

PHIL3110 - Exam

Maxwell Bo

June 21, 2018

Part A

Problem 1

1. Let $\beta = \{\varphi \mid \varphi \in \Gamma \wedge \varphi \in \Delta\}$.

ABORTIVE Say there's some φ such that $\beta \models \varphi$ but $\varphi \notin \beta$. If $\varphi \notin \beta$, then $\varphi \notin \Gamma$ and $\varphi \notin \Delta$. This didn't really go anywhere.

ABORTIVE I looked for a counter-example φ that could both be true in two separate sets, but when stripped of its environment (all the ψ s in which $\psi \in \Gamma$ but $\psi \notin \Delta$, and vice-versa) allowed a derivation to something that was not in Γ and Δ . I found this very difficult.

ABORTIVE I tried reductio ad absurdum.

Assume β is not a theory.

$\varphi \notin \beta$, and $\varphi \notin \Gamma$. If $\varphi \notin \Gamma$, then $\Gamma \models \neg\varphi$, which unless $\Gamma \models \perp$, Γ never $\models \varphi$ to begin with. This sort-of-not-really contradiction feels really flimsy. It feels more a proof of construction of what theories are, than anything about β . Nor does it say anything about the construction of β , critically the fact that $\varphi \in \Gamma \wedge \varphi \in \Delta$.

ABORTIVE I thought about a stage-based definition of β , and trying to prove that it's always a theory at every stage of construction. I figured that was more pain than it was worth, considering that there'd be something like 9 separate stages, when φ is in Γ , φ in Δ , and both, for 0, n and $n + 1$.

FINAL ANSWER

I settled on the following.

$$\begin{aligned}\varphi \in \beta &\Leftrightarrow \varphi \in \Gamma \text{ and } \varphi \in \Delta \\ &\Leftrightarrow \Gamma \models \varphi \text{ and } \Delta \models \varphi \\ &\Leftrightarrow \Gamma, \Delta \models \varphi \\ &\Leftrightarrow \Gamma \cap \Delta \models \varphi \\ &\Leftrightarrow \beta \models \varphi\end{aligned}$$

The first \Leftrightarrow is by the definition of β ; the second by the fact that Γ and Δ are both theories; and the fifth by the definition of β .

The critical leap is $\Gamma, \Delta \models \varphi \Leftrightarrow \Gamma \cap \Delta \models \varphi$.

2. Let Ξ be $\{\varphi\}$ such that for any ψ not equal to φ , $\varphi \not\models \psi$. In other words, φ only derives itself.

Ξ is a theory. By completeness, $\Xi \models \varphi$, and $\varphi \in \Xi$. Furthermore, also by completeness, there is no ψ such that $\Xi \models \psi$ and $\psi \notin \Xi$.

Choose some arbitrary χ . Let $\Gamma = \Xi \cup \{\chi\} \cup \{\text{sentences required to make } \Gamma \text{ a theory}\}$.

Ξ is a proper subset of Γ . Γ is a theory. Yet Ξ is a theory. This serves as a counter-example.

Problem 2

De Morgan's Law 1

$$\frac{\varphi^{(1)} \quad \psi^{(2)}}{\begin{array}{c} \varphi \vee \psi \quad \neg(\varphi \vee \psi) \\ \frac{\perp}{\neg\varphi} \text{ (1) } (\neg I) \end{array}} \quad \frac{\varphi \vee \psi \quad \neg(\varphi \vee \psi)}{\begin{array}{c} \perp \\ \frac{\perp}{\neg\psi} \text{ (2) } (\neg I) \end{array}}$$

$$\neg\varphi \wedge \neg\psi$$

$$(P \rightarrow Q) \rightarrow P \vdash P$$

I spent an entire day on this. The best I could do was a derivation of $Q \rightarrow P$ from the premise. I could complete the proof if I could get a proof of Q from the premise with all assumptions discharged, but, try as I might, I couldn't.

Often I'd want to create $P \rightarrow Q$ s, but I was unable to because I would have to assume the conclusion.

All the proofs I was doing "ran out of steam" very quickly. I'd rack up a lot of unjettisoned assumptions, and in order to jetison them I'd have to assume 1 or more new assumptions. This led me to believe that I had to be using $(\vee - E)$, so this didn't get out of hand. Often I find it difficult to prove the $\varphi \vee \psi$ part of that rule.

