NUMERICAL SIMULATION OF SDES WITH DISTRIBUTIONAL DRIFT

Maximilien Germain

May 2018

1 Introduction

We would like to simulate numerically sample paths of the solution of the stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\ \mathrm{d}t + \mathrm{d}W_t \tag{1}$$

where $b \in H_q^s(\mathbb{R}),\ s \in \left]-\frac{1}{2}, 0\right[,\ t \in [0,T]$, and W_t is a standard Brownian motion. This equation is studied in [1] in which the authors prove existence and unicity in law of a virtual solution for equation (1).

Example 1.1. *An example of such drift b is given by the derivative of a sample path of a fractional Brownian motion B_x^H with Hurst index $1/2 < H < 1$. These stochastic processes are gaussian processes verifying*

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} + |t-s|^{2H}\right).$$

We note $s = H - 1$. Given $B_x^H(\omega) \in H_q^{s+1}(\mathbb{R})$, we can take $b(x) = \frac{\partial}{\partial x} B_x^H(\omega) \in H_q^s(\mathbb{R})$. We will use this in our numerical simulations.

As far as the drift b is not a function but a distribution, it must be approximated if we want to evaluate it at points. In order to do so, we will use a series representation of b and truncate it. That is why we will consider two steps in our algorithm:

1. approximate the drift b by a function b^N meant to converge to b as $N \to \infty$.
2. approximate the solution X_t^N of the approximated SDE

   $$\mathrm{d}X_t^N = b^N\left(X_t^N\right)\ \mathrm{d}t + \mathrm{d}W_t \tag{2}$$

   by $X_t^{N,n}$ defined with the Euler-Maruyama scheme

   $$X_t^{N,n} = X_0 + \int_0^t b^N\left(X_{\eta_n(t)}^{N,n}\right)\mathrm{d}t + W_{\eta_n(t)}$$

   where $\eta_n(t) = t_k$ if $t \in [t_k, t_k + 1]$, for $t_k = \frac{k}{2^n}$ with $k \in [\![0, \lceil 2^n T\rceil]\!]$.

2 Numerical simulation of fractional Brownian motion

To simulate a sample path of a fractional brownian motion B^H_x on a finite grid $(x_k)_{k\in[\![1,n]\!]}$, we simulate n independent standard gaussian random variables $(X_k)_{k\in[\![1,n]\!]}$ and then correlate them with the definite positive correlation matrix

$$C_{k,s} = \mathbb{E}\left[B^H_{x_k} B^H_{x_s}\right] = \frac{1}{2}\left(x_k^{2H} + x_s^{2H} + |x_k - x_s|^{2H}\right).$$

To do so, we use the Cholesky decomposition method and calculate the triangular matrix M such that $C = MM^\top$. Therefore, defining

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ and } B^H = MX,$$

B^H contains the values of a fractional brownian motion evaluated on the grid $(x_k)_{k\in[\![1,n]\!]}$.

3 Approximation of the drift

3.1 Series representation

We use Haar wavelets to give a series representation of b. By doing so, we will be able to approximate it numerically by truncating the series.

Definition 3.1 (Haar wavelets)**.** *We define the Haar wavelets $h_{j,m}$ on $\mathbb{R}$ with $j \in \mathbb{N} \cup \{-1\}$ and $m \in \mathbb{Z}$ by:*

$$\begin{cases} h_M & : x \longmapsto \left(\mathbb{1}_{[0,\frac{1}{2}[} - \mathbb{1}_{[\frac{1}{2},1[}\right)(x) \\ h_{-1,m} & : x \longmapsto \sqrt{2}|h_M(x-m)| \\ h_{j,m} & : x \longmapsto h_M(2^j x - m) \end{cases}$$

Theorem 3.1 (See [2])**.** *Let $b \in H^s_q(\mathbb{R})$ for $2 \le q \le \infty$, and $s \in \left]-\frac{1}{2}, \frac{1}{q}\right[$. Therefore,*

$$b = \sum_{j=-1}^{+\infty} \sum_{m\in\mathbb{Z}} \mu_{j,m} h_{j,m} \tag{3}$$

where $\mu_{j,m} = 2^j \int_{\mathbb{R}} b(x) h_{j,m}(x)\,\mathrm{d}x$ in the sense of dual pairing.

