Linear Control Systems - 036012

Assignment #1

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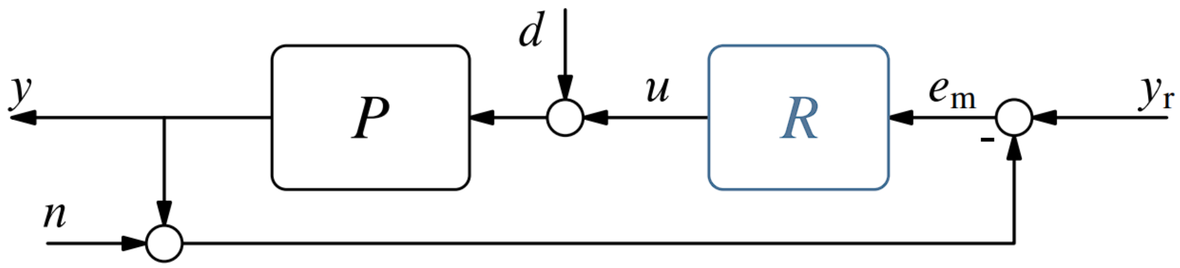
Problem 1 (25pt). Consider a plant P having the transfer function

$$P(s) = \frac{2.25(s+1)(s-2)}{s(s^2-9)}.$$

Design a stabilizing LTI controller *having an integral action* for it, by whatever method. Simulate the response of the resulted system (both the output and the control signal) to a step load disturbance for the setup in Fig. 1.4(c) in the Lecture Notes and present those plots. Submit also the stand-alone `*.m` file producing those plots (don't use Simulink, the step response of LTI systems can be simulated in Matlab).

We shall look at the following system:



Its response output is:

$$y = Ty_r + T_d d - Tn$$
$$u = T_c y_r - Td - T_c n$$
$$e = Sy_r - T_d d + Tn$$

Whereas its "gang of four" is:

$$\begin{bmatrix} S(s) & T_c(s) \\ T_d(s) & T(s) \end{bmatrix} = \frac{1}{1+P(s)R(s)} \begin{bmatrix} 1 & R(s) \\ P(s) & P(s)R(s) \end{bmatrix}$$

Thus, we receive:

$$y = \frac{P(s)R(s)}{1+P(s)R(s)} y_r + \frac{P(s)}{1+P(s)R(s)} d - \frac{P(s)R(s)}{1+P(s)R(s)} n$$
$$u = \frac{R(s)}{1+P(s)R(s)} y_r - \frac{P(s)R(s)}{1+P(s)R(s)} d - \frac{R(s)}{1+P(s)R(s)} n$$
$$e = \frac{1}{1+P(s)R(s)} y_r - \frac{P(s)}{1+P(s)R(s)} d + \frac{P(s)R(s)}{1+P(s)R(s)} n$$

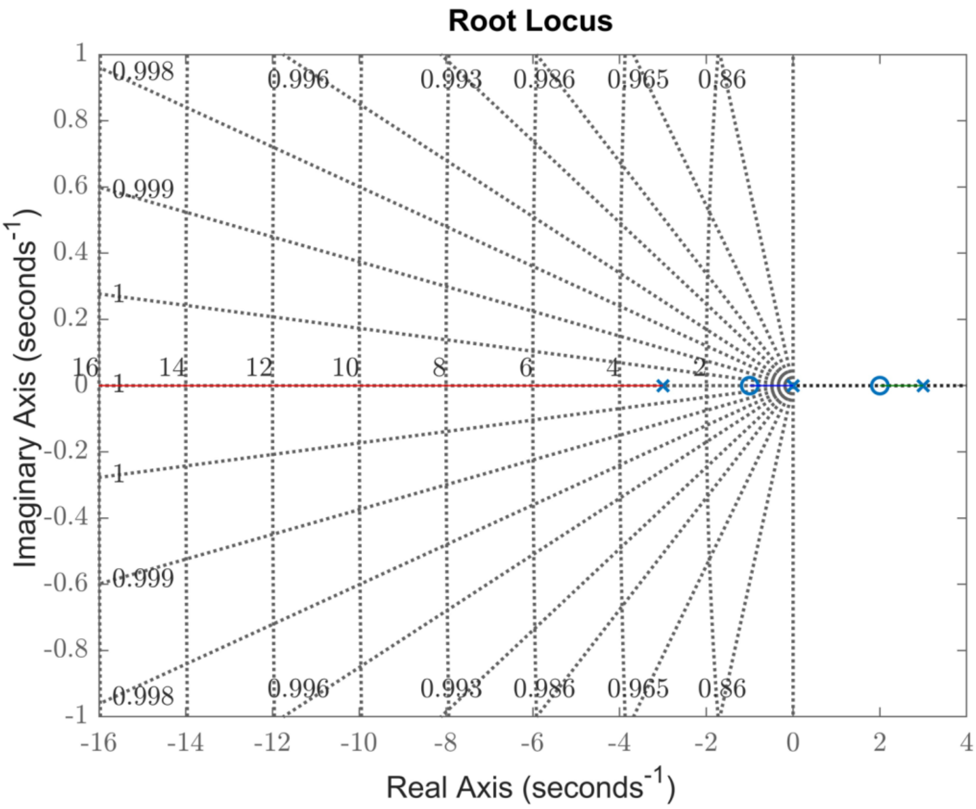
First, we need to make sure that all the sensitivity functions are stable – i.e. all poles of the transfer functions are on the OLHP.

We shall first look at the closed loop of the system without the disturbance and noise:

$$T(s) = \frac{P(s)R(s)}{1+P(s)R(s)}$$

We want to find out if the closed loop system is stable, for that we shall find the Root Locus of the system.

The Root Locus of the system represents the locations of the poles with regards to the controller (regulator) gain.



We see, however, that we no matter the gain that we use in the feedback we will still have poles in the ORHP, thus, the system won't be stable.

We know that the characteristic polynomial of the system symbolizes the roots of the closed loop system and for the following regulator and plant:

$$P(s) = \frac{N_P}{D_P} = \quad R(s) = \frac{N_R}{D_R}$$

the characteristic polynomial will be:

$$\Delta_s = N_P(s)N_R(s) + D_P(s)D_R(s)$$

With the roots being written as: χ_i.

We want to find a regulator that can place the roots in the region of the OLHP.

As we don't have any special requirements we shall choose arbitrary placements of the poles, to do that we shall first get the order of the characteristic polynomial.

The order of the plant is of the 3rd degree, so we shall decide that the controller will also be of the 3rd degree, therefore we shall decide on 6 arbitrary pole placements.
We choose 6 as we want later in the process to remove a degree of freedom to enforce an integral action to the regulator – resulting in a pole at the origin.

$$p_c = \begin{bmatrix} -2 \\ -20 \\ -5 \pm j \\ -2 \pm 3j \end{bmatrix}$$

The resultant design characteristic polynomial will be (with the pole at the origin):

$$\Delta_d = s(s+2)(s+20)((s+5)^2+1)((s+2)^2+9)$$

We shall want to achieve:

$$\Delta_s = \Delta_d$$

Where:

$$R(s) = \frac{\beta_3 s^3 + \beta_2 s^2 + \beta_1 s^1 + \beta_0}{\alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0}$$

And $\theta = \begin{bmatrix} \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \\ \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix}$

This will give us a system of linear equations:

$$M\theta = \chi$$

To find the matrix M we shall use Sylvester's theorem – will help up find a full rank matrix where the numerator and the denominator are relatively prime.

The resultant matrix is:

$$M_S = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 9 & 0 & 0 & 0 \\ -36 & 0 & 4 & 0 & -9 & 9 & 0 & 0 \\ 0 & -36 & 0 & 4 & -18 & -9 & 9 & 0 \\ 0 & 0 & -36 & 0 & 0 & -18 & -9 & 9 \\ 0 & 0 & 0 & -36 & 0 & 0 & -18 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -18 \end{bmatrix}$$

However, we want to exploit the degree of freedom that we have and enforce:

$$\alpha_0 = 0$$

This will result in us with an integral action.

