

NUMERICAL SIMULATION OF SDES WITH DISTRIBUTIONAL DRIFT

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1 Introduction

We would like to simulate numerically sample paths of the solution of the stochastic differential equation

$$dX_t = b(X_t) dt + dW_t \tag{1}$$

where $b \in H_q^{-\beta}(\mathbb{R})$, $\beta \in (0, \frac{1}{2})$, $q \in (\frac{1}{1-\beta}, \frac{1}{\beta})$, $t \in [0, T]$, and W_t is a standard Brownian motion. Equation (1) is studied by F. Flandoli, E. Issoglio, and F. Russo in [1] in which they define a concept of virtual solution. The authors prove then existence and unicity in law of this solution.

Example 1. *An example of such drift b is given by the derivative of a sample path of a fractional Brownian motion B_x^H with Hurst index $1/2 < H < 1$. These stochastic processes are gaussian processes verifying*

$$\mathbb{E} [B_t^H B_s^H] = \frac{1}{2} (t^{2H} + s^{2H} + |t - s|^{2H}).$$

We note $-\beta = H - 1$. Given $B_x^H(\omega) \in H_q^{1-\beta}(\mathbb{R})$, we can take $b(x) = \frac{\partial}{\partial x} B_x^H(\omega) \in H_q^{-\beta}(\mathbb{R})$. We will use this in our numerical simulations.

As far as the drift b is not a function but a distribution, it must be approximated if we want to evaluate it at points. In order to do so, we will use a series representation of b and truncate it. That is why we will consider two steps in our algorithm:

1. approximate the drift b by a function b^N meant to converge to b as $N \rightarrow \infty$.

2. approximate the solution X_t^N of the approximated SDE

$$dX_t^N = b^N(X_t^N) dt + dW_t \quad (2)$$

by $X_t^{N,n}$ defined with the Euler-Maruyama scheme

$$X_t^{N,n} = X_0 + \int_0^t b^N(X_{\eta_n(t)}^{N,n}) dt + W_{\eta_n(t)}$$

where $\eta_n(t) = t_k$ if $t \in [t_k, t_k + 1]$, for $t_k = \frac{k}{n}$ with $k \in \llbracket 0, \lceil nT \rceil \rrbracket$.

2 Numerical simulation of fractional Brownian motion

To simulate a sample path of a fractional brownian motion B_x^H on a finite grid $(x_k)_{k \in \llbracket 1, n \rrbracket}$, we simulate n independent standard gaussian random variables $(X_k)_{k \in \llbracket 1, n \rrbracket}$ and then correlate them with the definite positive correlation matrix

$$C_{k,s} = \mathbb{E}[B_{x_k}^H B_{x_s}^H] = \frac{1}{2} (x_k^{2H} + x_s^{2H} + |x_k - x_s|^{2H}).$$

To do so, we use the Cholesky decomposition method and calculate the triangular matrix M such that $C = MM^\top$. Therefore, defining

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ and } B^H = MX,$$

B^H contains the values of a fractional brownian motion evaluated on the grid $(x_k)_{k \in \llbracket 1, n \rrbracket}$.

3 Approximation of the drift

3.1 Series representation

We use Haar wavelets to give a series representation of b . By doing so, we will be able to approximate it numerically by truncating the series.

Definition 1 (Haar wavelets). *We define the Haar wavelets $h_{j,m}$ on \mathbb{R} with $j \in \mathbb{N} \cup \{-1\}$ and $m \in \mathbb{Z}$ by:*

$$\begin{cases} h_M & : x \mapsto \left(\mathbb{1}_{[0, \frac{1}{2}[} - \mathbb{1}_{[\frac{1}{2}, 1[} \right) (x) \\ h_{-1,m} & : x \mapsto \sqrt{2} |h_M(x - m)| \\ h_{j,m} & : x \mapsto h_M(2^j x - m) \end{cases}$$

Theorem 1 (See [2]). *Let $b \in H_q^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left] -\frac{1}{2}, \frac{1}{q} \right[$. Therefore,*

$$b = \sum_{j=-1}^{+\infty} \sum_{m \in \mathbb{Z}} \mu_{j,m} h_{j,m} \quad (3)$$

where $\mu_{j,m} = 2^j \int_{\mathbb{R}} b(x) h_{j,m}(x) \, dx$ in the sense of dual pairing.

