

NUMERICAL SIMULATION OF SDES WITH DISTRIBUTIONAL DRIFT

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1 Introduction

We would like to simulate numerically sample paths of the solution of the one-dimensional stochastic differential equation

$$dX_t = b(X_t) dt + dW_t \quad (1)$$

where $b \in H_q^{-\beta}(\mathbb{R})$, $\beta \in (0, \frac{1}{2})$, $q \in (\frac{1}{1-\beta}, \frac{1}{\beta})$, $t \in [0, T]$, and W_t is a standard Brownian motion. Equation (1) is studied by F. Flandoli, E. Issoglio, and F. Russo in [1] in which they define a concept of virtual solution. The authors prove then existence and unicity in law of this solution.

Example 1. *An example of such drift b is given by the derivative of a sample path of a fractional Brownian motion B_x^H with Hurst index $1/2 < H < 1$. These stochastic processes are gaussian processes verifying*

$$\mathbb{E} [B_t^H B_s^H] = \frac{1}{2} (t^{2H} + s^{2H} + |t - s|^{2H}).$$

We note $-\beta = H - 1$. Given $B_x^H(\omega) \in H_q^{1-\beta}(\mathbb{R})$, we can take $b(x) = \frac{\partial}{\partial x} B_x^H(\omega) \in H_q^{-\beta}(\mathbb{R})$. We will use this in our numerical simulations.

As far as the drift b is not a function but a distribution, it must be approximated if we want to evaluate it at points. In order to do so, we will use a series representation of b and truncate it. That is why we will consider two steps in our algorithm:

1. approximate the drift b by a function b^N meant to converge to b as $N \rightarrow \infty$.
2. approximate the solution X_t^N of the approximated SDE

$$dX_t^N = b^N(X_t^N) dt + dW_t \quad (2)$$

by $X_t^{N,n}$ defined with the Euler-Maruyama scheme

$$X_t^{N,n} = X_0 + \int_0^t b^N(X_{\eta_n(t)}^{N,n}) dt + W_{\eta_n(t)}$$

where $\eta_n(t) = t_k$ if $t \in [t_k, t_k + 1]$, for $t_k = \frac{k}{n}$ with $k \in \llbracket 0, \lceil nT \rceil \rrbracket$.

2 Numerical simulation of fractional Brownian motion

To simulate a sample path of a fractional brownian motion B_x^H on a finite grid $(x_k)_{k \in \llbracket 1, n \rrbracket}$, we simulate n independent standard gaussian random variables $(X_k)_{k \in \llbracket 1, n \rrbracket}$ and then correlate them with the definite positive correlation matrix

$$C_{k,s} = \mathbb{E} [B_{x_k}^H B_{x_s}^H] = \frac{1}{2} (x_k^{2H} + x_s^{2H} + |x_k - x_s|^{2H}).$$

To do so, we use the Cholesky decomposition method and calculate the triangular matrix M such that $C = MM^\top$. Therefore, defining

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ and } B^H = MX,$$

B^H contains the values of a fractional brownian motion evaluated on the grid $(x_k)_{k \in \llbracket 1, n \rrbracket}$.

3 Approximation of the drift

3.1 Series representation

We use Haar wavelets to give a series representation of b . By doing so, we will be able to approximate it numerically by truncating the series.

Definition 1 (Haar wavelets). *We define the Haar wavelets $h_{j,m}$ on \mathbb{R} with $j \in \mathbb{N} \cup \{-1\}$ and $m \in \mathbb{Z}$ by:*

$$\begin{cases} h_M & : x \mapsto \left(\mathbb{1}_{[0, \frac{1}{2}[} - \mathbb{1}_{[\frac{1}{2}, 1[} \right) (x) \\ h_{-1,m} & : x \mapsto \sqrt{2} |h_M(x - m)| \\ h_{j,m} & : x \mapsto h_M(2^j x - m) \end{cases}$$

Theorem 1 (See [2]). *Let $b \in H_q^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left] -\frac{1}{2}, \frac{1}{q} \right[$. Therefore,*

$$b = \sum_{j=-1}^{+\infty} \sum_{m \in \mathbb{Z}} \mu_{j,m} h_{j,m} \quad (3)$$

where $\mu_{j,m} = 2^j \int_{\mathbb{R}} b(x) h_{j,m}(x) dx$ in the sense of dual pairing.

Definition 2. *With the same notation $\mu_{j,m}$, let $b \in H_q^s(\mathbb{R})$ for $2 \leq q \leq \infty$, and $s \in \left] -\frac{1}{2}, \frac{1}{q} \right[$. Given $N \in \mathbb{N}^*$ we define b^N by:*

$$b^N = \sum_{m=N}^{N-1} \mu_{-1,m} h_{-1,m} + \sum_{j=0}^N \sum_{m=-N2^j}^{N2^j-1} \mu_{j,m} h_{j,m}. \quad (4)$$

Remark 1. *We can note that $\text{Supp } b^N \subset [-N, N]$. Moreover, we have:*

$$\|b - b^N\|_{H_q^s(\mathbb{R})} \xrightarrow{N \rightarrow +\infty} 0.$$

4 Convergence

4.1 Weak convergence of $X_t^{N,n}$ to X_t^N

Recently, Leobacher and Szölgyenyi proved in [3] the convergence of the Euler-Maruyama scheme for SDE with discontinuous but piecewise Lipschitz drift and with a degenerate diffusion coefficient. This framework applies to the scheme we use with piecewise constant drift b^N and a constant diffusion coefficient.

Theorem 2 (Theorem 3.1. in [3]). $\exists C_N > 0$ independent of n such that it holds $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$:

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t^{N,n} - X_t^N|^2 \right]^{1/2} \leq C_N \delta^{1/4-\varepsilon} \quad (5)$$

with $\delta = \frac{1}{n}$ the step size and C_N depending on $\|b^N\|_\infty$.

