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Numerical simulation of SDEs with distributional coefficients

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Abstract

Acknowledgment

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Introduction

Part I

Context

We would like to simulate numerically sample paths of the solution of the one-dimensional stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\ \mathrm{d}t + \mathrm{d}W_t \tag{I.1}$$

where $b \in H_q^{-\beta}(\mathbb{R})$, $\beta \in \left(0, \frac{1}{2}\right)$, $q \in \left(\frac{1}{1-\beta}, \frac{1}{\beta}\right)$, $t \in [0, T]$, and W_t is a standard Brownian motion. Equation (I.1) is studied by F. Flandoli, E. Issoglio, and F. Russo in [1] in which they define a concept of virtual solution. The authors prove then existence and unicity in law of this solution.

Example 1. *An example of such drift b is given by the derivative of a sample path of a fractional Brownian motion B_x^H with Hurst index $1/2 < H < 1$. These stochastic processes are gaussian processes verifying*

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} + |t-s|^{2H}\right).$$

We note $-\beta = H - 1$. Given $B_x^H(\omega) \in H_q^{1-\beta}(\mathbb{R})$, we can take $b(x) = \frac{\mathrm{d}}{\mathrm{d}x} B_x^H(\omega) \in H_q^{-\beta}(\mathbb{R})$. We will use this in our numerical simulations.

I.1 Principle of the approximation algorithm

As far as the drift b is not a function but a distribution, it must be approximated if we want to evaluate it at points. In order to do so, we will use a series representation of b and truncate it. That is why we will consider two steps in our algorithm:

1. approximate the drift b by a function b^N meant to converge to b in $H_q^{-\beta}(\mathbb{R})$ as $N \to \infty$. In practise we will choose $b^N \in L^p(\mathbb{R}) \cap H_q^{-\beta}(\mathbb{R})$.

2. approximate the solution X_t^N of the approximated SDE

$$\mathrm{d}X_t^N = b^N\left(X_t^N\right)\ \mathrm{d}t + \mathrm{d}W_t \tag{I.2}$$

   by $X_t^{N,n}$ defined with the Euler-Maruyama scheme

$$X_t^{N,n} = X_0 + \int_0^t b^N\left(X_{\eta_n(t)}^{N,n}\right) \mathrm{d}t + W_{\eta_n(t)}$$

   where $\eta_n(t) = t_k$ if $t \in [t_k, t_k + 1[$, for $t_k = \frac{k}{n}$ with $k \in [\![0, \lceil nT \rceil]\!]$.

I.2 Virtual solutions of the original SDE and its approximation

In order to approximate the solution of the SDE (I.1), we must go back to the definition of its virtual solution given in [1]. Let $(\delta, p) \in K(\beta, q) := \{(\delta, p) \mid \beta < \delta < 1-\beta,\ \frac{1}{\delta} < p < q\}$. The authors of [1] define the virtual solution of SDE (I.1) by X_t such that:

$$\begin{cases} Y_t = y + (\lambda+1)\int_0^t u(s, \Psi(s, Y_s))\, \mathrm{d}s + \int_0^t (\nabla u(s, \Psi(s, Y_s)) + 1)\, \mathrm{d}W_s \\ X_t = \Psi(t, Y_t) = \varphi^{-1}(t, Y_t) \end{cases} \tag{I.3}$$

where u is the mild solution in $H_p^{1+\delta}$ of the following parabolic PDE:

$$\begin{cases} \partial_t u + \frac{1}{2}\Delta u + b\nabla u - (\lambda+1)u = -b & \text{on } [0,T]\times\mathbb{R} \\ u(T) = 0 & \text{on } \mathbb{R} \end{cases} \tag{I.4}$$

with $\varphi(t,x) = x + u(t,x)$, and $y = \varphi(0,x)$.

We also define another similar PDE by replacing b by $b^N \in L^p(\mathbb{R}) \cap H_q^{-\beta}(\mathbb{R})$. We call u^N its mild solution in $H_p^{1+\delta}$:

$$\begin{cases} \partial_t u^N + \frac{1}{2}\Delta u^N + b^N\nabla u^N - (\lambda+1)u^N = -b^N & \text{on } [0,T]\times\mathbb{R} \\ u^N(T) = 0 & \text{on } \mathbb{R} \end{cases}. \tag{I.5}$$

Then we consider an approximated version of (I.3):

$$\begin{cases} Y_t^N = y^N + (\lambda+1)\int_0^t u^N\left(s, \Psi^N\left(s, Y_s^N\right)\right)\, \mathrm{d}s + \int_0^t \left(\nabla u^N\left(s, \Psi^N\left(s, Y_s^N\right)\right) + 1\right)\, \mathrm{d}W_s \\ X_t^N = \Psi^N(t, Y_t^N) = \left(\varphi^N\right)^{-1}(t, Y_t^N) \end{cases}. \tag{I.6}$$

with $\varphi^N(t,x) = x + u^N(t,x)$, and $y^N = \varphi^N(0,x)$.

Remark 1. *Proposition 26 in [1] assures us that the virtual solution of (I.6), X_t^N, defined above in (I.6) is in fact the classical solution of (I.2), as far as $b^N \in L^p$. That is why for each fixed N our Euler scheme is meant to converge to the virtual solution X_t^N.*

Part II

Numerical aspects

Our project concerns a large class of very irregular drifts. But in order to do numerical studies, we must select examples of such drifts. The easiest example we can think of is a sample path of a fractional brownian motion. That's why we will explain here how we can apply our approximation algorithm in this particular case.

II.1 Numerical simulation of fractional Brownian motion

To simulate a sample path of a fractional brownian motion B_x^H on a finite grid $(x_k)_{k\in[\![1,n]\!]}$, we simulate n independent standard gaussian random variables $(X_k)_{k\in[\![1,n]\!]}$ and then correlate them with the definite positive correlation matrix

$$C_{k,s} = \mathbb{E}\left[B_{x_k}^H B_{x_s}^H\right] = \frac{1}{2}\left(x_k^{2H} + x_s^{2H} + |x_k - x_s|^{2H}\right).$$

To do so, we use the Cholesky decomposition method and calculate the triangular matrix M such that $C = MM^\top$. Therefore, defining the multidimensional random values

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ and } B^H = MX,$$

B^H is a fractional brownian motion evaluated on the grid $(x_k)_{k\in[\![1,n]\!]}$. Indeed, numerically we will only consider some realizations of these random values.

