PHIL3110 - Assignment 1

Maxwell Bo

April 17, 2018

Part A

Problem 1

1. $$\dfrac{\dfrac{\dfrac{Pa \vee Qa^{(1)} \qquad (Pa \vee Qa) \to \bot}{\bot}}{\neg(Pa \vee Qa)}\;{\scriptstyle (1)\ (\neg I)}}{\neg Pa \wedge \neg Qa}\;{\scriptstyle (\text{DM-1})}$$

De Morgan's Law 1

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi^{(1)}}{\varphi \vee \psi} \qquad \neg(\varphi \vee \psi)}{\bot}}{\neg\varphi}\;{\scriptstyle (1)\ (\neg I)} \qquad \dfrac{\dfrac{\dfrac{\psi^{(2)}}{\varphi \vee \psi} \qquad \neg(\varphi \vee \psi)}{\bot}}{\neg\psi}\;{\scriptstyle (2)\ (\neg I)}}{\neg\varphi \wedge \neg\psi}$$

2. $$\dfrac{\dfrac{Qa \vee Ra^{(3)} \qquad \dfrac{Qa^{(1)} \qquad Qa \to Pa}{Pa} \qquad \dfrac{Ra^{(2)} \qquad Ra \to Pa}{Pa}}{Pa}\;{\scriptstyle (1)\ (2)\ (\vee E)}}{(Qa \vee Ra) \to Pa}\;{\scriptstyle (3)\ (\to I)}$$

3. $$\dfrac{\dfrac{\neg(\forall x Px \to \exists x Rx) \qquad \dfrac{\dfrac{\dfrac{\forall x Px^{(1)} \qquad \exists x Rx^{(2)}}{\forall x Px \wedge \exists x Rx}}{\exists x Rx}}{\forall x Px \to \exists x Rx}\;{\scriptstyle (1)\ (\to I)}}{\bot}}{\neg\exists x Rx}\;{\scriptstyle (2)\ (\neg I)}$$

Problem 2

1. $$\begin{aligned} Q^{\mathcal{M}} &= \{m_1\} \\ T^{\mathcal{M}} &= \{\langle m_1, m_1\rangle, \langle m_1, m_2\rangle, \langle m_2, m_2\rangle\} \end{aligned}$$

2. Distressingly, $\mathcal{L}$ does not define any constant symbols, nor does $\mathcal{M}$ provide interpretations of constant symbols in $\mathcal{M}$.

   Thus

   $$\mathcal{M} \nvDash \exists x \neg Txx$$

   However assuming $\mathcal{M}^+$, where $\mathcal{M}^+$ is the expanded model $\mathcal{M}$, where $m^{\mathcal{M}} = m$ for all $m \in M$, we see that

   $$M \models \exists x \neg Txx$$

   as

   $$\langle m_3^{\mathcal{M}}, m_3^{\mathcal{M}}\rangle \notin T^{\mathcal{M}}$$

3. No, as $\mathcal{M}$ does not define any constant symbols $\mathcal{M} \not\models \exists x\, \varphi$ for some arbitrary φ (as x will bind no constant symbols), and thus

$$\mathcal{M} \not\models \exists x \forall y\, (Qy \leftrightarrow Tyx)$$

Assuming $\mathcal{M}^+$,

$$\mathcal{M}^+ \not\models \exists x \forall y\, (Qy \leftrightarrow Tyx)$$

By fixing x to $m_1^{\mathcal{M}}$, we see that

$$\forall y \cdot y \in Q^{\mathcal{M}} \leftrightarrow \langle y, m_1^{\mathcal{M}} \rangle \in T^{\mathcal{M}}$$

as

$$\begin{gathered} m_1^{\mathcal{M}} \in Q^{\mathcal{M}} \text{ and } \langle m_1^{\mathcal{M}}, m_1^{\mathcal{M}} \rangle \in T^{\mathcal{M}} \\ m_2^{\mathcal{M}} \notin Q^{\mathcal{M}} \text{ and } \langle m_2^{\mathcal{M}}, m_1^{\mathcal{M}} \rangle \notin T^{\mathcal{M}} \\ m_3^{\mathcal{M}} \notin Q^{\mathcal{M}} \text{ and } \langle m_3^{\mathcal{M}}, m_1^{\mathcal{M}} \rangle \notin T^{\mathcal{M}} \end{gathered}$$

Part B

Problem 3

1. (a) $BHMB$ is not a good sandwich
   (b) $(((B)JB)HB)$ is a good sandwich
   (c) $BHBMBJM$ is not a good sandwich
2. **Definition 1** *A sandwich is good iff there is some stage n such that sandwich $\in$ Stage(n) where:*
   - *Stage$(0) = \{B\}$*
   - *Stage$(n+1)$ is the set of φ such that either:*
     (a) $\varphi \in$ Stage(n)
     (b) φ is of the form ψMB, ψHB, or ψJB, where $\psi \in$ Stage(n)
3. Make A the set of all n such that for each $\psi \in$ *Stage*(n), ψ does not contain any two instances of the same ingredient adjacently.

   Base

   $0 \in A$. As *Stage*$(0) = \{B\}$, all $\varphi \in$ *Stage*(0) do not contain any two instances of the same ingredient adjacently. We can also see that all $\varphi \in$ *Stage*(0) are terminated by B

   Induction Step

   Suppose $n \in A$.