Thus:

$$M_{S3} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 9 & 0 & 0 & 0 \\ -36 & 0 & 4 & -9 & 9 & 0 & 0 \\ 0 & -36 & 0 & -18 & -9 & 9 & 0 \\ 0 & 0 & -36 & 0 & -18 & -9 & 9 \\ 0 & 0 & 0 & 0 & 0 & -18 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 & -18 \end{bmatrix}$$

Now we shall solve the following system of equations:

$$M_{S3}\theta = \chi$$

For this result we shall find our regulator:

$$R(s) = \frac{-27772\mathrm{s}^3 - 68420\mathrm{s}^2 + 80288\mathrm{s} + 108160}{-36\mathrm{s}^3 + 61191\mathrm{s}^2 + 75762\mathrm{s}}$$

$$\boldsymbol{R(s) = \frac{4(6943s^3 + 17105s^2 - 20072s - 27040)}{9s(4s^2 - 6799s - 8418)}}$$

Now we shall test our regulator in the system shown at the start of this question.
We shall choose arbitrary amplitudes and noise characteristics:
(We've simulated the noise as a Gaussian distribution)

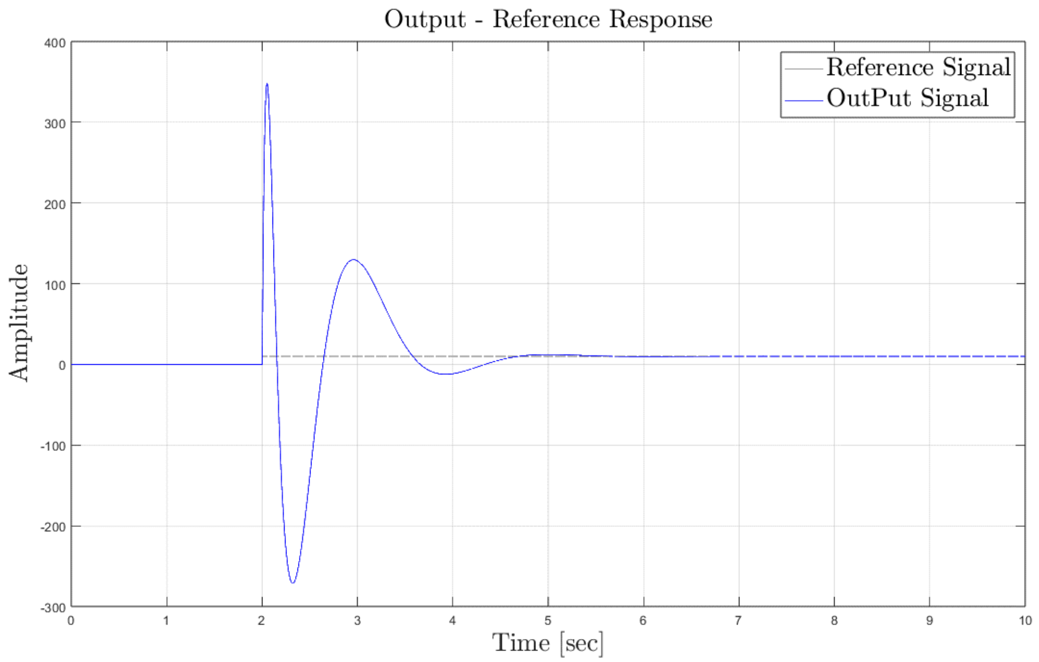
$$Test\ Period = 10\ [sec]$$
$$Start\ Time = 2\ [sec]$$
$$Reference\ Signal\ Amplitude = 10$$
$$Disturbance\ Signal\ Amplitude = -2$$
$$Disturbance\ Delay = 0.2\ [sec]$$
$$Noise\ STD = 0.5$$

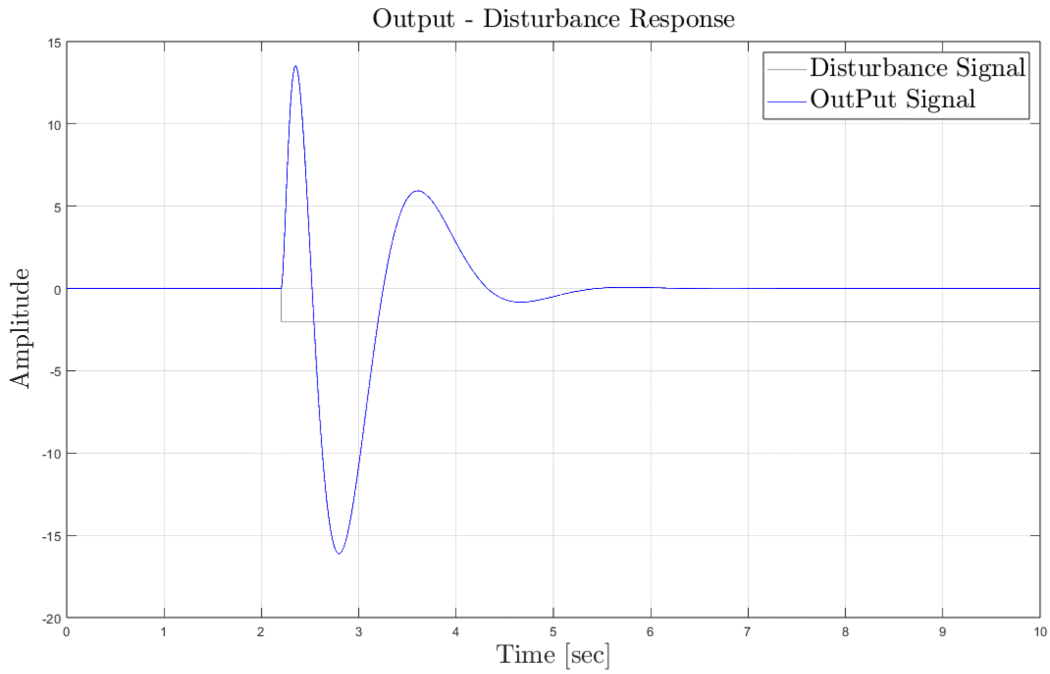
We shall show the simulation results for:

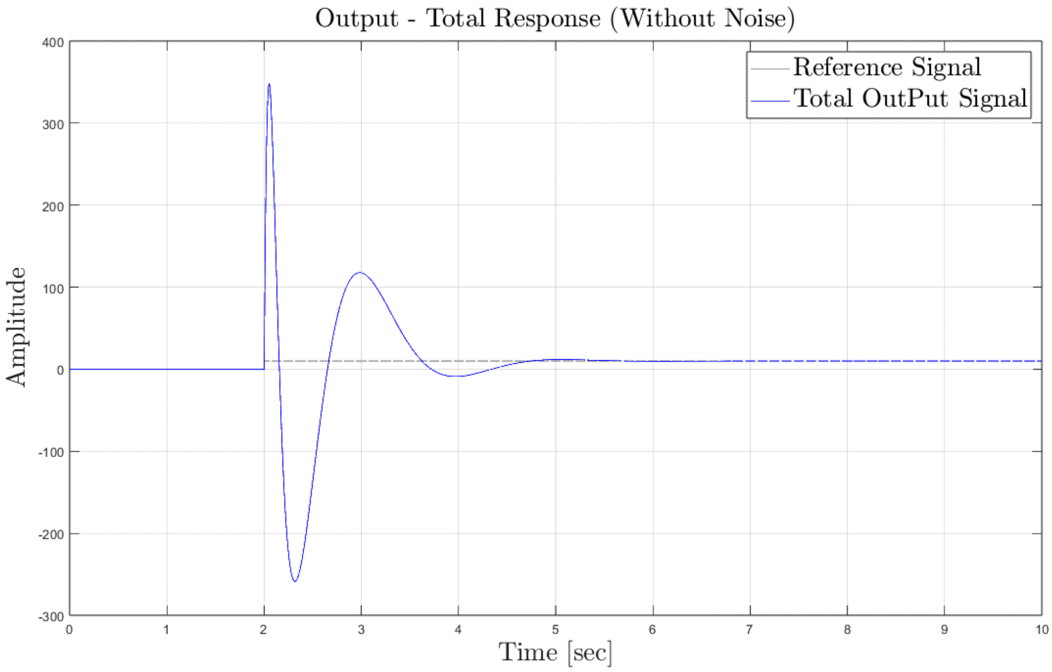
$$y = \frac{P(s)R(s)}{1 + P(s)R(s)}y_r + \frac{P(s)}{1 + P(s)R(s)}d - \frac{P(s)R(s)}{1 + P(s)R(s)}n$$
$$u = \frac{R(s)}{1 + P(s)R(s)}y_r - \frac{P(s)R(s)}{1 + P(s)R(s)}d - \frac{R(s)}{1 + P(s)R(s)}n$$