II.1.1 Refining a sample path of a fractional Brownian motion

II.2 Approximation of the drift

II.2.1 Series representation

We use Haar wavelets to give a series representation of b. By doing so, we will be able to approximate it numerically by truncating the series.

Definition 1 (Haar wavelets)**.** *We define the Haar wavelets $h_{j,m}$ on $\mathbb{R}$ with $j \in \mathbb{N} \cup \{-1\}$ and $m \in \mathbb{Z}$ by:*

$$\begin{cases} h_M & : x \longmapsto \left(\mathbb{1}_{[0,\frac{1}{2}[} - \mathbb{1}_{[\frac{1}{2},1[}\right)(x) \\ h_{-1,m} & : x \longmapsto \sqrt{2}|h_M(x-m)| \\ h_{j,m} & : x \longmapsto h_M(2^j x - m) \end{cases}$$

Theorem 1 (See [2])**.** *Let $b \in H_q^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left]-\frac{1}{2}, \frac{1}{q}\right[$. Therefore,*

$$b = \sum_{j=-1}^{+\infty} \sum_{m \in \mathbb{Z}} \mu_{j,m} h_{j,m} \tag{II.1}$$

where $\mu_{j,m} = 2^j \int_{\mathbb{R}} b(x) h_{j,m}(x)\,\mathrm{d}x$ in the sense of dual pairing.

Definition 2. *With the same notation $\mu_{j,m}$, let $b \in H_q^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left]-\frac{1}{2}, \frac{1}{q}\right[$. Given $N \in \mathbb{N}^*$ we define b^N by:*

$$b^N = \sum_{m=N}^{N-1} \mu_{-1,m} h_{-1,m} + \sum_{j=0}^{N} \sum_{m=-N2^j}^{N2^j-1} \mu_{j,m} h_{j,m}. \tag{II.2}$$

Remark 2. *We can note that* $\operatorname{Supp} b^N \subset [-N, N]$*. Moreover, we have:*

$$\left\|b - b^N\right\|_{H_q^s(\mathbb{R})} \xrightarrow[N \to +\infty]{} 0.$$

II.2.2 Computation of the coefficients for the series representation

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Part III

Convergence of the algorithm

III.1 Weak convergence rate of $X_t^{N,n}$ to X_t^N

Recently, Leobacher and Szölgyenyi proved in [3] the convergence of the Euler-Maruyama scheme for SDE with discontinuous but piecewise Lipschitz drift and with a degenerate diffusion coefficient. This framework applies to the scheme we use with piecewise constant drift b^N and a constant diffusion coefficient.

Theorem 2 (Theorem 3.1. in [3])**.** *$\exists C_N > 0$ independent of n such that it holds $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$:*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t^{N,n} - X_t^N\right|^2\right]^{1/2} \leq C_N \delta^{1/4-\varepsilon} \tag{III.1}$$

with $\delta = \frac{1}{n}$ the step size and C_N depending on $\left\|b^N\right\|_\infty$.

Theorem 3. *Let f be μ-Hölder with constant $C_f > 0$, $\mu \in (0,1]$ and $t \in [0,T]$. Then, exists $C'_N > 0$ independent of n such that it holds $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$:*

$$\left|\mathbb{E}\left[f\left(X_t^{N,n}\right) - f\left(X_t^N\right)\right]\right| \leq C'_N \delta^{\mu/4-\varepsilon} \tag{III.2}$$

with $\delta = \frac{1}{n}$ the step size.

Proof. By Jensen's inequality and the μ-Hölder property of f, we obtain:

$$\begin{aligned}\left|\mathbb{E}\left[f\left(X_t^{N,n}\right) - f\left(X_t^N\right)\right]\right| &\leq C_f \mathbb{E}\left[\left|X_t^{N,n} - X_t^N\right|^\mu\right] \\ &\leq C_f \mathbb{E}\left[\left|Y_t - Y_t^N\right|^2\right]^{\mu/2} \\ &\leq C_f C_N^\mu \delta^{\mu/4-\varepsilon}.\end{aligned}$$

□

III.2 Weak convergence rate of X_t^N to X_t

The goal of this section is to estimate the weak error $\left|\mathbb{E}\left[f\left(X_t\right) - f\left(X_t^N\right)\right]\right|$ with suitable functions f.

III.2.1 Useful lemmas

We first recall a useful lemma concerning the solutions of (I.4) and (I.5).

Lemma 4 (Lemma 20 in [1])**.** *Let $(\delta, p) \in K(\beta, q)$ and let u, u^N be the mild solutions to (I.4), (I.5) in $H_p^{1+\delta}$. Fix ρ such that the integral operator is a contraction and let $\lambda > \lambda^*$. Then $u(t)$, $u^N(t) \in \mathcal{C}^{1,\alpha}$ with $\alpha = \delta - 1/p$ for each fixed t and $\forall \varepsilon > 0$, $\exists \lambda_0 > 0$ such that*

$$\begin{cases} \forall \lambda \geq \lambda_0, & \sup\limits_{(t,x)\in[0,T]\times\mathbb{R}} |\nabla u(t,x)| \leq \varepsilon \\ \forall \lambda \geq \lambda_0, & \sup\limits_{(t,x)\in[0,T]\times\mathbb{R}} |\nabla u^N(t,x)| \leq \varepsilon \end{cases}$$

where the choice of λ_0 depends only on $\delta, \beta, \|b\|_{H_p^{-\beta}}$, and $\left\|b^N\right\|_{H_q^{-\beta}}$.

Lemma 5. *Let $(\delta, p) \in K(\beta, q)$ and let u, u^N be the mild solutions to (I.4),(I.5) in $H_p^{1+\delta}$, $\alpha = \delta - 1/p$. Exists $c, K > 0$ such that for both $N \in \mathbb{N}$ and $\rho > 1$ big enough, $\forall t \in [0,T]$,*

$$\begin{cases} \left\|u^N(t) - u(t)\right\|_{L^\infty} \leq \kappa \left\|b - b^N\right\|_{H_q^{-\beta}} \\ \left\|\nabla u^N(t) - \nabla u(t)\right\|_{L^\infty} \leq \kappa \left\|b - b^N\right\|_{H_q^{-\beta}} . \end{cases} . \tag{III.3}$$

with $\kappa = cKe^{\rho T}$.

