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Numerical schemes for multidimensional SDEs with distributional coefficients

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Author: Maximilien Germain

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ENSTA ParisTech Tutor:
Francesco Russo

Host Organism Tutors:
Elena Issoglio, Tiziano de Angelis

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Host organism: University of Leeds
Address: Leeds
LS2 9JT
United Kingdom

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Abstract

This report constructs an approximation method of a multidimensional stochastic differential equation with distributional drift. It applies the Euler-Maruyama scheme to a new class of problems with irregular coefficients, after approximating the drift by Haar wavelets. We study the convergence of our algorithm, give a rate in the one dimensional case, and apply it to the special case of the derivative of a fractional Brownian motion path as a drift.

Keywords and phrases: Stochastic differential equations; distributional drift; numerical simulation; fractional Brownian motion; Euler scheme; Haar wavelets.

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Introduction

When one wants to study a phenomenon which could be for instance a physical, biological or economical situation, it requires to produce a model. It is a simplification of reality which allows us to (at least partially) understand and represent the behavior of the system we look at. Most of these models rely on mathematical objects such as functions, which derivatives verify differential equations. Formally, you can look for instance to equations like

$$\mathrm{d}X_t = f(t, X_t)\ \mathrm{d}t \tag{1}$$

in which you express a link between time, the derivative of X_t and the values taken by X_t. If f is well-behaved, you can often show that it exists one and only one solution to this equation.

Unfortunately, in most models, f is not regular enough. Thus, you may only be able to show that some solutions exist but are not unique. This is a real problem because you would like to determine only one solution, it is to say the real description of the underlying phenomenon, directly from equation (1). For instance, if you want to study the effects of gravity on a system, you want to derive from your model equations only one trajectory for your system.

Nevertheless, when you add white noise, a random phenomenon which can be understood formally as the derivative of Brownian motion, you may be able to regain uniqueness in your equation[1], now understood as a Stochastic Differential Equation (SDE)

$$\mathrm{d}X_t = f(t, X_t)\ \mathrm{d}t + \mathrm{d}W_t. \tag{2}$$

You can understand this random noise as a perturbation modifying equation (1). In this project we will focus on such equations (2) for irregular f.

The motivation is the following. It is hopeless in general to search for explicit expressions for the solution to such equations. In fact, even for regular f, it is impossible in most cases to do this for equation (1), which is even easier to study than equation (2). But we can produce a numerical solution which is designed to behave as near as we want to the real abstract solution.

After the work of Flandoli, Issoglio and Russo studying and giving sense in [3] to equations like (2) when f is a distribution[2] in a fractional Sobolev space, I used their results during my research project in the School of Maths of the University of Leeds to construct a multidimensional simulation method, giving a numerical approximation of the solution.

Numerical approximation of SDEs with irregular coefficients has been studied by a lot of authors. The case of discontinuous coefficients on a set of Lebesgue measure zero is handled by Yan in [12], and the one of discontinuous monotone drift coefficient is investigated by Halidias and Kloeden in [4]. Another work from Ngo and Taguchi focuses on a drift satisfying a one-sided

[1] This phenomenon is called regularization by noise.
[2] It's a very irregular object which generalizes the notion of function.

Lipschitz condition in [9] and more recently, Leobacher and Szolgyenyi study in [8] a piecewise Lipschitz drift with a degenerate diffusion coefficient. Another approach from Ankirchner, Kruse and Urusov in the paper [1] requires no regularity or growth conditions. We can also cite the papers [7] of Kohatsu-Higa, Lejay and Yasuda, such as [13] written by Étoré and Martinez. Nevertheless, to the best of our knowledge, no result has been proven yet concerning the approximation of SDEs with distributional drift. Whereas all of the previous papers on this subject study coefficients which are functions, we provide the first approximation algorithm which applies to time-dependent drifts with Lipschitz regularity in time but which are distributions in space.

First we recall in Part I the notations and assumptions from [3] before giving the definition of Haar wavelets. These functions constitute a basis in distributional fractional Sobolev spaces and are used to construct the multidimensional approximation of the drift we are looking for. After defining an approximated SDE with the previous Haar approximation of the drift, we use in Part II the results of Leobacher and Szölgyenyi in [8] to prove the convergence of the Euler-Marayuma scheme for this approximated equation. After that step, a result from [3] shows the convergence in law of our approximated solution to the proper solution to (1). Our main result gives a convergence rate for the one-dimensional case. It makes use of a careful analysis of the local time of the difference between the solution to the transformed SDE and the solution to the transformed approximated SDE, via the use of the Meyer-Tanaka formula. After these theoretical aspects, we study in Part III and IV an example of application of our algorithm in the one-dimensional and time-homogeneous case. We apply our method to the special case of f as the realization of a random field, the derivative of fractional Brownian motion. We are able to compute the coefficients of this drift on the Haar basis, thanks to a Faber representation of the fractional Brownian motion path. Finally, we study the numerical convergence of our method in this case.

Part I

Context

I.1 Aims and notations

We would like to simulate numerically sample paths of the solution to the d-dimensional stochastic differential equation

$$\mathrm{d}X_t = b(t, X_t)\ \mathrm{d}t + \mathrm{d}W_t,\ \ X_0 = x \tag{I.1}$$

where $b : [0,T] \mapsto \mathcal{S}'(\mathbb{R}^d)$ is distribution valued, $T > 0$, $x \in \mathbb{R}^d$ and W is a standard d-dimensional Brownian motion. More precisely, we will use the following function spaces:

- the fractional Sobolev spaces $H^s_{p,q}(\mathbb{R}^d) = H^s_p(\mathbb{R}^d) \cap H^s_q(\mathbb{R}^d)$ with $s \in \mathbb{R}$, $p, q \in \mathbb{R}_+$;
- the Banach spaces $\mathcal{C}([0,T], B)$ of B-valued continuous functions, endowed with the norm $\|f\|_{\infty,B} = \sup\limits_{0\le t\le T} \|f(t)\|_B$ where B is a Banach space;
- the Banach spaces $\mathcal{C}^{1,\alpha}(\mathbb{R}^d) = \{f \in \mathcal{C}^{1,0}(\mathbb{R}^d, \mathbb{R}^d) \mid \|f\|_{\mathcal{C}^{1,\alpha}} < \infty\}$ endowed with the norm

$$\|f\|_{\mathcal{C}^{1,\alpha}} := \|f\|_{L^\infty} + \|\nabla f\|_{L^\infty} + \sup_{x\neq y\in\mathbb{R}^d} \frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\alpha}$$

  with $\mathcal{C}^{1,0}(\mathbb{R}^d, \mathbb{R}^d)$ defined as the closure of $\mathcal{S}$ with respect to the norm $\|f\|_{\mathcal{C}^{1,0}} := \|f\|_{L^\infty} + \|\nabla f\|_{L^\infty}$ and $\alpha > 0$;
- the spaces $L^q_t(L^p_x(\mathbb{R}^d))$ of mesurable functions $b : [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ verifying:

$$\int_0^T \left(\int_{\mathbb{R}^d} |b(t,x)|^p\ \mathrm{d}x \right)^{q/p} \mathrm{d}t < \infty$$

  with $d/p + 2/q < 1$.

We will often write H^s_q instead of $H^s_q(\mathbb{R}^d)$ (the same remark holds for $L^q_t(L^p_x)$ and $\mathcal{C}^{1,\alpha}$).

Assumption 1. *We will always assume that* $b \in \mathcal{C}\left([0,T], H^{-\beta}_{\tilde{q},q}(\mathbb{R}^d)\right)$, *with* $\beta \in \left(0, \frac{1}{2}\right)$, $\tilde{q} = \frac{d}{1-\beta}$, *and* $q \in \left(\tilde{q}, \frac{d}{\beta}\right)$. *Moreover, we assume that* $t \mapsto b(t)$ *is Lipschitz continuous.*

Equation (I.1) is studied by Franco Flandoli, Elena Issoglio, and Francesco Russo in [3] in which they define a concept of virtual solution. The authors prove then existence and unicity in law of this solution. Moreover, they show that this virtual solution in in fact the limit in law of classical solutions. We will develop here a numerical approximation method for this equation.

Example 2. *Let's take a look at an example of such drift b in 1D, in the time-homogeneous case. Fix a path of a fractional Brownian motion (fBm) B_x^H with Hurst index $1/2 < H < 1$ (for $H = 1/2$, we would retrieve the usual Brownian motion). We recall that these stochastic processes are centered gaussian processes verifying*

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} + |t-s|^{2H}\right). \tag{I.2}$$

Then, you smoothly cut this path to zero outside a compact and take the derivative in the distributional sense. We note $-\beta = H - 1 - \varepsilon$ with $\varepsilon > 0$ small. Given $B_x^H(\omega) \in H_{\tilde{q},q}^{1-\beta}(\mathbb{R})$ for $q > 2$ and $\tilde{q} = 2$, by taking the derivative we obtain $b = \frac{\mathrm{d}}{\mathrm{d}u} B_u^H(\omega) \in H_{\tilde{q},q}^{-\beta}(\mathbb{R})$.

I.2 Principle of the approximation algorithm

As far as the drift b is not a function but a distribution, it must be approximated if we want to evaluate it pointwise. In order to do so, we will use a series representation of b and truncate it. That is why we will consider two steps in our algorithm:

- approximate b by a function b^N meant to converge to b in $\mathcal{C}\left([0,T], H_{\tilde{q},q}^{-\beta}(\mathbb{R}^d)\right)$ as $N \to \infty$. In practise we will choose $b^N \in \mathcal{C}\left([0,T], H_{\tilde{q},q}^{-\beta}(\mathbb{R}^d)\right)$ piecewise constant with Supp $b^N \subset [-N,N]^d$. b^N will be defined as a truncated Haar representation. Therefore, by compact support, piecewise consistency, and continuity in time, we will have $b^N \in L_t^q(L_x^p(\mathbb{R}^d)) \cap \mathcal{C}\left([0,T], H_{\tilde{q},q}^{-\beta}(\mathbb{R}^d)\right)$ for any (q,p) satisfying $d/p + 2/q < 1$;
- approximate the solution X_t^N to the approximated SDE
$$\mathrm{d}X_t^N = b^N\left(t, X_t^N\right)\,\mathrm{d}t + \mathrm{d}W_t,\ X_0^N = x \tag{I.3}$$
by $X_t^{N,n}$ defined with the Euler-Maruyama scheme by
$$X_t^{N,n} = x + \int_0^t b^N\left(\eta_n(t), X_{\eta_n(t)}^{N,n}\right)\mathrm{d}t + W_{\eta_n(t)} \tag{I.4}$$
where $\eta_n(t) = t_k$ if and only if $t \in [t_k, t_{k+1}[$, for $t_k = \frac{k}{n}$ with $k \in [\![0, \lceil nT \rceil]\!]$.

I.3 Virtual solutions of the original SDE and its approximation

In order to approximate the solution to the SDE (I.1), we must go back to the definition of its virtual solution given in [3]. Let $(\delta,p) \in K(\beta,q) := \{(\delta,p) \mid \beta < \delta < 1-\beta,\ \frac{d}{\delta} < p < q\}$ (see Figure II.1). Let's fix ρ, λ big enough to apply Theorem 14 and Lemma 22 in [3].

Definition 3 (See [3])**.** *A stochastic basis is defined as a pentuple $(\Omega, \mathcal{F}, \mathbb{F}, P, W)$ where $(\Omega, \mathcal{F}, P)$ is a complete probability space with a completed filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ and W is a d-dimensional $\mathbb{F}$-Brownian motion.*

Flandoli, Issoglio and Russo define the virtual solution to SDE (I.1) as a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P, W)$ and a continuous stochastic process $X := (X_t)_{t\in[0,T]}$ on it, $\mathbb{F}$-adapted, shorten as $(X, \mathbb{F})$ such that the integral equation

$$X_t = x + u(0,x) - u(t,X_t) + (\lambda+1)\int_0^t u(s,X_s)\,\mathrm{d}s + \int_0^t \left(\nabla u(s,X_s) + \mathrm{I}_d\right)\mathrm{d}W_s \tag{I.5}$$

holds for all $t \in [0,T]$, with probability one, where u is the mild solution in $\mathcal{C}([0,T], H_p^{1+\delta}(\mathbb{R}^d))$ of the following parabolic PDE:

$$\begin{cases} \partial_t u + \frac{1}{2}\Delta u + b\nabla u - (\lambda+1)u = -b & \text{on } [0,T]\times\mathbb{R}^d \\ u(T) = 0 & \text{on } \mathbb{R}^d \end{cases}. \tag{I.6}$$

The authors of [3] show then existence and uniqueness in law of a virtual solution to (I.1).

Moreover, $(X,\mathbb{F})$ is solution to (I.5) if and only if $(Y,\mathbb{F})$ is solution to

$$\begin{cases} Y_t = y + (\lambda+1)\int_0^t u(s,\Psi(s,Y_s))\,\mathrm{d}s + \int_0^t (\nabla u(s,\Psi(s,Y_s)) + \mathrm{I}_d)\,\mathrm{d}W_s \\ X_t = \Psi(t,Y_t) \end{cases} \tag{I.7}$$

with $\varphi(t,x) = x + u(t,x)$, $y = \varphi(0,x)$ and $\Psi(t,\cdot) = \varphi^{-1}(t,\cdot)$.

We also define another similar PDE by replacing b by $b^N \in L_t^q(L_x^p(\mathbb{R}^d)) \cap \mathcal{C}\left([0,T], H_{\tilde{q},q}^{-\beta}(\mathbb{R}^d)\right)$. We call u^N its unique mild solution in $\mathcal{C}([0,T], H_p^{1+\delta}(\mathbb{R}^d))$:

$$\begin{cases} \partial_t u^N + \frac{1}{2}\Delta u^N + b^N \nabla u^N - (\lambda+1)u^N = -b^N & \text{on } [0,T]\times\mathbb{R}^d \\ u^N(T) = 0 & \text{on } \mathbb{R}^d \end{cases}. \tag{I.8}$$

Then we consider an approximated version of (I.7):

$$\begin{cases} Y_t^N = y^N + (\lambda+1)\int_0^t u^N\left(s,\Psi^N\left(s,Y_s^N\right)\right)\,\mathrm{d}s + \int_0^t \left(\nabla u^N\left(s,\Psi^N\left(s,Y_s^N\right)\right) + \mathrm{I}_d\right)\,\mathrm{d}W_s \\ X_t^N = \Psi^N(t,Y_t^N) \end{cases}. \tag{I.9}$$

with $\varphi^N(t,x) = x + u^N(t,x)$, $y^N = \varphi^N(0,x)$ and $\Psi^N(t,\cdot) = \left(\varphi^N\right)^{-1}(t,\cdot)$.

