NUMERICAL SIMULATION OF SDES WITH DISTRIBUTIONAL DRIFT

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1 Introduction

We would like to simulate numerically sample paths of the solution of the stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\ \mathrm{d}t + \mathrm{d}W_t \tag{1}$$

where $b \in H_q^{-\beta}(\mathbb{R}),\ \beta \in \left(0, \frac{1}{2}\right)$, $q \in \left(\frac{1}{1-\beta}, \frac{1}{\beta}\right)$, $t \in [0, T]$, and W_t is a standard Brownian motion. Equation (1) is studied by F. Flandoli, E. Issoglio, and F. Russo in [1] in which they define a concept of virtual solution. The authors prove then existence and unicity in law of this solution.

Example 1. *An example of such drift b is given by the derivative of a sample path of a fractional Brownian motion B_x^H with Hurst index $1/2 < H < 1$. These stochastic processes are gaussian processes verifying*

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} + |t-s|^{2H}\right).$$

We note $-\beta = H - 1$. Given $B_x^H(\omega) \in H_q^{1-\beta}(\mathbb{R})$, we can take $b(x) = \frac{\partial}{\partial x} B_x^H(\omega) \in H_q^{-\beta}(\mathbb{R})$. We will use this in our numerical simulations.

As far as the drift b is not a function but a distribution, it must be approximated if we want to evaluate it at points. In order to do so, we will use a series representation of b and truncate it. That is why we will consider two steps in our algorithm:

1. approximate the drift b by a function b^N meant to converge to b as $N \to \infty$.

2. approximate the solution X_t^N of the approximated SDE

$$\mathrm{d}X_t^N = b^N\left(X_t^N\right)\,\mathrm{d}t + \mathrm{d}W_t \tag{2}$$

by $X_t^{N,n}$ defined with the Euler-Maruyama scheme

$$X_t^{N,n} = X_0 + \int_0^t b^N\left(X_{\eta_n(t)}^{N,n}\right)\mathrm{d}t + W_{\eta_n(t)}$$

where $\eta_n(t) = t_k$ if $t \in [t_k, t_k + 1]$, for $t_k = \frac{k}{n}$ with $k \in [\![0, \lceil nT \rceil]\!]$.

2 Numerical simulation of fractional Brownian motion

To simulate a sample path of a fractional brownian motion B_x^H on a finite grid $(x_k)_{k\in[\![1,n]\!]}$, we simulate n independent standard gaussian random variables $(X_k)_{k\in[\![1,n]\!]}$ and then correlate them with the definite positive correlation matrix

$$C_{k,s} = \mathbb{E}\left[B_{x_k}^H B_{x_s}^H\right] = \frac{1}{2}\left(x_k^{2H} + x_s^{2H} + |x_k - x_s|^{2H}\right).$$

To do so, we use the Cholesky decomposition method and calculate the triangular matrix M such that $C = MM^\top$. Therefore, defining

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ and } B^H = MX,$$

B^H contains the values of a fractional brownian motion evaluated on the grid $(x_k)_{k\in[\![1,n]\!]}$.

3 Approximation of the drift

3.1 Series representation

We use Haar wavelets to give a series representation of b. By doing so, we will be able to approximate it numerically by truncating the series.

Definition 1 (Haar wavelets). *We define the Haar wavelets $h_{j,m}$ on $\mathbb{R}$ with $j \in \mathbb{N} \cup \{-1\}$ and $m \in \mathbb{Z}$ by:*

$$\begin{cases} h_M & : x \longmapsto \left(\mathbb{1}_{[0,\frac{1}{2}[} - \mathbb{1}_{[\frac{1}{2},1[}\right)(x) \\ h_{-1,m} & : x \longmapsto \sqrt{2}|h_M(x-m)| \\ h_{j,m} & : x \longmapsto h_M(2^j x - m) \end{cases}$$

Theorem 1 (See [2]). *Let $b \in H_q^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left]-\frac{1}{2}, \frac{1}{q}\right[$. Therefore,*

$$b = \sum_{j=-1}^{+\infty} \sum_{m \in \mathbb{Z}} \mu_{j,m} h_{j,m} \tag{3}$$

where $\mu_{j,m} = 2^j \int_{\mathbb{R}} b(x) h_{j,m}(x)\, \mathrm{d}x$ in the sense of dual pairing.