Theorem 3. Let f be μ -Hölder with constant $C_f > 0$, $\mu \in (0, 1]$ and $t \in [0, T]$. Then, exists $C'_N > 0$ independent of n such that it holds $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$:

$$\left| \mathbb{E} \left[f \left(X_t^{N,n} \right) - f \left(X_t^N \right) \right] \right| \leq C'_N \delta^{\mu/4-\varepsilon} \quad (6)$$

with $\delta = \frac{1}{n}$ the step size.

Proof. By Jensen's inequality and the μ -Hölder property of f , we obtain:

$$\begin{aligned} \left| \mathbb{E} \left[f \left(X_t^{N,n} \right) - f \left(X_t^N \right) \right] \right| &\leq C_f \mathbb{E} \left[|X_t^{N,n} - X_t^N|^\mu \right] \\ &\leq C_f \mathbb{E} \left[|Y_t - Y_t^N|^2 \right]^{\mu/2} \\ &\leq C_f C_N^\mu \delta^{\mu/4-\varepsilon}. \end{aligned}$$

□

4.2 Weak convergence of X_t^N to X_t

The goal of this section is to estimate the weak error $|\mathbb{E}[f(X_t) - f(X_t^N)]|$ with suitable functions f . In order to do so, we must go back to the definition of the virtual solution of the SDE (1) given in [1]. Let $(\delta, p) \in K(\beta, q) := \{(\delta, p) \mid \beta < \delta < 1 - \beta, \frac{1}{\delta} < p < q\}$. The authors of [1] define the virtual solution of SDE (1) by X_t such that:

$$\begin{cases} Y_t = y + (\lambda + 1) \int_0^t u(s, \Psi(s, Y_s)) \, ds + \int_0^t (\nabla u(s, \Psi(s, Y_s)) + 1) \, dW_s \\ X_t = \Psi(t, Y_t) = \varphi^{-1}(t, Y_t) \end{cases} \quad (7)$$

where u is the mild solution in $H_p^{1+\delta}$ of the following parabolic PDE:

$$\begin{cases} \partial_t u + \frac{1}{2} \Delta u + b \nabla u - (\lambda + 1)u = -b & \text{on } [0, T] \times \mathbb{R} \\ u(T) = 0 & \text{on } \mathbb{R} \end{cases} \quad (8)$$

with $\varphi(t, x) = x + u(t, x)$, and $y = \varphi(0, x)$.

We also define another similar PDE by replacing b by b^N . We call u^N its mild solution in $H_p^{1+\delta}$:

$$\begin{cases} \partial_t u^N + \frac{1}{2} \Delta u^N + b^N \nabla u^N - (\lambda + 1) u^N = -b^N & \text{on } [0, T] \times \mathbb{R} \\ u^N(T) = 0 & \text{on } \mathbb{R} \end{cases}. \quad (9)$$

Then we consider an approximated version of (7):

$$\begin{cases} Y_t^N = y^N + (\lambda + 1) \int_0^t u^N(s, \Psi^N(s, Y_s^N)) ds + \int_0^t (\nabla u^N(s, \Psi^N(s, Y_s^N)) + 1) dW_s \\ X_t^N = \Psi^N(t, Y_t^N) = (\varphi^N)^{-1}(t, Y_t^N) \end{cases}. \quad (10)$$

with $\varphi^N(t, x) = x + u^N(t, x)$, and $y^N = \varphi^N(0, x)$.

Remark 2. Proposition 26 in [1] assures us that X_t^N defined above in (10) is in fact the classical solution of (2), as far as $b^N \in L^p$. That is why for each fixed N our Euler scheme converges to the virtual solution X_t^N .

We also recall a useful lemma concerning the solutions of (8) and (9).

Lemma 4 (Lemma 20 in [1]). Let $(\delta, p) \in K(\beta, q)$ and let u, u^N be the mild solutions to (8), (9) in $H_p^{1+\delta}$. Fix ρ such that the integral operator is a contraction and let $\lambda > \rho$. Then $u(t), u^N(t) \in C^{1,\alpha}$ with $\alpha = \delta - 1/p$ for each fixed t and

$$\begin{cases} \sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla u(t, x)| \rightarrow 0, \text{ as } \lambda \rightarrow \infty \\ \sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla u^N(t, x)| \rightarrow 0, \text{ as } \lambda \rightarrow \infty \end{cases}$$

where the choice of λ depends only on $\delta, \beta, \|b\|_{H_p^{-\beta}}$, and $\|b^N\|_{H_q^{-\beta}}$.

Lemma 5. Let $(\delta, p) \in K(\beta, q)$ and let u, u^N be the mild solutions to (8), (9) in $H_p^{1+\delta}$, $\alpha = \delta - 1/p$. Exists $c, K > 0$ such that for both $N \in \mathbb{N}$ and $\rho > 1$ big enough, $\forall t \in [0, T]$,

$$\begin{cases} \|u^N(t) - u(t)\|_{L^\infty} \leq \kappa \|b - b^N\|_{H_q^{-\beta}} \\ \|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \kappa \|b - b^N\|_{H_q^{-\beta}} \end{cases}. \quad (11)$$

with $\kappa = cK e^{\rho T}$.

