}^{N2^j-1} \mu_{j,m} h_{j,m}. \quad (4)$$

Remark 3.1. *We can note that $\text{Supp } b^N \subset [-N, N]$. Moreover, we have:*

$$\|b - b^N\|_{H_q^s(\mathbb{R})} \xrightarrow{N \rightarrow +\infty} 0.$$

3.2 Computation of the coefficients $\mu_{j,m}$ when b is the derivative of a fractional brownian motion

Faber basis

4 Numerical results

5 Convergence

5.1 Convergence of $X_s^{N,n}$ to X_s^N in L^2

Recently, Leobacher and Szölgényi proved in [3] the convergence of the Euler-Maruyama scheme for SDE with discontinuous but piecewise Lipschitz drift and with a degenerate diffusion coefficient. This framework applies to the scheme we use with piecewise constant drift b^N and a constant diffusion coefficient.

Theorem 5.1 (Theorem 3.1. in [3]). $\forall \varepsilon > 0, \exists C_N > 0, \exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t^{N,n} - X_t^N|^2 \right]^{1/2} \leq C_N \delta^{1/4-\varepsilon} \quad (5)$$

with $\delta = \frac{1}{2^n}$ the step size.

TO DO: make C_N explicit.

5.2 Convergence of X_s^N to X_s

We want to estimate $w(y) := \mathbb{E} \left[\sup_{0 \leq t \leq y} |X_t^N - X_t|^2 \right]$ for $y \in [0, T]$. In order to do so, we must go back to the definition of the virtual solution of the SDE (1) given in [1]. The authors define the virtual solution of SDE (1) by:

$$X_t = X_0 + u(0, X_0) - u(t, X_t) + (\lambda + 1) \int_0^t u(s, X_s) ds + \int_0^t \nabla u(s, X_s) dW_s + W_t \quad (6)$$

where u is the mild solution in $H_p^{1+\delta}$ of the following parabolic PDE:

$$\begin{cases} \partial_t u + \frac{1}{2} \Delta u + b \nabla u - (\lambda + 1)u = -b & \text{on } [0, T] \times \mathbb{R} \\ u(T) = 0 & \text{on } \mathbb{R} \end{cases} \quad (7)$$

We also define another similar PDE by replacing b by b^N , and noting u^N its mild solution in $H_p^{1+\delta}$.

$$\begin{cases} \partial_t u^N + \frac{1}{2} \Delta u^N + b^N \nabla u^N - (\lambda + 1)u^N = -b^N & \text{on } [0, T] \times \mathbb{R} \\ u^N(T) = 0 & \text{on } \mathbb{R} \end{cases} \quad (8)$$

We recall a useful lemma concerning the solutions of (7) and (8).

Lemma 5.2 (Lemma 19 in [1]). *Let $(\delta, p) \in K(\beta, q)$ and let v_λ be the mild solution to (7) in $H_p^{1+\delta}$. Fix ρ such that the integral operator is a contraction and let $\lambda > \rho$. Then $v_\lambda(t) \in \mathcal{C}^{1,\alpha}$ with $\alpha = \delta - 1/p$ for each fixed t and*

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla v_\lambda(t, x)| \rightarrow 0, \text{ as } \lambda \rightarrow \infty \quad (9)$$

where the choice of λ depends only on $\delta, \beta, \|b\|_{H_p^{-\beta}}$, and $\|b\|_{H_q^{-\beta}}$.

By lemma 5.2, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{\sqrt{14}}$. λ can be chosen independently of N as far as $\|b - b^N\|_{H_q^s(\mathbb{R})} \xrightarrow{N \rightarrow \infty} 0$ (See Step 2 of the proof of Proposition 26 in [1]). Therefore u^N and u are $\frac{1}{\sqrt{14}}$ -lipschitz.