Definition 2. *With the same notation $\mu_{j,m}$, let $b \in H_q^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left]-\frac{1}{2}, \frac{1}{q}\right[$. Given $N \in \mathbb{N}^*$ we define b^N by:*

$$b^N = \sum_{m=N}^{N-1} \mu_{-1,m} h_{-1,m} + \sum_{j=0}^{N} \sum_{m=-N2^j}^{N2^j-1} \mu_{j,m} h_{j,m}. \tag{4}$$

Remark 1. *We can note that* $\operatorname{Supp} b^N \subset [-N, N]$. *Moreover, we have:*

$$\left\|b - b^N\right\|_{H_q^s(\mathbb{R})} \underset{N \to +\infty}{\longrightarrow} 0.$$

4 Convergence

4.1 Weak convergence of $X_T^{N,n}$ to X_T^N

Recently, Leobacher and Szölgyenyi proved in [3] the convergence of the Euler-Maruyama scheme for SDE with discontinuous but piecewise Lipschitz drift and with a degenerate diffusion coefficient. This framework applies to the scheme we use with piecewise constant drift b^N and a constant diffusion coefficient.

Theorem 2 (Theorem 3.1. in [3])**.** *$\exists C_N > 0$ independent of n such that it holds $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$:*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t^{N,n} - X_t^N\right|^2\right]^{1/2} \leq C_N \delta^{1/4-\varepsilon} \tag{5}$$

with $\delta = \frac{1}{n}$ the step size and C_N depending on $\left\|b^N\right\|_\infty$.

Theorem 3. *Let f be μ-Hölder with constant $C_f > 0$ and $\mu \in (0, 1]$. Then, exists $C'_N > 0$ independent of n such that it holds $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$:*

$$\left|\mathbb{E}\left[f\left(X_T^{N,n}\right) - f\left(X_T^N\right)\right]\right| \leq C'_N \delta^{\mu/4-\varepsilon} \tag{6}$$

with $\delta = \frac{1}{n}$ the step size.

Proof. By Jensen's inequality and the μ-Hölder property of f, we obtain:

$$\begin{aligned}\left|\mathbb{E}\left[f\left(X_T^{N,n}\right) - f\left(X_T^N\right)\right]\right| &\leq C_f \mathbb{E}\left[\left|X_T^{N,n} - X_T^N\right|^\mu\right] \\ &\leq C_f \mathbb{E}\left[\left|Y_T - Y_T^N\right|^2\right]^{\mu/2} \\ &\leq C_f C_N^\mu \delta^{\mu/4-\varepsilon}.\end{aligned}$$

□

4.2 Weak convergence of X_T^N to X_T

The goal of this section is to estimate the weak error $\left|\mathbb{E}\left[f\left(X_T\right) - f\left(X_T^N\right)\right]\right|$ with suitable functions f. In order to do so, we must go back to the definition of the virtual solution of the SDE (1) given in [1]. Let $(\delta, p) \in K(\beta, q) :=$

$\{(\delta, p) \mid \beta < \delta < 1-\beta,\ \frac{1}{\delta} < p < q\}$. The authors define the virtual solution of SDE (1) by X_t such that:

$$\begin{cases} Y_t = y + (\lambda+1)\int_0^t u(s,Y_s)\ \mathrm{d}s + \int_0^t (\nabla u(s,Y_s)+1)\ \mathrm{d}W_s \\ X_t = \Psi(t,Y_t) = \varphi^{-1}(t,Y_t) \end{cases} \tag{7}$$

where u is the mild solution in $H_p^{1+\delta}$ of the following parabolic PDE:

$$\begin{cases} \partial_t u + \frac{1}{2}\Delta u + b\nabla u - (\lambda+1)u = -b & \text{on } [0,T]\times\mathbb{R} \\ u(T) = 0 & \text{on } \mathbb{R} \end{cases} \tag{8}$$

and $\varphi(t,x) = x + u(t,x)$.

We also define another similar PDE by replacing b by b^N. We call u^N its mild solution in $H_p^{1+\delta}$:

$$\begin{cases} \partial_t u^N + \frac{1}{2}\Delta u^N + b^N\nabla u^N - (\lambda+1)u^N = -b^N & \text{on } [0,T]\times\mathbb{R} \\ u^N(T) = 0 & \text{on } \mathbb{R} \end{cases}. \tag{9}$$

Then we consider an approximated version of (7):

$$\begin{cases} Y_t^N = y + (\lambda+1)\int_0^t u^N\left(s,Y_s^N\right)\ \mathrm{d}s + \int_0^t \left(\nabla u^N\left(s,Y_s^N\right)+1\right)\ \mathrm{d}W_s \\ X_t^N = \Psi^N(t,Y_t^N) = (\varphi^N)^{-1}(t,Y_t^N) \end{cases}. \tag{10}$$

Remark 2. *Proposition 26 in [1] assures us that X_t^N defined above in (10) is in fact the classical solution of (2), as far as $b^N \in L^p$. That is why for each fixed N our Euler scheme converges to the virtual solution X_t^N.*

We also recall a useful lemma concerning the solutions of (8) and (9).