Definition 2. *With the same notation $\mu_{j,m}$, let $b \in H_q^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left] -\frac{1}{2}, \frac{1}{q} \right[$. Given $N \in \mathbb{N}^*$ we define b^N by:*

$$b^N = \sum_{m=N}^{N-1} \mu_{-1,m} h_{-1,m} + \sum_{j=0}^N \sum_{m=-N2^j}^{N2^j-1} \mu_{j,m} h_{j,m}. \quad (4)$$

Remark 1. *We can note that $\text{Supp } b^N \subset [-N, N]$. Moreover, we have:*

$$\|b - b^N\|_{H_q^s(\mathbb{R})} \xrightarrow{N \rightarrow +\infty} 0.$$

4 Convergence

4.1 Weak convergence of $X_t^{N,n}$ to X_t^N

Recently, Leobacher and Szölgyenyi proved in [3] the convergence of the Euler-Maruyama scheme for SDE with discontinuous but piecewise Lipschitz drift and with a degenerate diffusion coefficient. This framework applies to the scheme we use with piecewise constant drift b^N and a constant diffusion coefficient.

Theorem 2 (Theorem 3.1. in [3]). $\exists C_N > 0$ independent of n such that it holds $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0$:

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \left| X_t^{N,n} - X_t^N \right|^2 \right]^{1/2} \leq C_N \delta^{1/4-\varepsilon} \quad (5)$$

with $\delta = \frac{1}{n}$ the step size and C_N depending on $\|b^N\|_\infty$.

Theorem 3. Let f be μ -Hölder with constant $C_f > 0$, $\mu \in (0, 1]$ and $t \in [0, T]$. Then, exists $C'_N > 0$ independent of n such that it holds $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0$:

$$\left| \mathbb{E} \left[f \left(X_t^{N,n} \right) - f \left(X_t^N \right) \right] \right| \leq C'_N \delta^{\mu/4-\varepsilon} \quad (6)$$

with $\delta = \frac{1}{n}$ the step size.

Proof. By Jensen's inequality and the μ -Hölder property of f , we obtain:

$$\begin{aligned} \left| \mathbb{E} \left[f \left(X_t^{N,n} \right) - f \left(X_t^N \right) \right] \right| &\leq C_f \mathbb{E} \left[\left| X_t^{N,n} - X_t^N \right|^\mu \right] \\ &\leq C_f \mathbb{E} \left[\left| Y_t - Y_t^N \right|^2 \right]^{\mu/2} \\ &\leq C_f C_N^\mu \delta^{\mu/4-\varepsilon}. \end{aligned}$$

□

4.2 Weak convergence of X_t^N to X_t

The goal of this section is to estimate the weak error $\left| \mathbb{E} \left[f(X_t) - f(X_t^N) \right] \right|$ with suitable functions f . In order to do so, we must go back to the definition of the virtual solution of the SDE (1) given in [1]. Let $(\delta, p) \in K(\beta, q) :=$

$\{(\delta, p) \mid \beta < \delta < 1 - \beta, \frac{1}{\delta} < p < q\}$. The authors define the virtual solution of SDE (1) by X_t such that:

$$\begin{cases} Y_t = y + (\lambda + 1) \int_0^t u(s, \Psi(s, Y_s)) \, ds + \int_0^t (\nabla u(s, \Psi(s, Y_s)) + 1) \, dW_s \\ X_t = \Psi(t, Y_t) = \varphi^{-1}(t, Y_t) \end{cases} \quad (7)$$

where u is the mild solution in $H_p^{1+\delta}$ of the following parabolic PDE:

$$\begin{cases} \partial_t u + \frac{1}{2} \Delta u + b \nabla u - (\lambda + 1)u = -b & \text{on } [0, T] \times \mathbb{R} \\ u(T) = 0 & \text{on } \mathbb{R} \end{cases} \quad (8)$$

with $\varphi(t, x) = x + u(t, x)$, and $y = \varphi(0, x)$.