The derivation is as follows:

$$\frac{(P \rightarrow Q) \rightarrow P}{P}$$

$$\frac{(P \rightarrow Q)^{(1)} \quad (\rightarrow E)}{\begin{array}{c} \perp \\ \neg P^{(2)} \end{array}}$$

$$\frac{\perp}{\neg(P \rightarrow Q)^{(1)}} \quad (\neg E)$$

$$\frac{\perp}{\neg(P \rightarrow Q)^{(1)}} \quad (1).$$

$$\frac{\neg(P \rightarrow Q)^{(1)}}{\neg(\neg P \vee Q)} \quad (\text{disj. ad})$$

$$\frac{\neg(\neg P \vee Q)}{\begin{array}{c} (P \wedge \neg Q)^{(2)} \\ \frac{\begin{array}{c} \neg Q \\ \neg(\neg P) \end{array}}{\neg(\neg P)^{(3)}} \end{array}}$$

$$\frac{\neg(\neg P)^{(3)}}{P}$$

$$\frac{P}{Q \rightarrow P} \quad (3).$$

Problem 3

Let A be the range of the increasing total recursive function $f : \omega \rightarrow \omega$. We interpret "range" to mean the image of the function, not its co-domain.

Thus,

$$A = \text{im}(f) = \{f(a) \mid a \in \text{dom}(f)\}$$

Recalling DEFINITION 190¹ and DEFINTION 191², we'll construct a function that for any n will verify whether or not it is in A , and halts in finite time.

We'll reason informally about this function.

¹Page 124 of course notes

²Page 124 of course notes

First, we'll assume that $im(f)$ is finite. As f is increasing, it is injective. As it is injective, $dom(f)$ is finite. As f is a total recursive function, it halts in finite time for all inputs in its domain. Therefore, collecting all $f(n)$ for every $n \in dom(f)$ halts in finite time. Checking membership of a finite set halts in finite time. If a is in this set, $a \in A$, otherwise $a \notin A$.

Secondly, we'll assume that $im(f)$ is infinite. For every $n \in dom(f)$, we compute $f(n)$ (in finite time). If $f(n) = a$, we halt, and confirm that $a \in A$. If $f(n) > a$, we halt, and confirm that $a \notin A$. As f is increasing, we can assume that some successive n will never produce something that is equal to a - e.g. we are allowed to halt when $f(n) > a$. As $a \in A$ describes some finite point in the ordered ω , checking that $a \in A$ will halt in finite time.

Problem 4

Given:

- for $x \in M$, x is regular $\Leftrightarrow x \in R^M$
- for $x \in N$, x is perfect $\Leftrightarrow x \in P^N$
- Every derivation in M is regular
- Every derivation in N is perfect

We'll assume for both \mathcal{M} and \mathcal{N} , that for each constant symbol a , $a^{\mathcal{M}} = a$ and $a^{\mathcal{N}} = a$

We can thus conclude that $M = R^M$ and $N = P^N$.

We can say that $\mathcal{M} \models \forall x Rx$ and $\mathcal{N} \models \forall x Px$.

1. \mathcal{N} has a non-empty quantifiable domain, N . If $\mathcal{N} \models \forall x Px$, then $\mathcal{N} \models \exists x Px$. Therefore, $\mathcal{N} \models \exists x Px$.
2. Fix \mathcal{M}^+ such that there is some $m \in \mathcal{L}(C)$, such that $m^{\mathcal{M}^+} = m$, where $m \notin P^{\mathcal{M}^+}$. \mathcal{M}^+ is a valid substitute for \mathcal{M} , as it can be constructed using the rules specified prior.
Thus, $\mathcal{M} \not\models \forall x Px$.
3. Note that $\forall x Rx$ is a Π_1 sentence (a \forall -sentence). By THEOREM 139(2)³, if φ is Π_1 , and $\mathcal{N} \subseteq \mathcal{M}$, and $\mathcal{M} \models \varphi$, then $\mathcal{N} \models \varphi$. As specified prior $\mathcal{M} \models \forall x Rx$. Therefore $\mathcal{N} \models \forall x Rx$.
4. \mathcal{N} has a non-empty quantifiable domain, N . If $\mathcal{N} \models \forall x Px$, then $\mathcal{N} \models \exists x Px$. For $N \subseteq M$ to be true, if $m \in P^N$, then $m \in P^M$. Therefore $m \in P^M$. Therefore $\mathcal{M} \models \exists x Px$.