Proof. Applying fractional Morrey inequality, $\exists c > 0, \forall t \in [0, T]$:

$$\begin{cases} \|u^N(t) - u(t)\|_{L^\infty} \leq \|u^N(t) - u(t)\|_{C^{1,\alpha}} \leq c \|u^N(t) - u(t)\|_{H_p^{1+\delta}} \\ \|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \|u^N(t) - u(t)\|_{C^{1,\alpha}} \leq c \|u^N(t) - u(t)\|_{H_p^{1+\delta}} \end{cases}.$$

Now, we can conclude with

$$\|u^N - u\|_{\infty, H_p^{1+\delta}} \leq e^{\rho T} \|u^N - u\|_{\infty, H_p^{1+\delta}}^{(\rho)} \leq K e^{\rho T} \|b - b^N\|_{H_q^{-\beta}}$$

from Lemma 23 in [1], for both $N \in \mathbb{N}$ and $\rho > 1$ big enough, and where $\|f(t)\|_{\infty, X}^{(\rho)} := \sup_{0 \leq t \leq T} e^{-\rho t} \|f(t)\|_X$. \square

Lemma 6. *For λ big enough,*

$$|\Psi^N(t, Y_t^N) - \Psi(t, Y_t^N)| \leq 2\kappa \|b - b^N\|_{H_q^{-\beta}}$$

with $\kappa = cKe^{\rho T}$.

Proof. For λ big enough, by Lemma 4, $\forall t \in [0, T]$, $\sup_{x \in \mathbb{R}} |\nabla u(t, x)| \leq 1/2$, so we obtain with $\varphi(t, x) = x + u(t, x)$:

$$\begin{aligned} & |\varphi(t, \Psi^N(t, Y_t^N)) - \varphi(t, \Psi(t, Y_t^N))| \\ & \geq \inf_{x \in \mathbb{R}} |\nabla \varphi(t, x)| |\Psi^N(t, Y_t^N) - \Psi(t, Y_t^N)| \\ & \geq \frac{1}{2} |\Psi^N(t, Y_t^N) - \Psi(t, Y_t^N)| \end{aligned}$$

and

$$\begin{aligned} |\Psi^N(t, Y_t^N) - \Psi(t, Y_t^N)| & \leq 2 |\varphi(t, \Psi^N(t, Y_t^N)) - \varphi(t, \Psi(t, Y_t^N))| \\ & \leq 2 |\varphi(t, \Psi^N(t, Y_t^N)) - \varphi^N(t, \Psi^N(t, Y_t^N))| \\ & \leq 2 \|u(t) - u^N(t)\|_{\infty} \\ & \leq 2cKe^{\rho T} \|b - b^N\|_{H_q^{-\beta}} \end{aligned} \tag{12}$$

where we have used Lemma 5 and the fact that

$$\varphi^N(t, \Psi^N(t, Y_t^N)) = \varphi(t, \Psi(t, Y_t^N)) = Y_t^N.$$

□

We will need an adapted version of a local time inequality (Lemma 4.2 in [4]) from Liqing Yan:

Lemma 7. *Let X be a continuous semimartingale. For $\varepsilon > 0$ we define a double sequence of stopping times by $\sigma_1 = \inf\{t \geq 0 | X_t = 0\}$, $\tau_1 = \inf\{t > \sigma_1 | X_t = \varepsilon\}$, $\sigma_n = \inf\{t > \tau_{n-1} | X_t = 0\}$, $\tau_n = \inf\{t > \sigma_n | X_t = \varepsilon\}$. For any real function $F(\cdot) \in \mathcal{C}^2(\mathbb{R})$ with $F(0) = 0$, $F'(0) = 0$, $F(\cdot) > 0$ on $(0, \varepsilon_0)$ with some $\varepsilon_0 > 0$, and for any $0 < \varepsilon < \varepsilon_0$ we have*

$$0 \leq L_t^0(X) \leq 2\varepsilon - \frac{2}{F(\varepsilon)} \int_{\sigma_1 \wedge t}^t \theta_s(X) (F(\varepsilon) - \varepsilon F'(X_s^+)) dX_s + \frac{\varepsilon}{F(\varepsilon)} \int_{\sigma_1 \wedge t}^t \theta_s(X) F''(X_s^+) d[X]_s$$

with $\theta_s(X) = \sum_{n=1}^{\infty} \mathbf{1}_{\{\sigma_n < s \leq \tau_n, 0 < X_s \leq \varepsilon\}}(X)$.

Applying lemma 7 with $F : x \in \mathbb{R} \mapsto x^2$, it follows:

Corollary 8. *Let X be a continuous semimartingale. With the same notations as in lemma 7, for any $\varepsilon > 0$ we have*

$$0 \leq L_t^0(X) \leq 2\varepsilon - \frac{2}{\varepsilon} \int_{\sigma_1 \wedge t}^t \theta_s(X) (\varepsilon - 2X_s^+) dX_s + \frac{2}{\varepsilon} \int_{\sigma_1 \wedge t}^t \theta_s(X) d[X]_s \tag{13}$$

Proof of Lemma 7. We note $U_t(X) = \sup\{n \in \mathbb{N} | \tau_n < t\}$ and $n(t) = t \wedge \sigma_{U_t(X)+1}$. By Meyer-Tanaka's formula, $\forall i \in \mathbb{N}^*$:

$$X_{\tau_i \wedge t}^+ - X_{\sigma_i \wedge t}^+ = \int_{\sigma_i \wedge t}^{\tau_i \wedge t} \mathbb{1}_{\{X_s > 0\}} dX_s + \frac{1}{2} \{L_{\tau_i \wedge t}^0(X) - L_{\sigma_i \wedge t}^0(X)\}. \quad (14)$$

Because $\forall i \in \mathbb{N}^*$, $L_{\tau_i \wedge t}^0(X) = L_{\sigma_{i+1} \wedge t}^0(X)$ and $L_{\sigma_1 \wedge t}^0(X) = 0$, we have

$$\sum_{i=1}^{U_t(X)+1} (X_{\tau_i \wedge t}^+ - X_{\sigma_i \wedge t}^+) = \int_{\sigma_1 \wedge t}^t \theta_s(X) dX_s + \frac{1}{2} L_t^0(X).$$

The left term is equal to $\varepsilon U_t(X) + X_t^+ - X_{n(t)}^+$ so

$$\varepsilon U_t(X) = \int_{\sigma_1 \wedge t}^t \theta_s(X) dX_s + \frac{1}{2} L_t^0(X) - X_t^+ + X_{n(t)}^+. \quad (15)$$

