Assignment 3: Derivation

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1. (a) n is a **value** parameter. m is a **result** parameter.

   (b) Our post-condition, $mrun(A, n_0, m)$, expands to $lrun(A, n_0, m) \wedge (m < A.\mathrm{len} \Rightarrow A_{n_0} \neq A_m)$. This satisfies the form $Q_1 \wedge Q_2$. Q_1 was chosen as the invariant.

$$inv \triangleq lrun(A, n_0, m)$$

   Q_2 is chosen as the *negation* of the guard, such that the guard, by pattern matching

$$\begin{aligned} guard &\triangleq \neg(m < A.\mathrm{len} \Rightarrow A_{n_0} \neq A_m) \\ &\triangleq \neg(\neg(m < A.\mathrm{len}) \vee (A_{n_0} \neq A_m)) \\ &\triangleq (m < A.\mathrm{len}) \wedge \neg(A_{n_0} \neq A_m) \\ &\triangleq (m < A.\mathrm{len} \wedge A_{n_0} = A_m) \end{aligned}$$

   (c) Let

$$\begin{aligned} pre &\triangleq lrun(A, n, n+1) \\ post &\triangleq mrun(A, n_0, m) \end{aligned}$$

   s.t.

$$\begin{aligned} & n, m : [pre,\ post] \\ \sqsubseteq\ & \{\text{Composition: middle predicate is } inv\} \\ & n, m : [pre,\ inv]; \quad n, m : [inv,\ post] \\ \sqsubseteq\ & \{\text{Assignment: } pre \Rrightarrow inv[m \backslash n+1]\} \\ & m := n+1; \quad n, m : [inv,\ post] \end{aligned}$$

   $\because$

$$\begin{aligned} inv[m \backslash n+1] &\equiv lrun(A, n_0, m)[m \backslash n+1] \\ &\equiv lrun(A, n_0, n+1) \end{aligned}$$

   $\therefore$

$$lrun(A, n, n+1) \Rrightarrow lrun(A, n_0, n+1)$$

$$\begin{aligned} \sqsubseteq\ & \{\text{Strengthen post: } inv \wedge \neg guard \Rrightarrow post\} \\ & m := n+1; \quad n, m : [inv,\ inv \wedge \neg guard] \end{aligned}$$

   $\because$

$$\begin{aligned} & inv \wedge \neg guard \Rrightarrow post \\ \equiv\ & \{\text{Expansion of definitions}\} \\ & lrun(A, n_0, m) \wedge \neg(m < A.\mathrm{len} \wedge A_{n_0} = A_m) \Rrightarrow mrun(A, n_0, m) \\ \equiv\ & \{\text{Expansion of functions}\} \end{aligned}$$

$$
\begin{array}{ll}
 & lrun(A, n_0, m) \land \neg(m < A.\text{len} \land A_{n_0} = A_m) \;\Rrightarrow\; lrun(A, n_0, m) \land (m < A.\text{len} \Rightarrow A_{n_0} \neq A_m) \\
\equiv & \{\text{De Morgan's law - negation of conjunction}\} \\
 & lrun(A, n_0, m) \land (\neg(m < A.\text{len}) \lor \neg(A_{n_0} = A_m)) \;\Rrightarrow\; lrun(A, n_0, m) \land (m < A.\text{len} \Rightarrow A_{n_0} \neq A_m) \\
\equiv & \{P \Rightarrow Q \;\equiv\; \neg P \lor Q\} \\
 & lrun(A, n_0, m) \land (\neg(m < A.\text{len}) \lor \neg(A_{n_0} = A_m)) \;\Rrightarrow\; lrun(A, n_0, m) \land (\neg(m < A.\text{len}) \lor (A_{n_0} \neq A_m)) \\
\equiv & \{\} \\
 & \text{true}
\end{array}
$$

$$
\begin{array}{ll}
\sqsubseteq & \{\text{Repetition}\} \\
 & m := n + 1; \\
 & \textbf{do } (m < A.\text{len} \land A_{n_0} = A_m) \rightarrow \\
 & \qquad n, m : [inv \land guard,\; inv \land (0 \leqslant V < V_0)] \\
 & \textbf{od}
\end{array}
$$

