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Output Consensus of Heterogeneous Uncertain Linear Multi-Agent Systems

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***Abstract*—This technical note studies the *output consensus* problem for a class of *heterogeneous uncertain* linear multi-agent systems. All the agents can be of any order (which might widely differ among the agents) and possess parametric uncertainties that range over an arbitrarily large compact set. The controller uses only the output information of the plant; moreover, the delivered information throughout the communication network is also restricted to the output of each agent. Based on the output regulation theory, it is shown that the output consensus is reached if the (state) consensus is achieved within the internal models among the agent's controllers (even though the plant's outputs, rather than the internal model's outputs, are communicated). The internal models can be designed and embedded into the controller, which provides considerable flexibility to designers in terms of the type of signals that are agreed on among the agents.**

***Index Terms*—Heterogeneous multi-agent systems, internal models, output consensus, output feedback.**

I. Introduction

For a decade, the consensus problem is actively studied due to its numerous applications such as cooperative control of unmanned aerial vehicles, communication among sensor networks, and formation of mobile robots (see [2], [3], [8]–[12], [14] and the references therein). While most of the results have focused on the homogeneous multi-agent systems, only a few papers [2], [9], [10] and a book [8] considered heterogeneous cases. In particular, Chopra and Spong [2] defined the output synchronization problem and presented a solution for weakly minimum phase nonlinear systems having relative degree one, and Qu *et al.* [8]–[10] solved an output consensus problem under a dynamically changing communication network. Although they geared up for output consensus problem about nonlinear heterogeneous systems, there are also some limitations such as: i) the agreement achieved among the agents is on a constant value [2], [9], [10] or a ramp signal [8] (using a virtual leader apporach), ii) uncertainties are not taken into account [2], [8]–[10], and iii) state information should be available to local-level controllers [2], [8]–[10].

In this technical note, although limited to linear SISO systems and fixed network topology, we newly propose a different route to handle the problem. First, by introducing an internal model into the consensus problem, we extend the class of signals that will be agreed among the agents. The synchronized signal can be any time-varying one if it is the output of an autonomous linear system $\dot{w} = Sw$. Next, it is shown that large uncertainties and the heterogeneity can be effectively dealt with by the output regulation theory (that are well established in [1], [4], [5], [13], [14]). In particular, all the agents can be of any order which might even be different among the agents and have parametric

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uncertainties that range over an arbitrarily large compact set. Finally, each controller of an agent uses output information only (of its own agent or its neighbors) so that the pure output feedback is achieved.

The underlying philosophy is similar to the output regulation problem [4]. While output regulation theory deals with a tracking control problem for unknown reference that is determined by an external exo-system, we suppose that the unknown reference is the outcome of the consensus (i.e., a weighted average) among many agents participating in a group network. Depending on the network topology, the opinion (or the internal states) of each agent is reflected in the desired reference. Just like an internal model is necessary in the controller to solve the output regulation problem, we embed an internal model into the controller of each agent. We show that the consensus is achieved among those internal models, not necessarily by exchanging the states of internal models, but just by exchanging the outputs of the heterogeneous uncertain agents.

Notation: A directed graph is denoted by $\mathcal{G}=(\mathcal{N},\mathcal{E})$, where $\mathcal{N}=\{1,2,\ldots,N\}$ is a finite nonempty set of nodes and $\mathcal{E}\subset\mathcal{N}\times\mathcal{N}$ is an edge set of ordered pairs of nodes. The adjacency matrix $\mathcal{A}=[\alpha_{ij}]\in\mathbb{R}^{N\times N}$ of a directed graph with a node set $\mathcal{N}$ is defined such that α_{ij} is a positive weight if $(j,i)\in\mathcal{E}$, while $\alpha_{ij}=0$ if $(j,i)\notin\mathcal{E}$. The Laplacian matrix $L=[l_{ij}]\in\mathbb{R}^{N\times N}$ of a directed graph is defined as $l_{ii}:=\sum_{j\neq i}\alpha_{ij}$ and $l_{ij}:=-\alpha_{ij}$ for all $i\neq j$. For a square matrix A, $\lambda_i(A)$ denotes the ith eigenvalue of A. For a matrix having complex entries, A^* and A^T stand for the Hermitian and the transpose of A, respectively. If it is necessary to explicitly designate the dimension of the matrix then the subscript is used. For example, I_n is the $n\times n$ identity matrix and $0_{m\times n}$ the $m\times n$ zero matrix. By abuse of notation $0_{\bullet\times n}$ is the zero matrix which has n columns. The number of rows of $0_{\bullet\times n}$ is determined from the context and not of interest. Finally, $\mathrm{col}(x_1,\ldots,x_N):=\begin{bmatrix}x_1^T & \cdots & x_N^T\end{bmatrix}^T$ and $\mathrm{diag}(A_1,\ldots,A_N)$ is the block diagonal matrix of which ith diagonal block is A_i.

II. Formulation of Output Consensus Problem

Consider a group of heterogeneous uncertain N plants given by

$$\begin{aligned}\dot{x}^i&=A^i(\mu^i)x^i+B^i(\mu^i)u^i,\quad i=1,\ldots,N\\ y^i&=C^i(\mu^i)x^i\end{aligned}\tag{1}$$

where $x^i\in\mathbb{R}^{n^i}$ is the state, $u^i\in\mathbb{R}$ the control input, $y^i\in\mathbb{R}$ the output of the ith plant, and the parametric uncertain vector μ^i ranges over a (arbitrarily large but known) compact subset $\mathcal{M}^i$ of $\mathbb{R}^{m^i}$ for all $i=1,\ldots,N$. The uncertainties are denoted with the superscript "i" to emphasize they are different among the agents. The network topology is assumed to be given and is represented by the adjacency matrix $\mathcal{A}$ or the Laplacian L.

For the multi-agent system (1) and the network topology, we consider the design problem of the output feedback controller written as

$$\begin{aligned}\dot{\zeta}^i&=F^i\zeta^i+G_1^iy^i+G_2^iv^i,\quad i=1,\ldots,N\\ u^i&=H^i\zeta^i+J_1^iy^i+J_2^iv^i\end{aligned}\tag{2}$$

where $\zeta^i\in\mathbb{R}^{p^i}$ and, with the set of neighbors of the ith agent $\mathcal{N}_i=\{j\in\mathcal{N}:\alpha_{ij}\neq 0\}$, v^i is given by $v^i=\sum_{j\in\mathcal{N}_i}\alpha_{ij}(y^j-y^i)=-\sum_{j\in\mathcal{N}}l_{ij}y^j$. It should be noted that the ith controller (2) uses only the output information of its own agent (y^i) and its neighbors (y^j for $j\in\mathcal{N}_i$). The goal of the controller is to guarantee that $\lim_{t\to\infty}\{y^i(t)-y^j(t)\}=0$ for all $i,j=1,\ldots,N$ and for any initial conditions $x^i(0)=x_0^i$ and $\zeta^i(0)=\zeta_0^i$.[1] This in turn implies a certain signal $\phi(t)$ exists such that $\lim_{t\to\infty}\{y^i(t)-\phi(t)\}=0$ for all i, and $\phi(t)$ is the outcome of the on-line consensus mechanism among the agents. In general, $\phi(t)$ depends on the network topology, plants, controllers, and their initial conditions. Since the overall system is linear, the synchronized signal $\phi(t)$ needs to be the output of a certain linear system, namely, $\dot{w}=Sw$ and $\phi=Rw$ where $w\in\mathbb{R}^q$ and the pair (S,R) is observable. While the initial condition $w(0)\in\mathbb{R}^q$ is not restricted, the dynamics of w is constrained to $\mathrm{Re}\{\lambda_j(S)\}=0$ for $j=1,\ldots,q$ in this technical note. The exclusion of the stable mode (i.e., eigenvalues having negative real part) is natural because they do not contribute the persistent signal $\phi(t)$, but the exclusion of eigenvalues having positive real part is for technical reason since the proposed scheme relies on the result of [12]. Nevertheless, we can still deal with all the persistent signals such as constant, sinusoidal, and so on, and the diverging signals in the polynomial order of time. The problem formulated so far is called *Output Consensus Problem via Output Information* (OCP for short) throughout the technical note.

