PHIL3110 - Assignment 1

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Part A

Problem 1

1.

$$
\begin{prooftree}
\AxiomC{$Pa \lor Qa^{(1)}$}
\AxiomC{$(Pa \lor Qa) \to \bot$}
\BinaryInfC{$\bot$}
\RightLabel{(1) ($\neg I$)}
\UnaryInfC{$\neg(Pa \lor Qa)$}
\RightLabel{(DM-1)}
\UnaryInfC{$\neg Pa \land \neg Qa$}
\end{prooftree}
$$

De Morgan's Law 1

$$
\begin{prooftree}
\AxiomC{$\varphi^{(1)}$}
\UnaryInfC{$\varphi \lor \psi$}
\AxiomC{$\neg(\varphi \lor \psi)$}
\BinaryInfC{$\bot$}
\RightLabel{(1) ($\neg I$)}
\UnaryInfC{$\neg\varphi$}
\AxiomC{$\psi^{(2)}$}
\UnaryInfC{$\varphi \lor \psi$}
\AxiomC{$\neg(\varphi \lor \psi)$}
\BinaryInfC{$\bot$}
\RightLabel{(2) ($\neg I$)}
\UnaryInfC{$\neg\psi$}
\BinaryInfC{$\neg\varphi \land \neg\psi$}
\end{prooftree}
$$

2.

$$
\begin{prooftree}
\AxiomC{$Qa \lor Ra^{(3)}$}
\AxiomC{$Qa^{(1)}$}
\AxiomC{$Qa \to Pa$}
\BinaryInfC{Pa}
\AxiomC{$Ra^{(2)}$}
\AxiomC{$Ra \to Pa$}
\BinaryInfC{Pa}
\RightLabel{(1) (2) ($\lor E$)}
\TrinaryInfC{Pa}
\RightLabel{(3) ($\to I$)}
\UnaryInfC{$(Qa \lor Ra) \to Pa$}
\end{prooftree}
$$

3.

$$
\begin{prooftree}
\AxiomC{$\neg(\forall x Px \to \exists x Rx)$}
\AxiomC{$\forall x Px^{(1)}$}
\AxiomC{$\exists x Rx^{(2)}$}
\BinaryInfC{$\forall x Px \land \exists x Rx$}
\UnaryInfC{$\exists x Rx$}
\RightLabel{(1) ($\to I$)}
\UnaryInfC{$\forall x Px \to \exists x Rx$}
\BinaryInfC{$\bot$}
\RightLabel{(2) ($\neg I$)}
\UnaryInfC{$\neg\exists x Rx$}
\end{prooftree}
$$

Problem 2

1.

$$
\begin{aligned}
Q^{\mathcal{M}} &= \{m_1\} \\
T^{\mathcal{M}} &= \{\langle m_1, m_1\rangle, \langle m_1, m_2\rangle, \langle m_2, m_2\rangle\}
\end{aligned}
$$

2. Distressingly, $\mathcal{L}$ does not define any constant symbols, nor does $\mathcal{M}$ provide interpretations of constant symbols in $\mathcal{M}$.

   Thus

   $$\mathcal{M} \not\models \exists x \neg Txx$$

   However assuming $\mathcal{M}^+$, where $\mathcal{M}^+$ is the expanded model $\mathcal{M}$, where $m^{\mathcal{M}} = m$ for all $m \in M$, we see that

   $$M \models \exists x \neg Txx$$

   as

   $$\langle m_3^{\mathcal{M}}, m_3^{\mathcal{M}}\rangle \notin T^{\mathcal{M}}$$

3. No, as $\mathcal{M}$ does not define any constant symbols $\mathcal{M} \nvDash \exists x\, \varphi$ for some arbitrary φ (as x will bind no constant symbols), and thus