Lemma 4 (Lemma 20 in [1])**.** *Let $(\delta,p) \in K(\beta,q)$ and let u, u^N be the mild solutions to (8), (9) in $H_p^{1+\delta}$. Fix ρ such that the integral operator is a contraction and let $\lambda > \rho$. Then $u(t)$, $u^N(t) \in \mathcal{C}^{1,\alpha}$ with $\alpha = \delta - 1/p$ for each fixed t and*

$$\begin{cases} \sup_{(t,x)\in[0,T]\times\mathbb{R}} |\nabla u(t,x)| \to 0,\ \text{as } \lambda\to\infty \\ \sup_{(t,x)\in[0,T]\times\mathbb{R}} |\nabla u^N(t,x)| \to 0,\ \text{as } \lambda\to\infty \end{cases}$$

where the choice of λ depends only on $\delta, \beta, \|b\|_{H_p^{-\beta}}$, and $\left\|b^N\right\|_{H_q^{-\beta}}$.

Lemma 5. *Let $(\delta,p)\in K(\beta,q)$ and let $u,\ u^N$ be the mild solutions to (8),(9) in $H_p^{1+\delta}$, $\alpha=\delta-1/p$. Exists $c,K>0$ such that for both $N\in\mathbb{N}$ and $\rho>1$ big enough, $\forall t\in[0,T]$,*

$$\begin{cases}\left\|u^N(t)-u(t)\right\|_{L^\infty}\le cKe^{\rho T}\left\|b^N-b\right\|_{H_q^{-\beta}}\\ \left\|\nabla u^N(t)-\nabla u(t)\right\|_{L^\infty}\le cKe^{\rho T}\left\|b^N-b\right\|_{H_q^{-\beta}}.\end{cases}\qquad (11)$$

Proof. Applying fractional Morrey inequality, $\exists c>0,\ \forall t\in[0,T]$:

$$\begin{cases}\left\|u^N(t)-u(t)\right\|_{L^\infty}\le\left\|u^N(t)-u(t)\right\|_{\mathcal{C}^{1,\alpha}}\le c\left\|u^N(t)-u(t)\right\|_{H_p^{1+\delta}}\\ \left\|\nabla u^N(t)-\nabla u(t)\right\|_{L^\infty}\le\left\|u^N(t)-u(t)\right\|_{\mathcal{C}^{1,\alpha}}\le c\left\|u^N(t)-u(t)\right\|_{H_p^{1+\delta}}.\end{cases}$$

Now, we can conclude with

$$\left\|u^N-u\right\|_{\infty,H_p^{1+\delta}}\le e^{\rho T}\left\|u^N-u\right\|_{\infty,H_p^{1+\delta}}^{(\rho)}\le Ke^{\rho T}\left\|b^N-b\right\|_{H_q^{-\beta}}$$

from Lemma 23 in [1], for both $N\in\mathbb{N}$ and $\rho>1$ big enough, and where $\|f(t)\|_{\infty,X}^{(\rho)}:=\sup\limits_{0\le t\le T}e^{-\rho t}\|f(t)\|_X$. □

We will need the following local time inequality from Liqing Yan:

Lemma 6 (Lemma 4.2 in [4])**.** *Let X be a continuous semimartingale with $X_0=0$. For $\varepsilon>0$ we define a double sequence of stopping times by $\sigma_1=0$, $\tau_1=\inf\{t>0|X_t=\varepsilon\}$, $\sigma_n=\inf\{t>\tau_{n-1}|X_t=0\}$, $\tau_n=\inf\{t>\sigma_n|X_t=\varepsilon\}$. For any real function $F(\cdot)\in\mathcal{C}^2$ with $F(0)=0,\ F'(0)=0$ and $F(\cdot)>0$ on $(0,\varepsilon_0)$ for some $\varepsilon_0>0$, then for any $0<\varepsilon<\varepsilon_0$ we have*

$$0\le L_t^0(X)\le 2\varepsilon-\frac{2}{F(\varepsilon)}\int_0^t\theta_s(X)\left(F(\varepsilon)-\varepsilon F'\left(X_s^+\right)\right)\,\mathrm{d}X_s+\frac{1}{F(\varepsilon)}\int_0^t\theta_s(X)\varepsilon F''(X_s^+)\,\mathrm{d}[X]_s$$

with $\theta_s(X)=\sum_{n=1}^\infty\mathbb{1}_{\{\sigma_n<s\le\tau_n,\ 0<X_s\le\varepsilon\}}(X)$.