Part B

Problem 5

1. Let \mathcal{M} be a model with domain $M = \{\}$.

$\exists x(x = x)$ is a consequence.

Furthermore, we run into some strangeness when we find that $\forall x(x \neq x)$ is a consequence.

$\forall x\varphi(x) \Rightarrow \exists x\varphi(x)$ no longer holds either.

In other words, free logic admits existentially quantified formulas always being false in the empty domain, and universally quantified formulas always being true.

Furthermore, the notion that $\neg\exists x\varphi(x) \Rightarrow \forall x\neg\varphi(x)$ ceases to be meaningful.

³Page 85 of course notes

2. Rules

We're also going to consider \exists in the set logical vocabulary, because, why not?

$$\begin{array}{ccccc}
 \varphi \wedge \psi (\wedge) & \neg(\varphi \wedge \psi)(\neg\wedge) & \neg\neg\varphi(\neg\neg) & \forall x\varphi(x)(\forall) & \neg\forall x\varphi(x)(\neg\forall) \\
 | & / \backslash & | & / \backslash & | \\
 \varphi \quad \psi & \varphi \quad \psi & \varphi & \neg E!a \quad \varphi(a) & \exists x(\neg\varphi)(x) \\
 & & & & | \\
 & & & & \text{where } a \text{ is any name}
 \end{array}$$

$$\begin{array}{cc}
 \exists x\varphi(x)(\exists) & \neg\exists x\varphi(x)(\neg\exists) \\
 | & | \\
 E!a \quad \varphi(a) & \forall x(\neg\varphi)(x) \\
 | & \\
 \text{where } a \text{ is new}
 \end{array}$$

where $E!c$ expands to $\exists x(x = c)$ ⁴.

By new we mean that the name a has not occurred anywhere on the branches above.

3. Our goal is to show that if $\models_f \varphi$, then $\vdash_{Tab} \varphi$. In other words, if every model \mathcal{M} (of the language φ) is such that if $\mathcal{M} \models \varphi$ then the tableau commencing with $\neg\varphi$ is closed.

By contraposition, if $\not\vdash_{Tab} \varphi$ then $\not\models_f \varphi$.

In other words, if the tableau commencing with $\neg\varphi$ does not close, then there is some \mathcal{M} such that $M \models \neg\varphi$.

Rather than choosing some open branch B of a tableau commencing with $\neg\varphi$, and defining a \mathcal{M}^B , and reasoning with $+$ -complexity, we'll use a Δ that is maximally consistent⁵ and existentially witnessed⁶ set of sentences.

Let \mathcal{M}^Δ be such that:

- \mathcal{M}^Δ is the set of constant symbols a occurring in the stentences in Δ .
- for each constant symbol a , let $a^{\mathcal{M}} = a$
- for each n -ary relation symbol R of $\mathcal{L}(C)$ let $R^{\mathcal{M}}$ be the set of n -tuples $\langle a_1, \dots, a_n \rangle$ such that Ra_1, \dots, a_n is in Δ .

Reiterating

Claim 120 Suppose Δ is maximal consistent. Then $\varphi \in \Delta \Leftrightarrow \mathcal{M}^\Delta \models \varphi$

but eliding PROOF 120⁷.

We will show $\varphi \in \Delta \Rightarrow \mathcal{M}^\Delta \models \varphi$

We proceed by induction on the complexity of the formula.

Base

Suppose $\psi := \chi$. Then

$$\chi \in \Delta \Leftrightarrow \mathcal{M}^\Delta \models \chi$$

This flows trivially from CLAIM 120.

⁴I had to do this because the forest package wouldn't typeset leaves over a certain number of characters

⁵for an arbitrary φ either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$

⁶if $\exists x\varphi(x) \in \Delta$, then for some constant symbol a , $\varphi(a) \in \Delta$

⁷Page 77 of course notes

Induction Step

- Suppose $\psi := \chi \wedge \delta \in \Delta$.

We claim that $\chi \wedge \delta \in \Delta$ iff both $\chi \in \Delta$ and $\delta \in \Delta$.