For each output (system's output and controller signal), we shall show the reference signal and disturbance responses in their corresponding sensitivity functions and the total result without a noise.
Then we shall add noise to the system and show the response to the noise and the total response of the system's output and controller signal with the noise.

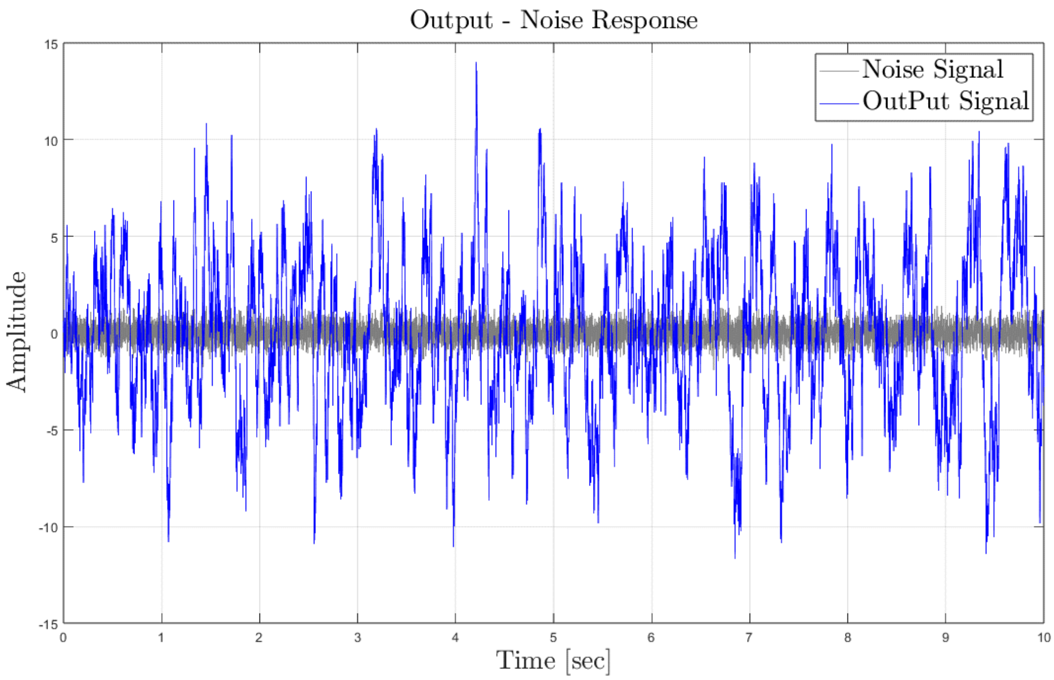
System’s Output:



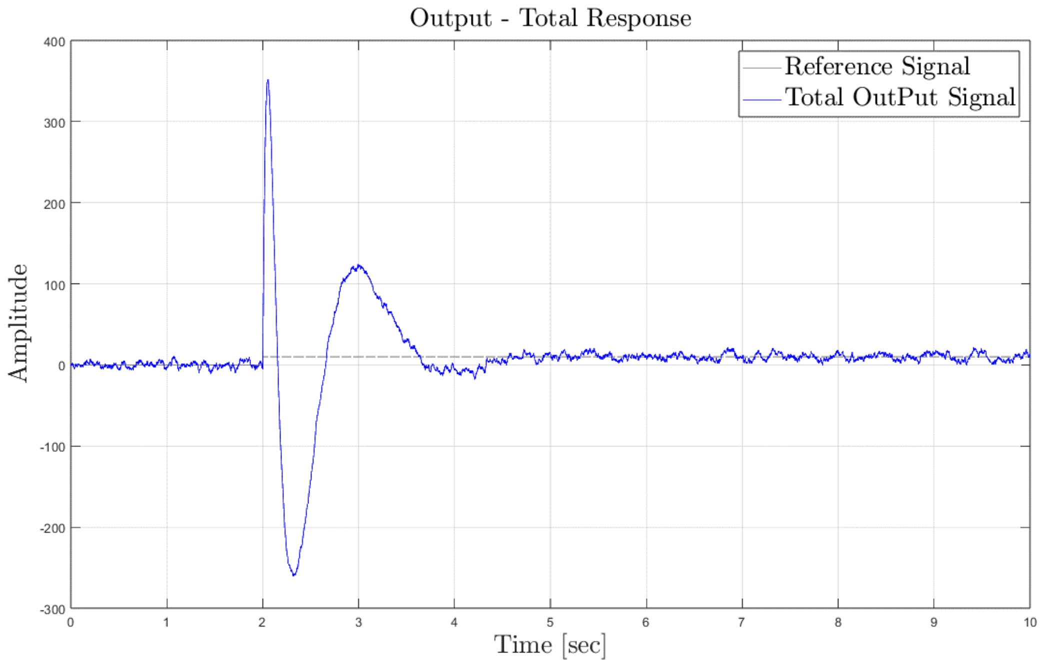




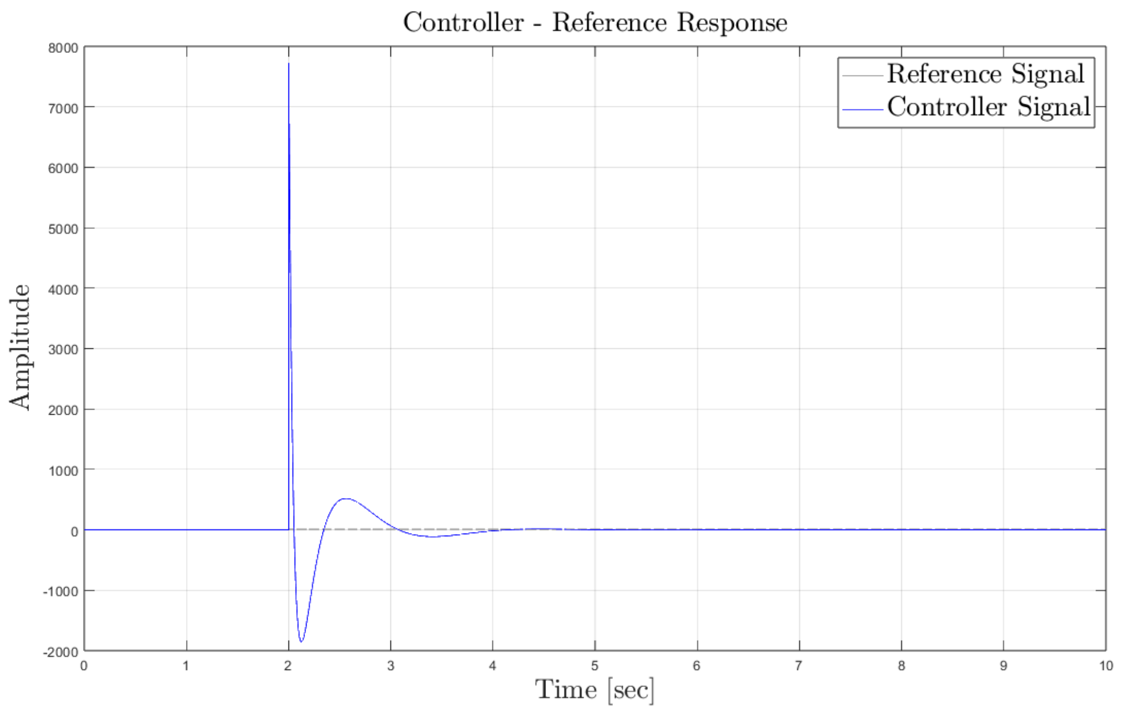
Now we shall add the noise:

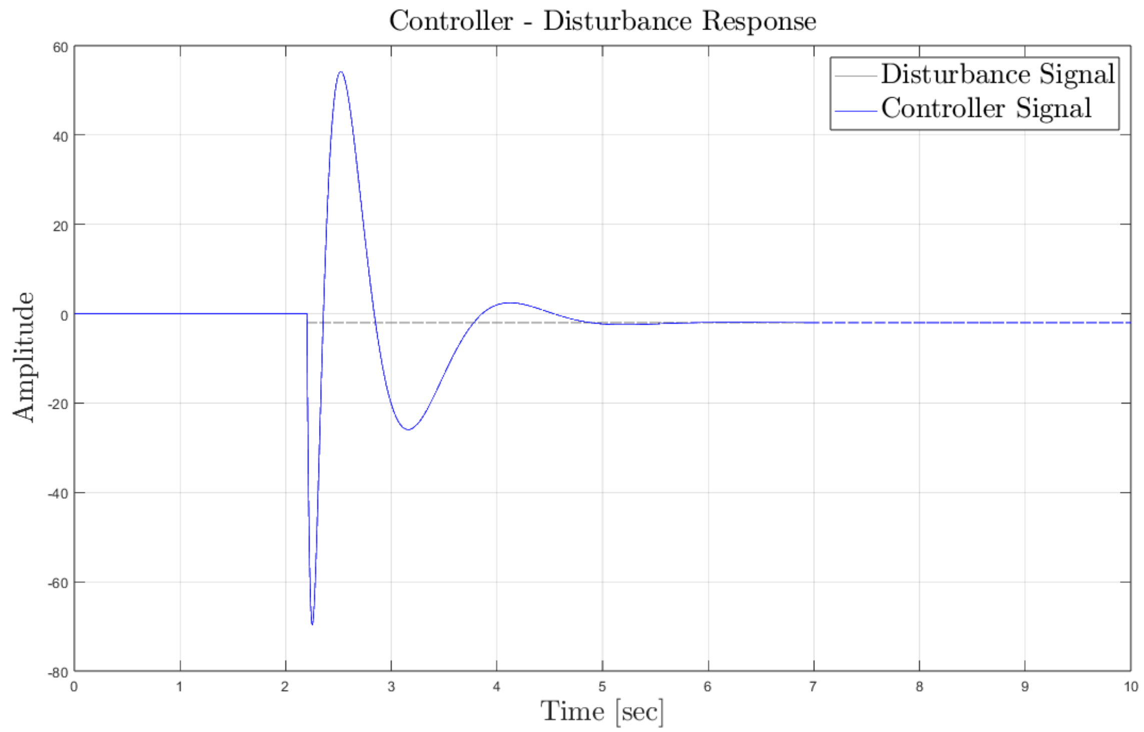


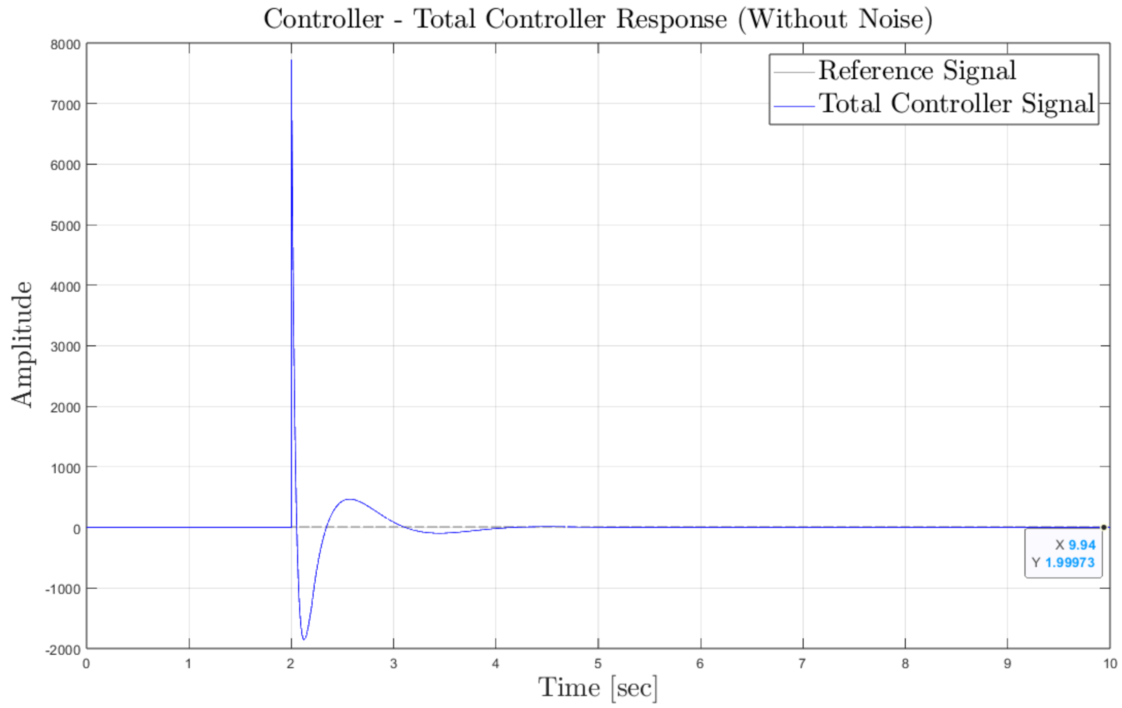
With the noise will shall have the following result:



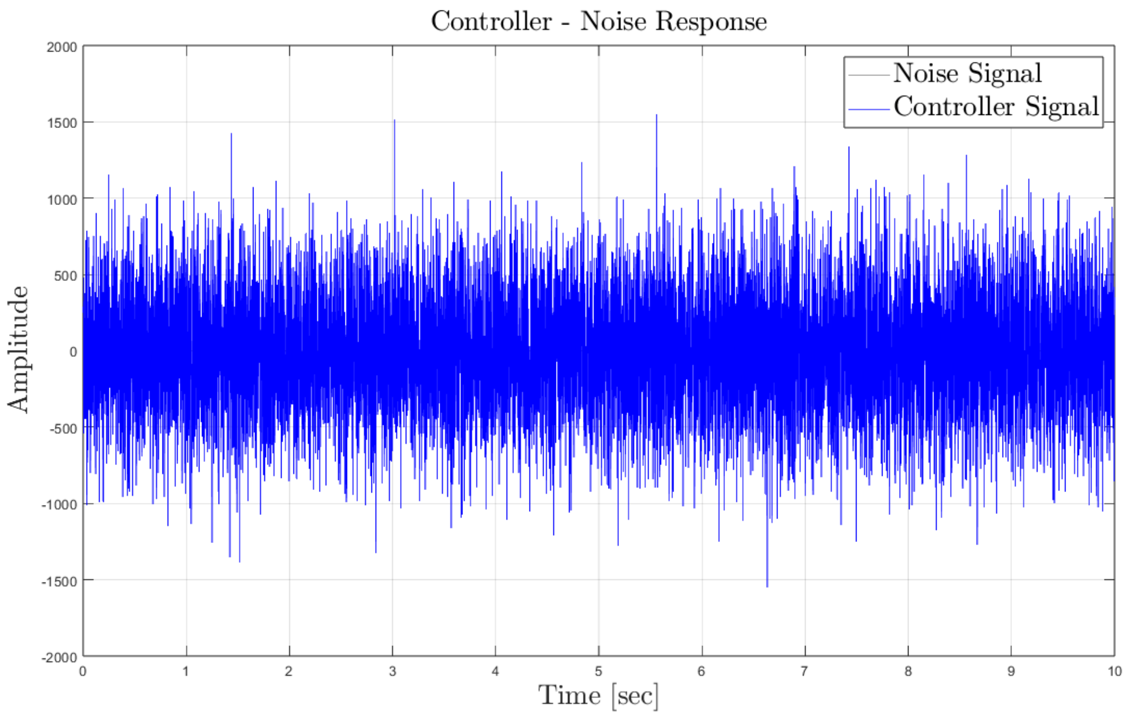
In the following page we shall show the result for the controller.