Applying lemma 6 with $F(x)=x^2$, it follows:

Corollary 7. *Let X be a continuous semimartingale with $X_0=0$. With the same notations as in lemma 6, for any $\varepsilon>0$ we have*

$$0\le L_t^0(X)\le 2\varepsilon-\frac{2}{\varepsilon}\int_0^t\theta_s(X)\left(\varepsilon-2X_s^+\right)\,\mathrm{d}X_s+\frac{2}{\varepsilon}\int_0^t\theta_s(X)\,\mathrm{d}[X]_s\qquad (12)$$

Lemma 8. *Let $(\delta,p)\in K(\beta,q)$, $\alpha=\delta-1/p<1$, u, u^N be the mild solutions to (8), (9) in $H_p^{1+\delta}$, and Y, Y^N solutions of the SDEs (7), (10). Then, if $\alpha>1/2$, for λ big enough we have $\forall\varepsilon\in(0,1]$,*

$$0\le \mathbb{E}\left[L_T^0(Y-Y^N)\right]\le g(\varepsilon).$$

where

$$g(\varepsilon)=2(\lambda+1)\ cTKe^{\rho T}\left\|b^N-b\right\|_{H_q^{-\beta}}+\left(4\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2T+2+(\lambda+1)T\right)\varepsilon^{2\alpha-1}\\ +4c^2TK^2e^{2\rho T}\left\|b^N-b\right\|_{H_q^{-\beta}}^2\varepsilon^{-1}$$

Proof. By Lemma 4, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac12$. λ can be chosen independently of N as far as $\left\|b-b^N\right\|_{H_q^s(\mathbb{R})}\underset{N\to\infty}{\longrightarrow}0$. Therefore $u^N(t,\cdot)$ and $u(t,\cdot)$ are $\frac12$-lipschitz. Let $\varepsilon\in(0,1]$. Corollary 7 gives us:

$$0\le L_T^0(Y-Y^N)\le 2\varepsilon-\frac{2}{\varepsilon}\int_0^T\theta_s(Y-Y^N)\left(\varepsilon-2(Y_s-Y_s^N)^+\right)\ \mathrm{d}(Y_s-Y_s^N)\\ +\frac{2}{\varepsilon}\int_0^T\theta_s(Y-Y^N)\ \mathrm{d}[(Y-Y^N)]_s.$$

Remark 3. *Note that $\theta_s(Y-Y^N)\left|\varepsilon-2(Y_s-Y_s^N)^+\right|\le\varepsilon\theta_s(Y-Y^N)$.*

Let $t\in[0,T]$.

$$Y_t-Y_t^N=(\lambda+1)\int_0^t\left\{u\left(s,Y_s\right)-u^N\left(s,Y_s^N\right)\right\}\ \mathrm{d}s\\ +\int_0^t\left\{\nabla u\left(s,Y_s\right)-\nabla u^N\left(t,Y_s^N\right)\right\}\ \mathrm{d}W_s.$$

∇u and ∇u^N are bounded so the Itô integral is a martingale. We take the expectation:

$$\mathbb{E}\left[L_T^0(Y-Y^N)\right]\le 2\varepsilon+2(\lambda+1)\,\mathbb{E}\left[\int_0^T\theta_s(Y-Y^N)\left\{u\left(s,Y_s\right)-u^N\left(s,Y_s^N\right)\right\}\mathrm{d}s\right]\\ +\frac{2}{\varepsilon}\,\mathbb{E}\left[\int_0^T\theta_s(Y-Y^N)\left\{\nabla u\left(s,Y_s\right)-\nabla u^N\left(s,Y_s^N\right)\right\}^2\ \mathrm{d}s\right]$$

$$
\begin{aligned}
&\leq 2\varepsilon + 2(\lambda+1)\ \mathbb{E}\left[\int_0^T \theta_s(Y-Y^N)\left\{u\left(s,Y_s\right)-u\left(s,Y_s^N\right)\right\}\ \mathrm{d}s\right]\\
&\qquad + 2(\lambda+1)\ \mathbb{E}\left[\int_0^T \theta_s(Y-Y^N)\left\{u\left(s,Y_s^N\right)-u^N\left(s,Y_s^N\right)\right\}\ \mathrm{d}s\right]\\
&\qquad\quad + \frac{4}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s(Y-Y^N)\left\{\nabla u\left(s,Y_s\right)-\nabla u\left(s,Y_s^N\right)\right\}^2\ \mathrm{d}s\right]\\
&\qquad\qquad + \frac{4}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s(Y-Y^N)\left\{\nabla u\left(s,Y_s^N\right)-\nabla u^N\left(s,Y_s^N\right)\right\}^2\ \mathrm{d}s\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2\varepsilon + (\lambda+1)\ \mathbb{E}\left[\int_0^T \theta_s(Y-Y^N)\left|Y_s-Y_s^N\right|\ \mathrm{d}s\right]\\
&\qquad + \frac{4\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s(Y-Y^N)\left|Y_s-Y_s^N\right|^{2\alpha}\ \mathrm{d}s\right]\\
&\qquad + 2(\lambda+1)\ cTKe^{\rho T}\left\|b^N-b\right\|_{H_q^{-\beta}} + 4c^2TK^2e^{2\rho T}\left\|b^N-b\right\|_{H_q^{-\beta}}^2\varepsilon^{-1}
\end{aligned}
$$