The following result from Flandoli, Issoglio and Russo expresses the link between classical and virtual solutions and justifies our approximation algorithm.

Proposition 4 (Proposition 26 in [3])**.** *If $b^N \in L_t^q(L_x^p(\mathbb{R}^d))$, then the classical solution $(X,\mathbb{F})$ to the SDE (I.3) is also a virtual solution.*

Remark 5. *Proposition 4 assures us that the virtual solution to (I.9), $(X^N,\mathbb{F})$, defined above in (I.9) is in fact the classical solution to (I.3), as far as $b^N \in L_t^q(L_x^p)$. That is why for each fixed N our Euler-Maruyuma approximation $X_t^{N,n}$ of the solution to (I.3) is meant to converge to the virtual solution X_t^N.*

Proposition 6. *Let u be the unique mild solution in $\mathcal{C}([0,T], H_p^{1+\delta}(\mathbb{R}^d))$ of equation (I.6) for $(\delta,p) \in K(\beta,q)$. Then $u \in \mathcal{C}([0,T], C^{1,\alpha}(\mathbb{R}^d))$ with $\alpha = \delta - d/p$.*

Proof. It's a direct consequence of the fractional Morrey inequality (See Theorem 16 in [3]). □

I.4 Approximation of the drift

We use Haar wavelets to give a series representation of b. By doing so, we will be able to approximate it numerically by truncating the series. We focus on the one dimensional case.

Definition 7 (Haar wavelets)**.** *We define the Haar wavelets $h_{j,m}^k$ on $\mathbb{R}$ with $j \in \mathbb{N}\cup\{-1\}$, $m \in \mathbb{N}$ and $k \in \mathbb{Z}$ by:*

$$\begin{cases} h_M & : x \longmapsto \left(\mathbb{1}_{[0,\frac{1}{2})} - \mathbb{1}_{[\frac{1}{2},1)}\right)(x) \\ h_{-1,0}^k & : x \longmapsto \sqrt{2}\,|h_M(x-k)| \\ h_{j,m} & : x \longmapsto h_M(2^j x - m) \\ h_{j,m}^k & : x \longmapsto h_{j,m}(x-k) \end{cases} \tag{I.10}$$

Theorem 8 (See Theorem 2.9 in [10] and Remark 3.4 in [11])**.** *Let $f \in H_q^s(\mathbb{R}) = F_{q2}^s(\mathbb{R})$ for $2 \le q < \infty$, and $s \in \left(-\frac{1}{2}, \frac{1}{q}\right)$. Therefore, we have the unique representation*

$$f = \sum_{j=-1}^{+\infty} \sum_{k\in\mathbb{Z}} \sum_{m=0}^{2^j-1} \mu_{j,m}^k 2^{-j\left(s-\frac{1}{q}\right)} h_{j,m}^k \tag{I.11}$$

with unconditional convergence in $S'(\mathbb{R})$*, where* $\mu_{j,m}^k = 2^{j\left(s-\frac{1}{q}+1\right)} \int_{\mathbb{R}} f(x) h_{j,m}^k(x)\, \mathrm{d}x$ *in the sense of dual pairing, and* $\sum_{m=0}^{2^j-1}$ *means* $m = 0$ *when* $j = -1$*.*

Now we are able to define how we approximate a distribution expressed on a Haar base.

Definition 9. *With the same notation* $\mu_{j,m}^k$*, let* $f \in H_q^s(\mathbb{R})$ *for* $2 \leq q < \infty$*, and* $s \in \left(-\frac{1}{2}, \frac{1}{q}\right)$*. Given* $N \in \mathbb{N}^*$ *we define* f^N *in* $H_q^s(\mathbb{R})$ *by:*

$$f^N = \sum_{j=0}^{N} \sum_{k=-N}^{N-1} \sum_{m=0}^{2^j-1} \mu_{j,m}^k 2^{-j\left(s-\frac{1}{q}\right)} h_{j,m}^k. \tag{I.12}$$

Remark 10. *We can note that* $\operatorname{Supp} f^N \subset [-N, N]$*. Moreover, we have:*

$$\left\| f - f^N \right\|_{H_q^s(\mathbb{R})} \xrightarrow[N \to +\infty]{} 0. \tag{I.13}$$

Remark 11. *The Haar system, defined here on* $\mathbb{R}$*, can be extended to* $\mathbb{R}^d$ *and remains an unconditional basis of* $H_q^s(\mathbb{R}^d)$ *for* $q \geq 2$ *and* $s \in \left(-\frac{1}{2}, \frac{1}{q}\right)$*. See Theorem 2.21 and Corollary 2.23 in [10]. Therefore our drift approximation with Haar wavelets is also relevant in* $\mathbb{R}^d$*. Nevertheless, in this case we don't know how to compute numerically the coefficients, even in the special case of the derivative of a fBm (or the derivative of any Hölder-continuous function actually). This is due to the lack of a proper Faber representation in* $\mathbb{R}^d$*. When* $d = 1$*, we are able to compute the coefficients for each given precision* N*. This will be done in Part III.*

Remark 12. *In our case,* $b \in \mathcal{C}\left([0,T], H_{\tilde{q},q}^{-\beta}(\mathbb{R}^d)\right)$*. Thus, we have for instance in dimension 1, to simplify notations but without any loss of generality, a Haar representation:*

$$\forall t \in [0,T],\ b(t) = \sum_{j=-1}^{+\infty} \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^j-1} \mu_{j,m}^k(t) 2^{-j\left(s-\frac{1}{q}\right)} h_{j,m}^k \tag{I.14}$$

where the coefficients $t \mapsto \mu_{j,m}^k(t) = 2^{j\left(s-\frac{1}{q}+1\right)} \int_{\mathbb{R}} b(t,x) h_{j,m}^k(x)\, \mathrm{d}x$ *are Lipschitz continuous, because* $t \mapsto b(t)$ *is. In the proof of Lemma 5.1 in [6], Issoglio and Russo proved in this case that*

$$b^N(\cdot) = \sum_{j=-1}^{N} \sum_{k=-N}^{N-1} \sum_{m=0}^{2^j-1} \mu_{j,m}^k(\cdot) 2^{-j\left(s-\frac{1}{q}\right)} h_{j,m}^k \tag{I.15}$$

belongs to $\mathcal{C}\left([0,T], H_{\tilde{q},q}^{-\beta}(\mathbb{R})\right)$ *and that* b^N *converges to* b *in* $\mathcal{C}\left([0,T], H_{\tilde{q},q}^{-\beta}(\mathbb{R})\right)$*. As they state, their argument extends to* $\mathcal{C}\left([0,T], H_{\tilde{q},q}^{-\beta}(\mathbb{R}^d)\right)$ *and the corresponding Haar basis by looking at each component of these functions in* $\mathcal{C}\left([0,T], H_{\tilde{q},q}^{-\beta}(\mathbb{R})\right)$*. Then we are able to construct with Haar wavelets a piecewise constant, with compact support and Lipschitz continuous in time bounded function* b^N *converging to* b *in* $\mathcal{C}\left([0,T], H_{\tilde{q},q}^{-\beta}(\mathbb{R}^d)\right)$*.*

Part II

Convergence of the algorithm

In this section, we present and prove convergence results for our approximation algorithm.

II.1 Weak convergence rate of the Euler approximation

Recently, Gunther Leobacher and Michaela Szölgyenyi proved in [8] the convergence of the Euler-Maruyama scheme for multidimensional SDEs with discontinuous but piecewise Lipschitz time-homogeneous drift and a degenerate diffusion coefficient. This framework applies to the scheme we use with piecewise constant in space and Lipschitz in time drift $b^N(t,x)$.

Let's define $\widetilde{X_t^N} := \begin{pmatrix} X_t^N \\ t \end{pmatrix}$, $\widetilde{b^N} : x \in \mathbb{R}^{d+1} \mapsto \begin{pmatrix} b^N(x_{d+1}, x_{|d}) \\ 1 \end{pmatrix} \in \mathbb{R}^{d+1}$,

$$\widetilde{\sigma} : x \in \mathbb{R}^{d+1} \mapsto \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \in \mathbb{R}^{(d+1)^2},$$

and $\widetilde{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^{d+1}$, with $x_{|d} := \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \in \mathbb{R}^d$ when $x \in \mathbb{R}^{d+1}$. Then $\widetilde{X_t^N}$ verifies

$$\mathrm{d}\widetilde{X_t^N} = \widetilde{b^N}\left(\widetilde{X_t^N}\right)\,\mathrm{d}t + \widetilde{\sigma}\left(\widetilde{X_t^N}\right)\,\mathrm{d}W_t,\ \widetilde{X_0^N} = \widetilde{x}. \tag{II.1}$$

Using notations from [8], the discontinuity set of the drift $\widetilde{b^N}$ is called Θ and $\forall \chi \in \Theta$, $n(\chi)$ is defined as the normalized normal vector to Θ at the point χ. Because [8] applies to degenerate diffusion coefficients, we are able to step up from time-homogeneous drifts to time-dependent drift. More precisely, the usual Zvonkin-Veretennikov uniform ellipticity condition on the diffusion coefficient σ is relaxed into the non-parallelity condition:

$$\forall \chi \in \Theta,\ \widetilde{\sigma}(\chi)^\top n(\chi) \neq 0.$$

As far as the discontinuity set Θ doesn't depend on time, we obtain $n(\chi) = \begin{pmatrix} n_1(\chi) \\ \vdots \\ n_d(\chi) \\ 0 \end{pmatrix}$. Then $\forall \chi \in \Theta,\ \sigma(\chi)^\top n(\chi) = \begin{pmatrix} n_1(\chi) \\ \vdots \\ n_d(\chi) \\ 0 \end{pmatrix} = n(\chi) \neq 0$ because $n(\chi)$ is normalized. Thus the non-parallelity condition is satisfied. Moreover, Θ is the finite union of orientable compact C^3-manifolds (we recall that Supp b^N is compact, and that these manifolds are subsets of lines, by definition of Haar wavelets). $\widetilde{b^N}$ is bounded, piecewise Lipschitz continuous and $\widetilde{\sigma}$ is bounded and Lipschitz. We are therefore satisfying all the conditions to apply Example 2.6 from [8].

Theorem 13 (Theorem 3.1 and Example 2.6 in [8])**.** *$\exists C_N > 0$ independent of n such that it holds $\forall \varepsilon > 0$ arbitrary small, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$:*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left\|X_t^{N,n} - X_t^N\right\|^2\right]^{1/2} \leq C_N \delta^{1/4-\varepsilon} \tag{II.2}$$

with $\delta = \frac{1}{n}$ the step size.

Remark 14. *It is unclear how C_N depends on N but nevertheless, the convergence of X_t^N to X_t will allow us to choose N big enough for the weak error to be small, and only then we will be able to choose n big enough in order to control the error between $X_t^{N,n}$ and X_t.*

This strong error obviously gives us a control of the weak error for μ-Hölder functions f with $\mu \in (0, 1]$.

Corollary 15. *Let f be μ-Hölder with constant $C_f > 0$, $\mu \in (0, 1]$ and $t \in [0, T]$. Then, exists $C'_N > 0$ independent of n such that it holds $\forall \varepsilon > 0$, arbitrary small, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$:*

$$\sup_{0\leq t\leq T}\left|\mathbb{E}\left[f\left(X_t^{N,n}\right) - f\left(X_t^N\right)\right]\right| \leq C'_N \delta^{\mu/4-\varepsilon} \tag{II.3}$$

with $\delta = \frac{1}{n}$ the step size.

Proof. By Jensen's inequality and the μ-Hölder property of f, we obtain:

$$\begin{aligned}
\sup_{0\leq t\leq T}\left|\mathbb{E}\left[f\left(X_t^{N,n}\right) - f\left(X_t^N\right)\right]\right| &\leq C_f \sup_{0\leq t\leq T}\mathbb{E}\left[\left\|X_t^{N,n} - X_t^N\right\|^\mu\right] \\
&\leq C_f \sup_{0\leq t\leq T}\mathbb{E}\left[\left\|X_t^{N,n} - X_t^N\right\|^2\right]^{\mu/2} \\
&\leq C_f \mathbb{E}\left[\sup_{0\leq t\leq T}\left\|X_t^{N,n} - X_t^N\right\|^2\right]^{\mu/2} \\
&\leq C_f C_N^\mu \delta^{\mu/4-\varepsilon}.
\end{aligned}$$

□

II.2 Convergence in law of the approximated solution to the virtual solution

We also have a result showing that our approximated solution X^N converges to the virtual solution X.

Proposition 16. *Let $b^N \xrightarrow[N\to\infty]{} b$ in $\mathcal{C}\left([0,T], H_{\tilde{q},q}^{-\beta}(\mathbb{R}^d)\right)$ with*

$$b^N \in L_t^q(L_x^p(\mathbb{R}^d)) \cap \mathcal{C}\left([0,T], H_{\tilde{q},q}^{-\beta}(\mathbb{R}^d)\right).$$

Then the unique strong solution X^N to the associated equation (I.3) *converges in law to the virtual solution* $(X, \mathbb{F})$ *of equation* (I.1)*.*

Proof. To prove the convergence in law of the solution X^N of (I.3) to the virtual solution X of (I.1), we use exactly the same arguments as in the Proposition 29 of [3] which uses the smoothness of the approximated drift only to apply Proposition 26 in [3], which extends to $L_t^q(L_x^p)$ drifts (See Proposition 4). $\square$

Therefore, the weak error of approximation converges to zero and our algorithm provides a weak approximation of the virtual solution to SDE (I.1).