Now we express differently $U_t(X)$. $F \in \mathcal{C}^2(\mathbb{R})$ so by Itô's formula:

$$F(X_{\tau_i \wedge t}^+) - F(X_{\sigma_i \wedge t}^+) = \int_{\sigma_i \wedge t}^{\tau_i \wedge t} F'(X_s^+) dX_s^+ + \frac{1}{2} \int_{\sigma_i \wedge t}^{\tau_i \wedge t} F''(X_s^+) d[X^+]_s.$$

By (14), $dX_s^+ = \mathbb{1}_{\{X_s > 0\}} dX_s + \frac{1}{2} dL_t^0(X)$ and $d[X^+]_s = \mathbb{1}_{\{X_s > 0\}} d[X]_s$. It follows

$$\begin{aligned} F(X_{\tau_i \wedge t}^+) - F(X_{\sigma_i \wedge t}^+) &= \int_{\sigma_i \wedge t}^{\tau_i \wedge t} F'(X_s^+) \mathbb{1}_{\{X_s > 0\}} dX_s + \frac{1}{2} \int_{\sigma_i \wedge t}^{\tau_i \wedge t} F'(X_s^+) dL_t^0(X) \\ &\quad + \frac{1}{2} \int_{\sigma_i \wedge t}^{\tau_i \wedge t} F''(X_s^+) \mathbb{1}_{\{X_s > 0\}} d[X]_s. \end{aligned}$$

Adding up for i , with $F(0) = 0$ we obtain

$$\begin{aligned} F(\varepsilon)U_t(X) + F(X_t^+) - F(X_{n(t)}^+) &= \sum_{i=1}^{U_t(X)+1} (F(X_{\tau_i \wedge t}^+) - F(X_{\sigma_i \wedge t}^+)) \\ &= \int_{\sigma_1 \wedge t}^t F'(X_s^+) \theta_s(X) dX_s + \frac{1}{2} \int_{\sigma_1 \wedge t}^t F'(X_s^+) \Xi_s dL_t^0(X) + \frac{1}{2} \int_{\sigma_1 \wedge t}^t F''(X_s^+) \theta_s(X) d[X]_s \end{aligned}$$

with $\Xi_s = \sum_{n=1}^{\infty} \mathbb{1}_{\{\sigma_n < s \leq \tau_n\}}$. The measure $dL_t^0(X)$ is almost surely carried by $\{t | X_t = 0\}$ so we can simplify $\int_{\sigma_1 \wedge t}^t F'(X_s^+) \Xi_s dL_t^0(X) = F'(0) \int_{\sigma_1 \wedge t}^t \Xi_s dL_t^0(X)$ in order to have, with $F'(0) = 0$:

$$F(\varepsilon)U_t(X) = -F(X_t^+) + F(X_{n(t)}^+) + \int_{\sigma_1 \wedge t}^t F'(X_s^+) \theta_s(X) dX_s + \frac{1}{2} \int_{\sigma_1 \wedge t}^t F''(X_s^+) \theta_s(X) d[X]_s. \quad (16)$$

Combining (15) and (16), it follows

$$\begin{aligned} L_t^0(X)F(\varepsilon) &= 2F(\varepsilon)(X_t^+ - X_{n(t)}^+) - 2\varepsilon(F(X_t^+) - F(X_{n(t)}^+)) \\ &\quad - 2 \int_{\sigma_1 \wedge t}^t (F(\varepsilon) - \varepsilon F'(X_s^+)) \theta_s(X) dX_s + \varepsilon \int_{\sigma_1 \wedge t}^t F''(X_s^+) \theta_s(X) d[X]_s. \end{aligned}$$

Then, if $n(t) = t$, the two first right terms of the equality are equal to zero. Else, if $n(t) = \sigma_{U_t(X)+1}$, $0 \leq F(\varepsilon)(X_t^+ - X_{n(t)}^+) = F(\varepsilon)X_t^+ \leq F(\varepsilon)\varepsilon$ and $-2\varepsilon(F(X_t^+) - F(X_{n(t)}^+)) = -2\varepsilon F(X_t^+) \leq 0$ because of the positivity of F . Finally we obtain:

$$L_t^0(X) \leq 2\varepsilon - \frac{2}{F(\varepsilon)} \int_{\sigma_1 \wedge t}^t (F(\varepsilon) - \varepsilon F'(X_s^+)) \theta_s(X) dX_s + \frac{\varepsilon}{F(\varepsilon)} \int_{\sigma_1 \wedge t}^t F''(X_s^+) \theta_s(X) d[X]_s.$$

□

Lemma 9. Let $(\delta, p) \in K(\beta, q)$, $\alpha = \delta - 1/p < 1$, u, u^N be the mild solutions to (8), (9) in $H_p^{1+\delta}$, and Y, Y^N solutions of the SDEs (7), (10). Then, if $\alpha > 1/2$, for λ big enough we have $\forall \varepsilon \in (0, 1]$,

$$0 \leq \mathbb{E} [L_T^0(Y - Y^N)] \leq g(\varepsilon).$$

where

$$g(\varepsilon) \leq 4(\lambda + 1)T\kappa \|b - b^N\|_{H_q^{-\beta}} + \left(2 + 2(\lambda + 1)T + 6\|u\|_{C^{1,\alpha}}^2 4^\alpha T\right) \varepsilon^{2\alpha-1} + 6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} \|b - b^N\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2 \|b - b^N\|_{H_q^{-\beta}}^2\right) \varepsilon^{-1}.$$