Definition 3.2. *With the same notation $\mu_{j,m}$, let $b \in H^s_q(\mathbb{R})$ for $2 \le q \le \infty$, and $s \in \left]-\frac{1}{2}, \frac{1}{q}\right[$. Given $N \in \mathbb{N}^*$ we define b^N by:*

$$b^N = \sum_{m=N}^{N-1} \mu_{-1,m} h_{-1,m} + \sum_{j=0}^{N} \sum_{m=-N2^j}^{N2^j-1} \mu_{j,m} h_{j,m}. \tag{4}$$

Remark 3.1. *We can note that* $\operatorname{Supp} b^N \subset [-N, N]$ *and* $b^N \in H_q^{1/q}(\mathbb{R})$. *Moreover, we have:*

$$\left\| b - b^N \right\|_{H_q^s(\mathbb{R})} \underset{N \to +\infty}{\longrightarrow} 0.$$

3.2 Computation of the coefficients $\mu_{j,m}$ when b is the derivative of a fractional brownian motion

Faber basis

4 Numerical results

5 Convergence

5.1 Convergence of $X_s^{N,n}$ to X_s^N in L^2

Recently, Leobacher and Szölgyenyi proved in [3] the convergence of the Euler-Maruyama scheme for SDE with discontinuous but piecewise Lipschitz drift and with a degenerate diffusion coefficient. This framework applies to the scheme we use with piecewise constant drift b^N and a constant diffusion coefficient.

Theorem 5.1 (Theorem 3.1. in [3]). *$\forall \varepsilon > 0$, $\exists C_N > 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t^{N,n} - X_t^N\right|^2\right]^{1/2} \leq C_N \delta^{1/4-\varepsilon} \tag{5}$$

with $\delta = \frac{1}{2^n}$ the step size.

TO DO: make C_N explicit.

5.2 Convergence of X_s^N to X_s

We want to estimate $\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t^N - X_t\right|^2\right]$. In order to do so, we must go back to the definition of the virtual solution of the SDE (1) given in [1]. The authors define the virtual solution of SDE (1) by:

$$X_t = X_0 + u(0,X_0) - u(t,X_t) + (\lambda+1)\int_0^t u(s,Xs)\,\mathrm{d}s + \int_0^t \nabla u(s,Xs)\,\mathrm{d}W_s + W_t \tag{6}$$

where u is the mild solution in $H_p^{1+\delta}$ of the following parabolic PDE:

$$\begin{cases} \partial_t u + \frac{1}{2}\Delta u + b\nabla u - (\lambda+1)u = -b & \text{on } [0,T]\times\mathbb{R} \\ u(T) = 0 & \text{on } \mathbb{R} \end{cases}. \tag{7}$$

We also define another similar PDE by replacing b by b^N, and noting u^N its mild solution in $H_p^{1+\delta}$.

$$\begin{cases} \partial_t u^N + \frac{1}{2}\Delta u^N + b^N\nabla u^N - (\lambda+1)u^N = -b^N & \text{on } [0,T]\times\mathbb{R} \\ u^N(T) = 0 & \text{on } \mathbb{R} \end{cases}. \tag{8}$$

We recall a useful lemma concerning the solutions of (7) and (8).

Lemma 5.2 (Lemma 19 in [1]). *Let $(\delta,p) \in K(\beta,q)$ and let v_λ be the mild solution to (7) in $H_p^{1+\delta}$. Fix ρ such that the integral operator is a contraction and let $\lambda > \rho$. Then $v_\lambda(t) \in \mathcal{C}^{1,\alpha}$ with $\alpha = \delta - 1/p$ for each fixed t and*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |\nabla v_\lambda(t,x)| \to 0, \text{ as } \lambda\to\infty \tag{9}$$

where the choice of λ depends only on $\delta, \beta, \|b\|_{H_p^{-\beta}}$, and $\|b\|_{H_q^{-\beta}}$.

By lemma 5.2, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2\sqrt{3}}$. λ can be chosen independently of N as far as $\left\|b - b^N\right\|_{H_q^s(\mathbb{R})} \underset{N\to\infty}{\longrightarrow} 0$ (See Step 2 of the proof of Proposition 26 in [1]). Therefore u^N and u are $\frac{1}{2\sqrt{3}}$-lipschitz.