One consequence of this formulation is that, the eigenvalues of the closed-loop system matrix

$$\bar{A}^i(\mu^i):=\begin{bmatrix}A^i(\mu^i)+B^i(\mu^i)J_1^iC^i(\mu^i) & B^i(\mu^i)H^i\\ G_1^iC^i(\mu^i) & F^i\end{bmatrix}$$

necessarily contain all the eigenvalues of S for all $\mu^i\in\mathcal{M}^i$ and $i=1,\ldots,N$ [7]. This fact in turn motivates us to embed the dynamics $\dot{w}=Sw$ into the controller (2) rather than require the containment of S within the uncertain plant. Indeed, it will be seen that this embedding strategy allows the large uncertainties and heterogeneity of the agents.

III. Main Result

First of all, we state two standing assumptions; one for each agent and the other for the network topology.

Assumption 1: (i) Each plant described by (1) has a well-defined relative degree r^i and is of minimum phase for all $\mu^i\in\mathcal{M}^i$. (ii) The weighted directed graph for the network topology contains a directed spanning tree [11].

From Assumption 1.(i), the high frequency gain $C^i(\mu^i)\{A^i(\mu^i)\}^{r^i-1}B^i(\mu^i)$ does not vanish for any $\mu^i\in\mathcal{M}^i$. Without loss of generality, we assume that the high frequency gain for each agent is positive (otherwise, reverse the sign of the input). Note that Assumption 1.(ii) is to make sure that the Laplacian L of the given network topology has only one zero eigenvalue, namely, $\lambda_1(L)=0$ and $\lambda_i(L)\neq 0$ for $i=2,\ldots,N$ [11].

For now, let us consider a group of auxiliary systems that is called as group of *type generators* throughout the technical note, dependent on the type of the signal $\phi(t)$ to be synchronized, and given by

$$\begin{aligned}\dot{w}^i&=Sw^i+QU^i,\quad i=1,\ldots,N\\ Y^i&=Rw^i\end{aligned}\tag{3}$$

where $w^i\in\mathbb{R}^q$, $U^i\in\mathbb{R}$, $Y^i\in\mathbb{R}$, and $\mathrm{Re}\{\lambda_j(S)\}=0$ for $j=1,\ldots,q$. The triplet (S,Q,R) is assumed to be controllable and observable. Here S and R are chosen (by the designer) from the specification of the problem, that is, from the type of signals to be made consensus on, while Q is freely chosen.

Assumption 2: For the given controllable and observable (S,Q,R), there exist matrices $A_\eta\in\mathbb{R}^{s\times s}$, $B_\eta\in\mathbb{R}^{s\times 1}$, $C_\eta\in\mathbb{R}^{1\times s}$, and $D_\eta\in\mathbb{R}^{1\times 1}$ such that A_η is Hurwitz and the complex matrix

$$A_{w,cl}^i:=\begin{bmatrix}S-\lambda_i(L)QD_\eta R & QC_\eta\\ -\lambda_i(L)B_\eta R & A_\eta\end{bmatrix}$$

is Hurwitz for all $\lambda_i(L)\neq 0$.

[1]Due to the uncertainty and heterogeneity, the state consensus (i.e., $\lim_{t\to\infty}\{x^i(t)-x^j(t)\}=0$) is generally impossible.

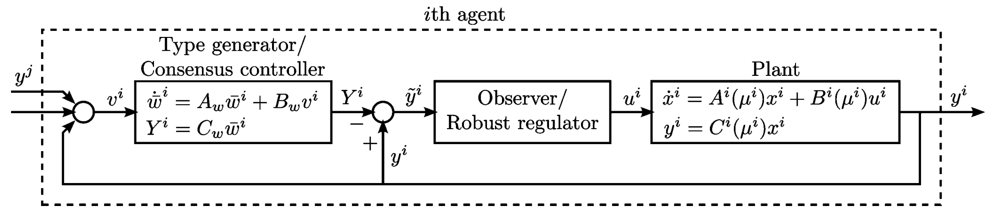

Fig. 1. Block diagram of the proposed scheme (7) consisting of type generator/consensus controller (7a), dirty derivative observer (7b), and robust regulator (7c).

We emphasize that Assumption 2 is not a restriction[2] because we can always take

$$\begin{aligned} A_\eta &= S - QQ^T P(\epsilon) + MR \\ B_\eta &= -M \\ C_\eta &= Q^T P(\epsilon) \\ D_\eta &= 0 \end{aligned} \tag{4}$$

where M is chosen so as to $(S + MR)$ being Hurwitz and $P(\epsilon) = P^T(\epsilon) > 0$ is the solution of $S^T P + PS - \lambda^\star PQQ^T P + \epsilon I = 0$ with $\lambda^\star := \min_{i=2,\ldots,N} \mathrm{Re}\{\lambda_i(L)\} > 0$ and $\epsilon > 0$ sufficiently small (see [12] for the proof). Although we use the choice of (4) in this technical note, we put it as an assumption in order to leave possibility to use different matrices.

Remark 1: The implication of the above argument is that, when the given Laplacian L represents a network topology that contains a directed spanning tree, the (state) consensus of the type generators (3) [not for (1)] is achieved with a stable compensator

$$\begin{aligned} \dot{\eta}^i &= A_\eta \eta^i + B_\eta V^i, \quad i = 1, \ldots, N \\ U^i &= C_\eta \eta^i + D_\eta V^i \end{aligned} \tag{5}$$

and with the interconnection $V^i = \sum_{j\in\mathcal{N}_i} \alpha_{ij}(Y^j - Y^i) = -\sum_{j\in\mathcal{N}} l_{ij} Y^j$. See [12] for details.

In the following, the type generator (3) and the compensator (5) are embedded into the individual agent's controller. However, it is noted that the signal V^i in (5) is not available to each controller because the controller of the ith agent can not access the signal w^j for $j \neq i$. Instead, the outputs of its neighboring agents y^j for $j \in \mathcal{N}_i$, as well as its own state w^i and output y^i, are available to the ith agent's controller. Therefore, what will be embedded into the final controller is the combination of (3) and (5) with V^i replaced by $v^i = -\sum_{j\in\mathcal{N}} l_{ij} y^j$

$$\begin{aligned} \dot{\bar{w}}^i &= A_w \bar{w}^i + B_w v^i := \begin{bmatrix} S & QC_\eta \\ 0 & A_\eta \end{bmatrix} \bar{w}^i + \begin{bmatrix} QD_\eta \\ B_\eta \end{bmatrix} v^i \\ Y^i &= C_w \bar{w}^i := [\,R \quad 0\,]\, \bar{w}^i \end{aligned} \tag{6}$$

for $i = 1, \ldots, N$, where $\bar{w}^i := \mathrm{col}(w^i, \eta^i) \in \mathbb{R}^{q+s}$. The mismatch caused by the replacement of V^i with v^i will be dealt with by the proposed controller in the following theorem.