where we have used Lemma 5, the $\frac{1}{2}$-lipschitz property of u and the α-Hölder property of ∇u (with constant $\|u\|_{\mathcal{C}^{1,\alpha}}$). As $\theta_s(Y-Y^N)\left|Y_s-Y_s^N\right| \leq \varepsilon$, we have

$$
\begin{aligned}
\mathbb{E}\left[L_T^0(Y-Y^N)\right] &\leq 2\varepsilon + (\lambda+1)\ T\varepsilon + 4\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2\ T\varepsilon^{2\alpha-1}\\
&\quad + 2(\lambda+1)\ cTKe^{\rho T}\left\|b^N-b\right\|_{H_q^{-\beta}} + 4c^2TK^2e^{2\rho T}\left\|b^N-b\right\|_{H_q^{-\beta}}^2\varepsilon^{-1}
\end{aligned}
$$

As $1 > 2\alpha-1 > 0$, the result follows from $\varepsilon \leq \varepsilon^{2\alpha-1}$ when $0 < \varepsilon \leq 1$.

□

Lemma 9. *With assumptions and notations of Lemma 8, and* $1 > \alpha > 1/2$ *we have* $\forall \varepsilon \in (0,1]$,

$$
g(\varepsilon) \leq \sigma\left\|b^N-b\right\|_{H_q^{-\beta}}^{2-1/\alpha} \tag{13}
$$

for $\left\|b^N-b\right\|_{H_q^{-\beta}}$ *small enough (it is to say* N *big enough) where*

$$
\sigma = 2(\lambda+1)\ cTKe^{\rho T} + \left(4\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + (\lambda+1)T\right)\omega^{2\alpha-1} + 4c^2TK^2e^{2\rho T}\omega^{-1}
$$

and

$$
\omega = \left(\frac{4c^2TK^2e^{2\rho T}}{(2\alpha-1)\left(4\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + (\lambda+1)T\right)}\right)^{\frac{1}{2\alpha}}.
$$

Proof. By Lemma 8,

$$0 \leq \mathbb{E}\left[L_T^0(Y - Y^N)\right] \leq g(\varepsilon)$$

where

$$g(\varepsilon) = 2(\lambda+1)\ cTKe^{\rho T}\left\|b^N - b\right\|_{H_q^{-\beta}} + \left(4\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + (\lambda+1)T\right)\varepsilon^{2\alpha-1} + 4c^2TK^2e^{2\rho T}\left\|b^N - b\right\|_{H_q^{-\beta}}^2 \varepsilon^{-1}.$$

With

$$g'(\varepsilon) = (2\alpha-1)\left(4\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + (\lambda+1)T\right)\varepsilon^{2\alpha-2} - 4c^2TK^2e^{2\rho T}\left\|b^N - b\right\|_{H_q^{-\beta}}^2 \varepsilon^{-2},$$

and

$$g''(\varepsilon) = (2\alpha-2)(2\alpha-1)\left(4\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + (\lambda+1)T\right)\varepsilon^{2\alpha-3} + 8c^2TK^2e^{2\rho T}\left\|b^N - b\right\|_{H_q^{-\beta}}^2 \varepsilon^{-3},$$

the minimum of g on $(0,1]$ is reached when N is big enough in

$$\varepsilon_0 = \left(\frac{4c^2TK^2e^{2\rho T}\left\|b^N - b\right\|_{H_q^{-\beta}}^2}{(2\alpha-1)\left(4\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + (\lambda+1)T\right)}\right)^{\frac{1}{2\alpha}} = \omega\left\|b^N - b\right\|_{H_q^{-\beta}}^{1/\alpha}.$$

where

$$g''(\varepsilon_0) = 8c^2TK^2e^{2\rho T}\left\|b^N - b\right\|_{H_q^{-\beta}}^2 \varepsilon_0^{-3}\left(1 - (1-\alpha)\right) > 0.$$

and

$$\omega = \left(\frac{4c^2TK^2e^{2\rho T}}{(2\alpha-1)\left(4\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + (\lambda+1)T\right)}\right)^{\frac{1}{2\alpha}}.$$