where

$$
V \;\triangleq\; A.\text{len} \;-\; m
$$

$$
\begin{array}{ll}
\sqsubseteq & \{\text{Assignment: } inv \land guard \;\Rrightarrow\; (inv \land (0 \leqslant V < V_0))[m \backslash m + 1]\} \\
 & m := n + 1; \\
 & \textbf{do } (m < A.\text{len} \land A_{n_0} = A_m) \rightarrow \\
 & \qquad m := m + 1 \\
 & \textbf{od}
\end{array}
$$

$\because$

$$
\begin{array}{rcl}
(inv \land (0 \leqslant V < V_0))[m \backslash m + 1] & \equiv & (lrun(A, n_0, m) \land (0 \leqslant (A.\text{len} - m) < (A.\text{len} - m_0)))[m, m_0 \backslash m + 1, m] \\
 & \equiv & lrun(A, n_0, m + 1) \land (0 \leqslant (A.\text{len} - (m + 1)) < (A.\text{len} - m))
\end{array}
$$

$\therefore$

$$
\begin{array}{rcl}
inv \land guard & \Rrightarrow & lrun(A, n_0, m + 1) \land (0 \leqslant (A.\text{len} - (m + 1)) < (A.\text{len} - m)) \\
lrun(A, n_0, m) \land (m < A.\text{len} \land A_{n_0} = A_m) & \Rrightarrow & lrun(A, n_0, m + 1) \land (0 \leqslant (A.\text{len} - (m + 1)) < (A.\text{len} - m))
\end{array}
$$

To justify this, we need to show that both conjuncts on the **RHS** are entailed by the **LHS**.

i.

$$
m < A.\text{len} \;\Rrightarrow\; 0 \leqslant (A.\text{len} - (m + 1)) < (A.\text{len} - m)
$$

is trivially *true*. The first conjunct is entailed by the **LHS**.

ii. Reiterating

$$
lrun(A, i, j) \;\triangleq\; run(A, i, j) \land (i > 0 \Rightarrow A_{i-1} \neq A_i)
$$

we can see that

$$
lrun(A, n_0, m) \land (m < A.\text{len} \land A_{n_0} = A_m) \;\Rrightarrow\; lrun(A, n_0, m + 1)
$$

holds because

A. $run(A, n_0, m + 1)$ describes a run up to, but not including index $m + 1$. Because we know that $A_{n_0} = A_m$, we are permitted absorb A_m into the run range by incrementing m to $m + 1$.
B. Due to $m < A.len$, A_m describes a valid array access.

The second conjunct is entailed by the **LHS**.

All conjuncts hold, and are entailed by the **LHS**.
□

2.

$$
\begin{array}{rcl}
pre & \triangleq & A.\text{len} > 0 \\
post & \triangleq & mrun(A, \ell, h) \land (\forall\, p, q \cdot mrun(A, p, q) \Rightarrow (h - \ell) \leqslant (q - p))
\end{array}
$$

$$\begin{aligned} & \ell, h : [\mathit{pre},\ \mathit{post}] \\ \sqsubseteq\ & \{\text{Composition: middle predicate is } \mathit{inv}\} \\ & \ell, h : [\mathit{pre},\ \mathit{inv}];\quad \ell, h : [\mathit{inv},\ \mathit{post}] \end{aligned}$$

where

$$\mathit{inv} \quad \triangleq \quad \mathit{mrun}(A_{[0,i)}, \ell, h) \wedge (\forall\, p, q \ \cdot\ \mathit{mrun}(A_{[0,i)}, p, q) \Rightarrow (h - \ell) \leqslant (q - p))$$

This invariant was chosen as the postcondition refers to a constant A, which can be written $A_{[0,A.\text{len})}$, which is of the form A^B, where B is A.len. We replace A.len with a program variable i, to create the invariant defined above. We can further derive the negation of our guard to be ($i = A$.len), such that the guard is ($i \neq A$.len).