Theorem 1: Under Assumptions 1 and 2, there exists a controller of the form (2) so that the solution of the closed-loop system satisfies that $\lim_{t\to\infty} \{y^i(t) - Re^{St} w_0\} = 0$ for all $i = 1, \ldots, N$, where $w_0 \in \mathbb{R}^q$ is a weighted average of the initial conditions $w^i(0)$ and $\eta^i(0)$. More specifically, the OCP is solved by the controller (see Fig. 1)

$$\dot{\bar{w}}^i = A_w \bar{w}^i + B_w v^i \tag{7a}$$

$$\dot{\hat{y}}^i = A_o^i \hat{y}^i + B_o^i \left(y^i - C_w \bar{w}^i - \hat{y}_1^i\right) \tag{7b}$$

$$\dot{\xi}^i = \Phi \xi^i - JK\Big(k^{r^i-1} c_0^i \hat{y}_1^i + k^{r^i-2} c_1^i \hat{y}_2^i + \cdots + k c_{r^i-2}^i \hat{y}_{r^i-1}^i + \hat{y}_{r^i}^i\Big) \tag{7c}$$

$$u^i = H\xi^i - K\Big(k^{r^i-1} c_0^i \hat{y}_1^i + k^{r^i-2} c_1^i \hat{y}_2^i + \cdots + k c_{r^i-2}^i \hat{y}_{r^i-1}^i + \hat{y}_{r^i}^i\Big) \tag{7d}$$

for all $k > k^\star$, $K > K^\star(k)$, and $g > g^\star(k, K)$, where $v^i = \sum_{j\in\mathcal{N}_i} \alpha_{ij}(y^j - y^i) = -\sum_{j\in\mathcal{N}} l_{ij} y^j$

$$\Phi = \begin{bmatrix} -\beta_{p-1} & 1 & 0 & \cdots & 0 \\ -\beta_{p-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\beta_1 & 0 & 0 & \cdots & 1 \\ -\beta_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T$$

$$A_o^i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B_o^i = \begin{bmatrix} g d_{r^i-1}^i \\ g^2 d_{r^i-2}^i \\ \vdots \\ g^{r^i-1} d_1^i \\ g^{r^i} d_0^i \end{bmatrix} \tag{8}$$

the vector J is chosen so that $(\Phi - JH)$ is Hurwitz, β_j's are the coefficients of the minimal polynomial of S, the polynomials $f_c^i(s) = s^{r^i-1} + c_{r^i-2}^i s^{r^i-2} + \cdots + c_1^i s + c_0^i$ and $f_d^i(s) = s^{r^i} + d_{r^i-1}^i s^{r^i-1} + \cdots + d_1^i s + d_0^i$ are Hurwitz for all $i = 1, \ldots, N$.

It will be seen from the proof that the proposed design provides much flexibility. Indeed, when all the plants (1) has (robust) relative degree 1, the use of the dirty derivative observer (7b) [6] can be avoided. In this case, the controller (7) is simply given by

$$\dot{\bar{w}}^i = A_w \bar{w}^i + B_w v^i \tag{9a}$$

$$\dot{\xi}^i = \Phi \xi^i - JK\left(y^i - C_w \bar{w}^i\right) \tag{9b}$$

$$u^i = H\xi^i - K\left(y^i - C_w \bar{w}^i\right). \tag{9c}$$

On the other hand, if there is no uncertainty (i.e., μ^i's are known) for the heterogeneous plants (1), the use of robust regulator (7c) can be avoided by replacing the control (7d) with

$$u^i = \Gamma^i w^i - K\Big(k^{r^i-1} c_0^i \hat{y}_1^i + k^{r^i-2} c_1^i \hat{y}_2^i + \cdots + k c_{r^i-2}^i \hat{y}_{r^i-1}^i + \hat{y}_{r^i}^i\Big) \tag{7d*}$$

[2]Note that, in OCP, we assumed $\mathrm{Re}\{\lambda_j(S)\} = 0$. This condition could be relaxed to $\mathrm{Re}\{\lambda_j(S)\} \geq 0$. But in this case, Assumption 2 is no longer ensured by (4), because the result of [12] holds for S having no eigenvalue in the open right-half complex plane $\mathcal{C}^+$. However once Assumption 2 holds with S, having eigenvalues in $\mathcal{C}^+$, then the result of this technical note still holds.

where $\Gamma^i = \Gamma^i(\mu^i)$ is obtained by solving the regulator equation $\Pi^i(\mu^i)S = A^i(\mu^i)\Pi^i(\mu^i) + B^i(\mu^i)\Gamma^i(\mu^i)$ and $C^i(\mu^i)\Pi^i(\mu^i) - R = 0$ for $\Pi^i(\mu^i)$ and $\Gamma^i(\mu^i)$.[3] Finally, if all the agents are identical (i.e., homogeneous) without any uncertainties, then the state consensus, as well as the output consensus, is achieved. Indeed, it holds that $x^i(t) \to \Pi e^{St} w_0$ where $\Pi = \Pi^i(\mu^i)$ for all i.

Remark 2: The constants k, K, and g in Theorem 1 are explicitly obtained from the norm bounds of certain quantities, e.g., the inequalities in (13). Since this is a tedious job, one may prefer to select them from repeated simulations by increasing them sequentially.

Remark 3: The statement of Theorem 1 indicates that the vector w_0 does not depend on the initial conditions of the plant, $x^i(0)$, and this may seem unnatural. It is because the controller enforces the output to track a certain command, which is not known *a priori* but is the outcome of the consensus on the state w^i and η^i among the agents. However, in order to truly reflect the initial conditions of the plants, the internal state w^i of each agent's controller can be set, at start-up, to be such that $y^i(0) = C^i(\mu^i)x^i(0) = Rw^i(0)$ with the initial measurement $y^i(0)$.[4] In fact, this is not a problem if each agent is working individually (before joining the network) under its own internal model $\dot{w}^i = Sw^i$ with $\eta^i(t) = 0$, and enters the network (at time $t = 0$) with their previous states for $w^i(0)$ and $\eta^i(0)$.

PROOF OF THEOREM 1

In order to present the main ingredient of Theorem 1 more effectively, we first consider the simpler case where all the agents have the same relative degree 1, so that the controller of the form (9) is enough. The proof for the case of higher relative degree is in Section III-B.