$$\mathcal{M} \nvDash \exists x \forall y\, (Qy \leftrightarrow Tyx)$$

Assuming $\mathcal{M}^+$,

$$\mathcal{M}^+ \nvDash \exists x \forall y\, (Qy \leftrightarrow Tyx)$$

By fixing x to $m_1^{\mathcal{M}}$, we see that

$$\forall y \cdot y \in Q^{\mathcal{M}} \leftrightarrow \langle y, m_1^{\mathcal{M}} \rangle \in T^{\mathcal{M}}$$

as

$$\begin{aligned} m_1^{\mathcal{M}} \in Q^{\mathcal{M}} &\text{ and } \langle m_1^{\mathcal{M}}, m_1^{\mathcal{M}} \rangle \in T^{\mathcal{M}} \\ m_2^{\mathcal{M}} \notin Q^{\mathcal{M}} &\text{ and } \langle m_2^{\mathcal{M}}, m_1^{\mathcal{M}} \rangle \notin T^{\mathcal{M}} \\ m_3^{\mathcal{M}} \notin Q^{\mathcal{M}} &\text{ and } \langle m_3^{\mathcal{M}}, m_1^{\mathcal{M}} \rangle \notin T^{\mathcal{M}} \end{aligned}$$

Part B

Problem 3

1. (a) $BHMB$ is not a good sandwich
   (b) $(((B)JB)HB)$ is a good sandwich
   (c) $BHBMBJM$ is not a good sandwich

2. **Definition 1** *A sandwich is good iff there is some stage n such that sandwich $\in$ Stage(n) where:*

   - *Stage*$(0) = \{B\}$
   - *Stage*$(n+1)$ *is the set of φ such that either:*
     (a) $\varphi \in$ Stage(n)
     (b) φ is of the form ψMB, ψHB, or ψJB, where $\psi \in$ Stage(n)

3. Make A the set of all n such that for each $\psi \in$ *Stage*(n), ψ does not contain any two instances of the same ingredient adjacently.

   Base

   $0 \in A$. As *Stage*$(0) = \{B\}$, all $\varphi \in$ *Stage*(0) do not contain any two instances of the same ingredient adjacently. We can also see that all $\varphi \in$ *Stage*(0) are terminated by B

   Induction Step

   Suppose $n \in A$.

   Lemma 1 *Since $n \in A$, and every $\varphi \in$ Stage(n) is of the form B, ψMB, ψHB, or ψJB, for $\psi \in$ Stage(m) for some $m < n$, all φ are terminated by B.*

   Lemma 2 *Since $n \in A$, every $\varphi \in$ Stage(n) does not contain any two instances of the same ingredient adjacently.*

   For each $\varphi \in$ *Stage*$(n+1) \setminus$ *Stage*(n), recalling the definition for *Stage*$(n+1)$, φ is of the form ψMB, ψHB, or ψJB, for some $\psi \in$ *Stage*(n)

   Given

- all $\varphi \in Stage(n)$ are terminated by B (Lemma 1)
- all $\varphi \in \{MB, HB, JB\}$ don't begin with B, and are not themselves adjacent ingredients
- Lemma 2

all $\varphi \in Stage(n+1) \setminus Stage(n)$ do not contain two instances of the same ingredient adjacently.

Thus, $n+1 \in A$. Then by induction, we see that every $n \in A$.

Problem 4

1. $(\to I)$ Suppose d_ψ is a derivation of $\Gamma, \varphi \vdash \psi$, with d_ψ being from stage n.

   The $(\to I)$ rule tells us that the $n+1^{\text{th}}$ stage contains a derivation d of $\Gamma, \varphi \vdash \varphi \to \psi$.

   We must show $\Gamma, \varphi \models \varphi \to \psi$.

   Since $d_\psi \in Stage_{Der}(n)$ we have $\Gamma, \varphi \models \psi$.

   Let $\mathcal{M}$ be a model which $\mathcal{M} \models \gamma$ for all $\gamma \in \Gamma \cup \varphi$.

   Thus we have $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \psi$, so $\mathcal{M} \models \varphi \to \psi$.