   Lemma 1 *Since $n \in A$, and every $\varphi \in$ Stage(n) is of the form B, ψMB, ψHB, or ψJB, for $\psi \in$ Stage(m) for some $m < n$, all φ are terminated by B.*

   Lemma 2 *Since $n \in A$, every $\varphi \in$ Stage(n) does not contain any two instances of the same ingredient adjacently.*

   For each $\varphi \in$ *Stage*$(n+1) \setminus$ *Stage*(n), recalling the definition for *Stage*$(n+1)$, φ is of the form ψMB, ψHB, or ψJB, for some $\psi \in$ *Stage*(n)

   Given

- all $\varphi \in Stage(n)$ are terminated by B (Lemma 1)
- all $\varphi \in \{MB, HB, JB\}$ don't begin with B, and are not themselves adjacent ingredients
- Lemma 2

all $\varphi \in Stage(n+1) \setminus Stage(n)$ do not contain two instances of the same ingredient adjacently.

Thus, $n+1 \in A$. Then by induction, we see that every $n \in A$.

Problem 4

1. $(\to I)$ Suppose d_ψ is a derivation of $\Gamma, \varphi \vdash \psi$, with d_ψ being from stage n.

   The $(\to I)$ rule tells us that the $n+1^{\text{th}}$ stage contains a derivation d of $\Gamma, \varphi \vdash \varphi \to \psi$.

   We must show $\Gamma, \varphi \models \varphi \to \psi$.

   Since $d_\psi \in Stage_{Der}(n)$ we have $\Gamma, \varphi \models \psi$.

   Let $\mathcal{M}$ be a model which $\mathcal{M} \models \gamma$ for all $\gamma \in \Gamma \cup \varphi$.

   Thus we have $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \psi$, so $\mathcal{M} \models \varphi \to \psi$.

   $(\exists E)$ Suppose:

   - $d_\exists$ is a derivation of $\Gamma \vdash \exists x \varphi(x)$
   - d_ψ is a derivation of $\Delta, \varphi(a) \vdash \psi$, where a does not occur in Δ or $\exists x \varphi(x)$

   where all are members of stage n. Then the $(\exists E)$ rule tells us that the $n+1^{\text{th}}$ stage contains a derivation d of $\Gamma, \Delta, \varphi(a) \vdash \psi$.

   We must show that $\Gamma, \Delta, \varphi(a) \models \psi$.

   Since $d_\exists \in Stage_{Der}(n)$, we have $\Gamma \models \exists x \varphi(x)$.

   Since $d_\psi \in Stage_{Der}(n)$, we have $\Delta, \varphi(a) \models \psi$.

   Let $\mathcal{M}$ be a model which $\mathcal{M} \models \gamma$ for all $\gamma \in \Delta \cup \varphi(a)$. Then $\mathcal{M} \models \psi$.

   Suppose that $m = a^{\mathcal{M}}$. Now consider any model $\mathcal{M}'$ which is exactly like $\mathcal{M}$ except that $a^{\mathcal{M}'} = m' \neq a^{\mathcal{M}} = m$; i.e., we let $a^{\mathcal{M}'}$ be some arbitrary other m' from the domain M. Since $\Delta \cup \varphi(a)$ has a sentence with the constant symbol a in it, $\mathcal{M}'$ only models ψ if it models $\varphi(a)$.

   But the fact that all models $\mathcal{M}'$ whose only difference from $\mathcal{M}$ is their interpretation of the symbol a are such that $\mathcal{M}' \models \psi$ just means that:

   $\exists m \in M \; \mathcal{M}' \models \exists x \varphi(x), \varphi(m) \Leftrightarrow \mathcal{M} \models \psi$.

2. Suppose $\psi := \chi \vee \delta \in \Delta$.

   We claim that $\chi \vee \delta$ iff $\chi \in \Delta$ or $\delta \in \Delta$.

   If $\chi \vee \delta \in \Delta$, then by Claim 120 and $(\vee I)$ we have either χ or δ in Δ, and by $(\vee E)$ we have χ *and* δ in Δ.

   If either χ or δ are in Δ then Claim 120 and $(\vee I)$ tell us that $\chi \vee \delta \in \Delta$.

   $$\begin{aligned} \chi \vee \delta \in \Delta &\Leftrightarrow \chi \in \Delta \mid\mid \delta \in \Delta \\ &\Leftrightarrow \mathcal{M}^\Delta \models \chi \mid\mid \mathcal{M}^\Delta \models \delta \\ &\Leftrightarrow \mathcal{M}^\Delta \models \chi \vee \delta \end{aligned}$$

   The first $\Leftrightarrow$ is via our claim; the second $\Leftrightarrow$ is by induction hypothesis; and the last $\Leftrightarrow$ is from the $\models$ definition.

3. Suppose $\psi := \forall x \chi(x)$.

   Then we claim $\forall x \chi(x) \in \Delta$ iff there is some constant symbol $a \in \mathcal{L}(C)$ such that $\chi(a) \in \Delta$.

   Suppose $\forall x \chi(x) \in \Delta$. Then by Claim 120 and $(\forall E)$, there is some $a \in \mathcal{L}(C)$ such that $\chi(a) \in \Delta$.

On the other hand, if for some some $a \in \mathcal{L}(C)$, $\chi(a) \in \Delta$, then by Claim 120 and $(\forall I)$, $\forall x \chi(x) \in \Delta$.

Then we have

$$\begin{aligned}
\forall x \chi(x) \in \Delta &\Leftrightarrow \text{there is some } a \in M^{\Delta} \text{ such that } \chi(a) \in \Delta \\
&\Leftrightarrow \text{there is some } a \in M^{\Delta} \text{ such that } \mathcal{M}^{\Delta} \models \chi(a) \\
&\Leftrightarrow \mathcal{M}^{\Delta} \models \forall x \chi(x)
\end{aligned}$$

The first $\Leftrightarrow$ is via our claim; the second $\Leftrightarrow$ is by induction hypothesis; and the final $\Leftrightarrow$ is was via the $\models$ definition.