II.3 Weak convergence rate of the approximated solution in 1D

The goal of this subsection, which is one of our contributions to this research project, is to give a convergence rate for the weak error in dimension one $\left|\mathbb{E}\left[f\left(X_t\right)-f\left(X_t^N\right)\right]\right|$ with suitable functions f. We are able to do so in the case where $\beta < 1/4$. Our main result relies on several lemmas. We will recall and use results from [3] where the stochastic differential equation (I.1) was first studied. Some of our technical lemmas proofs are developed in the appendix to simplify the reading of this report. Nevertheless, the proof of the main theorem is developped. We will assume that we have fixed a sequence b^N converging to b in $\mathcal{C}\left([0,T], H_{\tilde{q},q}^{-\beta}(\mathbb{R})\right)$.

II.3.1 Useful lemmas

Most of the following results allow us to study the behaviour of the solutions of PDEs (I.6) and (I.8) and then of the transformations φ and Ψ. We will use these lemmas to prove our main result, Theorem 27.

We first recall a useful lemma concerning the solutions of (I.6) and (I.8). It will allow us to choose λ big enough such that u and u^N are Lipschitz.

Lemma 17 (Lemma 20 in [3]). *Let $(\delta,p)\in K(\beta,q)$ and let u, u^N be the mild solutions to (I.6), (I.8) in $H_p^{1+\delta}$. Fix ρ big enough such that Theorem 14 in [3] applies and let $\lambda > \lambda^*$. Then $u(t)$, $u^N(t)\in\mathcal{C}^{1,\alpha}$ with $\alpha=\delta-1/p$ for each fixed t and $\forall\varepsilon>0$, $\exists\lambda_0>0$ such that*

$$\begin{cases}\forall\lambda\geq\lambda_0, & \sup\limits_{(t,x)\in[0,T]\times\mathbb{R}}|\nabla u(t,x)| \leq\varepsilon\\ \forall\lambda\geq\lambda_0, & \sup\limits_{(t,x)\in[0,T]\times\mathbb{R}}|\nabla u^N(t,x)| \leq\varepsilon\end{cases}\tag{II.4}$$

where the choice of λ^*, λ_0 depends only on $\delta,\beta,\|b\|_{\infty,H_{p,q}^{-\beta}}$, and $\|b^N\|_{\infty,H_{p,q}^{-\beta}}$.

Remark 18. *$\|b^N\|_{\infty,H_{p,q}^{-\beta}}\in(1/2\,\|b\|_{\infty,H_{p,q}^{-\beta}},2\,\|b\|_{\infty,H_{p,q}^{-\beta}})$ for N big enough so λ^*, λ_0 can be chosen independently of N for N big enough.*

The next lemma uses the embedding between $H_p^{1+\delta}$ and $\mathcal{C}^{1,\alpha}$ to control the error of approximation of u by u^N.

Lemma 19. *Let $(\delta,p)\in K(\beta,q)$ and let u, u^N be the mild solutions to (I.6),(I.8) in $H_p^{1+\delta}$, $\alpha=\delta-1/p$. Exists $c,K>0$ such that for both $N\in\mathbb{N}$ and ρ big enough (independently of N), $\forall t\in[0,T]$,*

$$\begin{cases}\left\|u^N(t)-u(t)\right\|_{L^\infty} & \leq\kappa\left\|b-b^N\right\|_{\infty,H_{\tilde{q},q}^{-\beta}}\\ \left\|\nabla u^N(t)-\nabla u(t)\right\|_{L^\infty} & \leq\kappa\left\|b-b^N\right\|_{\infty,H_{\tilde{q},q}^{-\beta}}.\end{cases}.\tag{II.5}$$

with $\kappa=cKe^{\rho T}$.

Proof. Applying fractional Morrey inequality, $\exists c > 0,\ \forall t \in [0, T]$:

$$\begin{cases} \left\| u^N(t) - u(t) \right\|_{L^\infty} & \leq \left\| u^N(t) - u(t) \right\|_{\mathcal{C}^{1,\alpha}} \leq c \left\| u^N(t) - u(t) \right\|_{H_p^{1+\delta}} \\ \left\| \nabla u^N(t) - \nabla u(t) \right\|_{L^\infty} & \leq \left\| u^N(t) - u(t) \right\|_{\mathcal{C}^{1,\alpha}} \leq c \left\| u^N(t) - u(t) \right\|_{H_p^{1+\delta}} . \end{cases}$$

Now, we can conclude with

$$\left\| u^N - u \right\|_{\infty, H_p^{1+\delta}} \leq e^{\rho T} \left\| u^N - u \right\|^{(\rho)}_{\infty, H_p^{1+\delta}} \leq K e^{\rho T} \left\| b - b^N \right\|_{\infty, H_{\tilde{q},q}^{-\beta}}$$

from Lemma 23 in [3], for both $N \in \mathbb{N}$ and $\rho > 1$ big enough, and where $\|f(t)\|^{(\rho)}_{\infty,B} := \sup_{0 \leq t \leq T} e^{-\rho t} \|f(t)\|_B$. ☐

Lemma 20. *For λ big enough, independently of N,*

$$\left| \Psi^N \left(t, Y_t^N\right) - \Psi \left(t, Y_t^N\right) \right| \leq 2\kappa \left\| b - b^N \right\|_{\infty, H_{\tilde{q},q}^{-\beta}} \tag{II.6}$$

with $\kappa = cKe^{\rho T}$.

Proof. For λ big enough, by Lemma 17, $\forall t \in [0, T]$, $\sup_{x \in \mathbb{R}} |\nabla u(t, x)| \leq 1/2$, so we obtain with $\varphi(t, x) = x + u(t, x)$:

$$\begin{aligned} \left| \varphi\left(t, \Psi^N \left(t, Y_t^N\right)\right) - \varphi\left(t, \Psi\left(t, Y_t^N\right)\right) \right| &\geq \inf_{x \in \mathbb{R}} |\nabla \varphi(t, x)| \left| \Psi^N \left(t, Y_t^N\right) - \Psi\left(t, Y_t^N\right) \right| \\ &\geq \frac{1}{2} \left| \Psi^N \left(t, Y_t^N\right) - \Psi\left(t, Y_t^N\right) \right| \end{aligned}$$

then

$$\begin{aligned} \left| \Psi^N \left(t, Y_t^N\right) - \Psi\left(t, Y_t^N\right) \right| &\leq 2 \ \left| \varphi\left(t, \Psi^N \left(t, Y_t^N\right)\right) - \varphi\left(t, \Psi\left(t, Y_t^N\right)\right) \right| \\ &= 2 \ \left| \varphi\left(t, \Psi^N \left(t, Y_t^N\right)\right) - \varphi^N \left(t, \Psi^N \left(t, Y_t^N\right)\right) \right| \\ &\leq 2 \ \left\| u(t) - u^N(t) \right\|_{L^\infty} \end{aligned}$$

where we used the fact that

$$\varphi^N \left(t, \Psi^N \left(t, Y_t^N\right)\right) = \varphi\left(t, \Psi\left(t, Y_t^N\right)\right) = Y_t^N. \tag{II.7}$$

With Lemma 19 we obtain

$$\left| \Psi^N \left(t, Y_t^N\right) - \Psi\left(t, Y_t^N\right) \right| \leq 2\kappa \left\| b - b^N \right\|_{\infty, H_{\tilde{q},q}^{-\beta}} . \tag{II.8}$$

☐

We will need an adapted version of a local time inequality (Lemma 4.2 in [12]) from Liqing Yan. The method he introduced will allow us to estimate the error $\mathbb{E}[|Y - Y^N|]$ and therefore calculate the weak error of approximation our algorithm commit. In his paper he studies the case where $X_0 = 0$ whereas in our case we don't assume anything about the initial value of the semimartingale X_t. That's why the proof is mainly the same. The difference is that υ_1 is now a stopping time when its value was 0 in the proof of Yan.

Lemma 21. *Let X_t be a continuous semimartingale. For $\varepsilon > 0$ we define a double sequence of stopping times by $\upsilon_1 = \inf\{t \geq 0 | X_t = 0\}$, $\tau_1 = \inf\{t > \upsilon_1 | X_t = \varepsilon\}$, $\upsilon_n = \inf\{t > \tau_{n-1} | X_t = 0\}$, $\tau_n = \inf\{t > \upsilon_n | X_t = \varepsilon\}$. For any real function $F(\cdot) \in \mathcal{C}^2(\mathbb{R})$ with $F(0) = 0,\ F'(0) = 0,\ F(\cdot) > 0$ on $(0, \varepsilon_0)$ with some $\varepsilon_0 > 0$, and for any $0 < \varepsilon < \varepsilon_0$ we have*

$$L_t^0(X) \leq 2\varepsilon - \frac{2}{F(\varepsilon)} \int_{\upsilon_1 \wedge t}^t \theta_s(X) \left(F(\varepsilon) - \varepsilon F'\left(X_s^+\right)\right) \mathrm{d}X_s + \frac{\varepsilon}{F(\varepsilon)} \int_{\upsilon_1 \wedge t}^t \theta_s(X) F''(X_s^+) \, \mathrm{d}[X]_s \tag{II.9}$$

with $\theta_s(X) = \sum_{n=1}^{\infty} \mathbb{1}_{\{\upsilon_n < s \leq \tau_n,\ 0 < X_s \leq \varepsilon\}}(X)$ and $t \in [0, T]$.

Applying lemma 21 with $F : x \in \mathbb{R} \mapsto x^2$, it follows:

Corollary 22. *Let X be a continuous semimartingale. With the same notations as in Lemma 21, for any $\varepsilon > 0$ we have*

$$0 \leq L_t^0(X) \leq 2\varepsilon - \frac{2}{\varepsilon}\int_{v_1\wedge t}^t \theta_s(X)\left(\varepsilon - 2X_s^+\right)\,\mathrm{d}X_s + \frac{2}{\varepsilon}\int_{v_1\wedge t}^t \theta_s(X)\,\mathrm{d}[X]_s \tag{II.10}$$

Remark 23. *For clarity and concision purposes, we will note $\tilde{u}(s,x) = u(s,\Psi(s,x))$, $\tilde{u}^N(s,x) = u^N(s,\Psi(s,x))$, and $\overline{u}^N(s,x) = u^N\left(s,\Psi^N(s,x)\right)$. The same notations will be used for the gradient.*

Lemma 24. *Let $(\delta,p) \in K(\beta,q)$, $\alpha = \delta - 1/p < 1$, u, u^N be the mild solutions to (I.6), (I.8) in $H_p^{1+\delta}$, and Y, Y^N solutions of the SDEs (I.5), (I.9). Then, if $1/2 < \alpha < 1$, for λ, N big enough we have $\forall\varepsilon \in (0,1]$,*

$$0 \leq \mathbb{E}\left[L_T^0(Y - Y^N)\right] \leq g(\varepsilon). \tag{II.11}$$

where

$$\begin{aligned} g(\varepsilon) &= 4(\lambda+1)T\kappa\left\|b - b^N\right\|_{\infty,H_{\tilde{q},q}^{-\beta}} + \left(2 + 2(\lambda+1)\,T + 6\left\|u\right\|_{\infty,\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right)\varepsilon^{2\alpha-1} \\ &+ 6T\left(\Lambda^2 4^\alpha \kappa^{2\alpha}\left\|b - b^N\right\|_{\infty,H_{\tilde{q},q}^{-\beta}}^{2\alpha} + \kappa^2\left\|b - b^N\right\|_{\infty,H_{\tilde{q},q}^{-\beta}}^2\right)\varepsilon^{-1}. \end{aligned}$$

We have been able with Lemma 24 to bound the expectation of the local time of $Y - Y^N$ by a function of $\varepsilon > 0$. Here we choose an optimal ε such that the bound is minimal.

Lemma 25. *With assumptions and notations of Lemma 24, and $1/2 < \alpha < 1$ we have*

$$\mathbb{E}\left[L_T^0(Y - Y^N)\right] \leq \Gamma\left\|b^N - b\right\|_{\infty,H_{\tilde{q},q}^{-\beta}}^{2\alpha-1} \tag{II.12}$$

for N big enough where

$$\Gamma = 4(\lambda+1)T\kappa + 6T\kappa^{2\alpha}\left(\frac{2\alpha+1}{2\alpha-1}\Lambda^2 4^\alpha + \kappa^{2(1-\alpha)}\right)\nu_\infty^{-1}$$

and

$$\nu_\infty = \left(\frac{6T\Lambda^2 4^\alpha\kappa^{2\alpha}}{(2\alpha-1)\left(2 + 2(\lambda+1)\,T + 6\left\|u\right\|_{\infty,\mathcal{C}^{1,\alpha}}^2 4^\alpha T\right)}\right)^{\frac{1}{2\alpha}}.$$

II.3.2 Main result

Remark 26. *For the proof of our theorem to hold, we will require $\alpha = \delta - \frac{1}{p}$ to be greater than $\frac{1}{2}$. Therefore, given the constraints on the choice of $(\delta,p) \in K(\beta,q) = \{(\delta,p) \mid \beta < \delta < 1-\beta,\ \frac{d}{\delta} < p < q\}$, we would like to know the greatest value α can take for a given β. First of all, $\delta < 1-\beta$. Then, $\frac{1}{p} > \frac{1}{q} > \beta$ so $-\frac{1}{p} < -\beta$. Thus, $\alpha < 1-2\beta$ and $\forall\varepsilon > 0$, we can take $\alpha = 1 - 2\beta - \frac{\varepsilon}{2}$. In figure II.1 page 24, the pair (δ,p) guaranteeing the maximal value for α corresponds to the top left corner of the grey triangle, indicated by a dot.*

Theorem 27. *Let f be μ-Hölder with constant $C_f > 0$ and $\mu \in (0,1]$. If $0 < \beta < 1/4$, $q \in \left(\tilde{q}, \frac{1}{\beta}\right)$, $\forall\varepsilon > 0$ arbitrarily small, with $(\delta,p) \in K(\beta,q)$ such that $\delta - 1/p = 1 - 2\beta - \varepsilon/2$, exists ξ independent of f such that for $N \in \mathbb{N}$, $\rho > 1$, λ big enough it holds:*

$$\sup_{0\leq t\leq T}\mathbb{E}\left[\left|f(X_t) - f\left(X_t^N\right)\right|\right] \leq \xi C_f\left\|b^N - b\right\|_{\infty,H_{\tilde{q},q}^{-\beta}}^{\mu(1-4\beta-\varepsilon)} \tag{II.13}$$

with $\xi = 2^\mu\left(\zeta^\mu e^{\mu(\lambda+1)T} + \kappa^\mu\right)$ and $\zeta = (2(\lambda+1)T + 1)\kappa + \Gamma$.