A. Case of Relative Degree One

Let us suppose that $r^1 = \cdots = r^N = 1$. Then, the plant (1) is put into the normal form

$$\begin{aligned}\dot{z}^i &= A_{00}^i(\mu^i)z^i + A_{01}^i(\mu^i)y^i, \quad i = 1,\ldots,N\\ \dot{y}^i &= A_{10}^i(\mu^i)z^i + a_{11}^i(\mu^i)y^i + b^i(\mu^i)u^i\end{aligned}$$

where $z^i \in \mathbb{R}^{n^i-1}$ and $b^i(\mu^i) \geq b_0 > 0$ for all $\mu^i \in \mathcal{M}^i$. Since the matrix $A_{00}^i(\mu^i)$ is Hurwitz for all $\mu^i \in \mathcal{M}^i$ by Assumption 1.(i), there is a matrix $\Pi_z^i(\mu^i) \in \mathbb{R}^{(n^i-1)\times q}$ such that it satisfies the Sylvester equation $\Pi_z^i(\mu^i)S = A_{00}^i(\mu^i)\Pi_z^i(\mu^i) + A_{01}^i(\mu^i)R$ for $i = 1,\ldots,N$. By defining the new coordinates $\tilde{y}^i := y^i - Y^i = y^i - C_w\bar{w}^i = y^i - Rw^i$ and $\tilde{z}^i := z^i - \Pi_z^i(\mu^i)w^i$ for $i = 1,\ldots,N$, the system is again represented by

$$\begin{aligned}\dot{\tilde{z}}^i &= A_{00}^i(\mu^i)\tilde{z}^i + A_{01}^i(\mu^i)\tilde{y}^i - \Pi_z^i(\mu^i)Q\left(C_\eta\eta^i + D_\eta v^i\right)\\ \dot{\tilde{y}}^i &= A_{10}^i(\mu^i)\tilde{z}^i + a_{11}^i(\mu^i)\tilde{y}^i + b^i(\mu^i)[u^i - \Gamma^i(\mu^i)w^i]\\ &\quad - RQ\left(C_\eta\eta^i + D_\eta v^i\right)\end{aligned} \tag{10}$$

where $\Gamma^i(\mu^i) = -(1/b^i(\mu^i))\left\{A_{10}^i(\mu^i)\Pi_z^i(\mu^i) + a_{11}^i(\mu^i)R - RS\right\}$. The coordinates $\tilde{z}^i$ and $\tilde{y}^i$ are nothing but the deviations of z^i and y^i from the signals $\Pi_z^i(\mu^i)w^i$ and Rw^i, respectively.

[3]It is well-known that the solution exists if Assumption 1.(i) holds [1], [5].

[4]Or, for example, if an agent such as a point-mass robot wants to take part in the network with its initial position $y^i(0)$ and velocity $\dot{y}^i(0)$ to be considered in the consensus signal $\phi(t)$, it is possible to run an estimator (like the dirty derivative observer [6]) to estimate $\dot{y}^i$ before entering the network at time $t = 0$, and set $w^i(0)$ so that $y^i(0) = Rw^i(0)$ and $\dot{y}^i(0) = RSw^i(0)$.

Now, let $\Pi_\xi^i(\mu^i) := \mathrm{col}\left\{\Gamma^i(\mu^i), \Gamma^i(\mu^i)(S + \beta_{p-1}I), \ldots, \Gamma^i(\mu^i)\left(S^{p-1} + \beta_{p-1}S^{p-2} + \cdots + \beta_1 I\right)\right\}$. Then, it follows from (8) that $\Pi_\xi^i(\mu^i)S = \Phi\Pi_\xi^i(\mu^i)$ and $\Gamma^i(\mu^i) = H\Pi_\xi^i(\mu^i)$.[5] Define a coordinate change $\tilde{\xi}^i := \xi^i - \Pi_\xi^i(\mu^i)w^i - (1/b^i(\mu^i))J\tilde{y}^i$, a new variable $\tilde{\sigma}^i := \mathrm{col}(\tilde{\xi}^i, \tilde{z}^i) \in \mathbb{R}^{p+n^i-1}$, and

$$\begin{aligned}A_\sigma^i(\mu^i) &= \begin{bmatrix}\Phi - JH & -\frac{1}{b^i(\mu^i)}JA_{10}^i(\mu^i)\\ 0_{(n-1)\times p} & A_{00}^i(\mu^i)\end{bmatrix}\\ B_{\sigma y}^i(\mu^i) &= \begin{bmatrix}\frac{1}{b^i(\mu^i)}\left\{\Phi - a_{11}^i(\mu^i)I - JH\right\}J\\ A_{01}^i(\mu^i)\end{bmatrix}\\ B_{\sigma w}^i(\mu^i) &= \begin{bmatrix}0_{\bullet\times q} & -\left\{\Pi_\xi^i(\mu^i) - \frac{1}{b^i(\mu^i)}JR\right\}QC_\eta\\ 0_{\bullet\times q} & -\Pi_z^i(\mu^i)QC_\eta\end{bmatrix}\\ B_\sigma^i(\mu^i) &= \begin{bmatrix}-\left\{\Pi_\xi^i(\mu^i) - \frac{1}{b^i(\mu^i)}JR\right\}QD_\eta\\ -\Pi_z^i(\mu^i)QD_\eta\end{bmatrix}\\ B_{y\sigma}^i(\mu^i) &= [\,b^i(\mu^i)H \quad A_{10}^i(\mu^i)\,]\\ a_y^i(\mu^i) &= a_{11}^i(\mu^i) + HJ\\ B_{yw} &= [\,0_{\bullet\times q} \quad -RQC_\eta\,], \quad b_y = -RQD_\eta.\end{aligned}$$

Then the overall dynamics of the ith agent, constituted by (9) and (10), is concisely written as

$$\begin{aligned}\dot{\bar{w}}^i &= A_w\bar{w}^i + B_w v^i\\ \dot{\tilde{\sigma}}^i &= B_{\sigma w}^i(\mu^i)\bar{w}^i + A_\sigma^i(\mu^i)\tilde{\sigma}^i + B_{\sigma y}^i(\mu^i)\tilde{y}^i + B_\sigma^i(\mu^i)v^i\\ \dot{\tilde{y}}^i &= B_{yw}\bar{w}^i + B_{y\sigma}^i(\mu^i)\tilde{\sigma}^i\\ &\quad + \left\{a_y^i(\mu^i) - b^i(\mu^i)K\right\}\tilde{y}^i + b_y v^i\end{aligned}$$

with $v^i = -\sum_{j\in\mathcal{N}} l_{ij}y^j = -\sum_{j\in\mathcal{N}} l_{ij}C_w\bar{w}^j - \sum_{j\in\mathcal{N}} l_{ij}\tilde{y}^j$.

Now let $T \in \mathbb{C}^{N\times N}$ be a nonsingular matrix such that $T^{-1}LT = \Lambda$ where Λ is in a Jordan form. Without loss of generality, assume the matrices T and T^{-1} are of the form $T = [\,1_N \quad T_1\,]$ and $T^{-1} = \mathrm{col}(r^T, T_2)$, where r^T is the left eigenvector of L corresponding to $\lambda_1(L) = 0$ with $r^T 1_N = 1$, 1_N is the $N \times 1$ vector of all ones, and the matrices T_1 and T_2 are appropriately defined. Note that $T^{-1}LT = \Lambda = \mathrm{diag}(0, \Lambda_1)$ and Λ_1 is upper triangular with nonzero diagonal elements which follows from Assumption 1.(ii). With the stacked coordinates $\bar{w} = \mathrm{col}(\bar{w}^1,\ldots,\bar{w}^N)$, $\tilde{\sigma} = \mathrm{col}(\tilde{\sigma}^1,\ldots,\tilde{\sigma}^N)$, and $\tilde{y} = \mathrm{col}(\tilde{y}^1,\ldots,\tilde{y}^N)$, consider a change of coordinate $\varphi = \mathrm{col}(\varphi^1,\ldots,\varphi^N) = (T^{-1}\otimes I_{q+s})\bar{w}$ where $\varphi^i = \mathrm{col}(\varphi_1^i, \varphi_2^i)$, $\varphi_1^i \in \mathbb{C}^q$, and $\varphi_2^i \in \mathbb{C}^s$. From $\dot{\bar{w}} = \{(I_N\otimes A_w) - (L\otimes B_wC_w)\}\bar{w} - (L\otimes B_w)\tilde{y}$, the dynamics of φ becomes

$$\begin{aligned}\dot{\varphi} &= \{(I_N\otimes A_w) - (\Lambda\otimes B_wC_w)\}\varphi - (T^{-1}L\otimes B_w)\tilde{y}\\ &= \begin{bmatrix}A_w & 0\\ 0 & (I_{N-1}\otimes A_w) - (\Lambda_1\otimes B_wC_w)\end{bmatrix}\varphi\\ &\quad - \left(\begin{bmatrix}0\\ \Lambda_1T_2\end{bmatrix}\otimes B_w\right)\tilde{y}\end{aligned}$$