Proof. Applying fractional Morrey inequality, $\exists c > 0$, $\forall t \in [0,T]$:

$$\begin{cases} \left\|u^N(t) - u(t)\right\|_{L^\infty} \leq \left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1,\alpha}} \leq c \left\|u^N(t) - u(t)\right\|_{H_p^{1+\delta}} \\ \left\|\nabla u^N(t) - \nabla u(t)\right\|_{L^\infty} \leq \left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1,\alpha}} \leq c \left\|u^N(t) - u(t)\right\|_{H_p^{1+\delta}} . \end{cases}$$

Now, we can conclude with

$$\left\|u^N - u\right\|_{\infty, H_p^{1+\delta}} \leq e^{\rho T} \left\|u^N - u\right\|_{\infty, H_p^{1+\delta}}^{(\rho)} \leq K e^{\rho T} \left\|b - b^N\right\|_{H_q^{-\beta}}$$

from Lemma 23 in [1], for both $N \in \mathbb{N}$ and $\rho > 1$ big enough, and where $\|f(t)\|_{\infty,X}^{(\rho)} := \sup\limits_{0\leq t\leq T} e^{-\rho t} \|f(t)\|_X$. □

Lemma 6. *For λ big enough,*

$$\left|\Psi^N\left(t, Y_t^N\right) - \Psi\left(t, Y_t^N\right)\right| \leq 2\kappa \left\|b - b^N\right\|_{H_q^{-\beta}}$$

with $\kappa = cKe^{\rho T}$.

Proof. For λ big enough, by Lemma 4, $\forall t \in [0,T]$, $\sup\limits_{x\in\mathbb{R}} |\nabla u(t,x)| \leq 1/2$, so we obtain with $\varphi(t,x) = x + u(t,x)$:

$$\begin{aligned} &\left|\varphi\left(t, \Psi^N\left(t, Y_t^N\right)\right) - \varphi\left(t, \Psi\left(t, Y_t^N\right)\right)\right| \\ &\geq \inf_{x\in\mathbb{R}} |\nabla\varphi(t,x)| \left|\Psi^N\left(t, Y_t^N\right) - \Psi\left(t, Y_t^N\right)\right| \\ &\geq \frac{1}{2}\left|\Psi^N\left(t, Y_t^N\right) - \Psi\left(t, Y_t^N\right)\right| \end{aligned}$$

and

$$\begin{aligned}\left|\Psi^N\left(t,Y_t^N\right)-\Psi\left(t,Y_t^N\right)\right| &\leq 2\ \left|\varphi\left(t,\Psi^N\left(t,Y_t^N\right)\right)-\varphi\left(t,\Psi\left(t,Y_t^N\right)\right)\right| \\ &= 2\ \left|\varphi\left(t,\Psi^N\left(t,Y_t^N\right)\right)-\varphi^N\left(t,\Psi^N\left(t,Y_t^N\right)\right)\right| \\ &\leq 2\ \left\|u(t)-u^N(t)\right\|_\infty \\ &\leq 2\kappa\left\|b-b^N\right\|_{H_q^{-\beta}} \end{aligned} \tag{III.4}$$

where we have used Lemma 5 and the fact that

$$\varphi^N\left(t,\Psi^N\left(t,Y_t^N\right)\right)=\varphi\left(t,\Psi\left(t,Y_t^N\right)\right)=Y_t^N.$$

□

We will need an adapted version of a local time inequality (Lemma 4.2 in [4]) from Liqing Yan:

Lemma 7. *Let X be a continuous semimartingale. For $\varepsilon>0$ we define a double sequence of stopping times by $\sigma_1=\inf\{t\geq 0|X_t=0\}$, $\tau_1=\inf\{t>\sigma_1|X_t=\varepsilon\}$, $\sigma_n=\inf\{t>\tau_{n-1}|X_t=0\}$, $\tau_n=\inf\{t>\sigma_n|X_t=\varepsilon\}$. For any real function $F(\cdot)\in\mathcal{C}^2(\mathbb{R})$ with $F(0)=0$, $F'(0)=0$, $F(\cdot)>0$ on $(0,\varepsilon_0)$ with some $\varepsilon_0>0$, and for any $0<\varepsilon<\varepsilon_0$ we have*

$$0\leq L_t^0(X)\leq 2\varepsilon-\frac{2}{F(\varepsilon)}\int_{\sigma_1\wedge t}^t\theta_s(X)\left(F\left(\varepsilon\right)-\varepsilon F'\left(X_s^+\right)\right)\,\mathrm{d}X_s+\frac{\varepsilon}{F(\varepsilon)}\int_{\sigma_1\wedge t}^t\theta_s(X)F''(X_s^+)\,\mathrm{d}[X]_s$$

with $\theta_s(X)=\sum_{n=1}^\infty \mathbb{1}_{\{\sigma_n<s\leq\tau_n,\ 0<X_s\leq\varepsilon\}}(X)$.

Applying lemma 7 with $F:x\in\mathbb{R}\mapsto x^2$, it follows:

Corollary 8. *Let X be a continuous semimartingale. With the same notations as in lemma 7, for any $\varepsilon>0$ we have*

$$0\leq L_t^0(X)\leq 2\varepsilon-\frac{2}{\varepsilon}\int_{\sigma_1\wedge t}^t\theta_s(X)\left(\varepsilon-2X_s^+\right)\,\mathrm{d}X_s+\frac{2}{\varepsilon}\int_{\sigma_1\wedge t}^t\theta_s(X)\,\mathrm{d}[X]_s \tag{III.5}$$

Proof of Lemma 7. We note $U_t(X)=\sup\{n\in\mathbb{N}|\tau_n<t\}$ and $n(t)=t\wedge\sigma_{U_t(X)+1}$. By Meyer-Tanaka's formula, $\forall i\in\mathbb{N}^*$:

$$X_{\tau_i\wedge t}^+-X_{\sigma_i\wedge t}^+=\int_{\sigma_i\wedge t}^{\tau_i\wedge t}\mathbb{1}_{\{X_s>0\}}\,\mathrm{d}X_s+\frac{1}{2}\left\{L_{\tau_i\wedge t}^0(X)-L_{\sigma_i\wedge t}^0(X)\right\}. \tag{III.6}$$

Because $\forall i\in\mathbb{N}^*$, $L_{\tau_i\wedge t}^0(X)=L_{\sigma_{i+1}\wedge t}^0(X)$ and $L_{\sigma_1\wedge t}^0(X)=0$, we have

$$\sum_{i=1}^{U_t(X)+1}\left(X_{\tau_i\wedge t}^+-X_{\sigma_i\wedge t}^+\right)=\int_{\sigma_1\wedge t}^t\theta_s(X)\,\mathrm{d}X_s+\frac{1}{2}\,L_t^0(X).$$