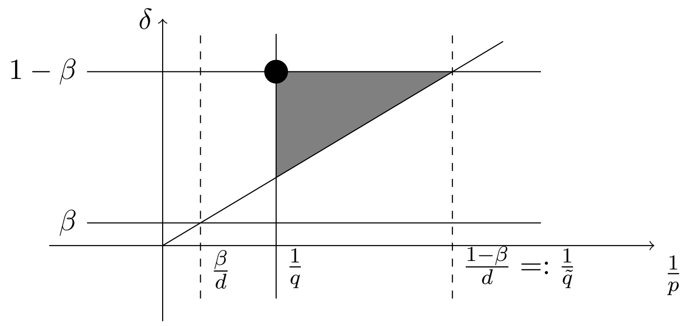

Figure II.1: The set $K(\beta, q)$. Figure taken from the paper [3] of Flandoli, Issoglio and Russo with the authorization of Elena Issoglio.

Proof of Theorem 27. We note as usual $\alpha = \delta - 1/p$. By Lemma 17, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\left\|b - b^N\right\|_{H_{\tilde{q}}^s(\mathbb{R})} \underset{N\to\infty}{\longrightarrow} 0$. Therefore $u^N(t,\cdot)$ and $u(t,\cdot)$ are $\frac{1}{2}$-Lipschitz. We recall that in this case, by Lemma 22 in [3], $\Psi(t,\cdot)$ and $\Psi^N(t,\cdot)$ are 2-Lipschitz. Therefore $\widetilde{u}^N(t,\cdot)$ and $\widetilde{u}(t,\cdot)$ are 1-Lipschitz. Let $t \in [0,T]$.

$$\begin{aligned}
&\mathbb{E}\left[\left|f\left(X_t\right) - f\left(X_t^N\right)\right|\right] \\
&= \mathbb{E}\left[\left|f\left(\Psi\left(t, Y_t\right)\right) - f\left(\Psi^N\left(t, Y_t^N\right)\right)\right|\right] \\
&\leq \mathbb{E}\left[\left|f\left(\Psi\left(t, Y_t\right)\right) - f\left(\Psi\left(t, Y_t^N\right)\right)\right|\right] + \mathbb{E}\left[\left|f\left(\Psi\left(t, Y_t^N\right)\right) - f\left(\Psi^N\left(t, Y_t^N\right)\right)\right|\right] \\
&\leq C_f\left(2^\mu \mathbb{E}\left[\left|Y_t - Y_t^N\right|^\mu\right] + \mathbb{E}\left[\left|\Psi\left(t, Y_t^N\right) - \Psi^N\left(t, Y_t^N\right)\right|^\mu\right]\right).
\end{aligned}$$

Finally we obtain by Jensen's inequality:

$$\leq C_f\left(2^\mu \mathbb{E}\left[\left|Y_t - Y_t^N\right|\right]^\mu + \mathbb{E}\left[\left|\Psi\left(t, Y_t^N\right) - \Psi^N\left(t, Y_t^N\right)\right|\right]^\mu\right). \tag{II.14}$$

Then we study the difference between Y_t^N and Y_t:

$$\begin{aligned}
Y_t - Y_t^N = y - y^N + (\lambda+1)\int_0^t &\left\{u\left(s, \Psi\left(s, Y_s\right)\right) - u^N\left(s, \Psi^N\left(s, Y_s^N\right)\right)\right\} \,\mathrm{d}s \\
+ \int_0^t &\left\{\nabla u\left(s, \Psi\left(s, Y_s\right)\right) - \nabla u^N\left(s, \Psi^N\left(s, Y_s^N\right)\right)\right\} \,\mathrm{d}W_s.
\end{aligned}$$

Remark 28. *We recall the notations* $\tilde{u}\left(s,x\right) = u\left(s, \Psi\left(s,x\right)\right)$*,* $\tilde{u}^N\left(s,x\right) = u^N\left(s, \Psi\left(s,x\right)\right)$*, and* $\overline{u}^N\left(s,x\right) = u^N\left(s, \Psi^N\left(s,x\right)\right)$*. The same notations are used for the gradient and the approximated mild solution.*

We apply Meyer-Tanaka's formula to obtain:

$$\begin{aligned}
\left|Y_t - Y_t^N\right| = \left|y - y^N\right| + (\lambda+1)\int_0^t &\operatorname{sign}(Y_s - Y_s^N)\left\{\widetilde{u}\left(s, Y_s\right) - \overline{u}^N\left(s, Y_s^N\right)\right\} \,\mathrm{d}s \\
+ \int_0^t &\operatorname{sign}(Y_s - Y_s^N)\left\{\widetilde{\nabla u}\left(s, Y_s\right) - \overline{\nabla u}^N\left(s, Y_s^N\right)\right\} \,\mathrm{d}W_s + L_t^0(Y - Y^N).
\end{aligned}$$

Taking the expectation leads to:

$$\begin{aligned}
\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] &= \left|u(0,x) - u^N(0,x)\right| + \mathbb{E}\left[L_t^0(Y - Y^N)\right] \\
&\quad + (\lambda+1)\,\mathbb{E}\left[\int_0^t \operatorname{sign}(Y_s - Y_s^N)\left\{\widetilde{u}\left(s, Y_s\right) - \overline{u}^N\left(s, Y_s^N\right)\right\} \,\mathrm{d}s\right]
\end{aligned}$$

because ∇u and ∇u^N are bounded so the Itô integral is a true martingale.

$$\begin{aligned}\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] \leq \left|u(0,x) - u^N(0,x)\right| + \mathbb{E}\left[L_t^0(Y - Y^N)\right] \\ + (\lambda+1)\ \mathbb{E}\left[\int_0^t \left|\widetilde{u}\left(s, Y_s\right) - \widetilde{u}\left(s, Y_s^N\right)\right| \mathrm{d}s\right] \\ + (\lambda+1)\ \mathbb{E}\left[\int_0^t \left|\widetilde{u}\left(s, Y_s^N\right) - \widetilde{u}^N\left(s, Y_s^N\right)\right| \mathrm{d}s\right] \\ + (\lambda+1)\ \mathbb{E}\left[\int_0^t \left|\widetilde{u}^N\left(s, Y_s^N\right) - \overline{u}^N\left(s, Y_s^N\right)\right| \mathrm{d}s\right].\end{aligned}$$

We use Lemma 19, the 1-Lipschitz property of $\widetilde{u}$, and the 1/2-Lipschitz property of u:

$$\begin{aligned}\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] &\leq \kappa \left\|b^N - b\right\|_{\infty, H_{\tilde{q},q}^{-\beta}} + (\lambda+1)\ \mathbb{E}\left[\int_0^t \left|Y_s - Y_s^N\right| \mathrm{d}s\right] + (\lambda+1)t\kappa \left\|b^N - b\right\|_{\infty, H_{\tilde{q},q}^{-\beta}} \\ &+ \frac{\lambda+1}{2}\ \mathbb{E}\left[\int_0^t \left|\Psi\left(s, Y_s^N\right) - \Psi^N\left(s, Y_s^N\right)\right| \mathrm{d}s\right] + \mathbb{E}\left[L_t^0(Y - Y^N)\right] \\ &\leq (\lambda+1) \int_0^t \mathbb{E}\left[\left|Y_s - Y_s^N\right|\right] \mathrm{d}s + (2(\lambda+1)T + 1)\kappa \left\|b^N - b\right\|_{\infty, H_{\tilde{q},q}^{-\beta}} \\ &+ \mathbb{E}\left[L_T^0(Y - Y^N)\right].\end{aligned}$$

where we used Lemma 20 and the fact that $L_t^0(Y - Y^N)$ is an increasing process.

By Gronwall's Lemma, it follows:

$$\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] \leq C(N)\ e^{(\lambda+1)t} \leq C(N)\ e^{(\lambda+1)T} \tag{II.15}$$

with $C(N) = (2(\lambda+1)T + 1)\kappa \left\|b^N - b\right\|_{\infty, H_{\tilde{q},q}^{-\beta}} + \mathbb{E}\left[L_T^0(Y - Y^N)\right]$.

With Lemma 24 and Lemma 25 we obtain

$$C(N) \leq (2(\lambda+1)T + 1)\kappa \left\|b^N - b\right\|_{\infty, H_{\tilde{q},q}^{-\beta}} + \Gamma \left\|b^N - b\right\|_{\infty, H_{\tilde{q},q}^{-\beta}}^{2\alpha-1} \leq \zeta \left\|b^N - b\right\|_{\infty, H_{\tilde{q},q}^{-\beta}}^{2\alpha-1}.$$

for N big enough where $\zeta = (2(\lambda+1)T + 1)\kappa + \Gamma$. It follows:

$$\mathbb{E}\left[\left|Y_t - Y_t^N\right|\right] \leq \zeta e^{(\lambda+1)T} \left\|b^N - b\right\|_{\infty, H_{\tilde{q},q}^{-\beta}}^{2\alpha-1}. \tag{II.16}$$

Finally, combining (II.14), (II.16) and Lemma 20 we obtain:

$$\begin{aligned}\left|\mathbb{E}\left[f\left(X_t\right) - f\left(X_t^N\right)\right]\right| &\leq C_f \left(2^\mu \mathbb{E}\left[\left|Y_t - Y_t^N\right|\right]^\mu + \mathbb{E}\left[\left|\Psi\left(t, Y_t^N\right) - \Psi^N\left(t, Y_t^N\right)\right|\right]^\mu\right) \\ &\leq C_f \left(2^\mu \zeta^\mu e^{\mu(\lambda+1)T} \left\|b^N - b\right\|_{\infty, H_{\tilde{q},q}^{-\beta}}^{\mu(2\alpha-1)} + 2^\mu \kappa^\mu \left\|b^N - b\right\|_{\infty, H_{\tilde{q},q}^{-\beta}}^{\mu}\right) \\ &\leq 2^\mu C_f \left(\zeta^\mu e^{\mu(\lambda+1)T} + \kappa^\mu\right) \left\|b^N - b\right\|_{\infty, H_{\tilde{q},q}^{-\beta}}^{\mu(2\alpha-1)}\end{aligned}$$

for N big enough, which is the expected result because none of the constants depend on t. □

Part III

Numerical aspects

Our project concerns a large class of very irregular drifts. But in order to do numerical studies, we must select examples of such drifts. If we want to use a drift selected as a realization of a random process, we can think of the derivative of a sample path of a fractional Brownian motion. That's why we will explain here how we can apply our approximation algorithm in this special case.

III.1 Numerical simulation of fractional Brownian motion

To simulate a sample path of a fBm B^H_x on a finite grid $(x_k)_{k\in[\![1,N]\!]}$, we simulate n independent standard gaussian random variables $(X_k)_{k\in[\![1,N]\!]}$ and then correlate them with the definite positive correlation matrix

$$C_{k,s} = \mathbb{E}\left[B^H_{x_k} B^H_{x_s}\right] = \frac{1}{2}\left(x_k^{2H} + x_s^{2H} + |x_k - x_s|^{2H}\right).$$

To do so, we use the Cholesky decomposition method and calculate the lower triangular matrix M such that $C = MM^\top$. Therefore, defining the multidimensional random values

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ and } B^H = MX,$$

B^H is a fractional Brownian motion evaluated on the grid $(x_k)_{k\in[\![1,n]\!]}$. Indeed, numerically we will only consider some realizations of these random values. Figure III.1 shows the result of such a simulation. We will always translate the path in order to have a symmetric domain. In fact, what does matter in our case is the regularity of a fixed path and not the domain of the underlying process.

III.1.1 Refining a sample path of a fractional Brownian motion

A first numerical concern was about the way to refine a path of fBm, it is to say to add new points in between the simulated path. For instance, if you have simulated a fBm on a regular grid $x_k = \frac{k}{N}$, you may wish to add new points to work on the grid 2 times more precise $\widetilde{x_k} = \frac{k}{2N}$ without changing the points already defined. Let X, Y be two vectors of N independent gaussian random variables and C the correlation matrix associated with the grid $(x_k)_{k\in[\![1,n]\!]}$. We note $\widetilde{X} = \begin{pmatrix} X \\ Y \end{pmatrix}$.

