# PHIL3110 - Exam

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## Part A

### Problem 1

1. Let  $\beta = \{ \varphi \mid \varphi \in \Gamma \land \varphi \in \Delta \}.$ 

Say there's some  $\varphi$  such that  $\beta \models \varphi$  but  $\varphi \notin \beta$ . If  $\varphi \notin \beta$ , then  $\varphi \notin \Gamma$  and  $\varphi \notin \Delta$ .

2. If there is no proper subset  $\Xi$  of  $\Gamma$  such that  $\Xi$  is a theory, there always exists a  $\varphi$  such that  $\Xi \models \varphi$  but  $\varphi \notin \Xi$ .

### Problem 2

$$(P \to Q) \to P$$

### Problem 3

### Problem 6

1. Let  $\mathcal{M}$  be a model with domain  $M = \{\}$ .

 $\not\exists x(x=x)$  is a consequence.

Furthermore, we run into some strangeness when we find that  $\forall x(x \neq x)$  is a consequence.

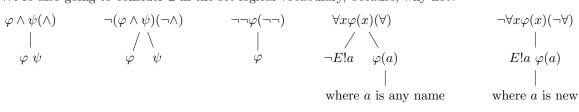
 $\forall x \varphi(x) \Rightarrow \exists x \varphi(x)$  no longer holds either.

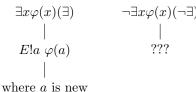
In other words, free logic admits existentially quantified formulas always being false in the empty domain, and universally quantified formulas always being true.

Furthermore, the notion that  $\neg \exists x \varphi(x) \Rightarrow \forall x \neg \varphi(x)$  ceases to be meaningful.

#### 2. Rules

We're also going to consider  $\exists$  in the set logical vocabulary, because, why not?





where E!c expands to  $\exists x(x=c)^1$ .

By new we mean that the name a has not occurred anywhere on the branches above.

3. Our goal is to show that if  $\models_f \varphi$ , then  $\vdash_{Tab} \varphi$ . In other words, if every model  $\mathcal{M}$  (of the language  $\varphi$ ) is such that if  $\mathcal{M} \models \varphi$  then the tableau commencing with  $\neg \varphi$  is closed.

By contraposition, if  $\not\vdash_{Tab} \varphi$  then  $\not\models_f \varphi$ .

In other words, if the tableau commencing with  $\neg \varphi$  does not close, then there is some  $\mathcal{M}$  such that  $M \models \neg \varphi$ .

Rather than choosing some open branch  $\mathcal{B}$  of a tableau commencing with  $\neg \varphi$ , and defining a  $\mathcal{M}^{\mathcal{B}}$ , and reasoning with +-complexity, we'll use a  $\Delta$  that is maximally consistent<sup>2</sup> and existentially witnessed<sup>3</sup> set of of sentences.

Let  $\mathcal{M}^{\Delta}$  be such that:

- $M^{\Delta}$  is the set of constant symbols a occurring in the stentences in  $\Delta$ .
- for each constant symbol a, let  $a^{\mathcal{M}} = a$
- for each n-ary relation symbol R of  $\mathcal{L}(C)$  let  $R^{\mathcal{M}}$  be the set of n-tuples  $\langle a_1, ..., a_n \rangle$  such that  $Ra_1, ..., a_n$  is in  $\Delta$ .

Reiterating

Claim 120 Suppose  $\Delta$  is maximal consistent. Then  $\varphi \in \Delta \Leftrightarrow \mathcal{M}^{\Delta} \models \varphi$ 

but eliding Proof  $120^4$ .

We will show  $\varphi \in \Delta \Rightarrow \mathcal{M}^{\Delta} \models \varphi$ 

We proceed by induction on the complexity of the forumula.

#### Base

Suppose  $\psi := \chi$ . Then

$$\chi \in \Delta \Leftrightarrow \mathcal{M}^{\Delta} \models \chi$$

This flows trivially from Claim 120.

### **Induction Step**

• Suppose  $\psi := \chi \wedge \delta \in \Delta$ .

We claim that  $\chi \wedge \delta \in \Delta$  iff both  $\chi \in \Delta$  and  $\delta \in \Delta$ .

If  $\chi \wedge \delta \in \Delta$ , then by Claim 120 and tableau rule ( $\wedge$ ), we have both  $\chi$  and  $\delta$  in  $\Delta$ .

$$\chi \wedge \delta \in \Delta \Rightarrow \chi \in \Delta \text{ and } \delta \in \Delta$$
  
 $\Leftrightarrow \mathcal{M}^{\Delta} \models \chi \text{ and } \mathcal{M}^{\Delta} \models \delta$   
 $\Leftrightarrow \mathcal{M}^{\Delta} \models \chi \wedge \delta$ 

The first  $\Rightarrow$  is via our claim; the second  $\Leftrightarrow$  is by induction hypothesis; and the last  $\Leftrightarrow$  is from the  $\models$  definition.