The left term is equal to $\varepsilon U_t(X)+X_t^+-X_{n(t)}^+$ so

$$\varepsilon U_t(X)=\int_{\sigma_1\wedge t}^t\theta_s(X)\,\mathrm{d}X_s+\frac{1}{2}\,L_t^0(X)-X_t^++X_{n(t)}^+. \tag{III.7}$$

Now we express differently $U_t(X)$. $F \in \mathcal{C}^2(\mathbb{R})$ so by Itô's formula:

$$F\left(X^+_{\tau_i\wedge t}\right) - F\left(X^+_{\sigma_i\wedge t}\right) = \int_{\sigma_i\wedge t}^{\tau_i\wedge t} F'\left(X^+_s\right) \mathrm{d}X^+_s + \frac{1}{2}\int_{\sigma_i\wedge t}^{\tau_i\wedge t} F''\left(X^+_s\right) \mathrm{d}[X^+]_s.$$

By (III.6), $\mathrm{d}X^+_s = \mathbb{1}_{\{X_s>0\}}\mathrm{d}X_s + \frac{1}{2}\,\mathrm{d}L^0_t(X)$ and $\mathrm{d}[X^+]_s = \mathbb{1}_{\{X_s>0\}}\mathrm{d}[X]_s$. It follows

$$\begin{aligned}F\left(X^+_{\tau_i\wedge t}\right) - F\left(X^+_{\sigma_i\wedge t}\right) = \int_{\sigma_i\wedge t}^{\tau_i\wedge t} F'\left(X^+_s\right) \mathbb{1}_{\{X_s>0\}}\, \mathrm{d}X_s + \frac{1}{2}\int_{\sigma_i\wedge t}^{\tau_i\wedge t} F'(X^+_s)\, \mathrm{d}L^0_t(X)\\ + \frac{1}{2}\int_{\sigma_i\wedge t}^{\tau_i\wedge t} F''\left(X^+_s\right) \mathbb{1}_{\{X_s>0\}}\, \mathrm{d}[X]_s.\end{aligned}$$

Adding up for i, with $F(0) = 0$ we obtain

$$\begin{aligned}&F(\varepsilon)U_t(X) + F\left(X^+_t\right) - F\left(X^+_{n(t)}\right) = \sum_{i=1}^{U_t(X)+1}\left(F\left(X^+_{\tau_i\wedge t}\right) - F\left(X^+_{\sigma_i\wedge t}\right)\right)\\ &= \int_{\sigma_1\wedge t}^{t} F'(X^+_s)\,\theta_s(X)\,\mathrm{d}X_s + \frac{1}{2}\int_{\sigma_1\wedge t}^{t} F'\left(X^+_s\right)\Xi_s\,\mathrm{d}L^0_t(X) + \frac{1}{2}\int_{\sigma_1\wedge t}^{t} F''\left(X^+_s\right)\theta_s(X)\,\mathrm{d}[X]_s\end{aligned}$$

with $\Xi_s = \sum_{n=1}^{\infty}\mathbb{1}_{\{\sigma_n<s\le\tau_n\}}$. The measure $\mathrm{d}L^0_t(X)$ is almost surely carried by $\{t|X_t = 0\}$ so we can simplify $\int_{\sigma_1\wedge t}^{t} F'\left(X^+_s\right)\Xi_s\,\mathrm{d}L^0_t(X) = F'(0)\int_{\sigma_1\wedge t}^{t}\Xi_s\,\mathrm{d}L^0_t(X)$ in order to have, with $F'(0) = 0$:

$$F(\varepsilon)U_t(X) = -F\left(X^+_t\right) + F\left(X^+_{n(t)}\right) + \int_{\sigma_1\wedge t}^{t} F'\left(X^+_s\right)\theta_s(X)\,\mathrm{d}X_s + \frac{1}{2}\int_{\sigma_1\wedge t}^{t} F''\left(X^+_s\right)\theta_s(X)\,\mathrm{d}[X]_s. \tag{III.8}$$

Combining (III.7) and (III.8), it follows

$$\begin{aligned}L^0_t(X)F(\varepsilon) = 2F(\varepsilon)(X^+_t - X^+_{n(t)}) - 2\varepsilon(F\left(X^+_t\right) - F\left(X^+_{n(t)}\right)\\ - 2\int_{\sigma_1\wedge t}^{t}\left(F(\varepsilon) - \varepsilon F'\left(X^+_s\right)\right)\theta_s(X)\,\mathrm{d}X_s + \varepsilon\int_{\sigma_1\wedge t}^{t} F''\left(X^+_s\right)\theta_s(X)\,\mathrm{d}[X]_s.\end{aligned}$$

Then, if $n(t) = t$, the two first right terms of the equality are equal to zero. Else, if $n(t) = \sigma_{U_t(X)+1}$, $0 \le F(\varepsilon)(X^+_t - X^+_{n(t)}) = F(\varepsilon)X^+_t \le F(\varepsilon)\varepsilon$ and $-2\varepsilon(F\left(X^+_t\right) - F\left(X^+_{n(t)}\right) = -2\varepsilon F\left(X^+_t\right) \le 0$ because of the positivity of F. Finally we obtain:

$$L^0_t(X) \le 2\varepsilon - \frac{2}{F(\varepsilon)}\int_{\sigma_1\wedge t}^{t}\left(F(\varepsilon) - \varepsilon F'\left(X^+_s\right)\right)\theta_s(X)\,\mathrm{d}X_s + \frac{\varepsilon}{F(\varepsilon)}\int_{\sigma_1\wedge t}^{t} F''\left(X^+_s\right)\theta_s(X)\,\mathrm{d}[X]_s.$$