We also define another similar PDE by replacing b by b^N . We call u^N its mild solution in $H_p^{1+\delta}$:

$$\begin{cases} \partial_t u^N + \frac{1}{2} \Delta u^N + b^N \nabla u^N - (\lambda + 1)u^N = -b^N & \text{on } [0, T] \times \mathbb{R} \\ u^N(T) = 0 & \text{on } \mathbb{R} \end{cases} \quad (9)$$

Then we consider an approximated version of (7):

$$\begin{cases} Y_t^N = y^N + (\lambda + 1) \int_0^t u^N(s, \Psi(s, Y_s^N)) \, ds + \int_0^t (\nabla u^N(s, \Psi(s, Y_s^N)) + 1) \, dW_s \\ X_t^N = \Psi(t, Y_t^N) = (\varphi^N)^{-1}(t, Y_t^N) \end{cases} \quad (10)$$

with $\varphi^N(t, x) = x + u^N(t, x)$, and $y^N = \varphi^N(0, x)$.

Remark 2. Proposition 26 in [1] assures us that X_t^N defined above in (10) is in fact the classical solution of (2), as far as $b^N \in L^p$. That is why for each fixed N our Euler scheme converges to the virtual solution X_t^N .

We also recall a useful lemma concerning the solutions of (8) and (9).

Lemma 4 (Lemma 20 in [1]). *Let $(\delta, p) \in K(\beta, q)$ and let u, u^N be the mild solutions to (8), (9) in $H_p^{1+\delta}$. Fix ρ such that the integral operator is a contraction and let $\lambda > \rho$. Then $u(t), u^N(t) \in C^{1,\alpha}$ with $\alpha = \delta - 1/p$ for each fixed t and*

$$\begin{cases} \sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla u(t, x)| \rightarrow 0, \text{ as } \lambda \rightarrow \infty \\ \sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla u^N(t, x)| \rightarrow 0, \text{ as } \lambda \rightarrow \infty \end{cases}$$

where the choice of λ depends only on $\delta, \beta, \|b\|_{H_p^{-\beta}}$, and $\|b^N\|_{H_q^{-\beta}}$.

Lemma 5. *Let $(\delta, p) \in K(\beta, q)$ and let u, u^N be the mild solutions to (8), (9) in $H_p^{1+\delta}$, $\alpha = \delta - 1/p$. Exists $c, K > 0$ such that for both $N \in \mathbb{N}$ and $\rho > 1$ big enough, $\forall t \in [0, T]$,*

$$\begin{cases} \|u^N(t) - u(t)\|_{L^\infty} \leq cKe^{\rho T} \|b^N - b\|_{H_q^{-\beta}} \\ \|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq cKe^{\rho T} \|b^N - b\|_{H_q^{-\beta}}. \end{cases} \quad (11)$$

Proof. Applying fractional Morrey inequality, $\exists c > 0, \forall t \in [0, T]$:

$$\begin{cases} \|u^N(t) - u(t)\|_{L^\infty} \leq \|u^N(t) - u(t)\|_{C^{1,\alpha}} \leq c \|u^N(t) - u(t)\|_{H_p^{1+\delta}} \\ \|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \|u^N(t) - u(t)\|_{C^{1,\alpha}} \leq c \|u^N(t) - u(t)\|_{H_p^{1+\delta}}. \end{cases}$$

Now, we can conclude with

$$\|u^N - u\|_{\infty, H_p^{1+\delta}} \leq e^{\rho T} \|u^N - u\|_{\infty, H_p^{1+\delta}}^{(\rho)} \leq Ke^{\rho T} \|b^N - b\|_{H_q^{-\beta}}$$

from Lemma 23 in [1], for both $N \in \mathbb{N}$ and $\rho > 1$ big enough, and where $\|f(t)\|_{\infty, X}^{(\rho)} := \sup_{0 \leq t \leq T} e^{-\rho t} \|f(t)\|_X$. \square

We will need the following local time inequality from Liqing Yan:

Lemma 6 (Lemma 4.2 in [4]). *Let X be a continuous semimartingale with $X_0 = 0$. For $\varepsilon > 0$ we define a double sequence of stopping times by $\sigma_1 = 0$, $\tau_1 = \inf\{t > 0 | X_t = \varepsilon\}$, $\sigma_n = \inf\{t > \tau_{n-1} | X_t = 0\}$, $\tau_n = \inf\{t > \sigma_n | X_t = \varepsilon\}$. For any real function $F(\cdot) \in C^2$ with $F(0) = 0$, $F'(0) = 0$ and $F(\cdot) > 0$ on $(0, \varepsilon_0)$ for some $\varepsilon_0 > 0$, then for any $0 < \varepsilon < \varepsilon_0$ we have*

$$\begin{aligned} 0 \leq L_t^0(X) &\leq 2\varepsilon - \frac{2}{F(\varepsilon)} \int_0^t \theta_s(X) (F(\varepsilon) - \varepsilon F'(X_s^+)) dX_s \\ &\quad + \frac{1}{F(\varepsilon)} \int_0^t \theta_s(X) \varepsilon F''(X_s^+) d[X]_s \end{aligned}$$

with $\theta_s(X) = \sum_{n=1}^{\infty} \mathbb{1}_{\{\sigma_n < s \leq \tau_n, 0 < X_s \leq \varepsilon\}}(X)$.

Applying lemma 6 with $F(x) = x^2$, it follows:

Corollary 7. *Let X be a continuous semimartingale with $X_0 = 0$. With the same notations as in lemma 6, for any $\varepsilon > 0$ we have*

$$0 \leq L_t^0(X) \leq 2\varepsilon - \frac{2}{\varepsilon} \int_0^t \theta_s(X) (\varepsilon - 2X_s^+) dX_s + \frac{2}{\varepsilon} \int_0^t \theta_s(X) d[X]_s \quad (12)$$

Lemma 8. *Let $(\delta, p) \in K(\beta, q)$, $\alpha = \delta - 1/p < 1$, u, u^N be the mild solutions to (8), (9) in $H_p^{1+\delta}$, and Y, Y^N solutions of the SDEs (7), (10). Then, if $\alpha > 1/2$, for λ big enough we have $\forall \varepsilon \in (0, 1]$,*

$$0 \leq \mathbb{E} [L_T^0(Y - Y^N)] \leq g(\varepsilon).$$

where

$$g(\varepsilon) = 2(\lambda+1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + \left(4^{\alpha+1} \|u\|_{C^{1,\alpha}}^2 T + 2 + 2(\lambda+1)T\right) \varepsilon^{2\alpha-1} + 4c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-1}$$

Proof. By Lemma 4, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\|b - b^N\|_{H_q^s(\mathbb{R})} \xrightarrow{N \rightarrow \infty} 0$. Therefore $u^N(t, \cdot)$ and $u(t, \cdot)$ are $\frac{1}{2}$ -lipschitz. Let $\varepsilon \in (0, 1]$. Corollary 7 gives us:

$$0 \leq L_T^0(Y - Y^N) \leq 2\varepsilon - \frac{2}{\varepsilon} \int_0^T \theta_s(Y - Y^N) (\varepsilon - 2(Y_s - Y_s^N)^+) d(Y_s - Y_s^N) + \frac{2}{\varepsilon} \int_0^T \theta_s(Y - Y^N) d[(Y - Y^N)]_s.$$

with

$$Y_T - Y_T^N = y - y^N + (\lambda + 1) \int_0^T \{u(s, \Psi(s, Y_s)) - u^N(s, \Psi(s, Y_s^N))\} ds + \int_0^T \{\nabla u(s, \Psi(s, Y_s)) - \nabla u^N(t, \Psi(s, Y_s^N))\} dW_s.$$

Remark 3. *Note that $\theta_s(Y - Y^N) |\varepsilon - 2(Y_s - Y_s^N)^+| \leq \varepsilon \theta_s(Y - Y^N)$.*

Remark 4. *For clarity purpose, we note $\tilde{u}(s, x) = u(s, \Psi(s, x))$ and use the same notation for the gradient and the approximated mild solution. We can notice that \tilde{u} is 1-lipschitz in space and $\widetilde{\nabla u}$ is α -Hölder with constant $2^\alpha \|u\|_{C^{1,\alpha}}$.*