$$\begin{aligned}
 |X_t^N - X_t|^2 &= \left| u^N(0, X_0) - u(0, X_0) + u(t, X_t) - u^N(t, X_t) + u^N(t, X_t) \right. \\
 &\quad \left. - u^N(t, X_t^N) + (\lambda + 1) \int_0^t \{u^N(s, X_s^N) - u^N(s, X_s) + u^N(s, X_s) - u(s, X_s)\} ds \right. \\
 &\quad \left. + \int_0^t \{\nabla u^N(s, X_s^N) - \nabla u^N(s, X_s) + \nabla u^N(s, X_s) - \nabla u(s, X_s)\} dW_s \right|^2 \\
 &\leq 7 \left(\left| u^N(0, X_0) - u(0, X_0) \right|^2 + \left| u(t, X_t) - u^N(t, X_t) \right|^2 + \left| u^N(t, X_t) - u^N(t, X_t^N) \right|^2 \right. \\
 &\quad \left. + \left| (\lambda + 1) \int_0^t \{u^N(s, X_s^N) - u^N(s, X_s)\} ds \right|^2 + \left| (\lambda + 1) \int_0^t \{u^N(s, X_s) - u(s, X_s)\} ds \right|^2 \right. \\
 &\quad \left. + \left| \int_0^t \{\nabla u^N(s, X_s^N) - \nabla u^N(s, X_s)\} dW_s \right|^2 + \left| \int_0^t \{\nabla u^N(s, X_s) - \nabla u(s, X_s)\} dW_s \right|^2 \right) \\
 &\leq 7 \left(\left| u^N(0, X_0) - u(0, X_0) \right|^2 + \left| u(t, X_t) - u^N(t, X_t) \right|^2 + \frac{1}{14} |X_t - X_t^N|^2 \right. \\
 &\quad \left. + (\lambda + 1) t \int_0^t |u^N(s, X_s^N) - u^N(s, X_s)|^2 ds + (\lambda + 1) t \int_0^t |u^N(s, X_s) - u(s, X_s)|^2 ds \right. \\
 &\quad \left. + \left| \int_0^t \{\nabla u^N(s, X_s^N) - \nabla u^N(s, X_s)\} dW_s \right|^2 + \left| \int_0^t \{\nabla u^N(s, X_s) - \nabla u(s, X_s)\} dW_s \right|^2 \right) \\
 &\leq 7 \left(\left| u^N(0, X_0) - u(0, X_0) \right|^2 + \left| u(t, X_t) - u^N(t, X_t) \right|^2 + \frac{1}{14} |X_t - X_t^N|^2 \right. \\
 &\quad \left. + \frac{\lambda + 1}{14} t \int_0^t |X_s^N - X_s|^2 ds + (\lambda + 1) t \int_0^t |u^N(s, X_s) - u(s, X_s)|^2 ds \right. \\
 &\quad \left. + \left| \int_0^t \{\nabla u^N(s, X_s^N) - \nabla u^N(s, X_s)\} dW_s \right|^2 + \left| \int_0^t \{\nabla u^N(s, X_s) - \nabla u(s, X_s)\} dW_s \right|^2 \right).
 \end{aligned}$$

It follows that:

$$\begin{aligned}
 \sup_{0 \leq t \leq y} |X_t^N - X_t|^2 &\leq 14 \sup_{0 \leq t \leq y} \left(\left| u^N(0, X_0) - u(0, X_0) \right|^2 + \left| u(t, X_t) - u^N(t, X_t) \right|^2 \right. \\
 &\quad + \frac{\lambda + 1}{14} t \int_0^t |X_s^N - X_s|^2 ds + (\lambda + 1) t \int_0^t |u^N(s, X_s) - u(s, X_s)|^2 ds \\
 &\quad \left. + \left| \int_0^t \{ \nabla u^N(s, X_s^N) - \nabla u^N(s, X_s) \} dW_s \right|^2 + \left| \int_0^t \{ \nabla u^N(s, X_s) - \nabla u(s, X_s) \} dW_s \right|^2 \right) \\
 &\leq 14 \left(\sup_{0 \leq t \leq y} \left| u^N(0, X_0) - u(0, X_0) \right|^2 + \sup_{0 \leq t \leq y} \left| u(t, X_t) - u^N(t, X_t) \right|^2 \right. \\
 &\quad + \frac{\lambda + 1}{14} y \int_0^y |X_s^N - X_s|^2 ds + (\lambda + 1) y \int_0^y |u^N(s, X_s) - u(s, X_s)|^2 ds \\
 &\quad + \sup_{0 \leq t \leq y} \left| \int_0^t \{ \nabla u^N(s, X_s^N) - \nabla u^N(s, X_s) \} dW_s \right|^2 + \\
 &\quad \left. \sup_{0 \leq t \leq y} \left| \int_0^t \{ \nabla u^N(s, X_s) - \nabla u(s, X_s) \} dW_s \right|^2 \right) \\
 &\leq 14 \left(\sup_{0 \leq t \leq T} \left| u^N(0, X_0) - u(0, X_0) \right|^2 + \sup_{0 \leq t \leq T} \left| u(t, X_t) - u^N(t, X_t) \right|^2 \right. \\
 &\quad + \frac{\lambda + 1}{14} T \int_0^y |X_s^N - X_s|^2 ds + (\lambda + 1) T \int_0^T |u^N(s, X_s) - u(s, X_s)|^2 ds \\
 &\quad + \sup_{0 \leq t \leq y} \left| \int_0^t \{ \nabla u^N(s, X_s^N) - \nabla u^N(s, X_s) \} dW_s \right|^2 + \\
 &\quad \left. \sup_{0 \leq t \leq T} \left| \int_0^t \{ \nabla u^N(s, X_s) - \nabla u(s, X_s) \} dW_s \right|^2 \right) \\
 &\leq 14 \left(\sup_{0 \leq t \leq T} \left| u^N(0, X_0) - u(0, X_0) \right|^2 + \sup_{0 \leq t \leq T} \left| u(t, X_t) - u^N(t, X_t) \right|^2 \right. \\
 &\quad + \frac{\lambda + 1}{14} T \int_0^y \sup_{0 \leq t \leq s} |X_s^N - X_s|^2 ds + (\lambda + 1) T \int_0^T |u^N(s, X_s) - u(s, X_s)|^2 ds \\
 &\quad + \sup_{0 \leq t \leq y} \left| \int_0^t \{ \nabla u^N(s, X_s^N) - \nabla u^N(s, X_s) \} dW_s \right|^2 + \\
 &\quad \left. \sup_{0 \leq t \leq T} \left| \int_0^t \{ \nabla u^N(s, X_s) - \nabla u(s, X_s) \} dW_s \right|^2 \right).
 \end{aligned}$$

Taking the expectation, we obtain:

$$\begin{aligned}
 w(y) &\leq 14 \left(\sup_{0 \leq t \leq T} |u^N(0, X_0) - u(0, X_0)|^2 + \mathbb{E} \left[\sup_{0 \leq t \leq T} |u(t, X_t) - u^N(t, X_t)|^2 \right] \right. \\
 &\quad + \frac{\lambda + 1}{14} T \int_0^y w(s) \, ds + (\lambda + 1) T \mathbb{E} \left[\int_0^T |u^N(s, X_s) - u(s, X_s)|^2 \, ds \right] \\
 &\quad + \mathbb{E} \left[\sup_{0 \leq t \leq y} \left| \int_0^t \{ \nabla u^N(s, X_s^N) - \nabla u^N(s, X_s) \} \, dW_s \right|^2 \right] + \\
 &\quad \left. \mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \int_0^t \{ \nabla u^N(s, X_s) - \nabla u(s, X_s) \} \, dW_s \right|^2 \right] \right).
 \end{aligned}$$