Controller - Reference Response
Reference Signal
Controller Signal
Amplitude
Time [sec]

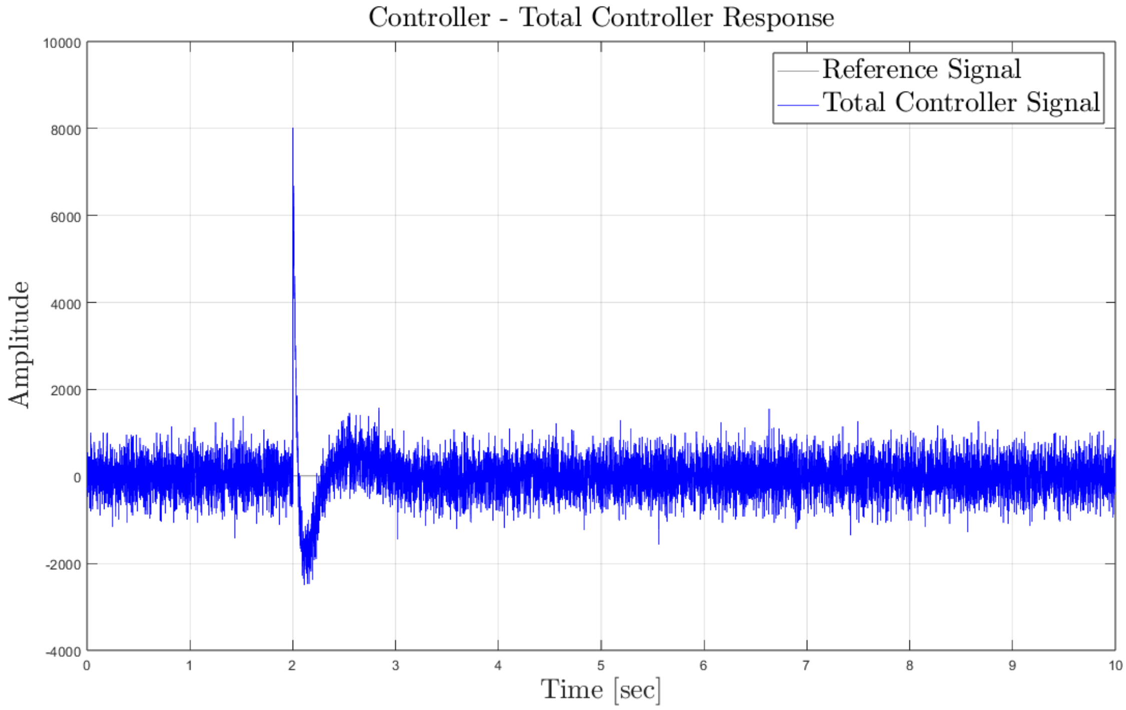

Controller - Disturbance Response
Disturbance Signal
Controller Signal
Amplitude
Time [sec]



Now we shall add the noise:



And with the noise we shall have the following response:



Problem 2 (15pt). Prove, using properties of the inner product, that $A'A \geq 0$ for all $A \in \mathbb{C}^{p\times m}$. What are conditions on A under which $A'A > 0$?

We know that:

$$A \in \mathbb{C}^{p\times m}$$

And can be written as such:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{p1} & \cdots & \cdots & a_{pm} \end{bmatrix}$$

We also know that the conjugate transpose is:

$$A' \in \mathbb{C}^{m\times p}$$

And can be written as such:

$$A' = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{p1}} \\ \overline{a_{12}} & \overline{a_{22}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \overline{a_{1m}} & \cdots & \cdots & \overline{a_{pm}} \end{bmatrix}$$

By using matrix multiplication we'll receive the following:

$$A'A = \begin{bmatrix} \overline{a_{11}}a_{11} + \overline{a_{21}}a_{21} + \cdots & \overline{a_{11}}a_{12} + \overline{a_{21}}a_{22} + \cdots & \cdots & \overline{a_{11}}a_{1m} + \overline{a_{21}}a_{2m} + \cdots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \overline{a_{1m}}a_{11} + \overline{a_{2m}}a_{12} + \cdots & \cdots & \cdots & \overline{a_{1m}}a_{1m} + \overline{a_{2m}}a_{2m} \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{i=1}^{p} \overline{a_{i1}}a_{i1} & \cdots & \sum_{i=1}^{p} \overline{a_{i1}}a_{im} \\ \vdots & \ddots & \vdots \\ \sum_{i,j=1}^{p,m} \overline{a_{im}}a_{1j} & \cdots & \sum_{i=1}^{p} \overline{a_{im}}a_{im} \end{bmatrix}$$

Whereas:

$$A'A \in \mathbb{C}^{m\times m}$$

We can see that every member of matrix $A'A$ is a sum of the inner product over the complex vector space:

$$\langle u, v\rangle = u\overline{v}$$

Because of the property that:

$$\langle u, v\rangle \geq 0$$

We can conclude that each member of the matrix, as we've mentioned before, is a sum of the inner products and each of them is greater than or equal to zero.
The following is, therefore, true:

$$A'A \geq 0$$

Let's call the matrix $A'A \rightarrow B$.
Now, to find the condition that will make B positive definite.
A matrix is called positive definite (in the complex vector space) if and only if:

$$x'Mx > 0$$

For all $x \in \mathbb{C}^n$ that are not zero.

The only way in which this will be possible is when the columns of B will be linear independent.

In this case no linear combination of columns can result in a zero vector (besides the trivial solution), meaning that:

$$x'Bx \neq 0$$

So:

$$B = A'A > 0$$

Problem 3 (15pt). Let $A_1 \in \mathbb{C}^{p_1 \times m}$ and $A_2 \in \mathbb{C}^{p_2 \times m}$. Prove that (here $\|\cdot\|$ stands for the spectral matrix norm)

- $\left\| \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \right\| < 1 \implies \|A_1\| < 1$ and $\|A_2\| < 1$
- $\|A_1\| < 1$ and $\|A_2\| < 1 \not\Longrightarrow \left\| \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \right\| < 1.$

Let's be reminded that the definition of the spectral matrix norm is the natural norm that is induced by the L2-norm.
More explicitly:
Which is:

$$\|A\|_2 = \sup_{\|u\|_2=1} \|Au\|_2 = \sqrt{\lambda_{max}(A'A)}$$

We shall define:

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = M \in \mathbb{C}^{p_1+p_2 \times m}$$

$$u \in \mathbb{C}^{m \times 1}$$

Where:

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{p_1 1} & a_{p_1 2} & \cdots & a_{p_1 m} \end{bmatrix} \quad A_2 = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ b_{p_2 1} & b_{p_2 2} & \cdots & b_{p_2 m} \end{bmatrix} \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

Now placing M at the norm definition results in:

$$\sup_{\|u\|_2=1} \left\| \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} u \right\|_2$$

Expanding it will result in:

$$\sup_{\|u\|_2=1} \left\| \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} u \right\|_2 = \sup_{\|u\|_2=1} \left\| \begin{bmatrix} A_1 u \\ A_2 u \end{bmatrix} \right\|_2$$