□

Lemma 9. *Let $(\delta, p) \in K(\beta, q)$, $\alpha = \delta - 1/p < 1$, u, u^N be the mild solutions to (I.4), (I.5) in $H^{1+\delta}_p$, and Y, Y^N solutions of the SDEs (I.3), (I.6). Then, if $\alpha > 1/2$, for λ big enough we have $\forall\varepsilon \in (0, 1]$,*

$$0 \le \mathbb{E}\left[L^0_T(Y - Y^N)\right] \le g(\varepsilon).$$

where

$$\begin{aligned}g(\varepsilon) \le 4(\lambda+1)T\kappa\left\|b - b^N\right\|_{H^{-\beta}_q} + \left(2 + 2(\lambda+1)\,T + 6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}}4^\alpha T\right)\varepsilon^{2\alpha-1}\\ + 6T\left(\Omega^2 4^\alpha\kappa^{2\alpha}\left\|b - b^N\right\|^{2\alpha}_{H^{-\beta}_q} + \kappa^2\left\|b - b^N\right\|^2_{H^{-\beta}_q}\right)\varepsilon^{-1}.\end{aligned}$$

Proof. By Lemma 4, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\left\|b - b^N\right\|_{H^s_q(\mathbb{R})} \underset{N\to\infty}{\longrightarrow} 0$. Therefore $u^N(t,\cdot)$ and $u(t,\cdot)$ are $\frac{1}{2}$-lipschitz. Let $\varepsilon \in (0,1]$. Corollary 8 gives us:

$$0 \leq L^0_T(Y-Y^N) \leq 2\varepsilon - \frac{2}{\varepsilon}\int_{\sigma_1\wedge T}^T \theta_s(Y-Y^N)\left(\varepsilon - 2(Y_s - Y^N_s)^+\right)\,\mathrm{d}\left(Y_s - Y^N_s\right) + \frac{2}{\varepsilon}\int_{\sigma_1\wedge T}^T \theta_s(Y-Y^N)\,\mathrm{d}\left[Y-Y^N\right]_s$$

with

$$Y_T - Y^N_T = y - y^N + (\lambda+1)\int_0^T \left\{u\left(s,\Psi\left(s,Y_s\right)\right) - u^N\left(s,\Psi^N\left(s,Y^N_s\right)\right)\right\}\,\mathrm{d}s + \int_0^T \left\{\nabla u\left(s,\Psi\left(s,Y_s\right)\right) - \nabla u^N\left(t,\Psi^N\left(s,Y^N_s\right)\right)\right\}\,\mathrm{d}W_s.$$

Remark 3. *Note that* $\theta_s(Y-Y^N)\left|\varepsilon - 2(Y_s-Y^N_s)^+\right| \leq \varepsilon\theta_s(Y-Y^N)$.

Remark 4. *For clarity purpose, we note* $\tilde{u}\left(s,x\right) = u\left(s,\Psi\left(s,x\right)\right)$, $\tilde{u}^N\left(s,x\right) = u^N\left(s,\Psi\left(s,x\right)\right)$, $\overline{u}^N\left(s,x\right) = u^N\left(s,\Psi^N\left(s,x\right)\right)$ *and use the same notations for the gradient and the approximated mild solution. We recall that in this case, by Lemma 22 in [1],* $\Psi(t,\cdot)$ *and* $\Psi^N(t,\cdot)$ *are 2-lipschitz. We can notice that* $\tilde{u}$ *is 1-lipschitz in space and* $\widetilde{\nabla u}$ *is* α*-Hölder with constant* $2^\alpha\left\|u\right\|_{\mathcal{C}^{1,\alpha}}$. *The same properties hold for* $\overline{u}$ *and* $\overline{\nabla u}$ *except that the Hölder constant for* $\overline{\nabla u}$ *can be bounded by a constant* Ω *for* N *big enough (see Lemma 24 in [1]).*

∇u and ∇u^N are bounded so the Itô integral is a martingale. We take the expectation:

$$\mathbb{E}\left[L^0_T(Y-Y^N)\right] \leq 2\varepsilon + 2(\lambda+1)\,\mathbb{E}\left[\int_{\sigma_1\wedge T}^T \theta_s\left(Y-Y^N\right)\left|\tilde{u}\left(s,Y_s\right) - \overline{u}^N\left(s,Y^N_s\right)\right|\mathrm{d}s\right] + \frac{2}{\varepsilon}\,\mathbb{E}\left[\int_{\sigma_1\wedge T}^T \theta_s\left(Y-Y^N\right)\left\{\widetilde{\nabla u}\left(s,Y_s\right) - \overline{\nabla u}\left(s,Y^N_s\right)\right\}^2\,\mathrm{d}s\right]$$

$$\begin{aligned}
&\leq 2\varepsilon + 2(\lambda+1)\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|\tilde{u}\left(s,Y_s\right) - \tilde{u}\left(s,Y^N_s\right)\right|\,\mathrm{d}s\right]\\
&\quad + 2(\lambda+1)\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|\tilde{u}\left(s,Y^N_s\right) - \tilde{u}^N\left(s,Y^N_s\right)\right|\,\mathrm{d}s\right]\\
&\quad + 2(\lambda+1)\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|\tilde{u}^N\left(s,Y^N_s\right) - \overline{u}^N\left(s,Y^N_s\right)\right|\mathrm{d}s\right]\\
&\quad + \frac{6}{\varepsilon}\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left\{\widetilde{\nabla u}\left(s,Y_s\right) - \widetilde{\nabla u}\left(s,Y^N_s\right)\right\}^2\,\mathrm{d}s\right]\\
&\quad + \frac{6}{\varepsilon}\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left\{\widetilde{\nabla u}\left(s,Y^N_s\right) - \widetilde{\nabla u}^N\left(s,Y^N_s\right)\right\}^2\,\mathrm{d}s\right]\\
&\quad + \frac{6}{\varepsilon}\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left\{\widetilde{\nabla u}^N\left(s,Y^N_s\right) - \overline{\nabla u}^N\left(s,Y^N_s\right)\right\}^2\,\mathrm{d}s\right]
\end{aligned}$$

$$\leq 2\varepsilon + 2(\lambda+1)\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|Y_s-Y_s^N\right|\ \mathrm{d}s\right] + 2(\lambda+1)T\kappa\left\|b^N-b\right\|_{H_q^{-\beta}}$$
$$+ (\lambda+1)\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|\Psi\left(s,Y_s^N\right)-\Psi^N\left(s,Y_s^N\right)\right|\mathrm{d}s\right]$$
$$+ \frac{6\times 4^\alpha \left\|u\right\|^2_{\mathcal{C}^{1,\alpha}}}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|Y_s-Y_s^N\right|^{2\alpha}\ \mathrm{d}s\right] + 6T\kappa^2\left\|b^N-b\right\|^2_{H_q^{-\beta}}\varepsilon^{-1}$$
$$+ \frac{6\Omega^2}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|\Psi\left(s,Y_s^N\right)-\Psi^N\left(s,Y_s^N\right)\right|^{2\alpha}\ \mathrm{d}s\right]$$