Proof. By Lemma 4, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\|b - b^N\|_{H_q^s(\mathbb{R})} \xrightarrow{N \rightarrow \infty} 0$. Therefore $u^N(t, \cdot)$ and $u(t, \cdot)$ are $\frac{1}{2}$ -lipschitz. Let $\varepsilon \in (0, 1]$. Corollary 8 gives us:

$$0 \leq L_T^0(Y - Y^N) \leq 2\varepsilon - \frac{2}{\varepsilon} \int_{\sigma_1 \wedge T}^T \theta_s(Y - Y^N) (\varepsilon - 2(Y_s - Y_s^N)^+) d(Y_s - Y_s^N) + \frac{2}{\varepsilon} \int_{\sigma_1 \wedge T}^T \theta_s(Y - Y^N) d[Y - Y^N]_s$$

with

$$Y_T - Y_T^N = y - y^N + (\lambda + 1) \int_0^T \{u(s, \Psi(s, Y_s)) - u^N(s, \Psi^N(s, Y_s^N))\} ds + \int_0^T \{\nabla u(s, \Psi(s, Y_s)) - \nabla u^N(t, \Psi^N(s, Y_s^N))\} dW_s.$$

Remark 3. Note that $\theta_s(Y - Y^N) |\varepsilon - 2(Y_s - Y_s^N)^+| \leq \varepsilon \theta_s(Y - Y^N)$.

Remark 4. For clarity purpose, we note $\tilde{u}(s, x) = u(s, \Psi(s, x))$, $\tilde{u}^N(s, x) = u^N(s, \Psi(s, x))$, $\bar{u}^N(s, x) = u^N(s, \Psi^N(s, x))$ and use the same notations for the gradient and the approximated mild solution. We recall that in this case, by Lemma 22 in [1], $\Psi(t, \cdot)$ and $\Psi^N(t, \cdot)$ are 2-lipschitz. We can notice that \tilde{u} is 1-lipschitz in space and $\nabla \tilde{u}$ is α -Hölder with constant $2^\alpha \|u\|_{C^{1,\alpha}}$. The same properties hold for \bar{u} and $\nabla \bar{u}$ except that the Hölder constant for $\nabla \bar{u}$ can be bounded by a constant Ω for N big enough (see Lemma 24 in [1]).

∇u and ∇u^N are bounded so the Itô integral is a martingale. We take the expectation:

$$\begin{aligned}
 \mathbb{E} [L_T^0(Y - Y^N)] &\leq 2\varepsilon + 2(\lambda + 1) \mathbb{E} \left[\int_{\sigma_1 \wedge T}^T \theta_s (Y - Y^N) |\tilde{u}(s, Y_s) - \bar{u}^N(s, Y_s^N)| \, ds \right] \\
 &\quad + \frac{2}{\varepsilon} \mathbb{E} \left[\int_{\sigma_1 \wedge T}^T \theta_s (Y - Y^N) \left\{ \widetilde{\nabla u}(s, Y_s) - \overline{\nabla u}(s, Y_s^N) \right\}^2 \, ds \right] \\
 &\leq 2\varepsilon + 2(\lambda + 1) \mathbb{E} \left[\int_0^T \theta_s (Y - Y^N) |\tilde{u}(s, Y_s) - \tilde{u}(s, Y_s^N)| \, ds \right] \\
 &\quad + 2(\lambda + 1) \mathbb{E} \left[\int_0^T \theta_s (Y - Y^N) |\tilde{u}(s, Y_s^N) - \tilde{u}^N(s, Y_s^N)| \, ds \right] \\
 &\quad + 2(\lambda + 1) \mathbb{E} \left[\int_0^T \theta_s (Y - Y^N) |\tilde{u}^N(s, Y_s^N) - \bar{u}^N(s, Y_s^N)| \, ds \right] \\
 &\quad + \frac{6}{\varepsilon} \mathbb{E} \left[\int_0^T \theta_s (Y - Y^N) \left\{ \widetilde{\nabla u}(s, Y_s) - \widetilde{\nabla u}(s, Y_s^N) \right\}^2 \, ds \right] \\
 &\quad + \frac{6}{\varepsilon} \mathbb{E} \left[\int_0^T \theta_s (Y - Y^N) \left\{ \widetilde{\nabla u}(s, Y_s^N) - \widetilde{\nabla u}^N(s, Y_s^N) \right\}^2 \, ds \right] \\
 &\quad + \frac{6}{\varepsilon} \mathbb{E} \left[\int_0^T \theta_s (Y - Y^N) \left\{ \widetilde{\nabla u}^N(s, Y_s^N) - \overline{\nabla u}^N(s, Y_s^N) \right\}^2 \, ds \right] \\
 &\leq 2\varepsilon + 2(\lambda + 1) \mathbb{E} \left[\int_0^T \theta_s (Y - Y^N) |Y_s - Y_s^N| \, ds \right] + 2(\lambda + 1) T \kappa \|b^N - b\|_{H_q^{-\beta}} \\
 &\quad + (\lambda + 1) \mathbb{E} \left[\int_0^T \theta_s (Y - Y^N) |\Psi(s, Y_s^N) - \Psi^N(s, Y_s^N)| \, ds \right] \\
 &\quad + \frac{6 \times 4^\alpha \|u\|_{\mathcal{C}^{1,\alpha}}^2}{\varepsilon} \mathbb{E} \left[\int_0^T \theta_s (Y - Y^N) |Y_s - Y_s^N|^{2\alpha} \, ds \right] + 6T\kappa^2 \|b^N - b\|_{H_q^{-\beta}}^2 \varepsilon^{-1} \\
 &\quad + \frac{6\Omega^2}{\varepsilon} \mathbb{E} \left[\int_0^T \theta_s (Y - Y^N) |\Psi(s, Y_s^N) - \Psi^N(s, Y_s^N)|^{2\alpha} \, ds \right]
 \end{aligned}$$