Where:

$$\begin{bmatrix} A_1 u \\ A_2 u \end{bmatrix} \in \mathbb{C}^{p_1+p_2 \times 1}$$

So:

$$\begin{bmatrix} A_1 u \\ A_2 u \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} a_{1i} u_i \\ \vdots \\ \sum_{i=1}^{m} a_{p_1 i} u_i \\ \hline \sum_{i=1}^{m} b_{1i} u_i \\ \vdots \\ \sum_{i=1}^{m} b_{p_2 i} u_i \end{bmatrix}$$

Let's mark the sums in the following way:

$$S_k^a = \sum_{i=1}^{m} a_{ki} u_i$$

$$S_k^b = \sum_{i=1}^{m} b_{ki} u_i$$

Using the definition of the L2-norm of the vector we can say that:

$$\sup_{\|u\|_2=1} \left\| \begin{bmatrix} A_1 u \\ A_2 u \end{bmatrix} \right\| = \sup_{\|u\|_2=1} \sqrt{S_1^{a2} + S_2^{a2} + \cdots + S_{p_1}^{a\ 2} + S_1^{b2} + S_2^{b2} + \cdots + S_{p_2}^{b\ 2}} < 1$$

As the right and the left sides are non-negative, we can square both sides, and use operation arithmetic:

$$\sup_{\|u\|_2=1} \underbrace{S_1^{a2} + S_2^{a2} + \cdots + S_{p_1}^{a\ 2}}_{\|A_1 u\|^2} + \sup_{\|u\|_2=1} \underbrace{S_1^{b2} + S_2^{b2} + \cdots + S_{p_2}^{b\ 2}}_{\|A_2 u\|^2} < 1$$

We achieve the following expression:

$$\sup_{\|u\|_2=1} \|A_1 u\|^2 + \sup_{\|u\|_2=1} \|A_2 u\|^2 < 1$$

Thus, we achieve:

$$\|A_1\|^2 + \|A_2\|^2 < 1$$

We got the unit circle; therefore, the following must be true:

$$\|A_1\| < 1 \ \& \ \|A_2\| < 1$$

To see that:

$$\|A_1\| < 1 \ \& \ \|A_2\| < 1$$

Doesn't mean that:

$$\left\| \begin{matrix} A_1 \\ A_2 \end{matrix} \right\| < 1$$

We'll find an example that it true, but to prove that it's not always the case we will find a counter example:

For:

$$A_1 = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.3 \end{bmatrix}$$

We get:

$$\|A_1\| = 0.316 < 1 \quad \|A_2\| = 0.51 < 1$$
$$\Downarrow$$
$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \left[\begin{array}{cc} 0.1 & 0.1 \\ 0.2 & 0.2 \\ \hline 0.2 & 0.2 \\ 0.3 & 0.3 \end{array} \right] \Longrightarrow \left\| \begin{matrix} A_1 \\ A_2 \end{matrix} \right\| = 0.6 < 1$$

But for:

$$A_1 = \begin{bmatrix} 0.6 & 0.6 \\ 0.1 & 0.1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.3 & 0.3 \\ 0.4 & 0.4 \end{bmatrix}$$

We get:

$$\|A_1\| = 0.86 < 1 \quad \|A_2\| = 0.707 < 1$$
$$\Downarrow$$
$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \left[\begin{array}{cc} 0.6 & 0.6 \\ 0.1 & 0.1 \\ \hline 0.3 & 0.3 \\ 0.4 & 0.4 \end{array} \right] \Longrightarrow \left\| \begin{matrix} A_1 \\ A_2 \end{matrix} \right\| = 1.1136 > 1$$

Problem 4 (21pt)**.** Consider the system $G : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -4 & 2 & 2 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

What is the

1. maximal achievable $\|y\|$ over all inputs in the region $\sqrt{4u_1^2 + u_2^2 + u_3^2/9} \le 1$?
2. maximal achievable $\|y\|_1$ over all inputs satisfying $|u_1 - u_2| + |u_2 - u_3| + |u_3| \le 1$?
3. minimal achievable $\|y\|$ over all inputs in the region $\sqrt{4u_1^2 + u_2^2 + u_3^2/9} = 1$?

Let's mark the system G, as such:

$$y = \underbrace{\begin{bmatrix} -4 & 2 & 2 \\ 2 & 1 & 0 \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{u}$$

1) We will be reminded first by the definition of the L2 induced norm:

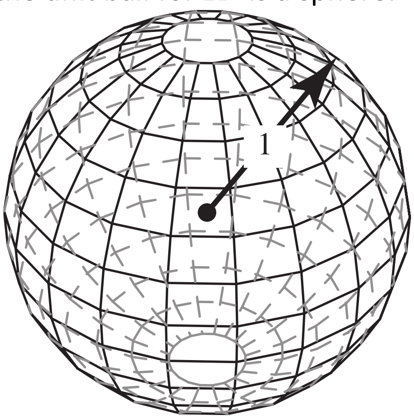
$$\|G\| = \sqrt{\lambda_{Max}(G'G)} = \sup_{\|v\|=1} \|Gv\|$$

Where:

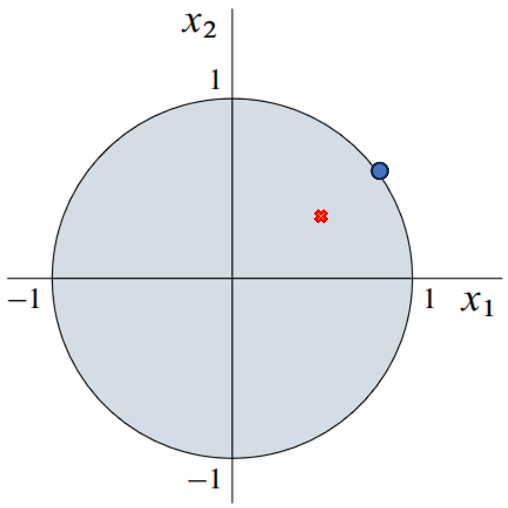
$$\|v\| = \sqrt{v_1^2 + v_2^2 + \dots + v_k^2} = 1$$

For our system $v = [v_1 \quad v_2 \quad v_3]^T$.

As per the definition (for v) the unit ball for $\mathbb{R}^3$ is a sphere:



We can use a section view of the sphere and see a circle:

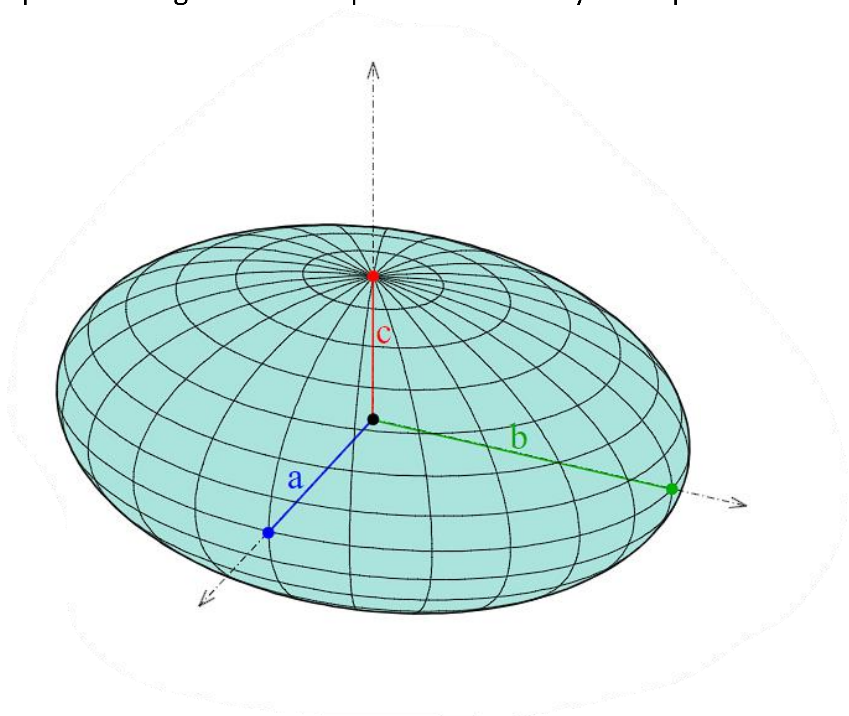


As we've seen from the lecture and from the definition of the L2 induced norm the maximal value is achieve by the constraint that the induced vector is on the circle and therefore generally on the surface of the sphere.