∇u and ∇u^N are bounded so the Itô integral is a martingale. We take the expectation:

$$\mathbb{E} [L_T^0(Y - Y^N)] \leq 2\varepsilon + 2(\lambda+1) \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) |\tilde{u}(s, Y_s) - \tilde{u}^N(s, Y_s^N)| ds \right] + \frac{2}{\varepsilon} \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) \left\{ \widetilde{\nabla u}(s, Y_s) - \widetilde{\nabla u}^N(s, Y_s^N) \right\}^2 ds \right]$$

$$\begin{aligned}
 &\leq 2\varepsilon + 2(\lambda + 1) \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) |\tilde{u}(s, Y_s) - \tilde{u}(s, Y_s^N)| \, ds \right] \\
 &\quad + 2(\lambda + 1) \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) |\tilde{u}(s, Y_s^N) - \tilde{u}^N(s, Y_s^N)| \, ds \right] \\
 &\quad + \frac{4}{\varepsilon} \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) \left\{ \widetilde{\nabla} u(s, Y_s) - \widetilde{\nabla} u(s, Y_s^N) \right\}^2 \, ds \right] \\
 &\quad + \frac{4}{\varepsilon} \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) \left\{ \widetilde{\nabla} u(s, Y_s^N) - \widetilde{\nabla} u^N(s, Y_s^N) \right\}^2 \, ds \right] \\
 &\leq 2\varepsilon + 2(\lambda + 1) \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) |Y_s - Y_s^N| \, ds \right] \\
 &\quad + \frac{4^{\alpha+1} \|u\|_{\mathcal{C}^{1,\alpha}}^2}{\varepsilon} \mathbb{E} \left[\int_0^T \theta_s(Y - Y^N) |Y_s - Y_s^N|^{2\alpha} \, ds \right] \\
 &\quad + 2(\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + 4c^2TK^2e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-1}
 \end{aligned}$$

where we have used Lemma 5, the 1-lipschitz property of \tilde{u} and the α -Hölder property of $\widetilde{\nabla} u$ (with constant $2^\alpha \|u\|_{\mathcal{C}^{1,\alpha}}$). As $\theta_s(Y - Y^N) |Y_s - Y_s^N| \leq \varepsilon$, we have

$$\begin{aligned}
 \mathbb{E} [L_T^0(Y - Y^N)] &\leq 2\varepsilon + 2(\lambda + 1) T\varepsilon + 4^{\alpha+1} \|u\|_{\mathcal{C}^{1,\alpha}}^2 T\varepsilon^{2\alpha-1} \\
 &\quad + 2(\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + 4c^2TK^2e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-1}
 \end{aligned}$$

As $1 > 2\alpha - 1 > 0$, the result follows from $\varepsilon \leq \varepsilon^{2\alpha-1}$ when $0 < \varepsilon \leq 1$. \square

Lemma 9. *With assumptions and notations of Lemma 8, and $1 > \alpha > 1/2$ we have*

$$\mathbb{E} [L_T^0(Y - Y^N)] \leq g(\varepsilon_N) = \sigma \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \quad (13)$$

for $\|b^N - b\|_{H_q^{-\beta}}$ small enough (it is to say N big enough) where

$$\sigma = 2(\lambda+1) cTK e^{\rho T} + \left(4^{\alpha+1} \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + 2(\lambda+1)T \right) \omega^{2\alpha-1} + 4c^2TK^2e^{2\rho T} \omega^{-1}$$

and

$$\omega = \left(\frac{4c^2TK^2e^{2\rho T}}{(2\alpha - 1) \left(4^{\alpha+1} \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + 2(\lambda+1)T \right)} \right)^{\frac{1}{2\alpha}}.$$

Proof. By Lemma 8,

$$0 \leq \mathbb{E} [L_T^0(Y - Y^N)] \leq g(\varepsilon)$$

where

$$\begin{aligned} g(\varepsilon) = & 2(\lambda+1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + \left(4^{\alpha+1} \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + 2(\lambda+1)T\right) \varepsilon^{2\alpha-1} \\ & + 4c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-1}. \end{aligned}$$

With

$$\begin{aligned} g'(\varepsilon) = & (2\alpha - 1) \left(4^{\alpha+1} \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + 2(\lambda+1)T\right) \varepsilon^{2\alpha-2} \\ & - 4c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-2}, \end{aligned}$$

and

$$\begin{aligned} g''(\varepsilon) = & (2\alpha - 2)(2\alpha - 1) \left(4^{\alpha+1} \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + 2(\lambda+1)T\right) \varepsilon^{2\alpha-3} \\ & + 8c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-3}, \end{aligned}$$

the minimum of g on $(0, 1]$ is reached when N is big enough in

$$\varepsilon_N = \left(\frac{4c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2}{(2\alpha - 1) \left(4^{\alpha+1} \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + 2(\lambda+1)T\right)} \right)^{\frac{1}{2\alpha}} = \omega \|b^N - b\|_{H_q^{-\beta}}^{1/\alpha}.$$

where

$$g''(\varepsilon_N) = 8c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon_N^{-3} \alpha > 0.$$

and

$$\omega = \left(\frac{4c^2TK^2 e^{2\rho T}}{(2\alpha - 1) \left(4^{\alpha+1} \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + 2(\lambda+1)T\right)} \right)^{\frac{1}{2\alpha}}.$$