where we have used Lemma 5, the 1-lipschitz property of $\widetilde{u}$, the 1/2-lipschitz property of u^N, the α-Hölder property of $\widetilde{\nabla u}$ (with constant $2^\alpha\left\|u\right\|_{\mathcal{C}^{1,\alpha}}$), and the α-Hölder property of ∇u^N (with constant Ω). Lemma 6 gives us:

$$\leq 2\varepsilon + 2(\lambda+1)\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|Y_s-Y_s^N\right|\ \mathrm{d}s\right] + 4(\lambda+1)T\kappa\left\|b^N-b\right\|_{H_q^{-\beta}}$$
$$+ \frac{6\times 4^\alpha \left\|u\right\|^2_{\mathcal{C}^{1,\alpha}}}{\varepsilon}\ \mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|Y_s-Y_s^N\right|^{2\alpha}\ \mathrm{d}s\right] + 6T\kappa^2\left\|b-b^N\right\|^2_{H_q^{-\beta}}\varepsilon^{-1}$$
$$+ 6\Omega^2 T 4^\alpha\kappa^{2\alpha}\left\|b-b^N\right\|^{2\alpha}_{H_q^{-\beta}}\varepsilon^{-1}.$$

As $\theta_s(Y-Y^N)\left|Y_s-Y_s^N\right|\leq\varepsilon$, we have

$$\mathbb{E}\left[L_T^0(Y-Y^N)\right] \leq 2\varepsilon + 2(\lambda+1)\ T\varepsilon + 4(\lambda+1)T\kappa\left\|b-b^N\right\|_{H_q^{-\beta}} + 6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}}4^\alpha T\varepsilon^{2\alpha-1}$$
$$+ 6T\left(\Omega^2 4^\alpha\kappa^{2\alpha}\left\|b-b^N\right\|^{2\alpha}_{H_q^{-\beta}} + \kappa^2\left\|b-b^N\right\|^2_{H_q^{-\beta}}\right)\varepsilon^{-1}.$$

As $1>2\alpha-1>0$, the result follows from $\varepsilon\leq\varepsilon^{2\alpha-1}$ when $0<\varepsilon\leq 1$.

□

Lemma 10. *With assumptions and notations of Lemma 9, and $1>\alpha>1/2$ we have*

$$\mathbb{E}\left[L_T^0(Y-Y^N)\right] \leq g(\varepsilon_N) = \sigma\left\|b^N-b\right\|^{2\alpha-1}_{H_q^{-\beta}} \tag{III.9}$$

for $\left\|b^N-b\right\|_{H_q^{-\beta}}$ small enough (it is to say N big enough) where

$$\sigma = 4(\lambda+1)T\kappa + \left(2+2(\lambda+1)\ T + 6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}}4^\alpha T\right)2\omega_\infty^{2\alpha-1} + 6T\left(\Omega^2 4^\alpha\kappa^{2\alpha}+\kappa^2\right)\omega_\infty^{-1}$$

and

$$\omega_\infty = \left(\frac{6T\Omega^2 4^\alpha\kappa^{2\alpha}}{(2\alpha-1)\left(2+2(\lambda+1)\ T+6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}}4^\alpha T\right)}\right)^{\frac{1}{2\alpha}}.$$

Proof. Let $\varepsilon\in(0,1]$, by Lemma 9,

$$0\leq\mathbb{E}\left[L_T^0(Y-Y^N)\right]\leq g(\varepsilon)$$

where

$$g(\varepsilon) = 4(\lambda+1)T\kappa\left\|b-b^N\right\|_{H_q^{-\beta}} + \left(2+2(\lambda+1)\ T+6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}}4^\alpha T\right)\varepsilon^{2\alpha-1}$$
$$+ 6T\left(\Omega^2 4^\alpha\kappa^{2\alpha}\left\|b-b^N\right\|^{2\alpha}_{H_q^{-\beta}} + \kappa^2\left\|b-b^N\right\|^2_{H_q^{-\beta}}\right)\varepsilon^{-1}.$$

With

$$g'(\varepsilon) = (2\alpha - 1)\left(2 + 2(\lambda+1)\ T + 6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}} 4^\alpha T\right)\varepsilon^{2\alpha-2} \\ - 6T\left(\Omega^2 4^\alpha \kappa^{2\alpha}\left\|b - b^N\right\|^{2\alpha}_{H_q^{-\beta}} + \kappa^2\left\|b - b^N\right\|^2_{H_q^{-\beta}}\right)\varepsilon^{-2},$$

and

$$g''(\varepsilon) = (2\alpha - 2)(2\alpha - 1)\left(2 + 2(\lambda+1)\ T + 6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}} 4^\alpha T\right)\varepsilon^{2\alpha-3} \\ + 12T\left(\Omega^2 4^\alpha \kappa^{2\alpha}\left\|b - b^N\right\|^{2\alpha}_{H_q^{-\beta}} + \kappa^2\left\|b - b^N\right\|^2_{H_q^{-\beta}}\right)\varepsilon^{-3},$$

the minimum of g on $(0, 1]$ is reached when N is big enough in

$$\varepsilon_N = \left(\frac{6T\left(\Omega^2 4^\alpha \kappa^{2\alpha}\left\|b - b^N\right\|^{2\alpha}_{H_q^{-\beta}} + \kappa^2\left\|b - b^N\right\|^2_{H_q^{-\beta}}\right)}{(2\alpha-1)\left(2 + 2(\lambda+1)\ T + 6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}} 4^\alpha T\right)}\right)^{\frac{1}{2\alpha}} = \omega_N\left\|b^N - b\right\|_{H_q^{-\beta}}.$$

where

$$g''(\varepsilon_N) = 12T\left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2\left\|b - b^N\right\|^{2(1-\alpha)}_{H_q^{-\beta}}\right)\varepsilon_N^{-3}\alpha > 0.$$

and

$$\omega_N = \left(\frac{6T\left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2\left\|b - b^N\right\|^{2(1-\alpha)}_{H_q^{-\beta}}\right)}{(2\alpha-1)\left(2 + 2(\lambda+1)\ T + 6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}} 4^\alpha T\right)}\right)^{\frac{1}{2\alpha}}$$

$$\geq \omega_\infty = \left(\frac{6T\Omega^2 4^\alpha \kappa^{2\alpha}}{(2\alpha-1)\left(2 + 2(\lambda+1)\ T + 6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}} 4^\alpha T\right)}\right)^{\frac{1}{2\alpha}}.$$