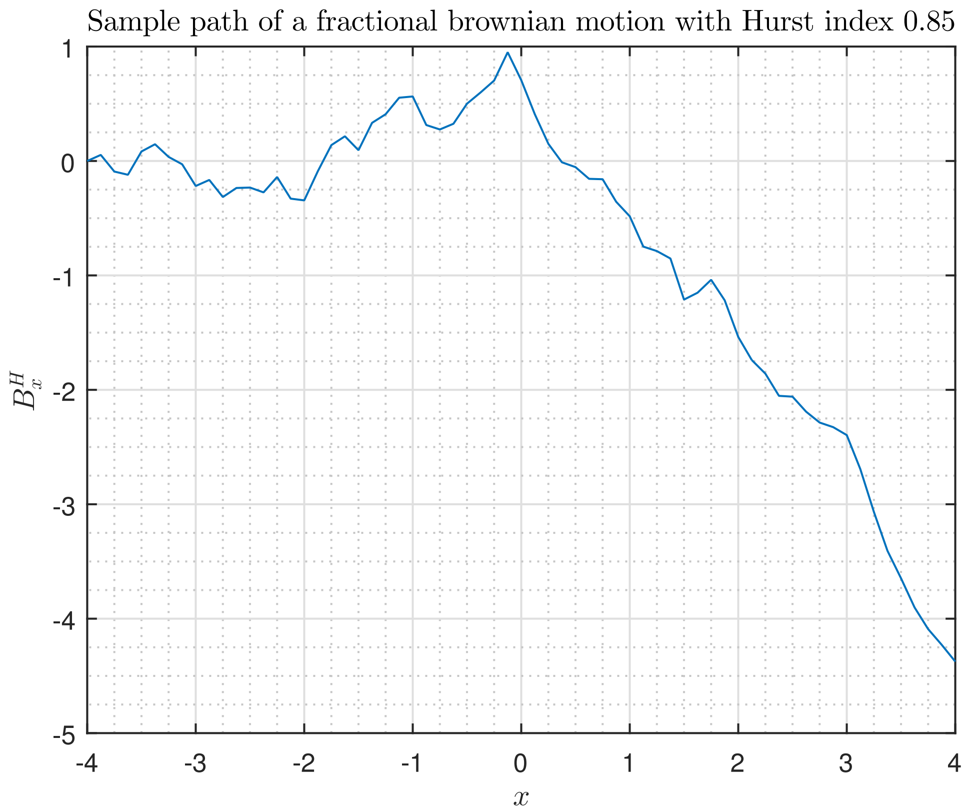

Figure III.1: Translated sample path of a fBm with 64 points

If you have already simulated B^H on the grid $(x_k)_{k\in[\![1,n]\!]}$ with the random variable X, it verifies $B^H = MX$ with $C = MM^\top$. Then, let's consider the new correlation matrix $\widetilde{C} = \begin{pmatrix} C & A^\top \\ A & B \end{pmatrix}$ with

$$\begin{aligned}
A_{k,s} &= \mathbb{E}\left[B^H_{\frac{x_k+x_{k-1}}{2}} B^H_{x_s}\right] = \frac{1}{2}\left(\left(\frac{x_k+x_{k-1}}{2}\right)^{2H} + x_s^{2H} + \left|\frac{x_k+x_{k-1}}{2} - x_s\right|^{2H}\right) \\
B_{k,s} &= \mathbb{E}\left[B^H_{\frac{x_k+x_{k-1}}{2}} B^H_{\frac{x_s+x_{s-1}}{2}}\right] \\
&= \frac{1}{2}\left(\left(\frac{x_k+x_{k-1}}{2}\right)^{2H} + \left(\frac{x_s+x_{s-1}}{2}\right)^{2H} + \left|\frac{x_k+x_{k-1}}{2} - \frac{x_s+x_{s-1}}{2}\right|^{2H}\right)
\end{aligned}$$

where we chose by convention $x_0 = 0$. Looking at $\widetilde{M}$, the Cholesky root of $\widetilde{C}$, we obtain:

$$\widetilde{M} = \begin{pmatrix} D & 0 \\ E & F \end{pmatrix} \tag{III.1}$$

with D lower triangular matrix verifying $DD^\top = C$ so $D = M$. Therefore, we will have

$$\widetilde{B}^H = \widetilde{M}\widetilde{X} = \begin{pmatrix} M & 0 \\ E & F \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} MX \\ EX+FY \end{pmatrix} = \begin{pmatrix} B^H \\ EX+FY \end{pmatrix} \tag{III.2}$$

and we obtain $\widetilde{B}^H$ on the grid $(x_k, \frac{x_k+x_{k-1}}{2})$ without changing the previous values. Therefore, one just have to reorder $\widetilde{B}^H$ to obtain a realization of a fBm on the grid

$$\widetilde{x_k} = \left(\frac{x_1}{2}, x_1, \frac{x_1+x_2}{2}, \cdots, x_{k-1}, \frac{x_k+x_{k-1}}{2}, x_k, \cdots, x_N\right).$$

Numerical results are shown in figures III.1 and III.2.

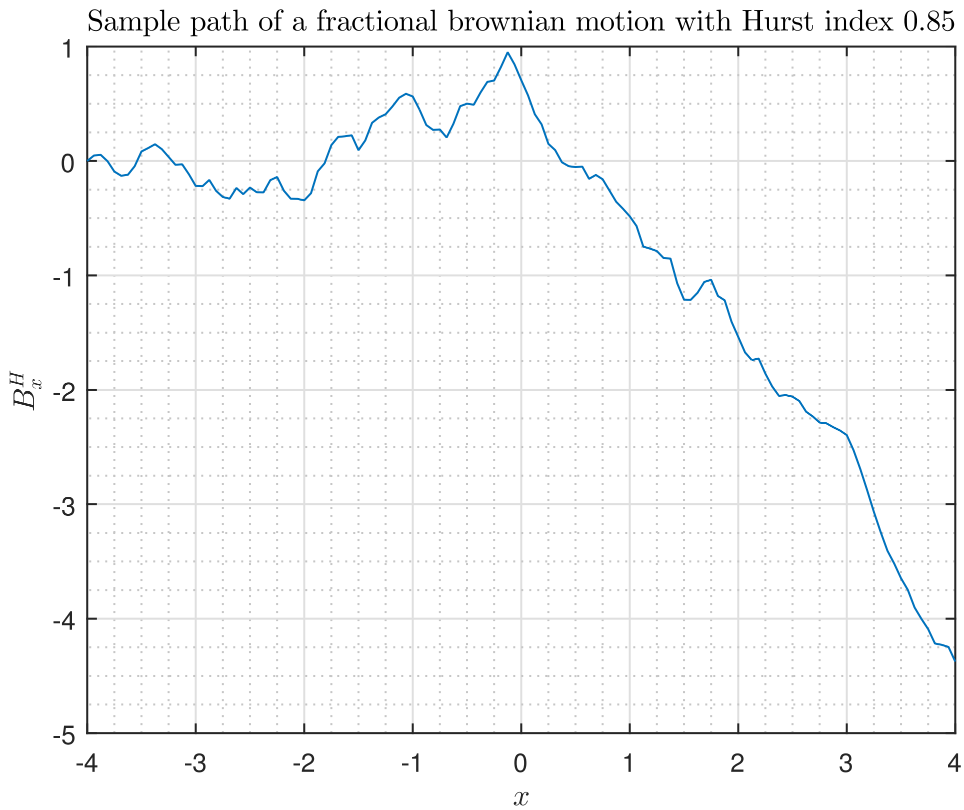

Figure III.2: Refined version of the previous fBm path with 128 points

III.2 Special case of the derivative of a fractional Brownian motion

This study of a special case for the approximation algorithm we developed is our other contribution to this research project. For our numerical studies, we will now fix the realization of a fractional Brownian motion B_x^H and cut its path to zero continuously. In order to apply Theorem 27, we suppose $H > \frac{3}{4}$. In this section, the drift will be time-homogeneous.

More precisely, let $K > 0$, $\gamma \in \mathcal{C}^\infty(\mathbb{R})$ such that $\forall x \in [-K, K]$, $\gamma(x) = 1$ and Supp $\gamma \subset D \subset [-K-1, K+1]$. Then for almost every $\omega \in \Omega$, $x \mapsto \gamma(x) B_x^H(\omega)$ is α-Hölder continuous for all $\alpha \in (0, H)$. We fix such an $\omega_0 \in \Omega$. To simplify notations, we still note B_x^H the cut to zero path $\gamma(x) B_x^H(\omega_0)$.

Remark 29. *It requires to define explicitly the smoothing function γ in order to simulate its values numerically. But one can also choose to restrict the domain of the drift to $[-K, K]$ and stop the process when it exits this compact set. Therefore the values of γ don't matter.*

We also recall the link between fractional Sobolev spaces and Triebel-Lizorkin spaces $F_{pq}^s(\mathbb{R})$.

Proposition 30 (See 4.1 in [5])**.** $\forall s \in \left(\frac{1}{2}, H\right)$, $\forall q \in [2, \infty)$

$$B_x^H \in F_{q2}^s(\mathbb{R}) = H_q^s(\mathbb{R}). \tag{III.3}$$

By Proposition 30, we fix $s = H - \varepsilon > \frac{3}{4}$ for $\varepsilon > 0$ small enough such that $B^H \in H_q^s(\mathbb{R})$ with $q \in \left(2, \frac{1}{1-s}\right)$, where we fixed $\tilde{q} = 2$.

Then, $b = \frac{\mathrm{d}}{\mathrm{d}u} B_u^H$ with support in D belongs to $H_q^{s-1}(\mathbb{R})$ so we have by Theorem 8 its Haar decomposition:

$$b = \sum_{j=-1}^{+\infty} \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^j-1} \mu_{j,m}^k 2^{-j\left(s-1-\frac{1}{q}\right)} h_{j,m}^k. \tag{III.4}$$

with $\mu_{j,m}^k = 2^{j\left(s-1-\frac{1}{q}\right)} \int_{\mathbb{R}} f(x) h_{j,m}^k(x)\ \mathrm{d}x$.

Then let $Z \in \mathbb{N}$ and b_Z defined with the same coefficients (μ_{jm}^k) by

$$b_Z = \sum_{j=-1}^{+\infty} \sum_{k=Z}^{Z-1} \sum_{m=0}^{2^j-1} \mu_{j,m}^k 2^{-j\left(s-1-\frac{1}{q}\right)} h_{j,m}^k\Big|_{(-Z,Z)}. \tag{III.5}$$

The next proposition shows that b_Z is in fact the restriction of b to $(-Z, Z)$. Therefore, by definition, it belongs to $H_q^{s-1}(-Z, Z)$.

Proposition 31. *Let $Z \in \mathbb{N}$. Then b_Z is the restriction of b to $(-Z, Z)$.*

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}) \subset S(\mathbb{R})$ with Supp $\varphi \subset (-Z, Z)$. Then

$$\begin{aligned}
\langle b, \varphi \rangle &= \left\langle \sum_{j=-1}^{\infty} \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^j-1} \mu_{j,m}^k 2^{-j\left(s-1-\frac{1}{q}\right)} h_{j,m}^k, \varphi \right\rangle \\
&= \lim_{M \to +\infty} \lim_{L \to +\infty} \sum_{j=-1}^{M} \sum_{k=-L}^{L-1} \sum_{m=0}^{2^j-1} \mu_{j,m}^k 2^{-j\left(s-1-\frac{1}{q}\right)} \left\langle h_{j,m}^k, \varphi \right\rangle
\end{aligned}$$

by unconditional convergence of the series in $S'(\mathbb{R})$ for the weak-$*$ topology (by Theorem 8). These dual pairings (which are integrals because Haar wavelets are locally integrable) are null if $k \notin [-Z, Z-1]$ so we obtain:

$$\begin{aligned}
\langle b, \varphi \rangle &= \lim_{M \to +\infty} \sum_{j=-1}^{M} \sum_{k=-Z}^{Z-1} \sum_{m=0}^{2^j-1} \mu_{j,m}^k 2^{-j\left(s-1-\frac{1}{q}\right)} \left\langle h_{j,m}^k, \varphi \right\rangle \\
&= \left\langle \sum_{j=-1}^{\infty} \sum_{k=-Z}^{Z-1} \sum_{m=0}^{2^j-1} \mu_{j,m}^k 2^{-j\left(s-1-\frac{1}{q}\right)} h_{j,m}^k, \varphi \right\rangle \\
&= \langle b_Z^N, \varphi \rangle
\end{aligned}$$

So by 1.86 in [10], b_Z is the restriction of b to $(-Z, Z)$. $\square$

Remark 32. *By definition, $b_{|(-Z,Z)} = \left(\frac{\mathrm{d}}{\mathrm{d}u} B_u^H\right)_{|(-Z,Z)}$ and $\left(\frac{\mathrm{d}}{\mathrm{d}u} B_u^H\right)_{|(-Z,Z)} = \frac{\mathrm{d}}{\mathrm{d}u} B^H_{u_{|(-Z,Z)}}$ as distributions, by definition of the derivative. That's why we are able to link $b_Z = b_{|(-Z,Z)}$ to B^H.*

Theorem 33 (See Remark 3.4 in [11] and Theorem 3.1 in [10])**.** *Let $Z \in \mathbb{N}$, $f \in H_q^s(-Z, Z)$, $g = f' \in H_q^{s-1}(-Z, Z) = F_{q2}^{s-1}(-Z, Z)$ for $2 \le q \le \infty$, and $s \in \left(\frac{1}{2}, 1 + \frac{1}{q}\right)$. Therefore, we have the unique Faber and Haar representations:*

$$f = \sum_{j=-1}^{+\infty} \sum_{k=-Z}^{Z-1} \sum_{m=0}^{2^j-1} \mu_{j,m}^k 2^{-j\left(s-\frac{1}{q}\right)-1} v_{j,m}^k \tag{III.6}$$

$$g = \sum_{j=-1}^{+\infty} \sum_{k=-Z}^{Z-1} \sum_{m=0}^{2^j-1} \mu_{j,m}^k 2^{-j\left(s-1-\frac{1}{q}\right)} h_{j,m}^k \tag{III.7}$$

with unconditional convergence of the second series in $D'(-Z,Z)$, where

$$\begin{cases} \mu^k_{j,m}(b) & = -2^{j\left(s-\frac{1}{q}\right)} \left(\Delta^2_{2^{-j-1}} f\right)(k+2^{-j}m) \\ \mu^k_{-1,0}(b) & = 2^{-s+\frac{1}{q}}(f(k+1)-f(k)) \end{cases}, \tag{III.8}$$

$-\frac{1}{2}(\Delta^2_{2^{-j-1}} f)(2^{-j}m) = f(2^{-j}m+2^{-j-1}) - \frac{1}{2}f(2^{-j}m) - \frac{1}{2}f(2^{-j}m+2^{-j})$ *and* $\sum_{m=0}^{2^j-1}$ *means* $m=0$ *when* $j=-1$*. Moreover,* $g \mapsto \mu(g)$ *is an isomorphism between* $H^{s-1}_q(\mathbb{R})$ *and* $f_{q2}(\mathbb{R})$*.*

Identifying the coefficients $(\mu^k_{jm})_{k\in[-N,N-1]}$ of $b_N = \frac{\mathrm{d}}{\mathrm{d}u} B^H_{u_{|(-N,N)}}$ with the ones in Theorem 33 by uniqueness, we obtain a formula to compute them numerically with the values taken by B^H.

Then, we return to the function

$$b^N = \sum_{j=0}^{N} \sum_{k=-N}^{N-1} \sum_{m=0}^{2^j-1} \mu^k_{j,m} 2^{-j\left(s-\frac{1}{q}\right)} h^k_{j,m} \tag{III.9}$$

converging to b in $H^s_{2,q}(\mathbb{R})$ for any $q \in \left(2, \frac{1}{1-s}\right)$. For each given precision $\varepsilon > 0$, we can take $N_0 \in \mathbb{N}$ such that $\forall N \geq N_0$, $\left\|b - b^N_N\right\|_{H^s_{2,q}(\mathbb{R})} \leq \varepsilon$. Then, for each fixed $N \geq N_0$, we are able to compute the coefficients $(\mu^k_{jm})_{k\in[-N,N-1]}$. Figures III.4 and III.5 show the result of such a approximation of the derivative of a fBm path by Haar wavelets.