Therefore $\mathbb{E} [L_T^0(Y - Y^N)] \leq g(\varepsilon_N)$

$$\begin{aligned} \leq & 2(\lambda+1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} \\ & + \left(4^{\alpha+1} \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + 2(\lambda+1)T\right) \omega^{2\alpha-1} \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \\ & + 4c^2TK^2 e^{2\rho T} \|b^N - b\|_{H_q^{-\beta}}^2 \omega^{-1} \|b^N - b\|_{H_q^{-\beta}}^{-1/\alpha} \end{aligned}$$

$$\begin{aligned}
 &\leq 2(\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} \\
 &+ \left(\left(4^{\alpha+1} \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + 2(\lambda + 1)T \right) \omega^{2\alpha-1} + 4c^2TK^2e^{2\rho T}\omega^{-1} \right) \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \\
 &\leq 2(\lambda + 1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \\
 &+ \left(\left(4^{\alpha+1} \|u\|_{\mathcal{C}^{1,\alpha}}^2 T + 2 + 2(\lambda + 1)T \right) \omega^{2\alpha-1} + 4c^2TK^2e^{2\rho T}\omega^{-1} \right) \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha}
 \end{aligned}$$

for N big enough. The result follows. \square

Theorem 10. *Let f be μ -Hölder with constant $C_f > 0$ and $\mu \in (0, 1]$. If $0 < \beta < 1/4$, $q \in \left(\frac{1}{1-\beta}, \frac{1}{\beta}\right)$, $\forall \varepsilon \in (0, \frac{1-4\beta}{2})$, with $(\delta, p) \in K(\beta, q)$ such that $\delta - 1/p = 1 - 2\beta - \varepsilon$, exists ξ independent of f such that for $N \in \mathbb{N}$, $\rho > 1$, λ big enough it holds:*

$$\sup_{0 \leq t \leq T} \mathbb{E} [|f(X_t) - f(X_t^N)|] \leq \xi C_f \|b^N - b\|_{H_q^{-\beta}}^{\mu \left(2 - \frac{1}{1-2\beta-\varepsilon}\right)}$$

Proof. We note as usual $\alpha = \delta - 1/p$. By Lemma 4, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\|b - b^N\|_{H_q^s(\mathbb{R})} \xrightarrow{N \rightarrow \infty} 0$. We recall that in this case, by Lemma 22 in [1], $\Psi(t, \cdot)$ and $\Psi^N(t, \cdot)$ are 2-lipschitz. Therefore $\tilde{u}^N(t, \cdot)$ and $\tilde{u}(t, \cdot)$ are 1-lipschitz. Let $t \in [0, T]$.

$$\begin{aligned}
 &\mathbb{E} [|f(X_t) - f(X_t^N)|] = \mathbb{E} [|f(\Psi(t, Y_t)) - f(\Psi^N(t, Y_t^N))|] \\
 &\leq \mathbb{E} [|f(\Psi(t, Y_t)) - f(\Psi(t, Y_t^N))|] + \mathbb{E} [|f(\Psi(t, Y_t^N)) - f(\Psi^N(t, Y_t^N))|] \\
 &\leq C_f \left(2^\mu \mathbb{E} [|Y_t - Y_t^N|^\mu] + \mathbb{E} [|\Psi(t, Y_t^N) - \Psi^N(t, Y_t^N)|^\mu] \right) \\
 &\leq C_f \left(2^\mu \mathbb{E} [|Y_t - Y_t^N|]^\mu + \mathbb{E} [|\Psi(t, Y_t^N) - \Psi^N(t, Y_t^N)|]^\mu \right) \quad (14)
 \end{aligned}$$

by Jensen's inequality.