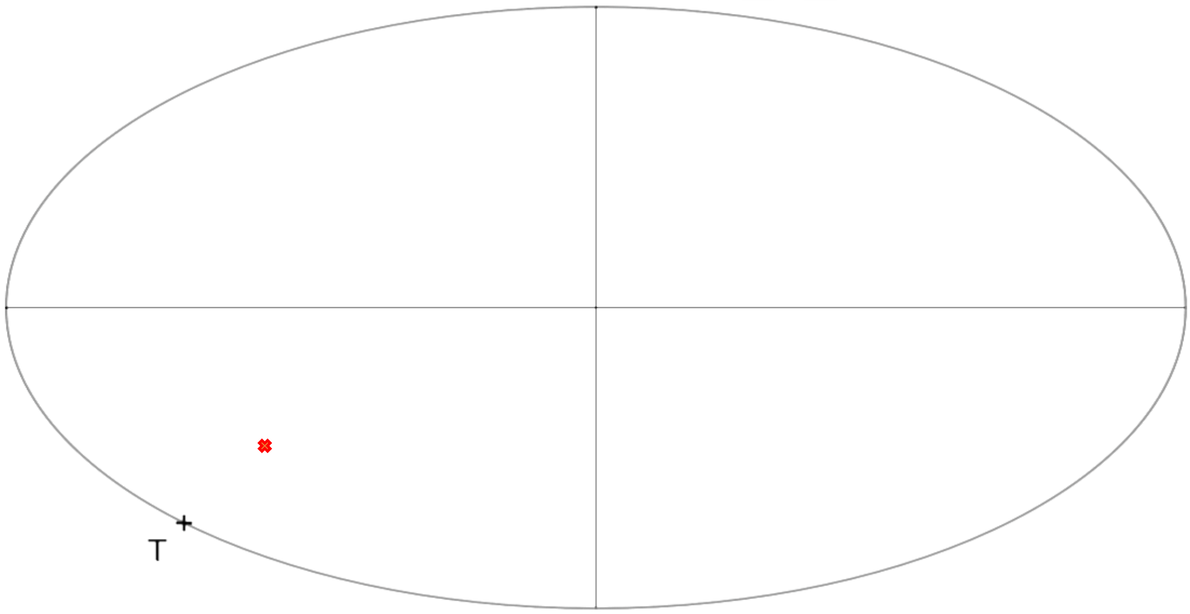
We see that if we choose a point that is "inside" the sphere (i.e. the red cross) it wont result with the maximal value of the norm, thus explaining why it has to be on the surface (the blue circle)

We will try to use the definition with the information that we've been provided with.

For u the shape of the region is not a sphere but actually an ellipsoid:



Let's see a section cut of the ellipsoid:



Again, from the case of the sphere and its section cut resulting in a circle, a section cut of the ellipsoid will give us an ellipse.

We can see that an induced vector that is chosen (i.e. the red cross) "inside" the ellipsoid ($\sqrt{4u_1^2 + u_2^2 + \frac{u_3^2}{9}} \leq 1$) doesn't give the maximal induced vector that will lead to the maximal value of the norm.

So, the maximal input region is on the surface of the ellipsoid, as such it needs to be:

$$\sqrt{4u_1^2 + u_2^2 + \frac{u_3^2}{9}} = 1$$

Now we can find a transformation matrix T that will transform our system to:

$$y = Gv$$

Where:

$$v = Tu$$

We see that when:

$$u_1 = \frac{1}{2}v_1 \quad u_2 = v_2 \quad u_3 = 3v_3$$

We get:

$$\sqrt{4u_1^2 + u_2^2 + \frac{u_3^2}{9}} = \sqrt{4\left(\frac{1}{2}v_1\right)^2 + v_2^2 + \frac{1}{9}(3v_3)^2} = \sqrt{v_1^2 + v_2^2 + v_3^2} = 1$$

So:

$$T = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Our entire system is then:

$$y = \tilde{G}v = GTv = \begin{bmatrix} -4 & 2 & 2 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Finally:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 6 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

And as we know that: $\|v\| = 1$

We can find the maximal eigenvalue of: $\tilde{G}'\tilde{G}$

Using MATLAB, we see that the system's eigenvalues are:

$$\lambda_1 = 0$$
$$\lambda_2 = 3.264$$
$$\lambda_3 = 25.73$$

According to the definition, the maximal achievable $\|y\|$ is:

$$\mathbf{max}\|\boldsymbol{y}\| = \sqrt{\mathbf{25.73}} = \mathbf{5.073}$$

2) The column sum ($\|G\|_1$) is defined as:

$$\|G\|_1 = \max_{1 \le j \le m} \sum_{i=1}^{p} |g_{ij}|$$

For our system it is:

$$\|y\|_1 = \left\| \begin{matrix} y_1 \\ y_2 \end{matrix} \right\|_1 = \left\| \begin{matrix} -4u_1 + 2u_2 + 2u_3 \\ 2u_1 + u_2 \end{matrix} \right\| = |-4u_1 + 2u_2 + 2u_3| + |2u_1 + u_2|$$

We know that $|u_1 - u_2| + |u_2 - u_3| + |u_3| \le 1$, let's use the following markings:

$$|u_1 - u_2| = a \ge 0$$
$$|u_2 - u_3| = b \ge 0$$
$$|u_3| = c \ge 0$$

So, we get:

$$a + b + c \le 1$$

We shall expand the expression:

$$\begin{aligned}
&= |-4u_1 - 4u_2 + 4u_2 + 2u_2 + 2u_3| + |2u_1 - 2u_2 + 2u_2 + u_2| = \\
&= |-4(u_1 - u_2) + 6u_2 - 6u_3 + 6u_3 + 2u_3| + |2(u_1 - u_2) + 3u_2 - 3u_3 + 3u_3| = \\
&= |-4(u_1 - u_2) + 6(u_2 - u_3) + 8u_3| + |2(u_1 - u_2) + 3(u_2 - u_3) + 3u_3| = \\
&= 4|u_1 - u_2| + 6|u_2 - u_3| + 8|u_3| + 2|u_1 - u_2| + 3|u_2 - u_3| + 3|u_3| = \\
&= 6a + 9b + 11c = 6a + 5a - 5a + 9b + 2b - 2b + 11c = \\
&11a + 11b + 11c - 5a - 2b \le 11a + 11b + 11c = 11\underbrace{(a + b + c)}_{\le 1} \le 11
\end{aligned}$$

As by the definition - $\|y\|_1 \ge 0$ and from what we've proven $\|y\|_1 \le 11$.

The following is true: $0 \le \|y\|_1 \le 11$.

So the maximal Achievable value is:

$$\|\boldsymbol{y}\|_{\mathbf{1}} = \mathbf{11}$$

3) We shall use the eigenvalues that we've found in the first clause.
It is known that the singular values of $\tilde{G}$ are the square roots of the eigenvalues of $\tilde{G}$.