Remark 34. *For a fixed N and for large values of k the coefficients μ^k_{jm} of b^N will all be zero because of the compact support of B^H. Then we can use instead*

$$b^N = \sum_{j=0}^{N} \sum_{k=-K-1}^{K} \sum_{m=0}^{2^j-1} \mu^k_{j,m} 2^{-j\left(s-\frac{1}{q}\right)} h^k_{j,m}. \tag{III.10}$$

But as explained in Remark 29, without choosing explicitly the smoothing function γ, one can use

$$b^N = \sum_{j=0}^{N} \sum_{k=-K}^{K-1} \sum_{m=0}^{2^j-1} \mu^k_{j,m} 2^{-j\left(s-\frac{1}{q}\right)} h^k_{j,m} \tag{III.11}$$

and stop the process X_t when it exits $[-K,K]$.

We are now able to numerically solve the SDE (I.3) with Euler-Marayuma scheme with drift b^N. Nevertheless, we still need to make sure that the original SDE (I.1) makes sense with respect to the framework developped in [3]. It is the object of the next subsection.

III.2.1 Applying the framework from Flandoli, Issoglio and Russo

We recall that in our special case, the drift b verifies $b = \frac{\mathrm{d}}{\mathrm{d}u} B^H_u \in H^{s-1}_q(\mathbb{R})$ for all $q \geq 2$ and a fortiori for all $q \in \left(2, \frac{1}{1-s}\right)$. Then we can use the results from [3] and Part II by choosing $\tilde{q} = 2$ and slightly modifying the set of parameters $K(\beta,q) = \{(\delta,p) \mid \beta < \delta < 1-\beta,\ \frac{1}{\delta} < p < q\}$ into $\widetilde{K}(\beta,q) = \{(\delta,p) \mid \beta < \delta < 1-\beta,\ \frac{1}{\delta} \vee 2 < p < q\}$ (see figure III.3, page 32 for the modified set and II.1, page 24 for the original set).

Eventually, we solve the PDEs (I.6) and (I.8) in $H^{1+\delta}_p(\mathbb{R})$ for a restricted set of parameters (δ,p) but the construction of the virtual solution to the SDE with distributional drift (I.1) still makes sense.

Remark 35. *We can still take $\alpha = 1-2\beta-\varepsilon$ with the new set $\widetilde{K}(\beta,q)$, and therefore apply Theorem 27. See Remark 26.*

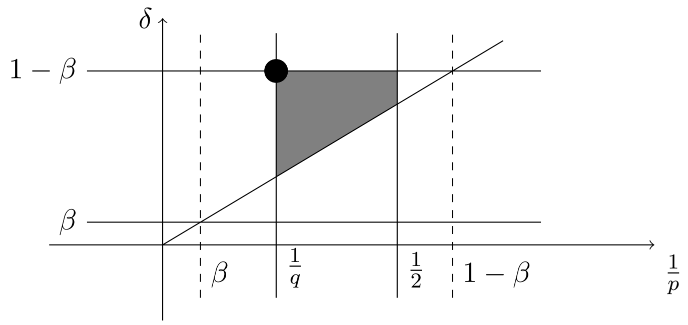

Figure III.3: The set $\widetilde{K}(\beta, q)$. Modified figure from the paper [3] of Flandoli, Issoglio and Russo.

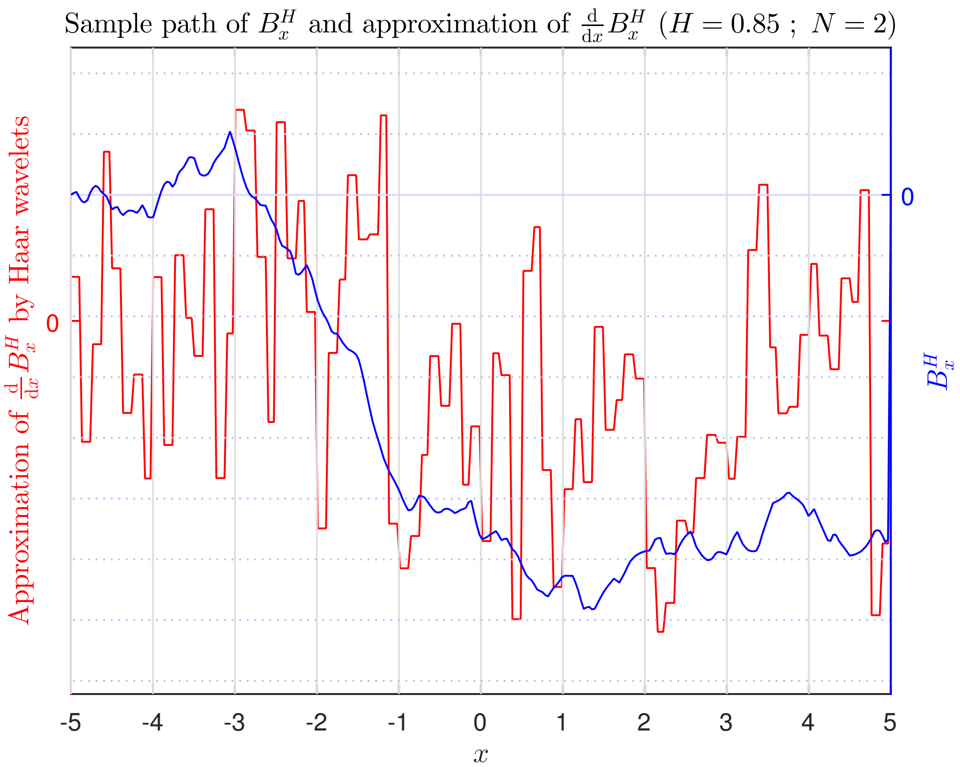

Figure III.4: Haar approximation of the derivative of a fBm path with $N = 2$.

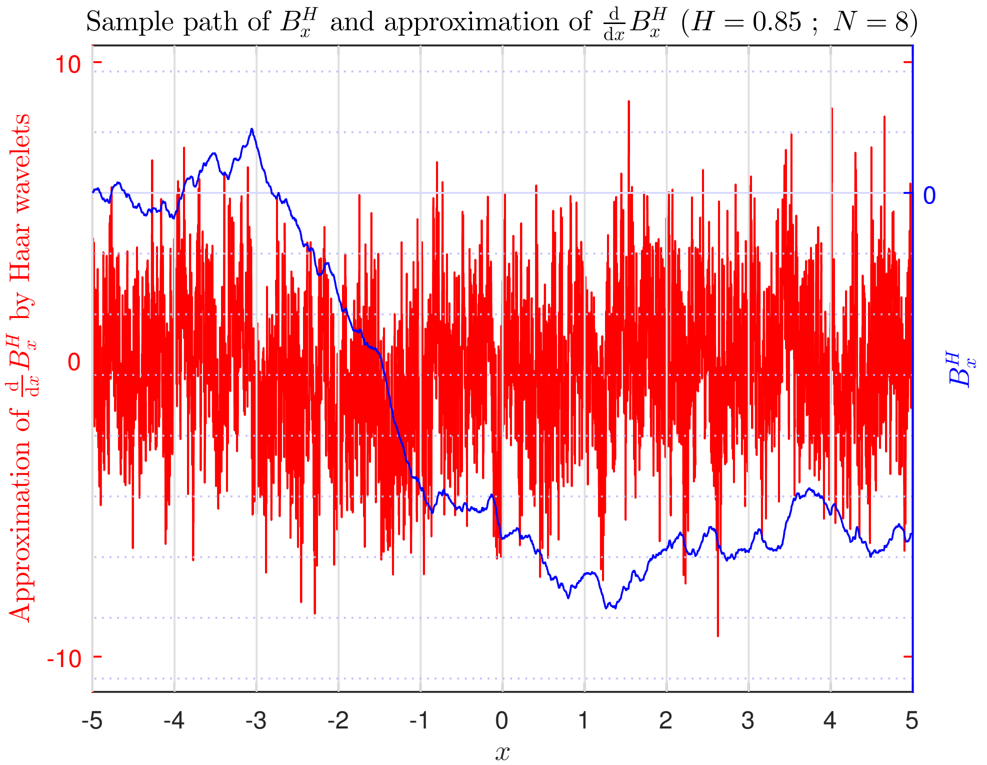

Figure III.5: Haar approximation of the derivative of a fBm path with $N = 8$.

Part IV

Numerical results

IV.1 Monte-Carlo method for error estimation

In order to study the numerical rates of convergence of our algorithm, we must numerically compute expectations. That's why we use the Monte-Carlo method which estimates expectations thanks to convergence results directly related to the Central Limit Theorem. This theorem expresses in fact the speed of convergence of the empirical mean to the expectation.

More precisely, if $X_1, \cdots, X_m$ are independent random variables with the same law as X, and $V_m = \frac{m}{m-1}\left(\frac{1}{m}\sum_{i=1}^m X_i^2 - \frac{1}{m^2}\left(\sum_{i=1}^m X_i\right)^2\right) = \frac{1}{m-1}\left(\sum_{i=1}^m X_i^2 - \frac{1}{m}\left(\sum_{i=1}^m X_i\right)^2\right)$ is the unbiased variance, the Central Limit Theorem and Slutsky's Lemma give the following convergence in law:

$$\frac{\sum_{i=1}^m (X_i - \mathbb{E}[X])}{\sqrt{V_m}\sqrt{m}} \xrightarrow[m\to\infty]{\mathcal{L}} \mathcal{N}(0,1). \tag{IV.1}$$

Then we can construct the asymptotic $100(1-\alpha)\%$ confidence interval such that:

$$\mathbb{P}\left(\mathbb{E}[X] \in \left[\frac{1}{m}\sum_{i=1}^m X_i - q\left(1-\frac{\alpha}{2}\right)\frac{\sqrt{V_m}}{\sqrt{m}}, \frac{1}{m}\sum_{i=1}^m X_i + q\left(1-\frac{\alpha}{2}\right)\frac{\sqrt{V_m}}{\sqrt{m}}\right]\right) \xrightarrow[m\to\infty]{} 1-\alpha. \tag{IV.2}$$

where q is the quantile of the normal distribution. We will apply this method to construct an asymptotic 95% confidence interval for the strong error $\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^{N,n} - X_t^N\right|^2\right]$ and the weak error $\left|\mathbb{E}\left[f\left(X_T\right) - f\left(X_T^N\right)\right]\right|$.

A key numerical concern here is to have the smallest variance as possible, in order to reduce the length of the confidence interval for a given number m of simulation. Therefore we obtain a more precise estimation of the expectation. To do so, a lot of variance reduction methods exist. We use the antithetic variates method, which exploits the fact that the Brownian motions W and $-W$ have the same law to reduce the variance. See [2] for a more detailed presentation. Numerically, it requires to simulate at each iteration two paths of the solution $X^{N,n}$ of (I.3) with both driving Brownian motions W and $-W$.

IV.2 Strong convergence of the Euler approximation

We study numerically the case $X_0 = 0$, the reference solution has $n_0 = 2^{12}$ time points, $n \in \{2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}\}$, and $N = 5$. We observe in figure IV.1 a numerical strong convergence rate of 0.81 when Theorem 13 shows a theoretical rate of $0.5 - \varepsilon$. This may be due to our special case, where the drift of the approximated SDE (I.3) is piecewise constant, which is more regular than a piecewise Lipschitz drift.

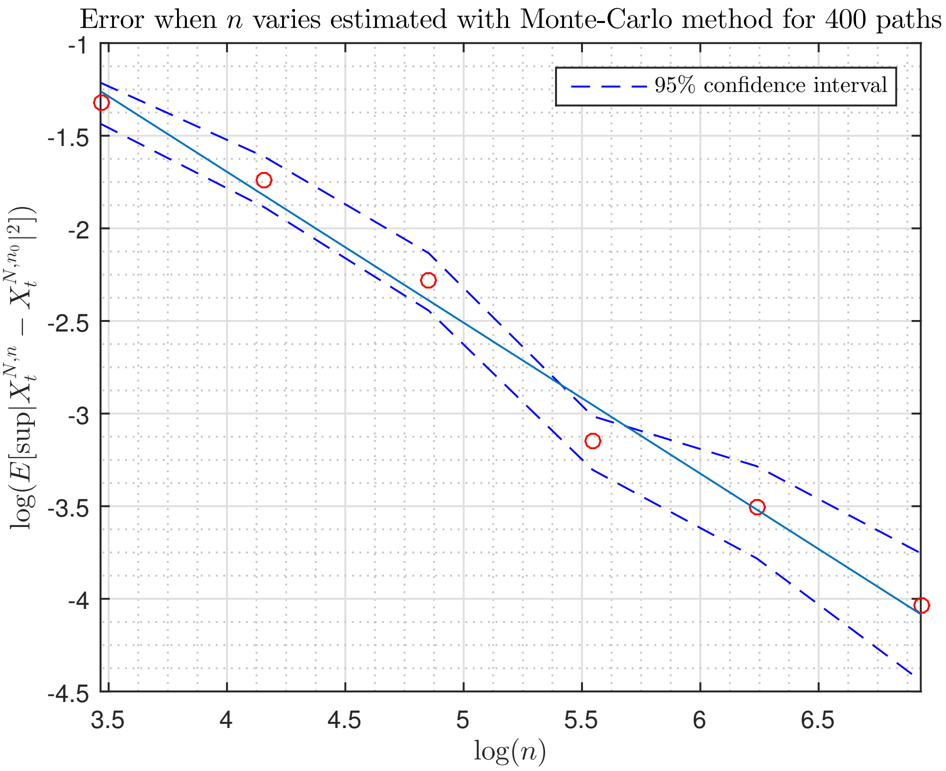

Figure IV.1: Estimation of the L^2 error of the Euler-Maruyama scheme with a Monte-Carlo method.