$$\begin{aligned}
 Y_t - Y_t^N &= y - y^N + (\lambda + 1) \int_0^t \{ \tilde{u}(s, Y_s) - \tilde{u}^N(s, Y_s^N) \} ds \\
 &\quad + \int_0^t \{ \widetilde{\nabla u}(s, Y_s) - \widetilde{\nabla u}^N(s, Y_s^N) \} dW_s.
 \end{aligned}$$

We apply Meyer-Tanaka's formula to obtain:

$$\begin{aligned} |Y_t - Y_t^N| &= |y - y^N| + (\lambda+1) \int_0^t \text{sign}(Y_s - Y_s^N) \{ \tilde{u}(s, Y_s) - \tilde{u}^N(s, Y_s^N) \} ds \\ &+ \int_0^t \text{sign}(Y_s - Y_s^N) \{ \widetilde{\nabla} u(s, Y_s) - \widetilde{\nabla} u^N(s, Y_s^N) \} dW_s + L_t^0(Y - Y^N). \end{aligned}$$

Taking the expectation leads to:

$$\begin{aligned} \mathbb{E}[|Y_t - Y_t^N|] &= (\lambda+1) \mathbb{E} \left[\int_0^t \text{sign}(Y_s - Y_s^N) \{ \tilde{u}(s, Y_s) - \tilde{u}^N(s, Y_s^N) \} ds \right] \\ &+ \mathbb{E}[L_t^0(Y - Y^N)] + |u(0, x) - u^N(0, x)| \end{aligned}$$

because ∇u and ∇u^N are bounded so the Itô integral is a martingale.

$$\begin{aligned} \mathbb{E}[|Y_t - Y_t^N|] &\leq (\lambda+1) \mathbb{E} \left[\int_0^t |\tilde{u}(s, Y_s) - \tilde{u}(s, Y_s^N)| ds \right] + |u(0, x) - u^N(0, x)| \\ &+ (\lambda+1) \mathbb{E} \left[\int_0^t |\tilde{u}(s, Y_s^N) - \tilde{u}^N(s, Y_s^N)| ds \right] + \mathbb{E}[L_t^0(Y - Y^N)]. \end{aligned}$$

We use Lemma 5 and the 1-lipschitz property of \tilde{u} :

$$\begin{aligned} \mathbb{E}[|Y_t - Y_t^N|] &\leq (\lambda+1) \mathbb{E} \left[\int_0^t |Y_s - Y_s^N| ds \right] + (\lambda+1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} \\ &+ cK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + \mathbb{E}[L_t^0(Y - Y^N)] \\ &\leq (\lambda+1) \int_0^t \mathbb{E}[|Y_s - Y_s^N|] ds + (\lambda+1) cTK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} \\ &+ cK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + \mathbb{E}[L_T^0(Y - Y^N)]. \end{aligned}$$

where we have used the fact that $L_t^0(Y - Y^N)$ is an increasing process.

By Gronwall's Lemma, it follows:

$$\mathbb{E}[|Y_t - Y_t^N|] \leq C(N) e^{(\lambda+1)t} \leq C(N) e^{(\lambda+1)T} \quad (15)$$

with $C(N) = cK e^{\rho T} \|b^N - b\|_{H_q^{-\beta}} ((\lambda+1)T + 1) + \mathbb{E}[L_T^0(Y - Y^N)]$.

With Lemma 8 and Lemma 9 we obtain

$$C(N) \leq ((\lambda+1)T+1)cKe^{\rho T} \|b^N - b\|_{H_q^{-\beta}} + \sigma \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \leq \zeta \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha}.$$

for $\|b^N - b\|_{H_q^{-\beta}}$ small enough where $\zeta = ((\lambda+1)T+1)cKe^{\rho T} + \sigma$. It follows:

$$\mathbb{E} [|Y_t - Y_t^N|] \leq \zeta e^{(\lambda+1)T} \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha}. \quad (16)$$