<sup>&</sup>lt;sup>1</sup>I had to do this because the forest package wouldn't typeset leaves over a certain number of characters

<sup>&</sup>lt;sup>2</sup> for an arbitrary  $\varphi$  either  $\varphi \in \Delta$  or  $\neg \varphi \in \Delta$ 

<sup>&</sup>lt;sup>3</sup>if  $\exists x \varphi(x) \in \Delta$ , then for some constant symbol  $a, \varphi(a) \in \Delta$ 

<sup>&</sup>lt;sup>4</sup>Page 77 of course notes

• Suppose  $\psi := \neg \chi$ .

If  $\chi \in \delta$ , then by Claim 120 and tableau rule  $(\neg)$ ,  $\chi \notin \Delta$ .

$$\neg \chi \in \Delta \Leftrightarrow \chi \not\in \Delta$$
$$\Leftrightarrow \mathcal{M}^{\Delta} \not\models \chi$$
$$\Leftrightarrow \mathcal{M}^{\Delta} \models \neg \chi$$

The first  $\Leftrightarrow$  is via the completeness of  $\Delta$ ; the second  $\Leftrightarrow$  is by the induction hypothesis; and the last is via the  $\models$  definition.

• Suppose  $\psi := \forall x \chi(x)$ .

We claim  $\forall x \chi(x) \in \Delta$  for every constant symbol  $a \in \mathcal{L}(C)$  such that  $\chi(a) \in \Delta$ .

Suppose  $\forall x \chi(x) \in \Delta$ .

Suppose  $\mathcal{L}(C) \neq \{\}$ , which given our definition of  $\mathcal{M}^{\Delta}$ , occurs when  $M^{\Delta} \neq \{\}$ .

Then by claim CLAIM 120 and tableau rule  $(\forall)$ , for all  $a \in \mathcal{L}(C)$ ,  $\chi(a) \in \Delta$ .

Suppose  $\mathcal{L}(C) = \{\}$ . We assert that there must be something in  $\Delta$  that says there is no  $a \in \mathcal{L}(C)$ .

Then by Claim 120 and tableau rule  $(\forall)$ , there is some  $\varphi$  of the form  $\neg \exists y (y = a)$  such that  $\varphi \in \Delta$ .

On the other hand, if for some some  $a \in \mathcal{L}(C)$ ,  $\chi(a) \in \Delta$ , then by Claim 120 and  $(\forall)$ ,  $\forall x \chi(x) \in \Delta$ .

Then we have

$$\forall x \chi(x) \in \Delta \Rightarrow \text{ for all } a \in M^{\Delta}, \ \chi(a) \in \Delta \text{ or } \neg \exists y (y = a) \in \Delta$$
  
 $\Leftrightarrow \text{ for all } a \in M^{\Delta}, \ \mathcal{M}^{\Delta} \models \chi(a) \in \Delta \text{ or } \mathcal{M}^{\Delta} \models \neg \exists y (y = a)$   
 $\Leftrightarrow \mathcal{M}^{\Delta} \models \forall x \chi(x)$ 

The first  $\Rightarrow$  is via our claim; the second  $\Leftrightarrow$  is by induction hypothesis; and the final  $\Leftrightarrow$  is was via the  $\models$  definition.

• Suppose  $\psi := \exists x \chi(x)$ .

Then we claim  $\exists x \chi(x) \in \Delta$  iff there is some constant symbol  $a \in \mathcal{L}(C)$  such that  $\chi(a) \in \Delta$ . Suppose  $\exists x \chi(x) \in \Delta$ .

By the construction of  $\Delta^5$ , Claim 120 and tableau rule ( $\exists$ ), there is some  $a \in \mathcal{L}(C)$  such that  $\chi(a) \in \Delta$ .

Furthermore we may also say that there exists some  $\varphi$  of the form  $\exists y(y=a)$ , such that  $\varphi \in \Delta$ . We would not be able to say this if  $\Delta$  were not existentially witnessed.

Then we have

$$\exists x \chi(x) \in \Delta \Leftrightarrow \text{there is some } a \in M^{\Delta} \text{ such that } \chi(a) \in \Delta \text{ and } \exists y(y=a) \in \Delta$$
  
  $\Leftrightarrow \text{ there is some } a \in M^{\Delta} \text{ such that } \mathcal{M}^{\Delta} \models \chi(a) \text{ and } \mathcal{M}^{\Delta} \models \exists y(y=a)$   
  $\Leftrightarrow \mathcal{M}^{\Delta} \models \exists x \chi(x)$ 

The first  $\Rightarrow$  is via our claim; the second  $\Leftrightarrow$  is by induction hypothesis; and the final  $\Leftrightarrow$  is was via the  $\models$  definition.

The other cases are left as exercises for the marker.

<sup>&</sup>lt;sup>5</sup>it is existentially witnessed