[5]The meaning of two equations is that the robust regulator (9b) and (9c) has the potential to generate the unknown signal $\Gamma^i(\mu^i)w^i$ in the steady-state. (In the steady-state, we will have that $\tilde{y}^i(t) = y^i(t) - C_w\bar{w}^i(t) = 0$, $\eta^i(t) = 0$, and $v^i(t) = 0$.) Indeed, if $\xi^i(0) = \Pi_\xi^i(\mu^i)w^i(0)$, then $\xi^i(t) = \Pi_\xi^i(\mu^i)w^i(t)$ in the steady-state (which is seen by taking time derivative under (9b) and (6)). Hence, we have that $u^i(t) = H\Pi_\xi^i(\mu^i)w^i(t) = \Gamma^i(\mu^i)w^i(t)$. This property in fact comes from the internal model principle [1], [4], [5], which will finally enable $\tilde{z}^i(t)$ and $\tilde{y}^i(t)$ to remain zero in (10).

whose second equation is obtained from $T^{-1}L = \Lambda T^{-1}$. Let l_j and $T_{1,j}$ denote the jth rows of L and T_1 respectively. Then, using $LT = T\Lambda$, the dynamics of $\tilde{\sigma}$ and $\tilde{y}$ are obtained by

$$\begin{aligned}\dot{\tilde{\sigma}} =& \operatorname{diag}\left\{B^1_{\sigma w}(\mu^1),\ldots,B^N_{\sigma w}(\mu^N)\right\}(T\otimes I_{q+s})\varphi\\ &+\operatorname{diag}\left\{A^1_{\sigma}(\mu^1),\ldots,A^N_{\sigma}(\mu^N)\right\}\tilde{\sigma}\\ &+\operatorname{diag}\left\{B^1_{\sigma y}(\mu^1),\ldots,B^N_{\sigma y}(\mu^N)\right\}\tilde{y}\\ &-\begin{bmatrix}[0\ T_{1,1}\Lambda_1]\otimes B^1_\sigma(\mu^1)C_w\\ \vdots\\ [0\ T_{1,N}\Lambda_1]\otimes B^N_\sigma(\mu^N)C_w\end{bmatrix}\varphi-\begin{bmatrix}l_1\otimes B^1_\sigma(\mu^1)\\ \vdots\\ l_N\otimes B^N_\sigma(\mu^N)\end{bmatrix}\tilde{y}\\ \dot{\tilde{y}} =& (I_N\otimes B_{yw})(T\otimes I_{q+s})\varphi\\ &+\operatorname{diag}\left\{B^1_{y\sigma}(\mu^1),\ldots,B^N_{y\sigma}(\mu^N)\right\}\tilde{\sigma}\\ &+\operatorname{diag}\left\{a^1_y(\mu^1)-b^1(\mu^1)K,\ldots,a^N_y(\mu^N)-b^N(\mu^N)K\right\}\tilde{y}\\ &-([0_{\bullet\times 1}\quad T_1\Lambda_1]\otimes b_yC_w)\varphi-(L\otimes b_y)\tilde{y}.\end{aligned}$$

Since the first q columns of $B^i_{\sigma w}(\mu^i)$ and B_{yw} are all zeros, so the first q columns of huge matrices $\operatorname{diag}\{B^1_{\sigma w}(\mu^1),\ldots,B^N_{\sigma w}(\mu^N)\}(T\otimes I_{q+s})$ and $(I_N\otimes B_{yw})(T\otimes I_{q+s})$ are, thanks to the identity matrix I_{q+s}. As a result there are the matrices $\bar{F}_1(\mu)\in\mathbb{R}^{E\times s}$, $\bar{F}_2(\mu)\in\mathbb{R}^{E\times(N-1)(q+s)}$, $\bar{F}_3\in\mathbb{R}^{N\times s}$, and $\bar{F}_4\in\mathbb{R}^{N\times(N-1)(q+s)}$ so that those two huge matrices are expressed as $[0_{\bullet\times q}\ \bar{F}_1(\mu)\ \bar{F}_2(\mu)]$ and $[0_{\bullet\times q}\ \bar{F}_3\ \bar{F}_4]$ respectively, where $E:=\sum_{i=1}^N(p+n^i-1)$ and $\mu:=\operatorname{col}(\mu^1,\ldots,\mu^N)\in\mathcal{M}:=\mathcal{M}^1\times\cdots\times\mathcal{M}^N$. With the new notation $\chi=\operatorname{col}\left(\chi_1,\chi_2,\chi_3,\chi_4,\chi_5\right):=\operatorname{col}(\varphi^1_1,\varphi^1_2,\bar{\varphi},\tilde{\sigma},\tilde{y})$ where $\bar{\varphi}=\operatorname{col}(\varphi^2,\ldots,\varphi^N)$, the overall dynamics is compactly written as

$$\dot{\chi}=\begin{bmatrix}S & QC_\eta & 0 & 0 & 0\\ 0 & A_\eta & 0 & 0 & 0\\ 0 & 0 & A_3 & 0 & B_{35}\\ 0 & B_{42}(\mu) & B_{43}(\mu) & A_4(\mu) & B_{45}(\mu)\\ 0 & B_{52} & B_{53} & B_{54}(\mu) & A_5(\mu)-B_5(\mu)K\end{bmatrix}\chi \quad (11)$$

in which $A_3=(I_{N-1}\otimes A_w)-(\Lambda_1\otimes B_wC_w)$, $B_{35}=-(\Lambda_1T_2\otimes B_w)$, $B_{42}=\bar{F}_1(\mu)$, $B_{43}(\mu)=\bar{F}_2(\mu)-\operatorname{col}\{T_{1,1}\Lambda_1\otimes B^1_\sigma(\mu^1)C_w,\ldots,T_{1,N}\Lambda_1\otimes B^N_\sigma(\mu^N)C_w\}$, $A_4(\mu)=\operatorname{diag}\{A^1_\sigma(\mu^1),\ldots,A^N_\sigma(\mu^N)\}$, $B_{45}(\mu)=\operatorname{diag}\{B^1_{\sigma y}(\mu^1),\ldots,B^N_{\sigma y}(\mu^N)\}-\operatorname{col}\{l_1\otimes B^1_\sigma(\mu^1),\ldots,l_N\otimes B^N_\sigma(\mu^N)\}$, $B_{52}=\bar{F}_3$, $B_{53}=\bar{F}_4-(T_1\Lambda_1\otimes b_yC_w)$, $B_{54}(\mu)=\operatorname{diag}\{B^1_{y\sigma}(\mu^1),\ldots,B^N_{y\sigma}(\mu^N)\}$, $A_5(\mu)=\operatorname{diag}\{a^1_y(\mu^1),\ldots,a^N_y(\mu^N)\}-(L\otimes b_y)$, and $B_5(\mu)=\operatorname{diag}\{b^1(\mu^1),\ldots,b^N(\mu^N)\}$.