where we have used Lemma 5, the 1-lipschitz property of \tilde{u} , the 1/2-lipschitz property of u^N , the α -Hölder property of $\widetilde{\nabla u}$ (with constant $2^\alpha \|u\|_{\mathcal{C}^{1,\alpha}}$), and the α -Hölder property of ∇u^N (with constant Ω). Lemma 6 gives us:

$$\begin{aligned}
 &\leq 2\varepsilon + 2(\lambda + 1) \mathbb{E} \left[\int_0^T \theta_s (Y - Y^N) |Y_s - Y_s^N| \, ds \right] + 4(\lambda + 1) T \kappa \|b^N - b\|_{H_q^{-\beta}} \\
 &\quad + \frac{6 \times 4^\alpha \|u\|_{\mathcal{C}^{1,\alpha}}^2}{\varepsilon} \mathbb{E} \left[\int_0^T \theta_s (Y - Y^N) |Y_s - Y_s^N|^{2\alpha} \, ds \right] + 6T\kappa^2 \|b - b^N\|_{H_q^{-\beta}}^2 \varepsilon^{-1} \\
 &\quad + 6\Omega^2 T 4^\alpha \kappa^{2\alpha} \|b - b^N\|_{H_q^{-\beta}}^{2\alpha} \varepsilon^{-1}.
 \end{aligned}$$

As $\theta_s(Y - Y^N) |Y_s - Y_s^N| \leq \varepsilon$, we have

$$\begin{aligned} \mathbb{E} [L_T^0(Y - Y^N)] &\leq 2\varepsilon + 2(\lambda + 1) T\varepsilon + 4(\lambda + 1)T\kappa \|b - b^N\|_{H_q^{-\beta}} + 6\|u\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T \varepsilon^{2\alpha-1} \\ &\quad + 6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} \|b - b^N\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2 \|b - b^N\|_{H_q^{-\beta}}^2 \right) \varepsilon^{-1}. \end{aligned}$$

As $1 > 2\alpha - 1 > 0$, the result follows from $\varepsilon \leq \varepsilon^{2\alpha-1}$ when $0 < \varepsilon \leq 1$. \square

Lemma 10. *With assumptions and notations of Lemma 9, and $1 > \alpha > 1/2$ we have*

$$\mathbb{E} [L_T^0(Y - Y^N)] \leq g(\varepsilon_N) = \sigma \|b^N - b\|_{H_q^{-\beta}}^{2-1/\alpha} \quad (17)$$

for $\|b^N - b\|_{H_q^{-\beta}}$ small enough (it is to say N big enough) where

$$\sigma = 4(\lambda + 1)T\kappa + \left(2 + 2(\lambda + 1) T + 6\|u\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T \right) 2\omega_\infty^{2\alpha-1} + 6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2 \right) \omega_\infty^{-1}$$

and

$$\omega_\infty = \left(\frac{6T\Omega^2 4^\alpha \kappa^{2\alpha}}{(2\alpha - 1) \left(2 + 2(\lambda + 1) T + 6\|u\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T \right)} \right)^{\frac{1}{2\alpha}}.$$

Proof. Let $\varepsilon \in (0, 1]$, by Lemma 9,

$$0 \leq \mathbb{E} [L_T^0(Y - Y^N)] \leq g(\varepsilon)$$

where

$$\begin{aligned} g(\varepsilon) &= 4(\lambda + 1)T\kappa \|b - b^N\|_{H_q^{-\beta}} + \left(2 + 2(\lambda + 1) T + 6\|u\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T \right) \varepsilon^{2\alpha-1} \\ &\quad + 6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} \|b - b^N\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2 \|b - b^N\|_{H_q^{-\beta}}^2 \right) \varepsilon^{-1}. \end{aligned}$$

With

$$\begin{aligned} g'(\varepsilon) &= (2\alpha - 1) \left(2 + 2(\lambda + 1) T + 6\|u\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T \right) \varepsilon^{2\alpha-2} \\ &\quad - 6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} \|b - b^N\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2 \|b - b^N\|_{H_q^{-\beta}}^2 \right) \varepsilon^{-2}, \end{aligned}$$

and

$$\begin{aligned} g''(\varepsilon) &= (2\alpha - 2)(2\alpha - 1) \left(2 + 2(\lambda + 1) T + 6\|u\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T \right) \varepsilon^{2\alpha-3} \\ &\quad + 12T \left(\Omega^2 4^\alpha \kappa^{2\alpha} \|b - b^N\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2 \|b - b^N\|_{H_q^{-\beta}}^2 \right) \varepsilon^{-3}, \end{aligned}$$

the minimum of g on $(0, 1]$ is reached when N is big enough in

$$\varepsilon_N = \left(\frac{6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} \|b - b^N\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2 \|b - b^N\|_{H_q^{-\beta}}^2 \right)}{(2\alpha - 1) \left(2 + 2(\lambda + 1) T + 6\|u\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T \right)} \right)^{\frac{1}{2\alpha}} = \omega_N \|b^N - b\|_{H_q^{-\beta}}.$$

where

$$g''(\varepsilon_N) = 12T \left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2 \|b - b^N\|_{H_q^{-\beta}}^{2(1-\alpha)} \right) \varepsilon_N^{-3} \alpha > 0.$$

and

$$\begin{aligned} \omega_N &= \left(\frac{6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2 \|b - b^N\|_{H_q^{-\beta}}^{2(1-\alpha)} \right)}{(2\alpha - 1) \left(2 + 2(\lambda + 1) T + 6 \|u\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T \right)} \right)^{\frac{1}{2\alpha}} \\ &\geq \omega_\infty = \left(\frac{6T \Omega^2 4^\alpha \kappa^{2\alpha}}{(2\alpha - 1) \left(2 + 2(\lambda + 1) T + 6 \|u\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T \right)} \right)^{\frac{1}{2\alpha}}. \end{aligned}$$