As (from the lecture notes) $\min_{\|w\|=1} \|Gw\| = \underline{\sigma}$, (for us $w = v$) we see that the singular values of the system are:

$$\overline{\sigma} = 5.073$$
$$\sigma_2 = \sqrt{3.264} = 1.806$$
$$\underline{\sigma} = 0$$

So, the minimal achievable $\|y\|$ is:

$$\mathbf{min}\|\boldsymbol{y}\| = \mathbf{0}$$

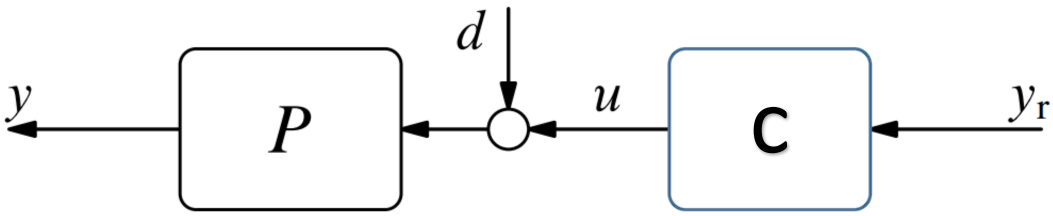
Problem 5 (24pt). Consider the control system in Fig. 1.4(b) of Lecture Notes for an $m \times m$ static plant P such that $\det P \neq 0$. Let the SVD of P be

$$P = U\Sigma V' \quad \text{for } U = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}, V = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix}, \text{and } \Sigma = \operatorname{diag}\{\sigma_i\}.$$

Suppose that signal sizes are measured by their Euclidean norms.

1. Design the controller C for which $y = r$ for all r whenever $d = 0$.
2. What is the direction of r, in terms of the singular vectors of P, for which the resulted control signal u is maximal? Minimal? Explain.
3. Let $r = 0$. What is the direction of d, in terms of the singular vectors of P, for which the resulted regulated output y is maximal? Minimal? Explain.
4. Let
$$P = \begin{bmatrix} 12 & 19 \\ 8 & 21 \end{bmatrix}.$$
Calculate normalized (i.e. such that $\|r\| = 1$) reference signals r for which the control signal u is maximal and minimal and present the resulted u's. Do the same for the disturbance signal with respect to maximal and minimal y and present the resulted y's.

We shall look at the following system:



1) We shall design the controller for this system using plant inversion.
The Control Law that we shall use is:
$$u = P^{-1}y_r$$
$(d = 0 \; and \; y_r = r)$

Therefore, the output of the system is:
$$y = Pu = PCr$$
Using the Control Law and to ensure that: $y = r$
The following controller we shall choose is:
$$C = P^{-1}$$
Using our knowledge on the SVD of P we shall find C in terms of the SVD.
$$C = P^{-1} = (U\Sigma V')^{-1} = V'^{-1}\Sigma^{-1}U^{-1}$$

Where (using unitary and diagonal properties):

$$V'^{-1} = V \;\; and \;\; U^{-1} = U'$$

$$\Sigma^{-1} = diag\left(\frac{1}{\sigma_m}\right)$$

However, we want to rearrange the singular values with accordance with their size, so:

$$\tilde{\Sigma}^{-1} = diag\left(\frac{1}{\sigma_m}, \frac{1}{\sigma_{m-1}}, \dots, \frac{1}{\sigma_1}\right)$$

As we've changed the order of the singular values we shall also change the order of the corresponding vectors of U and V, as such:

$$\tilde{V} = (v_k \quad v_{k-1} \quad \cdots \quad v_1) \quad \tilde{U} = (u_p \quad u_{p-1} \quad \cdots \quad u_1)$$

Our controller is:

$$\boldsymbol{C} = \boldsymbol{\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}'}$$

2) The control signal as we can see from the diagram is ($y_r = r$):

$$u = Cr$$

We've the knowledge that we've gained in the previous clause we can see that:

$$u = \tilde{V}\tilde{\Sigma}^{-1}\tilde{U}'r$$

We shall consider the gains and their direction by analyzing Σ^{-1} and its matching controls signal vector span.

Therefore, the resulted control signal is maximal for the largest singular value which is:

$$\frac{1}{\sigma_m}$$

It is to the direction of $span\{u_m\}$

Thus, with the same logic applied the signal is minimal for the smallest singular value which is:

$$\frac{1}{\sigma_1}$$

It is the direction of $span\{u_1\}$.

3) Now the output of the system is:

$$y = Pd$$

($r = 0$)

Again, with out knowledge of the SVD of P we see that:

$$y = U\Sigma V'd$$

As we've done in the previous clause, we can find the resulted output magnitude by analyzing the singular values and its matching disturbance signals.

Therefore, the maximal output's singular value is:

$$\sigma_1$$

It is in the direction of $span\{v_1\}$

The minimal output's singular value is:

$$\sigma_m$$

It is in the direction of $span\ \{v_m\}$.

4) We have the following plant:

$$P = \begin{bmatrix} 12 & 19 \\ 8 & 21 \end{bmatrix}$$

Using MATLAB, we will find the SVD of the plant:

$$U = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 31.62 & 0 \\ 0 & 3.162 \end{bmatrix}$$

$$V' = \begin{bmatrix} -04472 & -0.8944 \\ -0.8944 & 0.4472 \end{bmatrix}$$

We shall calculate as we've done in clause 1 the controller:

$$C = V\Sigma^{-1}U' = \tilde{V}\tilde{\Sigma}^{-1}\tilde{U}'$$

The corresponding matrices are:

$$V = \begin{bmatrix} -0.4472 & -0.8944 \\ -0.8944 & 0.4472 \end{bmatrix} \qquad \tilde{V} = \begin{bmatrix} -0.8944 & -0.4472 \\ 0.4472 & -0.8944 \end{bmatrix}$$

$$\Sigma^{-1} = \begin{bmatrix} 0.0316 & 0 \\ 0 & 0.3162 \end{bmatrix} \quad \Rightarrow \quad \tilde{\Sigma}^{-1} = \begin{bmatrix} 0.3162 & 0 \\ 0 & 0.0316 \end{bmatrix}$$

$$U' = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \tilde{U}' = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

With the conclusions we've made in the pervious clauses we can find the minimal and maximal signal and system outputs for $\|r\| = 1$.

For the maximal control signal for a normalized reference signal, for $u_2 = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 \\ -1 \end{bmatrix}$:

$$\boldsymbol{u}_{\mathbf{max}} = \begin{bmatrix} -0.8944 \\ 0.4472 \end{bmatrix} \cdot 0.3162 = \begin{bmatrix} \mathbf{-0.2828} \\ \mathbf{0.1414} \end{bmatrix}$$

We see that:

$$\|u_{\max}\| = 0.316$$

That corresponds to the singular value:

$$\frac{1}{\sigma_2} = 0.316$$

We shall calculate the minimal:

$$\boldsymbol{u}_{\mathbf{min}} = \begin{bmatrix} \mathbf{-0.0141} \\ \mathbf{-0.0283} \end{bmatrix}$$

Where:

$$\|u_{\min}\| = 0.03162$$

With:

$$\frac{1}{\sigma_1} = 0.03162$$

For $u_1 = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

We shall now find the output signal resulted maximal and minimal y, we remind that:

$$y = U\Sigma V'd$$

For the maximal output signal for a normalized disturbance signal, for $v_1 = \begin{bmatrix} -0.4472 \\ -0.8944 \end{bmatrix}$:

$$\boldsymbol{y}_{\mathbf{max}} = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 \\ -1 \end{bmatrix} \cdot 31.62 = \begin{bmatrix} \mathbf{-22.358} \\ \mathbf{-22.358} \end{bmatrix}$$

We see that:

$$\|y_{\max}\| = 31.622$$

That corresponds to the singular value:

$$\sigma_1 = 31.622$$

We shall calculate the minimal:

$$\boldsymbol{y}_{\mathbf{min}} = \begin{bmatrix} \mathbf{-2.236} \\ \mathbf{2.236} \end{bmatrix}$$

Where:

$$\|y_{\min}\| = 3.162$$

With:

$$\sigma_2 = 3.162$$

For $v_2 = \begin{bmatrix} -0.8944 \\ 0.4472 \end{bmatrix}$