Now we claim that the subsystem formed by the substate $\operatorname{col}(\chi_3,\chi_4,\chi_5)$ of (11), namely

$$\begin{bmatrix}\dot{\chi}_3\\ \dot{\chi}_4\\ \dot{\chi}_5\end{bmatrix}=\begin{bmatrix}A_3 & 0 & B_{35}\\ B_{43}(\mu) & A_4(\mu) & B_{45}(\mu)\\ B_{53} & B_{54}(\mu) & A_5(\mu)-B_5(\mu)K\end{bmatrix}\begin{bmatrix}\chi_3\\ \chi_4\\ \chi_5\end{bmatrix} \quad (12)$$

is exponentially stable. To see this, we first note that the matrices A_3 and $A_4(\mu)$ are exponentially stable by Assumptions 2 and 1.(i) respectively, so that there exist matrices $P_3=P_3^*>0$ and $P_4(\mu)=P_4^*(\mu)>0$ guaranteeing $A_3^*P_3+P_3A_3\le -I$ and $A_4^*(\mu)P_4(\mu)+P_4(\mu)A_4(\mu)\le -I$ for all $\mu\in\mathcal{M}$. We also note that $B_5(\mu)=B_5^*(\mu)\ge b_0I$. Let the Lyapunov function for (12) be $V(\chi_3,\chi_4,\chi_5)=\tau\chi_3^*P_3\chi_3+\chi_4^*P_4(\mu)\chi_4+\chi_5^*\chi_5$ where the positive constant τ will be chosen later. Then the time derivative of the Lyapunov function becomes

$$\begin{aligned}\dot{V}=&\,\tau\chi_3^*\left(A_3^*P_3+P_3A_3\right)\chi_3\\ &+\chi_4^*\left\{A_4(\mu)^*P_4(\mu)+P_4(\mu)A_4(\mu)\right\}\chi_4\\ &-2K\chi_5^*B_5(\mu)\chi_5+\tau\left(\chi_5^*B_{35}^*P_3\chi_3+\chi_3^*P_3B_{35}\chi_5\right)\\ &+\left\{\chi_3^*B_{43}^*(\mu)P_4(\mu)\chi_4+\chi_4^*P_4(\mu)B_{43}(\mu)\chi_3\right\}\\ &+\left\{\chi_5^*B_{45}^*(\mu)P_4(\mu)\chi_4+\chi_4^*P_4(\mu)B_{45}(\mu)\chi_5\right\}\\ &+\left(\chi_3^*B_{53}^*\chi_5+\chi_5^*B_{53}\chi_3\right)\\ &+\left\{\chi_4^*B_{54}^*(\mu)\chi_5+\chi_5^*B_{54}(\mu)\chi_4\right\}+2\chi_5^*A_5(\mu)\chi_5.\end{aligned}$$

By using Young's inequality, we get the positive constants c_i for $i=1,\ldots,6$ such that the following holds:

$$\begin{aligned}|\chi_3^*P_3B_{35}\chi_5|&\le\frac{1}{4}|\chi_3|^2+c_1|\chi_5|^2\\ |\chi_4^*P_4(\mu)B_{43}(\mu)\chi_3|&\le\frac{1}{12}|\chi_4|^2+c_2|\chi_3|^2\\ |\chi_4^*P_4(\mu)B_{45}(\mu)\chi_5|&\le\frac{1}{12}\left|\chi_4\right|^2+c_3\left|\chi_5\right|^2\\ |\chi_5^*B_{53}\chi_3|&\le c_4|\chi_5|^2+c_4|\chi_3|^2\\ |\chi_5^*B_{54}(\mu)\chi_4|&\le\frac{1}{12}|\chi_4|^2+c_5|\chi_5|^2\\ |\chi_5^*A_5(\mu)\chi_5|&\le c_6|\chi_5|^2\end{aligned} \quad (13)$$

for all $\mu\in\mathcal{M}$. These inequalities yield that

$$\dot{V}\le-\frac{1}{2}(\tau-4c_2-4c_4)|\chi_3|^2-\frac{1}{2}|\chi_4|^2 -2(b_0K-\tau c_1-c_3-c_4-c_5-c_6)|\chi_5|^2.$$

By selecting the design parameters as $\tau>4c_2+4c_4$ and $K>K^\star:=(1/b_0)(\tau c_1+c_3+c_4+c_5+c_6)$, the subsystem (12) is shown to be exponentially stable.

Moreover the subsystem formed by the substate $\operatorname{col}(\chi_2,\chi_3,\chi_4,\chi_5)$ is also exponentially stable because it is a cascade of two exponentially stable systems from the structure and Assumption 2. Thus, $\lim_{t\to\infty}\chi_i(t)=0$ for $i=2,\ldots,5$ and, from the structure again, $\operatorname{col}(\chi_1(t),\chi_2(t))=e^{A_wt}\operatorname{col}(\chi_1(0),\chi_2(0))$ for all $t\ge0$. Since $\chi=\operatorname{col}(\chi_1,\chi_2,\chi_3,\chi_4,\chi_5)=\operatorname{col}(\varphi^1_1,\varphi^1_2,\bar{\varphi},\tilde{\sigma},\tilde{y})$ and $\varphi=\operatorname{col}\left(\varphi^1_1,\varphi^1_2,\bar{\varphi}\right)=\left(T^{-1}\otimes I_{q+s}\right)\bar{w}$, it is seen that $\lim_{t\to\infty}\|\varphi(t)-\operatorname{diag}\left(e^{A_wt},0_{(q+s)(N-1)}\right)\varphi(0)\|=0$, $\lim_{t\to\infty}\|\tilde{\sigma}(t)\|=0$, and $\lim_{t\to\infty}\|\tilde{y}(t)\|=0$. This in turn implies that $\lim_{t\to\infty}\|\bar{w}(t)-\left(1_Nr^T\otimes e^{A_wt}\right)\bar{w}(0)\|=0$, or

$$\begin{aligned}\bar{w}^i(t)\longrightarrow&\left(r^T\otimes e^{A_wt}\right)\bar{w}(0)\\ =&\,e^{A_wt}\left[\bar{w}^1(0)\ \cdots\ \bar{w}^N(0)\right]r\\ =&\,\operatorname{col}\left(e^{St}w_0,0_{s\times1}\right)\end{aligned}$$

where $w_0:=\left[\mathcal{W}(0)+\int_0^\infty e^{-S\tau}QC_\eta e^{A_\eta\tau}d\tau\mathcal{H}(0)\right]r$, $\mathcal{W}(0):=[w^1(0)\cdots w^N(0)]$, and $\mathcal{H}(0):=[\eta^1(0)\cdots\eta^N(0)]$. The vector w_0 is obtained from the variation of constants formula and is well-defined since A_η is Hurwitz by Assumption 2. Finally we get $\operatorname{col}\left(\xi^i(t),z^i(t),y^i(t)\right)\longrightarrow\operatorname{col}\left(\Pi^i_\xi(\mu^i),\Pi^i_z(\mu^i),R\right)e^{St}w_0$, which follows from $\tilde{\xi}^i=\xi^i-\Pi^i_\xi(\mu^i)w^i-(1/b^i(\mu^i))J\tilde{y}^i$, $\tilde{z}^i=z^i-\Pi^i_z(\mu^i)w^i$, and $\tilde{y}^i=y^i-Rw^i$.