Therefore $\mathbb{E} [L_T^0(Y - Y^N)] \leq g(\varepsilon_N)$

$$\begin{aligned} &\leq 4(\lambda + 1)T\kappa \|b - b^N\|_{H_q^{-\beta}} + \left(2 + 2(\lambda + 1) T + 6 \|u\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T \right) \omega_N^{2\alpha-1} \|b - b^N\|_{H_q^{-\beta}}^{2\alpha-1} \\ &\quad + 6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} \|b - b^N\|_{H_q^{-\beta}}^{2\alpha} + \kappa^2 \|b - b^N\|_{H_q^{-\beta}}^2 \right) \omega_N^{-1} \|b - b^N\|_{H_q^{-\beta}}^{-1} \\ &\leq 4(\lambda + 1)T\kappa \|b - b^N\|_{H_q^{-\beta}} + \left(2 + 2(\lambda + 1) T + 6 \|u\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T \right) \omega_N^{2\alpha-1} \|b - b^N\|_{H_q^{-\beta}}^{2\alpha-1} \\ &\quad + 6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2 \|b - b^N\|_{H_q^{-\beta}}^{2(1-\alpha)} \right) \omega_\infty^{-1} \|b - b^N\|_{H_q^{-\beta}}^{2\alpha-1} \\ &\leq 4(\lambda + 1)T\kappa \|b - b^N\|_{H_q^{-\beta}}^{2\alpha-1} + \left(2 + 2(\lambda + 1) T + 6 \|u\|_{\mathcal{C}^{1,\alpha}}^2 4^\alpha T \right) 2\omega_\infty^{2\alpha-1} \|b - b^N\|_{H_q^{-\beta}}^{2\alpha-1} \\ &\quad + 6T \left(\Omega^2 4^\alpha \kappa^{2\alpha} + \kappa^2 \right) \omega_\infty^{-1} \|b - b^N\|_{H_q^{-\beta}}^{2\alpha-1} \end{aligned}$$

for N big enough. The result follows. \square

Theorem 11. *Let f be μ -Hölder with constant $C_f > 0$ and $\mu \in (0, 1]$. If $0 < \beta < 1/4$, $q \in \left(\frac{1}{1-\beta}, \frac{1}{\beta} \right)$, $\forall \varepsilon \in (0, 1 - 4\beta)$, with $(\delta, p) \in K(\beta, q)$ such that $\delta - 1/p = 1 - 2\beta - \varepsilon/2$, exists ξ independent of f such that for $N \in \mathbb{N}$, $\rho > 1$, λ big enough it holds:*

$$\sup_{0 \leq t \leq T} \mathbb{E} [|f(X_t) - f(X_t^N)|] \leq \xi C_f \|b^N - b\|_{H_q^{-\beta}}^{\mu(1-4\beta-\varepsilon)}$$

Proof. We note as usual $\alpha = \delta - 1/p$. By Lemma 4, we choose λ big enough for ∇u and ∇u^N to be bounded by $\frac{1}{2}$. λ can be chosen independently of N as far as $\|b - b^N\|_{H_q^s(\mathbb{R})} \xrightarrow{N \rightarrow \infty} 0$.

We recall that in this case, by Lemma 22 in [1], $\Psi(t, \cdot)$ and $\Psi^N(t, \cdot)$ are 2-lipschitz. Therefore $\tilde{u}^N(t, \cdot)$ and $\tilde{u}(t, \cdot)$ are 1-lipschitz. Let $t \in [0, T]$.

$$\begin{aligned} &\mathbb{E} [|f(X_t) - f(X_t^N)|] = \mathbb{E} [|f(\Psi(t, Y_t)) - f(\Psi^N(t, Y_t^N))|] \\ &\leq \mathbb{E} [|f(\Psi(t, Y_t)) - f(\Psi(t, Y_t^N))|] + \mathbb{E} [|f(\Psi(t, Y_t^N)) - f(\Psi^N(t, Y_t^N))|] \\ &\leq C_f \left(2^\mu \mathbb{E} [|Y_t - Y_t^N|^\mu] + \mathbb{E} [|\Psi(t, Y_t^N) - \Psi^N(t, Y_t^N)|^\mu] \right) \end{aligned}$$

$$\leq C_f \left(2^\mu \mathbb{E} [|Y_t - Y_t^N|]^\mu + \mathbb{E} [|\Psi(t, Y_t^N) - \Psi^N(t, Y_t^N)|]^\mu \right) \quad (18)$$

by Jensen's inequality.

$$\begin{aligned} Y_T - Y_T^N &= y - y^N + (\lambda + 1) \int_0^T \{u(s, \Psi(s, Y_s)) - u^N(s, \Psi^N(s, Y_s^N))\} \, ds \\ &\quad + \int_0^T \{\nabla u(s, \Psi(s, Y_s)) - \nabla u^N(t, \Psi^N(s, Y_s^N))\} \, dW_s. \end{aligned}$$

We apply Meyer-Tanaka's formula to obtain:

$$\begin{aligned} |Y_t - Y_t^N| &= |y - y^N| + (\lambda + 1) \int_0^t \text{sign}(Y_s - Y_s^N) \{\tilde{u}(s, Y_s) - \bar{u}^N(s, Y_s^N)\} \, ds \\ &\quad + \int_0^t \text{sign}(Y_s - Y_s^N) \{\widetilde{\nabla u}(s, Y_s) - \overline{\nabla u}^N(s, Y_s^N)\} \, dW_s + L_t^0(Y - Y^N). \end{aligned}$$

Taking the expectation leads to:

$$\begin{aligned} \mathbb{E} [|Y_t - Y_t^N|] &= |u(0, x) - u^N(0, x)| \\ &\quad + (\lambda + 1) \mathbb{E} \left[\int_0^t \text{sign}(Y_s - Y_s^N) \{\tilde{u}(s, Y_s) - \bar{u}^N(s, Y_s^N)\} \, ds \right] + \mathbb{E} [L_t^0(Y - Y^N)] \end{aligned}$$

because ∇u and ∇u^N are bounded so the Itô integral is a martingale.

