ASSIGNMENT 1
PHIL3110
ADVANCED LOGIC

PART A

Attempt <u>**all**</u> problems in Part A.

Problem 1. (5 marks) Prove the following using natural deduction:

(1) $(Pa \vee Qa) \rightarrow \bot \vdash \neg Pa \wedge \neg Qa$.
(2) $Qa \rightarrow Pa, Ra \rightarrow Pa \vdash (Qa \vee Ra) \rightarrow Pa$.
(3) $\neg(\forall xPx \rightarrow \exists xRx) \vdash \neg\exists xRx$
(4) $\vdash \exists x(Px \wedge Qx) \rightarrow \exists x(Qx \wedge Px)$
(5) $\vdash \neg\exists x\neg Px \rightarrow \forall xPx$.

Problem 2. (5 marks) Let $\mathscr{L} = \{Q, T\}$ where Q is a one-place relation symbol and T is a two-place relation symbol. Let $\mathscr{M}$ be a model of $\mathscr{L}$ with domain $M = \{m_1, m_2, m_3\}$; and let the interpretations of Q and T be described by tables as follows:

$Q^{\mathscr{M}}$	
m_1	1
m_2	0
m_3	0

  and
$T^{\mathscr{M}}$	m_1	m_2	m_3
m_1	1	0	0
m_2	1	1	0
m_3	0	0	0

where the T table is to be read anti-clockwise.

(1) Represent the interpretations of Q and T using set theoretic notation (i.e., using the symbols $\langle, \rangle, \{$ and $\}$).
(2) Is it correct that $\mathscr{M} \models \exists x\neg Txx$? Show full working.
(3) Is it correct that $\mathscr{M} \models \exists x\forall y(Qy \leftrightarrow Tyx)$? Provide an informal explanation of your answer (i.e., you don't need to give the full working, but you need to provide an argument as to why your answer is right).

PART B

Attempt **two** of the following problems:

Problem 3. (5 Marks) The local shop has branched out into sandwich production. You've been asked to stand in for a shift behind the counter and are given the following instructions for making *good sandwiches*.

- A piece of bread is a good sandwich.
- If you have a good sandwich and you spread marmite over the top piece of bread and cover it with another piece of bread then you have another good sandwich.
- If you have a good sandwich and you put some ham on the top piece of bread and cover that with another piece of bread then you get a good sandwich.
- If you have a good sandwich and you put some jam on the top piece of bread and cover it with another piece of bread then you have a good sandwich.

It's not a great menu, but with more diversity the proofs get longer, so that's probably a good thing. Let's adopt a simple notation system to represent the ingredients and their combination. Let's write:

- B for a piece of bread;
- M for a spreading of marmite;
- H for a covering of ham; and
- J for a covering jam.

Then write symbols representing the ingredients left to right to represent the combination of the actual ingredients from bottom to top. So, for example, BHM would represent a piece of bread covered by some ham with marmite spread over it.

Complete the following tasks:

(1) Which of the following are good sandwiches:
   (a) $BHMB$;
   (b) $BJBHB$; and
   (c) $BHBMBJM$.
(2) Write out a stage-based definition of a good sandwich.
(3) Prove by induction that no good sandwich contains any two instances of the same ingredient adjacently.

Problem 4. (5 Marks) Complete the following:

(1) In Theorem 108 in the notes, a number of cases are left as exercises. Complete the cases for $(\rightarrow E)$ and $(\exists E)$.
(2) In Theorem 118 in the notes, a number of cases are left as exercises. Complete the cases for $\psi := \chi \vee \delta$ and $\psi := \forall x \chi(x)$.

Problem 5. (5 Marks) Suppose we remove the $(\neg\neg E)$ rule from our natural deduction system. Let us write $\Gamma \vdash_{Nat^-} \varphi$ to mean there is a proof in the natural deduction system without double negation elimination using assumptions among Γ and resulting in φ. Complete the following:

(1) Is the system still sound? I.e., do we we have $\Gamma \vdash_{Nat^-} \varphi \Rightarrow \Gamma \models \varphi$? Justify your answer. If the answer is yes, explain why it is so. If not, indicate where the proof of soundness breaks down.

(2) Is the system still complete? I.e., do we have $\Gamma \models \varphi \Rightarrow \Gamma \vdash_{Nat^-} \varphi$? Justify your answer. If the answer is yes, explain why it is so. If not, indicate where the proof of completeness breaks down.