Therefore $\mathbb{E}\left[L_T^0(Y - Y^N)\right] \leq g(\varepsilon_0)$

$$\leq 2(\lambda+1)\ cTKe^{\rho T}\left\|b^N - b\right\|_{H_q^{-\beta}} + \left(4\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + (\lambda+1)T\right)\omega^{2\alpha-1}\left\|b^N - b\right\|_{H_q^{-\beta}}^{2-1/\alpha} + 4c^2TK^2e^{2\rho T}\left\|b^N - b\right\|_{H_q^{-\beta}}^2 \omega^{-1}\left\|b^N - b\right\|_{H_q^{-\beta}}^{-1/\alpha}$$

$$\leq 2(\lambda+1)\ cTKe^{\rho T}\left\|b^N-b\right\|_{H_q^{-\beta}}$$
$$+\left(\left(4\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 T+2+(\lambda+1)T\right)\omega^{2\alpha-1}+4c^2TK^2e^{2\rho T}\omega^{-1}\right)\left\|b^N-b\right\|_{H_q^{-\beta}}^{2-1/\alpha}$$

$$\leq 2(\lambda+1)\ cTKe^{\rho T}\left\|b^N-b\right\|_{H_q^{-\beta}}^{2-1/\alpha}$$
$$+\left(\left(4\left\|u\right\|_{\mathcal{C}^{1,\alpha}}^2 T+2+(\lambda+1)T\right)\omega^{2\alpha-1}+4c^2TK^2e^{2\rho T}\omega^{-1}\right)\left\|b^N-b\right\|_{H_q^{-\beta}}^{2-1/\alpha}$$

for N big enough. The result follows. $\square$

Theorem 10. *Let f be μ-Hölder with constant $C_f>0$ and $\mu\in(0,1]$. If $0<\beta<1/4$, $q\in\left(\frac{1}{1-\beta},\frac{1}{\beta}\right)$, $\forall\varepsilon\in(0,\frac{1-4\beta}{2})$, with $(\delta,p)\in K(\beta,q)$ such that $\delta-1/p=1-2\beta-\varepsilon$, exists ξ independent of f such that for $N\in\mathbb{N}$, $\rho>1$, λ big enough it holds:*

$$\left|\mathbb{E}\left[f\left(X_T\right)-f\left(X_T^N\right)\right]\right|\leq\xi C_f\left\|b^N-b\right\|_{H_q^{-\beta}}^{\mu\left(2-\frac{1}{1-2\beta-\varepsilon}\right)}$$

Proof. We note as usual $\alpha=\delta-1/p$. By Lemma 4, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\left\|b-b^N\right\|_{H_q^s(\mathbb{R})}\underset{N\to\infty}{\longrightarrow}0$ (See Step 2 of the proof of Proposition 29 in [1]). Therefore $u^N(t,\cdot)$ and $u(t,\cdot)$ are $\frac{1}{2}$-lipschitz. We recall that in this case, by Lemma 22 in [1], $\Psi(t,\cdot)$ and $\Psi^N(t,\cdot)$ are 2-lipschitz.

$$\left|\mathbb{E}\left[f\left(X_T\right)-f\left(X_T^N\right)\right]\right|=\left|\mathbb{E}\left[f\left(\Psi\left(T,Y_T\right)\right)-f\left(\Psi^N\left(T,Y_T^N\right)\right)\right]\right|$$
$$\leq\mathbb{E}\left[\left|f\left(\Psi\left(T,Y_T\right)\right)-f\left(\Psi\left(T,Y_T^N\right)\right)\right|\right]+\mathbb{E}\left[\left|f\left(\Psi\left(T,Y_T^N\right)\right)-f\left(\Psi^N\left(T,Y_T^N\right)\right)\right|\right]$$
$$\leq C_f\left(2^\mu\mathbb{E}\left[\left|Y_T-Y_T^N\right|^\mu\right]+\mathbb{E}\left[\left|\Psi\left(T,Y_T^N\right)-\Psi^N\left(T,Y_T^N\right)\right|^\mu\right]\right)$$
$$\leq C_f\left(2^\mu\mathbb{E}\left[\left|Y_T-Y_T^N\right|\right]^\mu+\mathbb{E}\left[\left|\Psi\left(T,Y_T^N\right)-\Psi^N\left(T,Y_T^N\right)\right|\right]^\mu\right)\tag{14}$$

by Jensen's inequality. Let $t\in[0,T]$.