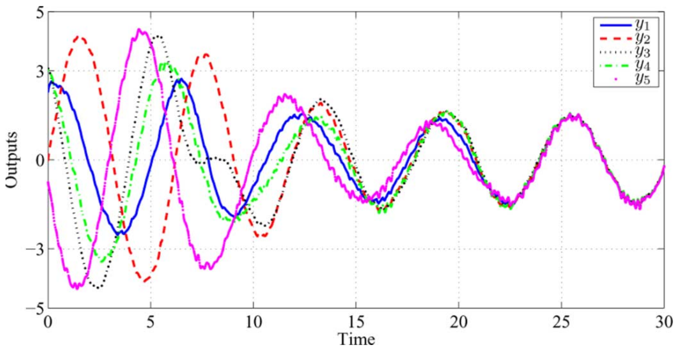

Fig. 2. Simulation result.

B. Case of Higher Relative Degrees

When the N plants (1) have arbitrary relative degrees, the problem can be dealt with by employing the idea of the nested high-gain backstepping and the dirty derivative observer (see [6] and [13]).

Under Assumption 1.(i), the ith plant (1) is now put into the normal form

$$\begin{aligned}\dot{z}^i &= A_{00}^i(\mu^i)z^i + A_{01}^i(\mu^i)y_1^i\\ \dot{y}_j^i &= y_{j+1}^i, \qquad j=1,\ldots,r^i-1\\ \dot{y}_{r^i}^i &= A_{10}^i(\mu^i)z^i + a_{11}^i(\mu^i)y_1^i + \cdots + a_{1r^i}^i(\mu^i)y_{r^i}^i\\ &\quad + b^i(\mu^i)u^i\end{aligned}$$

where $z^i \in \mathbb{R}^{n^i-r^i}$, $y^i = y_1^i$, and $b^i(\mu^i) \geq b_0 > 0$ for all $i = 1,\ldots,N$. Define the changes of coordinates $\tilde{z}^i := z^i - \Pi_z^i(\mu^i)w^i$ and $\tilde{y}_j^i := y_j^i - RS^{j-1}w^i$ for $j=1,\ldots,r^i$, where $\Pi_z^i(\mu^i)$ is a solution of $\Pi_z^i(\mu^i)S = A_{00}^i(\mu^i)\Pi_z^i(\mu^i) + A_{01}^i(\mu^i)R$ and the dynamics of w^i is given in (7a) [or (6)]. Then the system becomes

$$\begin{aligned}\dot{\tilde{z}}^i &= A_{00}^i(\mu^i)\tilde{z}^i + A_{01}^i(\mu^i)\tilde{y}_1^i - \Pi_z^i(\mu^i)Q\left(C_\eta\eta^i + D_\eta v^i\right)\\ \dot{\tilde{y}}_j^i &= \tilde{y}_{j+1}^i - RS^{j-1}Q\left(C_\eta\eta^i + D_\eta v^i\right) \qquad j=1,\ldots,r^i-1\\ \dot{\tilde{y}}_{r^i}^i &= A_{10}^i(\mu^i)\tilde{z}^i + a_{11}^i(\mu^i)\tilde{y}_1^i + \cdots + a_{1r^i}^i(\mu^i)\tilde{y}_{r^i}^i\\ &\quad + b^i(\mu^i)[u^i - \Gamma^i(\mu^i)w^i] - RS^{r^i-1}Q\left(C_\eta\eta^i + D_\eta v^i\right)\end{aligned}$$

where $v^i = -\sum_{j\in\mathcal{N}} l_{ij}y_1^j$ and $\Gamma^i(\mu^i) = -(1/b^i(\mu^i))\left\{A_{10}^i\Pi_z^i(\mu^i) + \sum_{j=1}^{r^i} a_{1j}^i(\mu^i)RS^{j-1} - RS^{r^i}\right\}$.

Suppose temporarily $\tilde{y}_j^i$ for $j=1,\ldots,r^i$ are measurable to the ith agent's controller and consider the regulator given in (7c) and (7d) with the estimates $\hat{y}_j^i$ replaced by $\tilde{y}_j^i$ for $j=1,\ldots,r^i$. Let the coordinate changes be $\theta^i := k^{r^i-1}c_0^i\tilde{y}_1^i + k^{r^i-2}c_1^i\tilde{y}_2^i + \cdots + kc_{r^i-2}^i\tilde{y}_{r^i-1}^i + \tilde{y}_{r^i}^i$, $\tilde{\xi}^i := \xi^i - \Pi_\xi^i(\mu^i)w^i - (1/b^i(\mu^i))J\theta^i$, $\tilde{\sigma}^i := \mathrm{col}(\tilde{\xi}^i, \tilde{z}^i)$, and $\bar{y}_j^i := \tilde{y}_j^i/k^{j-1}$ for $j=1,\ldots,r^i-1$. Let the matrix $\Pi_\xi^i(\mu^i) \in \mathbb{R}^{p\times q}$ be the same as in Section III-A, which satisfies $\Pi_\xi^i(\mu^i)S = \Phi\Pi_\xi^i(\mu^i)$ and $\Gamma^i(\mu^i) = H\Pi_\xi^i(\mu^i)$. Then the ith agent is written as

$$\begin{aligned}\dot{\bar{w}}^i &= A_w\bar{w}^i + B_wv^i\\ \dot{\tilde{\sigma}}^i &= B_{\sigma w}^i(\mu^i,[k]_0^{r^i-1})\bar{w}^i + A_\sigma^i(\mu^i)\tilde{\sigma}^i + B_{\sigma y}^i(\mu^i,[k]_0^{r^i})\bar{y}^i\\ &\quad + B_{\sigma\theta}^i(\mu^i,[k]_0^1)\theta^i + B_\sigma^i(\mu^i,[k]_0^{r^i-1})v^i\\ \dot{\bar{y}}^i &= B_{yw}^i([k]_{2-r^i}^0)\bar{w}^i + kA_c^i\bar{y}^i\\ &\quad + B_{y\theta}^i([k]_{2-r^i}^{2-r^i})\theta^i + B_y^i([k]_{2-r^i}^0)v^i\\ \dot{\theta}^i &= B_{\theta w}^i([k]_1^{r^i-1})\bar{w}^i + B_{\theta\sigma}^i(\mu^i)\tilde{\sigma}^i + B_{\theta y}^i(\mu^i,[k]_0^{r^i})\bar{y}^i\\ &\quad + \left\{a_\theta^i(\mu^i,[k]_0^1) - b^i(\mu^i)K\right\}\theta^i + B_\theta^i([k]_1^{r^i-1})v^i\end{aligned}$$