$$\begin{aligned}
\left|X_t^N - X_t\right|^2 = \Big| & u^N(0,X_0) - u(0,X_0) + u(t,X_t) - u^N(t,X_t) + u^N(t,X_t) \\
& - u^N(t,X_t^N) + (\lambda+1)\int_0^t \{u^N(s,X_s^N) - u^N(s,X_s) + u^N(s,X_s) - u(s,X_s)\}\,\mathrm{d}s \\
& + \int_0^t \{\nabla u^N(s,X_s^N) - \nabla u(s,X_s)\}\,\mathrm{d}W_s\Big|^2
\end{aligned}$$

$$\begin{aligned}
\le 6\Bigg(& \left|u^N(0,X_0) - u(0,X_0)\right|^2 + \left|u(t,X_t) - u^N(t,X_t)\right|^2 + \left|u^N(t,X_t) - u^N(t,X_t^N)\right|^2 \\
& + \left|(\lambda+1)\int_0^t \{u^N(s,X_s^N) - u^N(s,X_s)\}\,\mathrm{d}s\right|^2 + \left|(\lambda+1)\int_0^t \{u^N(s,X_s) - u(s,X_s)\}\,\mathrm{d}s\right|^2 \\
& + \left|\int_0^t \{\nabla u^N(s,X_s^N) - \nabla u(s,X_s)\}\,\mathrm{d}W_s\right|^2\Bigg)
\end{aligned}$$

$$\begin{aligned}
\le 6\Bigg(& \left|u^N(0,X_0) - u(0,X_0)\right|^2 + \left|u(t,X_t) - u^N(t,X_t)\right|^2 + \frac{1}{12}\left|X_t - X_t^N\right|^2 \\
& + (\lambda+1)\, t\int_0^t \left|u^N(s,X_s^N) - u^N(s,X_s)\right|^2 \mathrm{d}s + (\lambda+1)\, t\int_0^t \left|u^N(s,X_s) - u(s,X_s)\right|^2 \mathrm{d}s \\
& + \left|\int_0^t \{\nabla u^N(s,X_s^N) - \nabla u(s,X_s)\}\,\mathrm{d}W_s\right|^2\Bigg)
\end{aligned}$$

$$\begin{aligned}
\le 6\Bigg(& \left|u^N(0,X_0) - u(0,X_0)\right|^2 + \left|u(t,X_t) - u^N(t,X_t)\right|^2 + \frac{1}{12}\left|X_t - X_t^N\right|^2 \\
& + \frac{\lambda+1}{12}\, t\int_0^t \left|X_s^N - X_s\right|^2 \mathrm{d}s + (\lambda+1)\, t\int_0^t \left|u^N(s,X_s) - u(s,X_s)\right|^2 \mathrm{d}s \\
& + \left|\int_0^t \{\nabla u^N(s,X_s^N) - \nabla u(s,X_s)\}\,\mathrm{d}W_s\right|^2\Bigg).
\end{aligned}$$

It follows that:

$$\sup_{0\le t\le T}\left|X_t^N-X_t\right|^2 \le 12\sup_{0\le t\le T}\Bigg(\left|u^N(0,X_0)-u(0,X_0)\right|^2+\left|u(t,X_t)-u^N(t,X_t)\right|^2 + \frac{\lambda+1}{12}\;t\int_0^t\left|X_s^N-X_s\right|^2\,\mathrm{d}s+(\lambda+1)\;t\int_0^t\left|u^N(s,X_s)-u(s,X_s)\right|^2\,\mathrm{d}s + \left|\int_0^t\{\nabla u^N(s,X_s^N)-\nabla u(s,X_s)\}\,\mathrm{d}W_s\right|^2\Bigg)$$

$$\le 12\Bigg(\sup_{0\le t\le T}\left|u^N(0,X_0)-u(0,X_0)\right|^2+\sup_{0\le t\le T}\left|u(t,X_t)-u^N(t,X_t)\right|^2 + \frac{\lambda+1}{12}\;T\int_0^T\left|X_s^N-X_s\right|^2\,\mathrm{d}s+(\lambda+1)\;T\int_0^T\left|u^N(s,X_s)-u(s,X_s)\right|^2\,\mathrm{d}s + \sup_{0\le t\le T}\left|\int_0^t\{\nabla u^N(s,X_s^N)-\nabla u(s,X_s)\}\,\mathrm{d}W_s\right|^2\Bigg)$$

$$\le 12\Bigg(\sup_{0\le t\le T}\left|u^N(0,X_0)-u(0,X_0)\right|^2+\sup_{0\le t\le T}\left|u(t,X_t)-u^N(t,X_t)\right|^2 + \frac{\lambda+1}{12}\;T\int_0^T\sup_{0\le t\le s}\left|X_s^N-X_s\right|^2\,\mathrm{d}s+(\lambda+1)\;T\int_0^T\left|u^N(s,X_s)-u(s,X_s)\right|^2\,\mathrm{d}s + \sup_{0\le t\le T}\left|\int_0^t\{\nabla u^N(s,X_s^N)-\nabla u(s,X_s)\}\,\mathrm{d}W_s\right|^2\Bigg).$$