Moreover, using $\forall t \in [0, T], \sup_{x \in \mathbb{R}} |\nabla u(t, x)| \leq 1/2$, we obtain with $\varphi(t, x) = x + u(t, x)$:

$$\begin{aligned} & |\varphi(t, \Psi^N(t, Y_t^N)) - \varphi(t, \Psi(t, Y_t^N))| \\ & \geq \inf_{x \in \mathbb{R}} |\nabla \varphi(t, x)| |\Psi^N(t, Y_t^N) - \Psi(t, Y_t^N)| \\ & \geq \frac{1}{2} |\Psi^N(t, Y_t^N) - \Psi(t, Y_t^N)| \end{aligned}$$

and

$$\begin{aligned} |\Psi^N(t, Y_t^N) - \Psi(t, Y_t^N)| & \leq 2 |\varphi(t, \Psi^N(t, Y_t^N)) - \varphi(t, \Psi(t, Y_t^N))| \\ & \leq 2 |\varphi(t, \Psi^N(t, Y_t^N)) - \varphi^N(t, \Psi^N(t, Y_t^N))| \\ & \leq 2 \|u(t) - u^N(t)\|_{\infty} \\ & \leq 2cKe^{\rho T} \|b - b^N\|_{H_q^{-\beta}} \end{aligned} \quad (17)$$

where we have used Lemma 5 and the fact that $\varphi^N(t, \Psi^N(t, Y_t^N)) = \varphi(t, \Psi(t, Y_t^N)) = Y_t^N$.

Finally, combining (14), (16) and (17) we obtain:

$$\begin{aligned} & |\mathbb{E} [f(X_t) - f(X_t^N)]| \\ & \leq C_f \left(2^\mu \mathbb{E} [|Y_t - Y_t^N|]^\mu + \mathbb{E} [|\Psi(t, Y_t^N) - \Psi^N(t, Y_t^N)|]^\mu \right) \\ & \leq C_f \left(2^\mu \zeta^\mu e^{\mu(\lambda+1)T} \|b^N - b\|_{H_q^{-\beta}}^{\mu(2-1/\alpha)} + 2^\mu c^\mu K^\mu e^{\mu\rho T} \|b^N - b\|_{H_q^{-\beta}}^\mu \right) \\ & \leq 2^\mu C_f \left(\zeta^\mu e^{\mu(\lambda+1)T} + c^\mu K^\mu e^{\mu\rho T} \right) \|b^N - b\|_{H_q^{-\beta}}^{\mu(2-1/\alpha)} \end{aligned}$$

for N big enough, which is the expected result. \square

5 Numerical results

5.1 Strong convergence of the Euler scheme

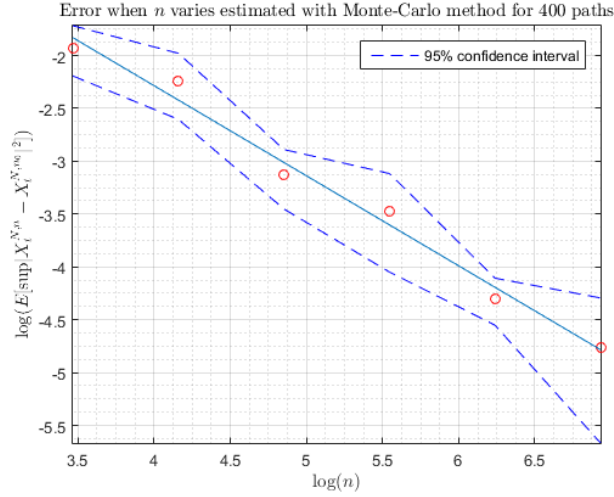


Figure 1: Estimation of the L^2 error of the Euler-Maruyama scheme with a Monte-Carlo method. 400 paths, $N = 5$, $n \in \{2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}\}$, reference solution with $n_0 = 2^{12}$ points.

We observe a numerical convergence rate of 0.85 when Theorem 2 shows a theoretical rate of $0.5 - \varepsilon$.

References

- [1] F. Flandoli, E. Issoglio, and F. Russo. Multidimensional stochastic differential equations with distributional drift. *Transactions of the American Mathematical Society*, 369 (3):1655–1688, 3 2017.
- [2] E. Issoglio and F. Russo. On a class of Markov BSDEs with generalized driver. submitted. arXiv:1805.02466v1.
- [3] G. Leobacher and M. Szölgényi. Convergence of the Euler–Maruyama method for multidimensional SDEs with discontinuous drift and degenerate diffusion coefficient. preprint. arXiv:1610.07047v5.
- [4] L. Yan. The Euler Scheme with Irregular Coefficients. *The Annals of Probability*, 30(3):1172–1194, 7 2002.