IV.3 Weak convergence of the approximated to the virtual solution

We study numerically the case $X_0 = 0$, solutions have $n_0 = 2^{12}$ time points, $N \in \{4, 5, 6, 7, 8\}$, $n \in \{2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}\}$, and $N = 5$. We observe in figure IV.2 a numerical weak convergence rate of 0.13. The simulation was done with a drift of regularity $H_{2,q}^{-0.15}$ for $q \geq 2$. Therefore, we would expect from Theorem 27 a convergence rate of $1 - 4\beta = 0.4$.

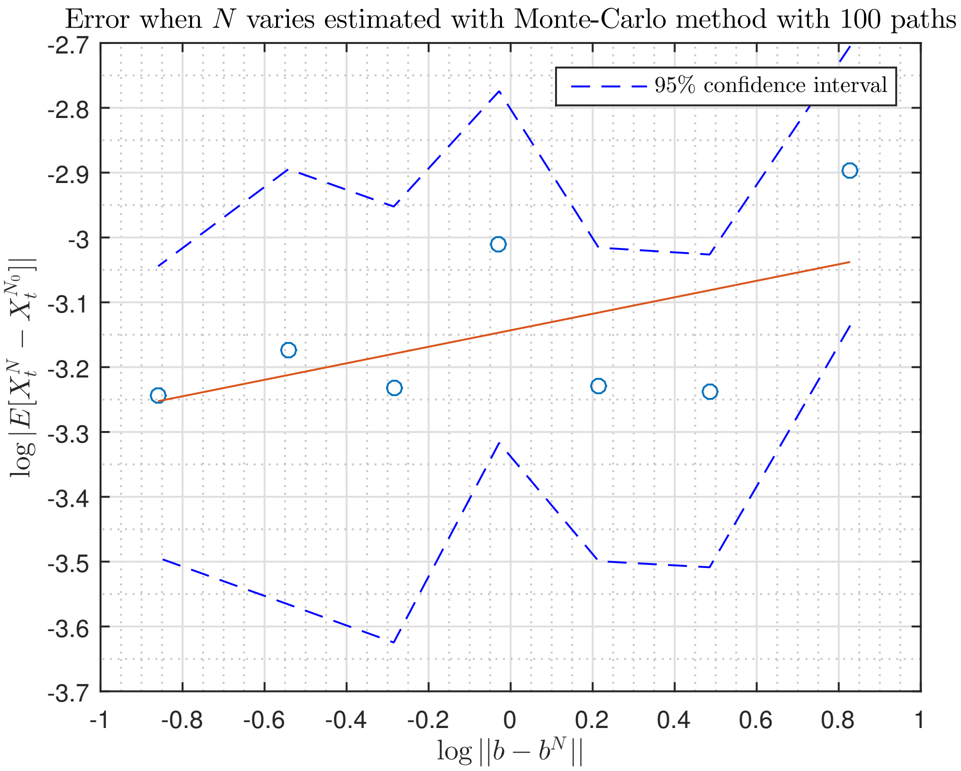

Figure IV.2: Estimation of the weak error of the Euler-Maruyama scheme with a Monte-Carlo method.

Internship's organisation

Dates	Research work
May 14th–May 21st	Study of the paper [3], numerical simulation of fBm. Beginning of the implementation of the Euler-Maruyama method.
May 21st–May 28th	Haar approximation in the case of a regular drift (identity), plots of the Euler method error and reading of the paper [8].
May 28th–June 4th	Improvement of the algorithm, numerical and theoretical study of the convergence rate.
June 4th–June 11th	Study of Yan's article [12] and understanding how to use his results to find a convergence rate for our algorithm in 1D.
June 11th–June 18th	Careful writing and correction of lemmas, proof of the convergence rate result.
June 18th–June 25th	Understanding and adapting the lemma from Yan bounding the expectation of the local time.
June 25th–July 2nd	Convergence in law of the approximating process to the virtual solution with results from [3], and writing of the report
July 2nd–July 9th	Writing of the report, numerical studies of convergence rates and attempts to justify the computation of the Haar coefficients on $\mathbb{R}$.
July 9th–July 16th	Reading Triebel books in order to understand Haar and Faber representations, study of fBm's regularity with [5].
July 16th–July 23rd	Writing of the details concerning the Haar approximation in the case of the derivative of a fractional Brownian motion.
July 23rd–August 3rd	Study of Haar wavelets in $\mathbb{R}^d$ and writing of the report. Extension of the results to time-dependent drift

Conclusion

My work at the University of Leeds has been a very stimulating first research experience. During this project, I contributed to the numerical study of irregular stochastic differential equations, by designing a new algorithm which applies to distributional drifts. My main contribution was obtaining a rate in 1D of $1-4\beta-\varepsilon$ for the convergence of the solution to the approximated SDE to the virtual solution. Moreover, I was able to apply my approximation method in the case of the derivative of a fBm path. This required to produce an approximation method using Haar and Faber wavelets in order to approximate a distributional drift by a function. On this occasion, I developped various Matlab scripts and realized numerical simulations. That's why this internship allowed me to deepen my knowledge in stochastic analysis, function spaces and numerical methods. Moreover, working in a international environment was a fascinating experience.

Several points of this project could call for future work. For instance, it would be great to investigate a 1D rate when $\beta > 1/4$, and to find a way to obtain a convergence rate in any finite dimension. Nevertheless, no counterpart of local time exist in $\mathbb{R}^d$ thus our proof cannot extend directly to higher dimensions. Another possible work can be dedicated to find a way to compute the Haar coefficients in $\mathbb{R}^d$ when the drift is the derivative of a Hölder-continuous function, like a fBm path. In fact, in 1D, we were able to use a Faber representation for this path, and therefore deduce the Haar coefficients. But Triebel explains in [10] that Faber basis cannot be extended to $\mathbb{R}^d$ as easily as the Haar basis is. It should thus be interesting to look for an alternative method to compute these coefficients.

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Acronyms

fBm: fractional Brownian motion.
See the definition in Remark 2.

SDE: Stochastic Differential Equation.
Differential equation involving stochastic processes.

PDE: Partial Derivative Equation.
Differential equation involving partial derivatives of multivariate functions.

Appendix A

Technical proofs

A.1 Lemma 21

As explained before, this proof is an adaptation of the one of L. Yan in [12]. The only difference is the fact that in our case, we don't assume that the process starts at 0.

Proof of Lemma 21. We note $U_t(X) = \sup\{n \in \mathbb{N} | \tau_n < t\}$ and $n(t) = t \wedge \upsilon_{U_t(X)+1}$. By Meyer-Tanaka's formula, $\forall i \in \mathbb{N}^*$:

$$X^+_{\tau_i \wedge t} - X^+_{\upsilon_i \wedge t} = \int_{\upsilon_i \wedge t}^{\tau_i \wedge t} \mathbb{1}_{\{X_s > 0\}} \, \mathrm{d}X_s + \frac{1}{2} \left\{ L^0_{\tau_i \wedge t}(X) - L^0_{\upsilon_i \wedge t}(X) \right\}. \tag{A.1}$$

Because $\forall i \in \mathbb{N}^*, \; L^0_{\tau_i \wedge t}(X) = L^0_{\upsilon_{i+1} \wedge t}(X)$ and $L^0_{\upsilon_1 \wedge t}(X) = 0$, we have

$$\sum_{i=1}^{U_t(X)+1} \left(X^+_{\tau_i \wedge t} - X^+_{\upsilon_i \wedge t} \right) = \int_{\upsilon_1 \wedge t}^{t} \theta_s(X) \, \mathrm{d}X_s + \frac{1}{2} \, L^0_t(X).$$

The left term is equal to $\varepsilon U_t(X) + X^+_t - X^+_{n(t)}$ so

$$\varepsilon U_t(X) = \int_{\upsilon_1 \wedge t}^{t} \theta_s(X) \, \mathrm{d}X_s + \frac{1}{2} \, L^0_t(X) - X^+_t + X^+_{n(t)}. \tag{A.2}$$

Now we express differently $U_t(X)$. $F \in \mathcal{C}^2(\mathbb{R})$ so by Itô's formula:

$$F\left(X^+_{\tau_i \wedge t}\right) - F\left(X^+_{\upsilon_i \wedge t}\right) = \int_{\upsilon_i \wedge t}^{\tau_i \wedge t} F'\left(X^+_s\right) \mathrm{d}X^+_s + \frac{1}{2} \int_{\upsilon_i \wedge t}^{\tau_i \wedge t} F''\left(X^+_s\right) \mathrm{d}[X^+]_s.$$

By (A.1), $\mathrm{d}X^+_s = \mathbb{1}_{\{X_s>0\}} \mathrm{d}X_s + \frac{1}{2} \, \mathrm{d}L^0_t(X)$ and $\mathrm{d}[X^+]_s = \mathbb{1}_{\{X_s>0\}} \mathrm{d}[X]_s$. It follows

$$\begin{aligned} F\left(X^+_{\tau_i \wedge t}\right) - F\left(X^+_{\upsilon_i \wedge t}\right) = &\int_{\upsilon_i \wedge t}^{\tau_i \wedge t} F'\left(X^+_s\right) \mathbb{1}_{\{X_s>0\}} \, \mathrm{d}X_s + \frac{1}{2} \int_{\upsilon_i \wedge t}^{\tau_i \wedge t} F'(X^+_s) \, \mathrm{d}L^0_t(X) \\ &+ \frac{1}{2} \int_{\upsilon_i \wedge t}^{\tau_i \wedge t} F''\left(X^+_s\right) \mathbb{1}_{\{X_s>0\}} \, \mathrm{d}[X]_s. \end{aligned}$$

Adding up for i, with $F(0) = 0$ we obtain

$$\begin{aligned} F(\varepsilon)U_t(X) + F\left(X^+_t\right) - F\left(X^+_{n(t)}\right) &= \sum_{i=1}^{U_t(X)+1} \left(F\left(X^+_{\tau_i \wedge t}\right) - F\left(X^+_{\upsilon_i \wedge t}\right) \right) \\ &= \int_{\upsilon_1 \wedge t}^{t} F'(X^+_s) \, \theta_s(X) \, \mathrm{d}X_s + \frac{1}{2} \int_{\upsilon_1 \wedge t}^{t} F'\left(X^+_s\right) \Xi_s \, \mathrm{d}L^0_t(X) \\ &+ \frac{1}{2} \int_{\upsilon_1 \wedge t}^{t} F''\left(X^+_s\right) \theta_s(X) \, \mathrm{d}[X]_s \end{aligned}$$

with $\Xi_s = \sum_{n=1}^{\infty} \mathbb{1}_{\{\upsilon_n < s \leq \tau_n\}}$. The measure $\mathrm{d}L_t^0(X)$ is almost surely carried by $\{t | X_t = 0\}$ so we can simplify $\int_{\upsilon_1 \wedge t}^t F'(X_s^+)\,\Xi_s \,\mathrm{d}L_t^0(X) = F'(0) \int_{\upsilon_1 \wedge t}^t \Xi_s \,\mathrm{d}L_t^0(X)$ in order to have, with $F'(0) = 0$:

$$F(\varepsilon) U_t(X) = -F\left(X_t^+\right) + F\left(X_{n(t)}^+\right) + \int_{\upsilon_1 \wedge t}^t F'\left(X_s^+\right) \theta_s(X) \,\mathrm{d}X_s \tag{A.3}$$

$$+ \frac{1}{2} \int_{\upsilon_1 \wedge t}^t F''\left(X_s^+\right) \theta_s(X) \,\mathrm{d}[X]_s. \tag{A.4}$$

Combining (A.2) and (A.3), it follows

$$\begin{aligned} L_t^0(X) F(\varepsilon) &= 2F(\varepsilon)(X_t^+ - X_{n(t)}^+) - 2\varepsilon(F\left(X_t^+\right) - F\left(X_{n(t)}^+\right) \\ &\quad - 2\int_{\upsilon_1 \wedge t}^t \left(F(\varepsilon) - \varepsilon F'\left(X_s^+\right)\right) \theta_s(X) \,\mathrm{d}X_s + \varepsilon \int_{\upsilon_1 \wedge t}^t F''\left(X_s^+\right) \theta_s(X) \,\mathrm{d}[X]_s. \end{aligned}$$

Then, if $n(t) = t$, the two first right terms of the equality are equal to zero. Else, if $n(t) = \upsilon_{U_t(X)+1}$, $0 \leq F(\varepsilon)(X_t^+ - X_{n(t)}^+) = F(\varepsilon) X_t^+ \leq F(\varepsilon)\varepsilon$ and $-2\varepsilon(F\left(X_t^+\right) - F\left(X_{n(t)}^+\right)$ $= -2\varepsilon F\left(X_t^+\right) \leq 0$ because of the positivity of F. Finally we obtain:

$$L_t^0(X) \leq 2\varepsilon - \frac{2}{F(\varepsilon)} \int_{\upsilon_1 \wedge t}^t \left(F(\varepsilon) - \varepsilon F'\left(X_s^+\right)\right) \theta_s(X) \,\mathrm{d}X_s + \frac{\varepsilon}{F(\varepsilon)} \int_{\upsilon_1 \wedge t}^t F''\left(X_s^+\right) \theta_s(X) \,\mathrm{d}[X]_s.$$

□

A.2 Lemma 24

Proof of Lemma 24. We will use the notations introduced in Remark 23. By Lemma 17, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\left\|b - b^N\right\|_{\infty, H_{\tilde{q},q}^s(\mathbb{R})} \underset{N \to \infty}{\longrightarrow} 0$. Therefore $u^N(t, \cdot)$ and $u(t, \cdot)$ are $\frac{1}{2}$-Lipschitz. We recall that in this case, by Lemma 22 in [3], $\Psi(t, \cdot)$ and $\Psi^N(t, \cdot)$ are 2-Lipschitz. Therefore $\widetilde{u}^N(t, \cdot)$ and $\widetilde{u}(t, \cdot)$ are 1-Lipschitz.