Using Burkholder-Davis-Gundy inequality, it follows that:

$$\begin{aligned}
 w(y) &\leq 14 \left(\sup_{0 \leq t \leq T} |u^N(0, X_0) - u(0, X_0)|^2 + \mathbb{E} \left[\sup_{0 \leq t \leq T} |u(t, X_t) - u^N(t, X_t)|^2 \right] \right. \\
 &\quad + \frac{\lambda + 1}{14} T \int_0^y w(s) \, ds + (\lambda + 1) T \mathbb{E} \left[\int_0^T |u^N(s, X_s) - u(s, X_s)|^2 \, ds \right] \\
 &\quad + 2 \int_0^y \mathbb{E} \left[\left| \nabla u^N(s, X_s^N) - \nabla u^N(s, X_s) \right|^2 \right] \, ds + 2 \int_0^T \mathbb{E} \left[\left| \nabla u^N(s, X_s) - \nabla u(s, X_s) \right|^2 \right] \, ds \Bigg) \\
 &\leq 14 \left(\sup_{0 \leq t \leq T} |u^N(0, X_0) - u(0, X_0)|^2 + \mathbb{E} \left[\sup_{0 \leq t \leq T} |u(t, X_t) - u^N(t, X_t)|^2 \right] \right. \\
 &\quad + \frac{\lambda + 1}{14} T \int_0^y w(s) \, ds + (\lambda + 1) T \mathbb{E} \left[\int_0^T |u^N(s, X_s) - u(s, X_s)|^2 \, ds \right] \\
 &\quad + \frac{1}{7} \int_0^y \mathbb{E} \left[|X_s^N - X_s|^{2\alpha} \right] \, ds + 2 \int_0^T \mathbb{E} \left[\left| \nabla u^N(s, X_s) - \nabla u(s, X_s) \right|^2 \right] \, ds \Bigg)
 \end{aligned}$$

Applying fractional Morrey inequality, we obtain $\forall t \in [0, T]$:

$$\begin{cases} \|u^N(t) - u(t)\|_{L^\infty} \leq \|u^N(t) - u(t)\|_{C^{1,\alpha}} \leq c \|u^N(t) - u(t)\|_{H_{\bar{q},q}^{-\beta}} \\ \|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \|u^N(t) - u(t)\|_{C^{1,\alpha}} \leq c \|u^N(t) - u(t)\|_{H_{\bar{q},q}^{-\beta}}. \end{cases}$$

Now, with

$$\|u^N(t) - u(t)\|_{H_{\bar{q},q}^{-\beta}} \leq e^T \|u^N(t) - u(t)\|_{H_{\bar{q},q}^{-\beta}}^{(\rho)} \leq K e^T \|b^N - b\|_{H_q^{-\beta}}$$

from lemma 22 in [1], we have:

$$\begin{aligned}
 w(y) &\leq 14 \left(2c^2 K^2 e^{2T} \|b^N - b\|_{H_q^{-\beta}}^2 + \frac{\lambda+1}{14} T \int_0^y w(s) \, ds \right. \\
 &\quad \left. + \frac{1}{7} \int_0^y \mathbb{E} \left[|X_s^N - X_s|^{2\alpha} \right] \, ds + (\lambda+1) T^2 c^2 K^2 e^{2T} \|b^N - b\|_{H_q^{-\beta}}^2 \right. \\
 &\quad \left. + 2c^2 K^2 T e^{2T} \|b^N - b\|_{H_q^{-\beta}}^2 \right) \\
 &\leq 14 [2 + T + (\lambda+1) T^2] c^2 K^2 e^{2T} \|b^N - b\|_{H_q^{-\beta}}^2 + (\lambda+1) T \int_0^y w(s) \, ds \\
 &\quad + 2 \int_0^y \mathbb{E} \left[\sup_{0 \leq t \leq s} |X_s^N - X_s|^{2\alpha} \right] \, ds \\
 &\leq 14 [2 + T + (\lambda+1) T^2] c^2 K^2 e^{2T} \|b^N - b\|_{H_q^{-\beta}}^2 + (\lambda+1) T \int_0^y w(s) \, ds \\
 &\quad + 2 \int_0^y \mathbb{E} \left[\sup_{0 \leq t \leq s} |X_s^N - X_s|^2 \right]^\alpha \, ds \\
 &\leq 14 [2 + T + (\lambda+1) T^2] c^2 K^2 e^{2T} \|b^N - b\|_{H_q^{-\beta}}^2 + (\lambda+1) T \int_0^y w(s) \, ds \\
 &\quad + 2 \int_0^y w(s)^\alpha \, ds
 \end{aligned}$$

References

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