Therefore $\mathbb{E}\left[L_T^0(Y - Y^N)\right] \leq g(\varepsilon_N)$

$$\leq 4(\lambda+1)T\kappa\left\|b - b^N\right\|_{H_q^{-\beta}} + \left(2 + 2(\lambda+1)\ T + 6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}} 4^\alpha T\right)\omega_N^{2\alpha-1}\left\|b - b^N\right\|^{2\alpha-1}_{H_q^{-\beta}} \\ + 6T\left(\Omega^2 4^\alpha \kappa^{2\alpha}\left\|b - b^N\right\|^{2\alpha}_{H_q^{-\beta}} + \kappa^2\left\|b - b^N\right\|^2_{H_q^{-\beta}}\right)\omega_N^{-1}\left\|b - b^N\right\|^{-1}_{H_q^{-\beta}}$$

$$\leq 4(\lambda+1)T\kappa\left\|b - b^N\right\|_{H_q^{-\beta}} + \left(2 + 2(\lambda+1)\ T + 6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}} 4^\alpha T\right)\omega_N^{2\alpha-1}\left\|b - b^N\right\|^{2\alpha-1}_{H_q^{-\beta}} \\ + 6T\left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2\left\|b - b^N\right\|^{2(1-\alpha)}_{H_q^{-\beta}}\right)\omega_\infty^{-1}\left\|b - b^N\right\|^{2\alpha-1}_{H_q^{-\beta}}$$

$$\leq 4(\lambda+1)T\kappa\left\|b - b^N\right\|^{2\alpha-1}_{H_q^{-\beta}} + \left(2 + 2(\lambda+1)\ T + 6\left\|u\right\|^2_{\mathcal{C}^{1,\alpha}} 4^\alpha T\right)2\omega_\infty^{2\alpha-1}\left\|b - b^N\right\|^{2\alpha-1}_{H_q^{-\beta}} \\ + 6T\left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2\right)\omega_\infty^{-1}\left\|b - b^N\right\|^{2\alpha-1}_{H_q^{-\beta}}$$

for N big enough. The result follows. □

III.2.2 Main result

Theorem 11. *Let f be μ-Hölder with constant $C_f > 0$ and $\mu \in (0,1]$. If $0 < \beta < 1/4$, $q \in \left(\frac{1}{1-\beta}, \frac{1}{\beta}\right)$, $\forall \varepsilon \in (0, 1-4\beta)$, with $(\delta, p) \in K(\beta, q)$ such that $\delta - 1/p = 1 - 2\beta - \varepsilon/2$, exists ξ independent of f such that for $N \in \mathbb{N}$, $\rho > 1$, λ big enough it holds:*

$$\sup_{0\le t\le T} \mathbb{E}\left[\left|f\left(X_t\right) - f\left(X_t^N\right)\right|\right] \le \xi C_f \left\|b^N - b\right\|_{H_q^{-\beta}}^{\mu(1-4\beta-\varepsilon)}$$

Proof. We note as usual $\alpha = \delta - 1/p$. By Lemma 4, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\left\|b - b^N\right\|_{H_q^s(\mathbb{R})} \underset{N\to\infty}{\longrightarrow} 0$. We recall that in this case, by Lemma 22 in [1], $\Psi(t,\cdot)$ and $\Psi^N(t,\cdot)$ are 2-lipschitz. Therefore $\widetilde{u}^N(t,\cdot)$ and $\widetilde{u}(t,\cdot)$ are 1-lipschitz. Let $t \in [0,T]$.

$$\mathbb{E}\left[\left|f\left(X_t\right) - f\left(X_t^N\right)\right|\right] = \mathbb{E}\left[\left|f\left(\Psi\left(t, Y_t\right)\right) - f\left(\Psi^N\left(t, Y_t^N\right)\right)\right|\right]$$

$$\begin{aligned}
&\le \mathbb{E}\left[\left|f\left(\Psi\left(t, Y_t\right)\right) - f\left(\Psi\left(t, Y_t^N\right)\right)\right|\right] + \mathbb{E}\left[\left|f\left(\Psi\left(t, Y_t^N\right)\right) - f\left(\Psi^N\left(t, Y_t^N\right)\right)\right|\right] \\
&\le C_f\left(2^\mu \mathbb{E}\left[\left|Y_t - Y_t^N\right|^\mu\right] + \mathbb{E}\left[\left|\Psi\left(t, Y_t^N\right) - \Psi^N\left(t, Y_t^N\right)\right|^\mu\right]\right) \\
&\le C_f\left(2^\mu \mathbb{E}\left[\left|Y_t - Y_t^N\right|\right]^\mu + \mathbb{E}\left[\left|\Psi\left(t, Y_t^N\right) - \Psi^N\left(t, Y_t^N\right)\right|\right]^\mu\right)
\end{aligned} \tag{III.10}$$

by Jensen's inequality.

$$\begin{aligned}
Y_T - Y_T^N = y - y^N + (\lambda+1)\int_0^T &\left\{u\left(s, \Psi\left(s, Y_s\right)\right) - u^N\left(s, \Psi^N\left(s, Y_s^N\right)\right)\right\} \, \mathrm{d}s \\
&+ \int_0^T \left\{\nabla u\left(s, \Psi\left(s, Y_s\right)\right) - \nabla u^N\left(t, \Psi^N\left(s, Y_s^N\right)\right)\right\} \, \mathrm{d}W_s.
\end{aligned}$$

We apply Meyer-Tanaka's formula to obtain:

$$\begin{aligned}
\left|Y_t - Y_t^N\right| = \left|y - y^N\right| + (\lambda+1)\int_0^t &\operatorname{sign}(Y_s - Y_s^N)\left\{\widetilde{u}\left(s, Y_s\right) - \overline{u}^N\left(s, Y_s^N\right)\right\} \, \mathrm{d}s \\
&+ \int_0^t \operatorname{sign}(Y_s - Y_s^N)\left\{\widetilde{\nabla u}\left(s, Y_s\right) - \overline{\nabla u}^N\left(s, Y_s^N\right)\right\} \, \mathrm{d}W_s + L_t^0(Y - Y^N).
\end{aligned}$$