Taking the expectation, we obtain:

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^N-X_t\right|^2\right] \le 12\Bigg(\sup_{0\le t\le T}\left|u^N(0,X_0)-u(0,X_0)\right|^2 + \mathbb{E}\left[\sup_{0\le t\le T}\left|u(t,X_t)-u^N(t,X_t)\right|^2\right]+\frac{\lambda+1}{12}\;T\;\mathbb{E}\left[\int_0^T\sup_{0\le t\le s}\left|X_s^N-X_s\right|^2\,\mathrm{d}s\right] + (\lambda+1)\;T\;\mathbb{E}\left[\int_0^T\left|u^N(s,X_s)-u(s,X_s)\right|^2\,\mathrm{d}s\right] + \mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t\{\nabla u^N(s,X_s^N)-\nabla u(s,X_s)\}\,\mathrm{d}W_s\right|^2\right]\Bigg).$$

Using Burkholder-Davis-Gundy inequality, it follows that:

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^N - X_t\right|^2\right] \le 12\Bigg(\sup_{0\le t\le T}\left|u^N(0,X_0) - u(0,X_0)\right|^2 + \mathbb{E}\Bigg[\sup_{0\le t\le T}\left|u(t,X_t) - u^N(t,X_t)\right|^2\Bigg] + \frac{\lambda+1}{12}\; T\;\mathbb{E}\Bigg[\int_0^T \sup_{0\le t\le s}\left|X_s^N - X_s\right|^2\,\mathrm{d}s\Bigg] + (\lambda+1)\; T\;\mathbb{E}\Bigg[\int_0^T \left|u^N(s,X_s) - u(s,X_s)\right|^2\,\mathrm{d}s\Bigg] + 2\int_0^T\left|\nabla u^N(s,X_s^N) - \nabla u(s,X_s)\right|^2\,\mathrm{d}s\Bigg)$$

Applying fractional Morrey inequality, we obtain $\forall t\in[0,T]$:

$$\begin{cases} \left\|u^N(t) - u(t)\right\|_{L^\infty} \le \left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1,\alpha}} \le c\left\|u^N(t) - u(t)\right\|_{H^{-\beta}_{\tilde{q},q}} \\ \left\|\nabla u^N(t) - \nabla u(t)\right\|_{L^\infty} \le \left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1,\alpha}} \le c\left\|u^N(t) - u(t)\right\|_{H^{-\beta}_{\tilde{q},q}}. \end{cases}$$

Now, with

$$\left\|u^N(t) - u(t)\right\|_{H^{-\beta}_{\tilde{q},q}} \le e^T\left\|u^N(t) - u(t)\right\|^{(\rho)}_{H^{-\beta}_{\tilde{q},q}} \le Ke^T\left\|b^N(t) - b(t)\right\|_{H^{-\beta}_q}$$

from lemma 22 in [1], we have:

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^N - X_t\right|^2\right] \le 12\Bigg(2c^2K^2e^{2T}\left\|b^N(t) - b(t)\right\|^2_{H^{-\beta}_q} + \frac{\lambda+1}{12}\; T\;\mathbb{E}\Bigg[\int_0^T \sup_{0\le t\le s}\left|X_s^N - X_s\right|^2\,\mathrm{d}s\Bigg] + (\lambda+1)\; T^2c^2K^2e^{2T}\left\|b^N(t) - b(t)\right\|^2_{H^{-\beta}_q} + 2c^2K^2Te^{2T}\left\|b^N(t) - b(t)\right\|^2_{H^{-\beta}_q}\Bigg)$$

$$\le 12\left(2+T+(\lambda+1)\;T^2\right)c^2K^2e^{2T}\left\|b^N(t) - b(t)\right\|^2_{H^{-\beta}_q} + (\lambda+1)\; T\;\mathbb{E}\Bigg[\int_0^T \sup_{0\le t\le s}\left|X_s^N - X_s\right|^2\,\mathrm{d}s\Bigg]$$

Finally Gronwall inequality gives:

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^N - X_t\right|^2\right] \le C\exp\left((\lambda+1)\;T^2\right)\left\|b^N(t) - b(t)\right\|^2_{H^{-\beta}_q} \tag{10}$$

with $C = \left(12\left(2+T+(\lambda+1)\;T^2\right)c^2K^2e^{2T}\right)$.

References

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