$$Y_t-Y_t^N=(\lambda+1)\int_0^t\left\{u\left(s,Y_s\right)-u^N\left(s,Y_s^N\right)\right\}\ \mathrm{d}s$$
$$+\int_0^t\left\{\nabla u\left(s,Y_s\right)-\nabla u^N\left(t,Y_s^N\right)\right\}\ \mathrm{d}W_s.$$

We apply Meyer-Tanaka's formula to obtain:

$$
\begin{aligned}
\left|Y_t - Y_t^N\right| &= (\lambda+1) \int_0^t \operatorname{sign}(Y_s - Y_s^N) \left\{u\left(s, Y_s\right) - u^N\left(s, Y_s^N\right)\right\} \, \mathrm{d}s \\
&+ \int_0^t \operatorname{sign}(Y_s - Y_s^N) \left\{\nabla u\left(s, Y_s\right) - \nabla u^N\left(s, Y_s^N\right)\right\} \, \mathrm{d}W_s + L_t^0(Y - Y^N).
\end{aligned}
$$

Taking the expectation leads to:

$$
\begin{aligned}
\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] = (\lambda+1)\, \mathbb{E}\left[\int_0^t \operatorname{sign}(Y_s - Y_s^N) \left\{u\left(s, Y_s\right) - u^N\left(s, Y_s^N\right)\right\} \, \mathrm{d}s\right] \\
+ \mathbb{E}\left[L_t^0(Y - Y^N)\right]
\end{aligned}
$$

because ∇u and ∇u^N are bounded so the Itô integral is a martingale.

$$
\begin{aligned}
\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] &\le (\lambda+1)\, \mathbb{E}\left[\int_0^t \left\{u\left(s, Y_s\right) - u\left(s, Y_s^N\right)\right\} \mathrm{d}s\right] \\
&+ (\lambda+1)\, \mathbb{E}\left[\int_0^t \left\{u\left(s, Y_s^N\right) - u^N\left(s, Y_s^N\right)\right\} \mathrm{d}s\right] + \mathbb{E}\left[L_t^0(Y - Y^N)\right].
\end{aligned}
$$

We use Lemma 5 and the $\frac{1}{2}$-lipschitz property of u:

$$
\begin{aligned}
\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] \le \frac{\lambda+1}{2}\, \mathbb{E}\left[\int_0^t \left|Y_s - Y_s^N\right| \mathrm{d}s\right] + (\lambda+1)\; ctKe^{\rho T} \left\|b^N - b\right\|_{H_q^{-\beta}} \\
+ \mathbb{E}\left[L_t^0(Y - Y^N)\right]
\end{aligned}
$$

$$
\begin{aligned}
\le \frac{\lambda+1}{2} \int_0^t \mathbb{E}\left[\left|Y_s - Y_s^N\right|\right] \mathrm{d}s + (\lambda+1)\; cTKe^{\rho T} \left\|b^N - b\right\|_{H_q^{-\beta}} \\
+ \mathbb{E}\left[L_T^0(Y - Y^N)\right].
\end{aligned}
$$

where we have used the fact that $L_t^0(Y - Y^N)$ is an increasing process.

By Gronwall's Lemma, it follows:

$$
\mathbb{E}\left[\left|Y_T - Y_T^N\right|\right] \le C(N)\; e^{(\lambda+1)T/2} \tag{15}
$$

with $C(N) = (\lambda+1)\; cTKe^{\rho T} \left\|b^N - b\right\|_{H_q^{-\beta}} + \mathbb{E}\left[L_T^0(Y - Y^N)\right]$.

With Lemma 8 and Lemma 9 we obtain

$$C(N) \le (\lambda+1)\ cTKe^{\rho T}\left\|b^N-b\right\|_{H_q^{-\beta}} + \sigma\left\|b^N-b\right\|_{H_q^{-\beta}}^{2-1/\alpha} \le \zeta\left\|b^N-b\right\|_{H_q^{-\beta}}^{2-1/\alpha}.$$

for $\left\|b^N-b\right\|_{H_q^{-\beta}}$ small enough where $\zeta=(\lambda+1)\ cTKe^{\rho T}+\sigma$. It follows:

$$\mathbb{E}\left[\left|Y_T-Y_T^N\right|\right] \le \zeta e^{(\lambda+1)T/2}\left\|b^N-b\right\|_{H_q^{-\beta}}^{2-1/\alpha}. \tag{16}$$