$$\begin{aligned} \mathbb{E} [|Y_t - Y_t^N|] &\leq |u(0, x) - u^N(0, x)| + (\lambda + 1) \mathbb{E} \left[\int_0^t |\tilde{u}(s, Y_s) - \tilde{u}(s, Y_s^N)| \, ds \right] + \\ &+ (\lambda + 1) \mathbb{E} \left[\int_0^t |\tilde{u}(s, Y_s^N) - \tilde{u}^N(s, Y_s^N)| \, ds \right] + (\lambda + 1) \mathbb{E} \left[\int_0^t |\tilde{u}^N(s, Y_s^N) - \bar{u}^N(s, Y_s^N)| \, ds \right] \\ &\quad + \mathbb{E} [L_t^0(Y - Y^N)]. \end{aligned}$$

We use Lemma 5, the 1-lipschitz property of \tilde{u} , and the 1/2-lipschitz property of u :

$$\begin{aligned} \mathbb{E} [|Y_t - Y_t^N|] &\leq \kappa \|b^N - b\|_{H_q^{-\beta}} + (\lambda + 1) \mathbb{E} \left[\int_0^t |Y_s - Y_s^N| \, ds \right] + (\lambda + 1)t\kappa \|b^N - b\|_{H_q^{-\beta}} \\ &\quad + \frac{\lambda + 1}{2} \mathbb{E} \left[\int_0^t |\Psi(s, Y_s^N) - \Psi^N(s, Y_s^N)| \, ds \right] + \mathbb{E} [L_t^0(Y - Y^N)] \\ &\leq (\lambda + 1) \int_0^t \mathbb{E} [|Y_s - Y_s^N|] \, ds + (2(\lambda + 1)T + 1)\kappa \|b^N - b\|_{H_q^{-\beta}} + \mathbb{E} [L_T^0(Y - Y^N)]. \end{aligned}$$

where we have used the fact that $L_t^0(Y - Y^N)$ is an increasing process and Lemma 6.

By Gronwall's Lemma, it follows:

$$\mathbb{E} [|Y_t - Y_t^N|] \leq C(N) e^{(\lambda+1)t} \leq C(N) e^{(\lambda+1)T} \quad (19)$$

with $C(N) = (2(\lambda + 1)T + 1)\kappa \|b^N - b\|_{H_q^{-\beta}} + \mathbb{E} [L_T^0(Y - Y^N)]$.

With Lemma 9 and Lemma 10 we obtain

$$C(N) \leq (2(\lambda + 1)T + 1)\kappa \|b^N - b\|_{H_q^{-\beta}} + \sigma \|b^N - b\|_{H_q^{-\beta}}^{2\alpha-1} \leq \zeta \|b^N - b\|_{H_q^{-\beta}}^{2\alpha-1}.$$

for $\|b^N - b\|_{H_q^{-\beta}}$ small enough where $\zeta = (2(\lambda + 1)T + 1)\kappa + \sigma$. It follows:

$$\mathbb{E} [|Y_t - Y_t^N|] \leq \zeta e^{(\lambda+1)T} \|b^N - b\|_{H_q^{-\beta}}^{2\alpha-1}. \quad (20)$$

Finally, combining (18), (20) and Lemma 6 we obtain:

$$\begin{aligned} & |\mathbb{E} [f(X_t) - f(X_t^N)]| \\ & \leq C_f \left(2^\mu \mathbb{E} [|Y_t - Y_t^N|]^\mu + \mathbb{E} [|\Psi(t, Y_t^N) - \Psi^N(t, Y_t^N)|]^\mu \right) \\ & \leq C_f \left(2^\mu \zeta^\mu e^{\mu(\lambda+1)T} \|b^N - b\|_{H_q^{-\beta}}^{\mu(2\alpha-1)} + 2^\mu \kappa^\mu \|b^N - b\|_{H_q^{-\beta}}^\mu \right) \\ & \leq 2^\mu C_f \left(\zeta^\mu e^{\mu(\lambda+1)T} + 2^\mu \kappa^\mu \right) \|b^N - b\|_{H_q^{-\beta}}^{\mu(2\alpha-1)} \end{aligned}$$

for N big enough, which is the expected result. \square

5 Numerical results

5.1 Strong convergence of the Euler scheme

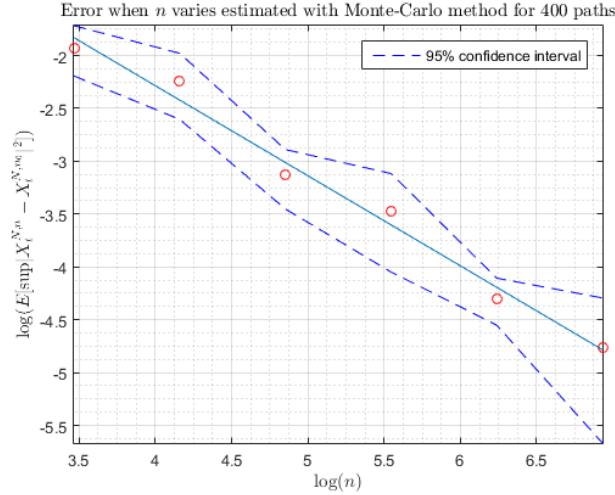


Figure 1: Estimation of the L^2 error of the Euler-Marayuma scheme with a Monte-Carlo method. 400 paths, $N = 5$, $n \in \{2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}\}$, reference solution with $n_0 = 2^{12}$ points.

We observe a numerical convergence rate of 0.85 when Theorem 2 shows a theoretical rate of $0.5 - \varepsilon$.

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