We can notice that $\widetilde{\nabla u}$ is α-Hölder with constant $2^\alpha \|u\|_{\infty, \mathcal{C}^{1,\alpha}}$ (See Proposition 6) and ∇u^N is α-Hölder with a constant which can be bounded by Λ independently of N (see the proof of Lemma 24 in [3]). Let $\varepsilon \in (0, 1]$. Corollary 22 gives us:

$$\begin{aligned} 0 \leq L_T^0(Y - Y^N) &\leq 2\varepsilon - \frac{2}{\varepsilon} \int_{\upsilon_1 \wedge T}^T \theta_s(Y - Y^N) \left(\varepsilon - 2(Y_s - Y_s^N)^+\right) \,\mathrm{d}\left(Y_s - Y_s^N\right) \\ &\quad + \frac{2}{\varepsilon} \int_{\upsilon_1 \wedge T}^T \theta_s(Y - Y^N) \,\mathrm{d}\left[Y - Y^N\right]_s \end{aligned}$$

with

$$\begin{aligned} Y_t - Y_t^N &= y - y^N + (\lambda + 1) \int_0^t \left\{u\left(s, \Psi\left(s, Y_s\right)\right) - u^N\left(s, \Psi^N\left(s, Y_s^N\right)\right)\right\} \,\mathrm{d}s \\ &\quad + \int_0^t \left\{\nabla u\left(s, \Psi\left(s, Y_s\right)\right) - \nabla u^N\left(s, \Psi^N\left(s, Y_s^N\right)\right)\right\} \,\mathrm{d}W_s. \end{aligned}$$

Remark 36. *Note that* $\theta_s(Y - Y^N)\left|\varepsilon - 2(Y_s - Y_s^N)^+\right| \leq \varepsilon \theta_s(Y - Y^N)$.

∇u and ∇u^N are bounded so the Itô integral is a true martingale. We take the expectation:

$$
\begin{aligned}
\mathbb{E}\left[L_T^0(Y-Y^N)\right] &\le 2\varepsilon + 2(\lambda+1)\,\mathbb{E}\left[\int_{\upsilon_1\wedge T}^T \theta_s\left(Y-Y^N\right)\left|\widetilde{u}\left(s,Y_s\right)-\overline{u}^N\left(s,Y_s^N\right)\right|\mathrm{d}s\right] \\
&\quad + \frac{2}{\varepsilon}\,\mathbb{E}\left[\int_{\upsilon_1\wedge T}^T \theta_s\left(Y-Y^N\right)\left\{\widetilde{\nabla u}\left(s,Y_s\right)-\overline{\nabla u}\left(s,Y_s^N\right)\right\}^2\,\mathrm{d}s\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le 2\varepsilon + 2(\lambda+1)\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|\widetilde{u}\left(s,Y_s\right)-\widetilde{u}\left(s,Y_s^N\right)\right|\,\mathrm{d}s\right] \\
&+ 2(\lambda+1)\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|\widetilde{u}\left(s,Y_s^N\right)-\widetilde{u}^N\left(s,Y_s^N\right)\right|\,\mathrm{d}s\right] \\
&+ 2(\lambda+1)\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|\widetilde{u}^N\left(s,Y_s^N\right)-\overline{u}^N\left(s,Y_s^N\right)\right|\mathrm{d}s\right] \\
&+ \frac{6}{\varepsilon}\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left\{\widetilde{\nabla u}\left(s,Y_s\right)-\widetilde{\nabla u}\left(s,Y_s^N\right)\right\}^2\,\mathrm{d}s\right] \\
&+ \frac{6}{\varepsilon}\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left\{\widetilde{\nabla u}\left(s,Y_s^N\right)-\widetilde{\nabla u}^N\left(s,Y_s^N\right)\right\}^2\,\mathrm{d}s\right] \\
&+ \frac{6}{\varepsilon}\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left\{\widetilde{\nabla u}^N\left(s,Y_s^N\right)-\overline{\nabla u}^N\left(s,Y_s^N\right)\right\}^2\,\mathrm{d}s\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le 2\varepsilon + 2(\lambda+1)\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|Y_s-Y_s^N\right|\,\mathrm{d}s\right] + 2(\lambda+1)T\kappa\left\|b^N-b\right\|_{\infty,H_{\tilde{q},q}^{-\beta}} \\
&+ (\lambda+1)\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|\Psi\left(s,Y_s^N\right)-\Psi^N\left(s,Y_s^N\right)\right|\mathrm{d}s\right] \\
&+ \frac{6\times 4^\alpha\left\|u\right\|_{\infty,\mathcal{C}^{1,\alpha}}^2}{\varepsilon}\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|Y_s-Y_s^N\right|^{2\alpha}\,\mathrm{d}s\right] + 6T\kappa^2\left\|b^N-b\right\|_{\infty,H_{\tilde{q},q}^{-\beta}}^2\,\varepsilon^{-1} \\
&+ \frac{6\Lambda^2}{\varepsilon}\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|\Psi\left(s,Y_s^N\right)-\Psi^N\left(s,Y_s^N\right)\right|^{2\alpha}\,\mathrm{d}s\right]
\end{aligned}
$$

where we used Lemma 19, the 1-Lipschitz property of $\widetilde{u}$, the 1/2-Lipschitz property of u^N, the α-Hölder property of $\widetilde{\nabla u}$ (with constant $2^\alpha\left\|u\right\|_{\infty,\mathcal{C}^{1,\alpha}}$), and the α-Hölder property of ∇u^N (with constant Λ). Lemma 20 gives us:

$$
\begin{aligned}
&\mathbb{E}\left[L_T^0(Y-Y^N)\right] \\
&\le 2\varepsilon + 2(\lambda+1)\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|Y_s-Y_s^N\right|\,\mathrm{d}s\right] + 4(\lambda+1)T\kappa\left\|b^N-b\right\|_{\infty,H_{\tilde{q},q}^{-\beta}} \\
&+ \frac{6\times 4^\alpha\left\|u\right\|_{\infty,\mathcal{C}^{1,\alpha}}^2}{\varepsilon}\,\mathbb{E}\left[\int_0^T \theta_s\left(Y-Y^N\right)\left|Y_s-Y_s^N\right|^{2\alpha}\,\mathrm{d}s\right] + 6T\kappa^2\left\|b-b^N\right\|_{\infty,H_{\tilde{q},q}^{-\beta}}^2\,\varepsilon^{-1} \\
&+ 6\Lambda^2 T 4^\alpha\kappa^{2\alpha}\left\|b-b^N\right\|_{\infty,H_{\tilde{q},q}^{-\beta}}^{2\alpha}\,\varepsilon^{-1}.
\end{aligned}
$$

As $\theta_s(Y-Y^N)\left|Y_s-Y_s^N\right|\le\varepsilon$ and $\theta_s(Y-Y^N)\left|Y_s-Y_s^N\right|^{2\alpha}\le\varepsilon^{2\alpha}$, we have

$$
\begin{aligned}
\mathbb{E}\left[L_T^0(Y-Y^N)\right] &\le 2\varepsilon + 2(\lambda+1)\,T\varepsilon + 4(\lambda+1)T\kappa\left\|b-b^N\right\|_{\infty,H_{\tilde{q},q}^{-\beta}} + 6\left\|u\right\|_{\infty,\mathcal{C}^{1,\alpha}}^2 4^\alpha T\varepsilon^{2\alpha-1} \\
&\quad + 6T\left(\Lambda^2 4^\alpha\kappa^{2\alpha}\left\|b-b^N\right\|_{\infty,H_{\tilde{q},q}^{-\beta}}^{2\alpha} + \kappa^2\left\|b-b^N\right\|_{\infty,H_{\tilde{q},q}^{-\beta}}^2\right)\varepsilon^{-1}.
\end{aligned}
$$

As $0<2\alpha-1<1$, the result follows from $\varepsilon\le\varepsilon^{2\alpha-1}$ when $0<\varepsilon\le 1$.

□

A.3 Lemma 25

Proof of Lemma 25. Let $\varepsilon \in (0,1]$, by Lemma 24,

$$0 \leq \mathbb{E}\left[L_T^0(Y - Y^N)\right] \leq g(\varepsilon)$$

where

$$\begin{aligned} g(\varepsilon) = 4(\lambda+1)T\kappa \left\|b - b^N\right\|_{\infty, H_{\tilde{q},q}^{-\beta}} + \left(2 + 2(\lambda+1)\; T + 6\left\|u\right\|^2_{\infty,\mathcal{C}^{1,\alpha}} 4^\alpha T\right) \varepsilon^{2\alpha-1} \\ + 6T\left(\Lambda^2 4^\alpha \kappa^{2\alpha} \left\|b - b^N\right\|^{2\alpha}_{\infty, H_{\tilde{q},q}^{-\beta}} + \kappa^2 \left\|b - b^N\right\|^2_{\infty, H_{\tilde{q},q}^{-\beta}}\right) \varepsilon^{-1}. \end{aligned}$$

With

$$\begin{aligned} g'(\varepsilon) = (2\alpha - 1)\left(2 + 2(\lambda+1)\; T + 6\left\|u\right\|^2_{\infty,\mathcal{C}^{1,\alpha}} 4^\alpha T\right) \varepsilon^{2\alpha-2} \\ - 6T\left(\Lambda^2 4^\alpha \kappa^{2\alpha} \left\|b - b^N\right\|^{2\alpha}_{\infty, H_{\tilde{q},q}^{-\beta}} + \kappa^2 \left\|b - b^N\right\|^2_{\infty, H_{\tilde{q},q}^{-\beta}}\right) \varepsilon^{-2}, \end{aligned}$$

the minimum of g on $(0,1]$ is reached when N is big enough in

$$\varepsilon_N = \left(\frac{6T\left(\Lambda^2 4^\alpha \kappa^{2\alpha} \left\|b - b^N\right\|^{2\alpha}_{\infty, H_{\tilde{q},q}^{-\beta}} + \kappa^2 \left\|b - b^N\right\|^2_{\infty, H_q^{-\beta}}\right)}{(2\alpha-1)\left(2 + 2(\lambda+1)\; T + 6\left\|u\right\|^2_{\infty,\mathcal{C}^{1,\alpha}} 4^\alpha T\right)}\right)^{\frac{1}{2\alpha}} = \nu_N \left\|b^N - b\right\|_{\infty, H_{\tilde{q},q}^{-\beta}}.$$

where

$$\begin{aligned} \nu_N = \left(\frac{6T\left(\Lambda^2 4^\alpha \kappa^{2\alpha} + \kappa^2 \left\|b - b^N\right\|^{2(1-\alpha)}_{\infty, H_{\tilde{q},q}^{-\beta}}\right)}{(2\alpha-1)\left(2 + 2(\lambda+1)\; T + 6\left\|u\right\|^2_{\infty,\mathcal{C}^{1,\alpha}} 4^\alpha T\right)}\right)^{\frac{1}{2\alpha}} \\ \geq \nu_\infty = \left(\frac{6T\Lambda^2 4^\alpha \kappa^{2\alpha}}{(2\alpha-1)\left(2 + 2(\lambda+1)\; T + 6\left\|u\right\|^2_{\infty,\mathcal{C}^{1,\alpha}} 4^\alpha T\right)}\right)^{\frac{1}{2\alpha}}. \end{aligned}$$

Therefore $\mathbb{E}\left[L_T^0(Y - Y^N)\right] \leq g(\varepsilon_N)$

$$\begin{aligned} &\leq 4(\lambda+1)T\kappa \left\|b - b^N\right\|_{\infty, H_{\tilde{q},q}^{-\beta}} + \left(2 + 2(\lambda+1)\; T + 6\left\|u\right\|^2_{\infty,\mathcal{C}^{1,\alpha}} 4^\alpha T\right) \nu_N^{2\alpha-1} \left\|b - b^N\right\|^{2\alpha-1}_{\infty, H_{\tilde{q},q}^{-\beta}} \\ &+ 6T\left(\Lambda^2 4^\alpha \kappa^{2\alpha} \left\|b - b^N\right\|^{2\alpha}_{\infty, H_{\tilde{q},q}^{-\beta}} + \kappa^2 \left\|b - b^N\right\|^2_{\infty, H_{\tilde{q},q}^{-\beta}}\right) \nu_N^{-1} \left\|b - b^N\right\|^{-1}_{\infty, H_{\tilde{q},q}^{-\beta}} \\ &\leq 4(\lambda+1)T\kappa \left\|b - b^N\right\|_{\infty, H_{\tilde{q},q}^{-\beta}} + \left(2 + 2(\lambda+1)\; T + 6\left\|u\right\|^2_{\infty,\mathcal{C}^{1,\alpha}} 4^\alpha T\right) \nu_N^{2\alpha-1} \left\|b - b^N\right\|^{2\alpha-1}_{\infty, H_{\tilde{q},q}^{-\beta}} \\ &+ 6T\left(\Lambda^2 4^\alpha \kappa^{2\alpha} + \kappa^2 \left\|b - b^N\right\|^{2(1-\alpha)}_{\infty, H_{\tilde{q},q}^{-\beta}}\right) \nu_\infty^{-1} \left\|b - b^N\right\|^{2\alpha-1}_{\infty, H_{\tilde{q},q}^{-\beta}} \\ &\leq 4(\lambda+1)T\kappa \left\|b - b^N\right\|^{2\alpha-1}_{\infty, H_{\tilde{q},q}^{-\beta}} + \left(2 + 2(\lambda+1)\; T + 6\left\|u\right\|^2_{\infty,\mathcal{C}^{1,\alpha}} 4^\alpha T\right) 2\nu_\infty^{2\alpha-1} \left\|b - b^N\right\|^{2\alpha-1}_{\infty, H_{\tilde{q},q}^{-\beta}} \\ &+ 6T\left(\Lambda^2 4^\alpha \kappa^{2\alpha} + \kappa^2\right) \nu_\infty^{-1} \left\|b - b^N\right\|^{2\alpha-1}_{\infty, H_{\tilde{q},q}^{-\beta}} \\ &\leq \left(4(\lambda+1)T\kappa + \nu_\infty^{-1}\left(\frac{12}{2\alpha-1} T\Lambda^2 4^\alpha \kappa^{2\alpha} + 6T\left(\Lambda^2 4^\alpha \kappa^{2\alpha} + \kappa^2\right)\right)\right) \left\|b - b^N\right\|^{2\alpha-1}_{\infty, H_{\tilde{q},q}^{-\beta}} \end{aligned}$$

for N big enough $\left(\nu_N \xrightarrow[N\to\infty]{} \nu_\infty \text{ so } \nu_N^{2\alpha-1} \leq 2\nu_\infty^{2\alpha-1} \text{ when } N \text{ is big enough}\right)$. The result follows. $\square$