where $\bar{y}^i := \mathrm{col}(\bar{y}_1^i,\ldots,\bar{y}_{r^i-1}^i)$, $A_\sigma^i(\mu^i)$ is Hurwitz for all $\mu^i \in \mathcal{M}^i$, A_c^i is a companion matrix induced by the Hurwitz polynomial $f_c^i(s)$ in Theorem 1, the first q columns of $B_{\sigma w}^i(\cdot)$, $B_{yw}^i(\cdot)$, and $B_{\theta w}^i(\cdot)$ are all zeros, and other matrices are appropriately defined. For the integers $a \leq b$, the gain vector $[k]_a^b$ is defined as $[k]_a^b := [k^a\ k^{a+1}\ \cdots\ k^b]$. Under the coordinates $\bar{y} := \mathrm{col}(\bar{y}^1,\ldots,\bar{y}^N)$, $\theta := \mathrm{col}(\theta^1,\ldots,\theta^N)$, and those in Section III-A, the overall dynamics is given by (14), shown at the bottom of the page, where $\chi = \mathrm{col}(\chi_1,\chi_2,\chi_3,\chi_4,\chi_5,\chi_6) := \mathrm{col}(\varphi_1^1, \varphi_2^1, \bar{\varphi}, \tilde{\sigma}, \bar{y}, \theta)$, $B_6(\mu) = B_6^*(\mu) \geq b_0I$, $\bar{r} := \max(r^1,\ldots,r^N)$, and $\underline{r} := \min(r^1,\ldots,r^N)$. Since A_3, $A_4(\mu)$, and A_5 are exponentially stable for all $\mu \in \mathcal{M}$, there exist positive Hermitian matrices P_3, $P_4(\mu)$, and P_5 so that $A_i^*P_i + P_iA_i \leq -I$ holds for $i = 3, 4, 5$. By selecting a Lyapunov function as $V(\chi_3,\chi_4,\chi_5,\chi_6) = \tau\chi_3^*P_3\chi_3 + \delta\chi_4^*P_4(\mu)\chi_4 + \chi_5^*P_5\chi_5 + \chi_6^*\chi_6$, it can be shown, with the sufficiently small δ and large τ, that the subsystem formed by the substate $\mathrm{col}(\chi_3,\chi_4,\chi_5,\chi_6)$ is exponentially stable for all $k > k^\star$ and $K > K^\star(k)$ with certain large constants $k^\star$ and $K^\star(k)$. Therefore it is concluded that the output consensus is reached, with measurements $\tilde{y}_j^i$, for such k and K. The detailed proof, which is similar as in Section III-A, is omitted due to the page limit, but available from the authors.

Finally, the dirty derivative observer (7b) is employed to estimate $\tilde{y}_j^i$'s from the information $\tilde{y}_1^i = y^i - C_w\bar{w}^i$ which is available to the ith agent's controller. In fact, it can be shown that there exists a constant $g^\star(k,K)$ such that the overall stability is recovered with the observer (7b) for all $g > g^\star(k,K)$. The proof proceeds as in [13], and thus, omitted.

$$\dot{\chi} = \begin{bmatrix} S & QC_\eta & 0 & 0 & 0 & 0\\ 0 & A_\eta & 0 & 0 & 0 & 0\\ 0 & 0 & A_3 & 0 & B_{35} & 0\\ 0 & B_{42}\left(\mu,[k]_0^{\bar{r}-1}\right) & B_{43}\left(\mu,[k]_0^{\bar{r}-1}\right) & A_4(\mu) & B_{45}(\mu,[k]_0^{\bar{r}}) & B_{46}\left(\mu,[k]_0^1\right)\\ 0 & B_{52}\left([k]_{2-\bar{r}}^0\right) & B_{53}\left([k]_{2-\bar{r}}^0\right) & 0 & kA_5 + B_{55}\left([k]_{2-\bar{r}}^0\right) & B_{56}\left([k]_{2-\bar{r}}^{2-\underline{r}}\right)\\ 0 & B_{62}([k]_1^{\bar{r}-1}) & B_{63}\left([k]_1^{\bar{r}-1}\right) & B_{64}(\mu) & B_{65}\left(\mu,[k]_0^{\bar{r}}\right) & A_6\left(\mu,[k]_0^1\right) - B_6(\mu)K \end{bmatrix}\chi \qquad (14)$$

IV. EXAMPLE

Consider a group of five agents which is categorized by the two subgroups

$$\begin{aligned} \dot{z}^i &= \mu_1^i z^i + \mu_2^i y^i & \dot{y}_1^j &= y_2^j \\ \dot{y}^i &= \mu_3^i z^i + \mu_4^i y^i + \mu_5^i u^i & \dot{y}_2^j &= y_1^j - y_2^j + 3u^j \end{aligned} \tag{15}$$

where $i \in \{1,2,3\}$, $j \in \{4,5\}$, μ^i's are the uncertain parameters such that $\mu_1^i \in [-2,-1]$, $\mu_5^i \in [1,3]$, and others belong to $[-2,2]$. For this example, we consider the two network topologies which are described by their Laplacians L_1 and L_2. Here $L_1 = [1.5\ 0 - 1\ 0 - 0.5; -1\ 1\ 0\ 0\ 0; 0 - 2\ 3 - 1\ 0; -2\ 0 - 2\ 4\ 0; -0.5\ 0\ 0 - 1\ 1.5]$ and L_2 represents the strongly connected graph [11] with all the weights being 1. Hence Assumption 1.(ii) holds in both cases.

For the systems (15), suppose that the synchronized output signal $\phi(t)$ needs to be the output of $\dot{w} = Sw$ and $\phi = Rw$, where $S = [0\ 1; -1\ 0]$ and $R = [1\ 0]$. Then Assumption 2 holds with both Laplacians if we take the controllers (5) as in (4) with $Q = [0;1]$, $M = [-5;-5]$, $\lambda^\star = 2$, and $\epsilon = 0.1$. Thus the final controllers for the first three agents are given by (9) while the controllers for the other two agents are (7a), (7b), and (7d*), where $\Phi = S$, $H = R$, $J = [2\ 1]^T$, $f_c^j(s) = s+1$, $f_d^j(s) = s^2+2s+1$, and $\Gamma^j = (R - RS - RS^2)/(-3)$ for $j = 4, 5$. Note that (7d*) is used instead of (7c) and (7d) since the fourth and fifth agents do not have any uncertainties. Finally the output consensus is achieved in both cases of L_1 and L_2 if we select $k = 5$, $K = 8$, and $g = 50$ which are easily obtained according to Remark 2.

Fig. 2 is the simulation result. All the uncertainties and initial conditions $w^i(0)$ for $i = 1, \ldots, 5$ are chosen randomly within their bounds, while other initial conditions are set to zero. In the simulation, the network is disconnected (i.e., $v^i = 0$) for $t \in [0,5)$ and $t \in [15,20)$, and is connected through L_1 and L_2 during the time intervals $[5,15)$ and $[20,30)$ respectively. Moreover, the uniform random noises with magnitude between $+0.1$ and -0.1 are added on the measurements to make the simulation more interesting. It is observed from the simulation that the multi-agent system (15) eventually reaches the output consensus. Note that the outputs of fourth and fifth agents are sensitive to measurement noises as seen from the figure, since the high-gain observers are utilized for their controllers.

V. CONCLUSION

In this technical note, we have solved the output consensus problem for a class of heterogeneous uncertain linear multi-agent systems. Our approach is to embed an internal model, which plays a role of command generator, and the command is determined as time goes since it is the outcome of on-line consensus by communicating the outputs of the agents. The agreed signal may consist of constant, ramp, parabola, sinusoidal, harmonics, and their combinations.

On the other hand, some limitation may arise when the relative degree of the plant goes high. First, the implementation complexity of the proposed controller (7) increases. Second, as the order of the high-gain observer increases, the proposed controller becomes sensitive to measurement noise. Finally, consideration of the problem under the time-varying network topology as well as multi-output extension is under further study.

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