Moreover, using $\sup\limits_{x\in\mathbb{R}}\left|\nabla u^N(T,x)\right| \le 1/2$, we obtain with $\varphi(t,x)=x+u(t,x)$:

$$\begin{aligned}
&\left|\varphi^N\left(T,\Psi^N\left(T,Y_T^N\right)\right)-\varphi^N\left(T,\Psi\left(t,Y_T^N\right)\right)\right| \\
&\ge \inf_{x\in\mathbb{R}}\left|\nabla\varphi^N(T,x)\right|\left|\Psi^N\left(T,Y_T^N\right)-\Psi\left(T,Y_T^N\right)\right| \\
&\ge \frac{1}{2}\left|\Psi^N\left(T,Y_T^N\right)-\Psi\left(T,Y_T^N\right)\right|
\end{aligned}$$

and

$$\begin{aligned}
\left|\Psi^N\left(T,Y_T^N\right)-\Psi\left(T,Y_T^N\right)\right| &\le 2\ \left|\varphi^N\left(T,\Psi^N\left(T,Y_T^N\right)\right)-\varphi^N\left(t,\Psi\left(T,Y_T^N\right)\right)\right| \\
\le 2\ \Big|\varphi^N\left(T,\Psi^N\left(t,Y_T^N\right)\right)&-\varphi\left(T,\Psi\left(T,Y_T^N\right)\right)\Big| \\
&+\left|\varphi\left(T,\Psi\left(T,Y_T^N\right)\right)-\varphi^N\left(T,\Psi\left(t,Y_T^N\right)\right)\right| \\
&\le 2\ \left|\varphi\left(T,\Psi\left(T,Y_T^N\right)\right)-\varphi^N\left(T,\Psi\left(T,Y_T^N\right)\right)\right| \\
&\le 2\ \left\|u^N(T)-u(T)\right\|_\infty \\
&\le 2cKe^{\rho T}\ \left\|b^N-b\right\|_{H_q^{-\beta}}
\end{aligned} \tag{17}$$

where we have used Lemma 5 and the fact that $\varphi^N\left(T,\Psi^N\left(T,Y_T^N\right)\right) = \varphi\left(T,\Psi\left(T,Y_T^N\right)\right)=Y_T^N$.

Finally, combining (14), (16) and (17) we obtain:

$$\begin{aligned}
&\left|\mathbb{E}\left[f\left(X_T\right)-f\left(X_T^N\right)\right]\right| \\
&\le C_f\left(2^\mu\mathbb{E}\left[\left|Y_T-Y_T^N\right|\right]^\mu+\mathbb{E}\left[\left|\Psi\left(T,Y_T^N\right)-\Psi^N\left(T,Y_T^N\right)\right|\right]^\mu\right) \\
&\le C_f\left(2^\mu\zeta^\mu e^{\mu(\lambda+1)T/2}\left\|b^N-b\right\|_{H_q^{-\beta}}^{\mu(2-1/\alpha)}+2^\mu c^\mu K^\mu e^{\mu\rho T}\ \left\|b^N-b\right\|_{H_q^{-\beta}}^\mu\right) \\
&\le 2^\mu C_f\left(\zeta^\mu e^{\mu(\lambda+1)T/2}+c^\mu K^\mu e^{\mu\rho T}\right)\left\|b^N-b\right\|_{H_q^{-\beta}}^{\mu(2-1/\alpha)}
\end{aligned}$$

for N big enough, which is the expected result.

□

5 Numerical results

5.1 Strong convergence of the Euler scheme

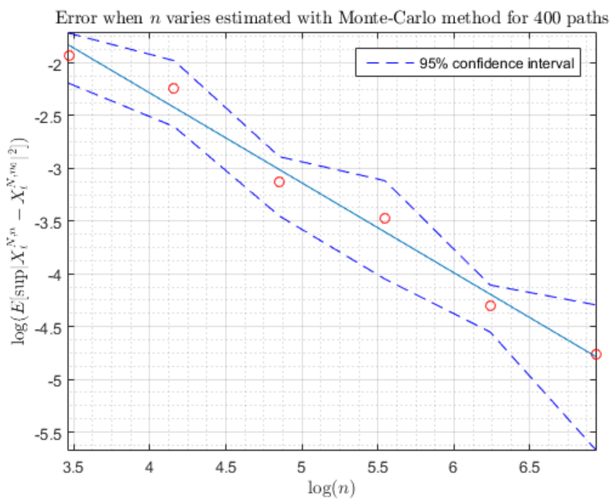

Figure 1: Estimation of the L^2 error of the Euler-Marayuma scheme with a Monte-Carlo method. 400 paths, $N = 5$, $n \in \{2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}\}$, reference solution with $n_0 = 2^{12}$ points.

We observe a numerical convergence rate of 0.85 when Theorem 2 shows a theoretical rate of $0.5 - \varepsilon$.

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