$$\begin{aligned} \sqsubseteq\ & \{\text{Assignment: } \mathit{pre} \Rrightarrow \mathit{inv}[i, \ell, h \backslash 1, 0, 1]\} \\ & i, \ell, h := 1, 0, 1;\quad \ell, h : [\mathit{inv},\ \mathit{post}] \end{aligned}$$

$\because$

$$\mathit{inv}[i, \ell, h \backslash 1, 0, 1] \quad \equiv \quad \mathit{mrun}(A_{[0,1)}, 0, 1) \wedge (\forall\, p, q \ \cdot\ \mathit{mrun}(A_{[0,1)}, p, q) \Rightarrow (1 - 0) \leqslant (q - p))$$

$\therefore$

$$A.\text{len} > 0 \quad \Rrightarrow \quad \mathit{mrun}(A_{[0,1)}, 0, 1) \wedge (\forall\, p, q \ \cdot\ \mathit{mrun}(A_{[0,1)}, p, q) \Rightarrow (1 - 0) \leqslant (q - p))$$

The first conjunct is intuitively *true*, as the maximal run of an array of len $= 1$ is itself.
The **LHS** of the implication in the second conjunct is *true* only when $p = 0$ and $q = 1$. The **RHS** is then $(1 - 0) \leqslant (1 - 0)$. Thus, the implication is *true* for these values of p and q. All other values of p and q cause the **LHS** of the implication to be *false*, and thus the implication to be *true*. Thus, the second conjunct is *true*. Thus, the entailment holds as

$$A.\text{len} > 0 \quad \Rrightarrow \quad \text{true} \wedge \text{true}$$

Let

$$\mathit{guard} \quad \triangleq \quad (i \neq A.\text{len})$$

s.t.

$$\begin{aligned} \sqsubseteq\ & \{\text{Strengthen post: } \mathit{inv} \wedge \neg \mathit{guard} \ \Rrightarrow\ \mathit{post}\} \\ & i, \ell, h := 1, 0, 1;\quad \ell, h : [\mathit{inv},\ \mathit{inv} \wedge \neg \mathit{guard}] \end{aligned}$$

$\because$

$$\begin{aligned} \mathit{inv} \wedge \neg \mathit{guard} \ \Rrightarrow\ \mathit{post} \quad \equiv \quad & \mathit{mrun}(A_{[0,i)}, \ell, h) \wedge (\forall\, p, q \ \cdot\ \mathit{mrun}(A_{[0,i)}, p, q) \Rightarrow (h - \ell) \leqslant (q - p)) \wedge \neg(i \neq A.\text{len}) \\ & \Rrightarrow\ \mathit{mrun}(A, \ell, h) \wedge (\forall\, p, q \ \cdot\ \mathit{mrun}(A, p, q) \Rightarrow (h - \ell) \leqslant (q - p)) \end{aligned}$$

The third conjunct $\neg(i \neq A.\text{len})$, is equivalent to $i = A$.len. We can absorb this into the first and second conjuncts to give

$$\begin{aligned} & \mathit{mrun}(A_{[0,A.len)}, \ell, h) \wedge (\forall\, p, q \ \cdot\ \mathit{mrun}(A_{[0,A.len)}, p, q) \Rightarrow (h - \ell) \leqslant (q - p)) \\ & \Rrightarrow \mathit{mrun}(A, \ell, h) \wedge (\forall\, p, q \ \cdot\ \mathit{mrun}(A, p, q) \Rightarrow (h - \ell) \leqslant (q - p)) \end{aligned}$$

As $A_{[0,A.len)} \equiv A$, the entailment holds.

$$\begin{aligned} \sqsubseteq\ & \{\text{Repetition}\} \\ & \ i, \ell, h := 1, 0, 1; \\ & \ \mathbf{do}\ (i \neq A.\text{len}) \rightarrow \\ & \qquad \ell, h : [\mathit{inv} \wedge \mathit{guard},\ \mathit{inv} \wedge (0 \leqslant V < V_0)] \\ & \ \mathbf{od} \end{aligned}$$

where

$$V \quad \triangleq \quad A.\text{len} \ - \ i$$