If $\chi \wedge \delta \in \Delta$, then by Claim 120 and tableau rule (\wedge), we have both χ and δ in Δ .

$$\begin{aligned} \chi \wedge \delta \in \Delta &\Rightarrow \chi \in \Delta \text{ and } \delta \in \Delta \\ &\Leftrightarrow \mathcal{M}^\Delta \models \chi \text{ and } \mathcal{M}^\Delta \models \delta \\ &\Leftrightarrow \mathcal{M}^\Delta \models \chi \wedge \delta \end{aligned}$$

The first \Rightarrow is via our claim; the second \Leftrightarrow is by induction hypothesis; and the last \Leftrightarrow is from the \models definition.

- Suppose $\psi := \neg\chi$.

If $\chi \in \Delta$, then by CLAIM 120 and tableau rule (\neg), $\chi \notin \Delta$.

$$\begin{aligned} \neg\chi \in \Delta &\Leftrightarrow \chi \notin \Delta \\ &\Leftrightarrow \mathcal{M}^\Delta \not\models \chi \\ &\Leftrightarrow \mathcal{M}^\Delta \models \neg\chi \end{aligned}$$

The first \Leftrightarrow is via the completeness of Δ ; the second \Leftrightarrow is by the induction hypothesis; and the last is via the \models definition.

- Suppose $\psi := \forall x\chi(x)$.

We claim $\forall x\chi(x) \in \Delta$ for every constant symbol $a \in \mathcal{L}(C)$ such that $\chi(a) \in \Delta$.

Suppose $\forall x\chi(x) \in \Delta$.

Suppose $\mathcal{L}(C) \neq \{\}$, which given our definiton of \mathcal{M}^Δ , occurs when $M^\Delta \neq \{\}$.

Then by claim CLAIM 120 and tableau rule (\forall), for all $a \in \mathcal{L}(C)$, $\chi(a) \in \Delta$.

Suppose $\mathcal{L}(C) = \{\}$. We assert that there must be something in Δ that says there is no $a \in \mathcal{L}(C)$.

Then by CLAIM 120 and tableau rule (\forall), there is some φ of the form $\neg\exists y(y = a)$ such that $\varphi \in \Delta$.

On the other hand, if for some some $a \in \mathcal{L}(C)$, $\chi(a) \in \Delta$, then by CLAIM 120 and (\forall), $\forall x\chi(x) \in \Delta$.

Then we have

$$\begin{aligned} \forall x\chi(x) \in \Delta &\Rightarrow \text{for all } a \in M^\Delta, \chi(a) \in \Delta \text{ or } \neg\exists y(y = a) \in \Delta \\ &\Leftrightarrow \text{for all } a \in M^\Delta, \mathcal{M}^\Delta \models \chi(a) \in \Delta \text{ or } \mathcal{M}^\Delta \models \neg\exists y(y = a) \\ &\Leftrightarrow \mathcal{M}^\Delta \models \forall x\chi(x) \end{aligned}$$

The first \Rightarrow is via our claim; the second \Leftrightarrow is by induction hypothesis; and the final \Leftrightarrow is was via the \models definition.

- Suppose $\psi := \exists x\chi(x)$.

Then we claim $\exists x\chi(x) \in \Delta$ iff there is some constant symbol $a \in \mathcal{L}(C)$ such that $\chi(a) \in \Delta$.

Suppose $\exists x\chi(x) \in \Delta$.

By the construction of Δ ⁸, CLAIM 120 and tableau rule (\exists), there is some $a \in \mathcal{L}(C)$ such that $\chi(a) \in \Delta$.

Furthermore we may also say that there exists some φ of the form $\exists y(y = a)$, such that $\varphi \in \Delta$.

We would not be able to say this if Δ were not existentially witnessed.

⁸it is existentially witnessed

Then we have

$$\begin{aligned}\exists x\chi(x) \in \Delta &\Rightarrow \text{there is some } a \in M^\Delta \text{ such that } \chi(a) \in \Delta \text{ and } \exists y(y = a) \in \Delta \\ &\Leftrightarrow \text{there is some } a \in M^\Delta \text{ such that } \mathcal{M}^\Delta \models \chi(a) \text{ and } \mathcal{M}^\Delta \models \exists y(y = a) \\ &\Leftrightarrow \mathcal{M}^\Delta \models \exists x\chi(x)\end{aligned}$$

The first \Rightarrow is via our claim; the second \Leftrightarrow is by induction hypothesis; and the final \Leftrightarrow is via the \models definition.

The other cases are left as exercises for the marker.