Taking the expectation leads to:

$$\begin{aligned}
\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] &= \left|u(0,x) - u^N(0,x)\right| \\
&+ (\lambda+1)\,\mathbb{E}\left[\int_0^t \operatorname{sign}(Y_s - Y_s^N)\left\{\widetilde{u}\left(s, Y_s\right) - \overline{u}^N\left(s, Y_s^N\right)\right\} \, \mathrm{d}s\right] + \mathbb{E}\left[L_t^0(Y - Y^N)\right]
\end{aligned}$$

because ∇u and ∇u^N are bounded so the Itô integral is a martingale.

$$\begin{aligned}
\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] &\le \left|u(0,x) - u^N(0,x)\right| + (\lambda+1)\,\mathbb{E}\left[\int_0^t \left|\widetilde{u}\left(s, Y_s\right) - \widetilde{u}\left(s, Y_s^N\right)\right| \mathrm{d}s\right] + \\
&+(\lambda+1)\,\mathbb{E}\left[\int_0^t \left|\widetilde{u}\left(s, Y_s^N\right) - \widetilde{u}^N\left(s, Y_s^N\right)\right| \mathrm{d}s\right] + (\lambda+1)\,\mathbb{E}\left[\int_0^t \left|\widetilde{u}^N\left(s, Y_s^N\right) - \overline{u}^N\left(s, Y_s^N\right)\right| \mathrm{d}s\right] \\
&+ \mathbb{E}\left[L_t^0(Y - Y^N)\right].
\end{aligned}$$

We use Lemma 5, the 1-lipschitz property of $\widetilde{u}$, and the 1/2-lipschitz property of u:

$$\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] \leq \kappa \left\|b^N - b\right\|_{H_q^{-\beta}} + (\lambda+1)\,\mathbb{E}\left[\int_0^t \left|Y_s - Y_s^N\right| \mathrm{d}s\right] + (\lambda+1)t\kappa \left\|b^N - b\right\|_{H_q^{-\beta}}$$
$$+ \frac{\lambda+1}{2}\,\mathbb{E}\left[\int_0^t \left|\Psi\left(s, Y_s^N\right) - \Psi^N\left(s, Y_s^N\right)\right| \mathrm{d}s\right] + \mathbb{E}\left[L_t^0(Y - Y^N)\right]$$

$$\leq (\lambda+1) \int_0^t \mathbb{E}\left[\left|Y_s - Y_s^N\right|\right] \mathrm{d}s + (2(\lambda+1)T+1)\kappa \left\|b^N - b\right\|_{H_q^{-\beta}} + \mathbb{E}\left[L_T^0(Y - Y^N)\right].$$

where we have used the fact that $L_t^0(Y - Y^N)$ is an increasing process and Lemma 6.

By Gronwall's Lemma, it follows:

$$\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] \leq C(N)\, e^{(\lambda+1)t} \leq C(N)\, e^{(\lambda+1)T} \tag{III.11}$$

with $C(N) = (2(\lambda+1)T+1)\kappa \left\|b^N - b\right\|_{H_q^{-\beta}} + \mathbb{E}\left[L_T^0(Y - Y^N)\right]$.

With Lemma 9 and Lemma 10 we obtain

$$C(N) \leq (2(\lambda+1)T+1)\kappa \left\|b^N - b\right\|_{H_q^{-\beta}} + \sigma \left\|b^N - b\right\|_{H_q^{-\beta}}^{2\alpha-1} \leq \zeta \left\|b^N - b\right\|_{H_q^{-\beta}}^{2\alpha-1}.$$

for $\left\|b^N - b\right\|_{H_q^{-\beta}}$ small enough where $\zeta = (2(\lambda+1)T+1)\kappa + \sigma$. It follows:

$$\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] \leq \zeta e^{(\lambda+1)T} \left\|b^N - b\right\|_{H_q^{-\beta}}^{2\alpha-1}. \tag{III.12}$$

Finally, combining (III.10), (III.12) and Lemma 6 we obtain:

$$\left|\mathbb{E}\left[f\left(X_t\right) - f\left(X_t^N\right)\right]\right|$$
$$\leq C_f\left(2^\mu \mathbb{E}\left[\left|Y_t - Y_t^N\right|\right]^\mu + \mathbb{E}\left[\left|\Psi\left(t, Y_t^N\right) - \Psi^N\left(t, Y_t^N\right)\right|\right]^\mu\right)$$
$$\leq C_f\left(2^\mu \zeta^\mu e^{\mu(\lambda+1)T} \left\|b^N - b\right\|_{H_q^{-\beta}}^{\mu(2\alpha-1)} + 2^\mu \kappa^\mu \left\|b^N - b\right\|_{H_q^{-\beta}}^{\mu}\right)$$
$$\leq 2^\mu C_f\left(\zeta^\mu e^{\mu(\lambda+1)T} + 2^\mu \kappa^\mu\right) \left\|b^N - b\right\|_{H_q^{-\beta}}^{\mu(2\alpha-1)}$$

for N big enough, which is the expected result. □

III.3 Convergence in law of X_t^N to X_t

Part IV

Numerical results

IV.1 Strong convergence of the Euler scheme

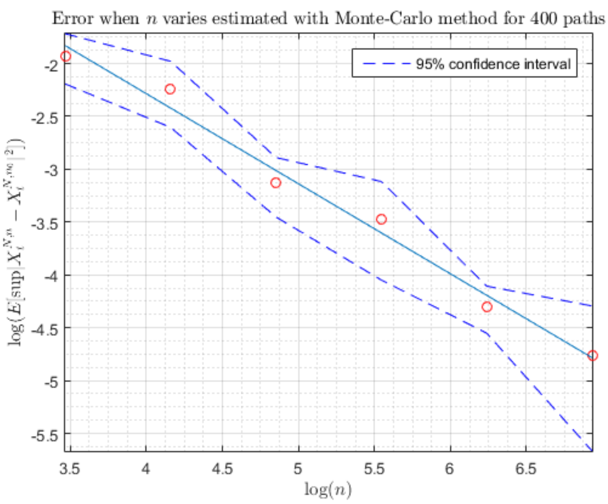

Figure IV.1: Estimation of the L^2 error of the Euler-Marayuma scheme with a Monte-Carlo method. 400 paths, $N = 5$, $n \in \{2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}\}$, reference solution with $n_0 = 2^{12}$ points.

We observe a numerical convergence rate of 0.85 when Theorem 2 shows a theoretical rate of $0.5 - \varepsilon$.

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