   $(\exists E)$ Suppose:

   - $d_{\exists x \varphi(x)}$ is a derivation of $\Gamma \vdash \exists x \varphi(x)$
   - d_ψ is a derivation of $\Delta, \varphi(a) \vdash \psi$, where a does not occur in Δ or $\exists x \varphi(x)$

   where all are members of stage n. Then the $(\exists E)$ rule tells us that the $n+1^{\text{th}}$ stage contains a derivation d of $\Gamma, \Delta, \varphi(a) \vdash \psi$.

   We must show that $\Gamma, \Delta, \varphi(a) \models \psi$.

   Since $d_{\exists x \psi(x)} \in Stage_{Der}(n)$, we have $\Gamma \models \exists x \varphi(x)$.

   Since $d_\psi \in Stage_{Der}(n)$, we have $\Delta, \varphi(a) \models \psi$.

   Let $\mathcal{M}$ be a model which $\mathcal{M} \models \gamma$ for all $\gamma \in \Gamma \cup \Delta$, with some $a^{\mathcal{M}} = m$.

   $\mathcal{M} \models \psi$ for some arbitrary interpretation of a, as a does not occur in Δ, ψ, or $\varphi(x)$.

   Thus, some $\mathcal{M}'$ with a differing interpretation of a to an object in M than $\mathcal{M}$, $\mathcal{M}' \models \psi$ iff $\mathcal{M}' \models \exists x \varphi(x)$.

   Therefore

   $\exists m \in M \ \mathcal{M}' \models \exists x \varphi(x), \varphi(m) \Leftrightarrow \mathcal{M} \models \psi$.

2. Suppose $\psi := \chi \vee \delta \in \Delta$.

   We claim that $\chi \vee \delta$ iff $\chi \in \Delta$ or $\delta \in \Delta$.

   If $\chi \vee \delta \in \Delta$, then by Claim 120 and $(\vee I)$ we have either χ or δ in Δ, and by $(\vee E)$ we have χ *and* δ in Δ.

   If either χ or δ are in Δ then Claim 120 and $(\vee I)$ tell us that $\chi \vee \delta \in \Delta$.

$$\begin{aligned} \chi \vee \delta \in \Delta &\Leftrightarrow \chi \in \Delta \text{ or } \delta \in \Delta \\ &\Leftrightarrow \mathcal{M}^\Delta \models \chi \text{ or } \mathcal{M}^\Delta \models \delta \\ &\Leftrightarrow \mathcal{M}^\Delta \models \chi \vee \delta \end{aligned}$$

   The first $\Leftrightarrow$ is via our claim; the second $\Leftrightarrow$ is by induction hypothesis; and the last $\Leftrightarrow$ is from the $\models$ definition.

3. Suppose $\psi := \forall x \chi(x)$.

   Then we claim $\forall x \chi(x) \in \Delta$ iff there is some constant symbol $a \in \mathcal{L}(C)$ such that $\chi(a) \in \Delta$.

   Suppose $\forall x \chi(x) \in \Delta$. Then by Claim 120 and $(\forall E)$, there is some $a \in \mathcal{L}(C)$ such that $\chi(a) \in \Delta$.

   On the other hand, if for some some $a \in \mathcal{L}(C)$, $\chi(a) \in \Delta$, then by Claim 120 and $(\forall I)$, $\forall x \chi(x) \in \Delta$.

Then we have

$$\begin{aligned}\forall x \chi(x) \in \Delta &\Leftrightarrow \text{there is some } a \in M^{\Delta} \text{ such that } \chi(a) \in \Delta \\ &\Leftrightarrow \text{there is some } a \in M^{\Delta} \text{ such that } \mathcal{M}^{\Delta} \models \chi(a) \\ &\Leftrightarrow \mathcal{M}^{\Delta} \models \forall x \chi(x)\end{aligned}$$

The first $\Leftrightarrow$ is via our claim; the second $\Leftrightarrow$ is by induction hypothesis; and the final $\Leftrightarrow$ is was via the $\models$ definition.