PHIL3110 - Exam

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Part A

Problem 1

1. Let $\beta = \{\varphi \mid \varphi \in \Gamma \land \varphi \in \Delta\}$.

   Say there's some φ such that $\beta \models \varphi$ but $\varphi \notin \beta$. If $\varphi \notin \beta$, then $\varphi \notin \Gamma$ and $\varphi \notin \Delta$.

2. If there is no proper subset Ξ of Γ such that Ξ is a theory, there always exists a φ such that $\Xi \models \varphi$ but $\varphi \notin \Xi$.

Problem 2

$$(P \to Q) \to P$$

Problem 3

Problem 6

1. Let $\mathcal{M}$ be a model with domain $M = \{\}$.

   $\nexists x(x = x)$ is a consequence.

   Furthermore, we run into some strangeness when we find that $\forall x(x \neq x)$ is a consequence.

   $\forall x \varphi(x) \Rrightarrow \exists x \varphi(x)$ no longer holds either.

   In other words, free logic admits existentially quantified formulas always being false in the empty domain, and universally quantified formulas always being true.

   Furthermore, the notion that $\neg\exists x \varphi(x) \Rrightarrow \forall x \neg\varphi(x)$ ceases to be meaningful.

2. **Rules**

   We're also going to consider $\exists$ in the set logical vocabulary, because, why not?

$\varphi \land \psi (\land)$
|
φ ψ

$\neg(\varphi \land \psi)(\neg\land)$
/ \
φ ψ

$\neg\neg\varphi(\neg\neg)$
|
φ

$\forall x \varphi(x)(\forall)$
/ \
$\neg E!a$ $\varphi(a)$
|
where a is any name

$\neg\forall x \varphi(x)(\neg\forall)$
|
$E!a$ $\varphi(a)$
|
where a is new

$\exists x \varphi(x)(\exists)$
|
$E!a$ $\varphi(a)$
|
where a is new

$\neg\exists x \varphi(x)(\neg\exists)$
|
???

where $E!c$ expands to $\exists x(x = c)$[1].

By new we mean that the name a has not occured anywhere on the branches above.

3. Our goal is to show that if $\models_f \varphi$, then $\vdash_{Tab} \varphi$. In other words, if every model $\mathcal{M}$ (of the language φ) is such that if $\mathcal{M} \models \varphi$ then the tableau commencing with $\neg\varphi$ is closed.

   By contraposition, if $\not\vdash_{Tab} \varphi$ then $\not\models_f \varphi$.

   In other words, if the tableau commencing with $\neg\varphi$ does not close, then there is some $\mathcal{M}$ such that $M \models \neg\varphi$.

   Rather than choosing some open branch $\mathcal{B}$ of a tableau commencing with $\neg\varphi$, and defining a $\mathcal{M}^{\mathcal{B}}$, and reasoning with $+$-complexity, we'll use a Δ that is maximally consistent[2] and existentially witnessed[3] set of of sentences.

   Let $\mathcal{M}^{\Delta}$ be such that:

   - M^{Δ} is the set of constant symbols a occurring in the stentences in Δ.
   - for each constant symbol a, let $a^{\mathcal{M}} = a$
   - for each n-ary relation symbol R of $\mathcal{L}(C)$ let $R^{\mathcal{M}}$ be the set of n-tuples $\langle a_1, ..., a_n \rangle$ such that $Ra_1, ..., a_n$ is in Δ.

   Reiterating

   Claim 120 *Suppose Δ is maximal consistent. Then $\varphi \in \Delta \Leftrightarrow \mathcal{M}^{\Delta} \models \varphi$*

   but eliding Proof 120[4].

   We will show $\varphi \in \Delta \Rightarrow \mathcal{M}^{\Delta} \models \varphi$

   We proceed by induction on the complexity of the forumula.

   Base

   Suppose $\psi := \chi$. Then

   $$\chi \in \Delta \Leftrightarrow \mathcal{M}^{\Delta} \models \chi$$

   This flows trivially from Claim 120.

   Induction Step

   - Suppose $\psi := \chi \wedge \delta \in \Delta$.
     We claim that $\chi \wedge \delta \in \Delta$ iff both $\chi \in \Delta$ and $\delta \in \Delta$.
     If $\chi \wedge \delta \in \Delta$, then by Claim 120 and tableau rule $(\wedge)$, we have both χ and δ in Δ.

     $$\begin{aligned} \chi \wedge \delta \in \Delta &\Rightarrow \chi \in \Delta \text{ and } \delta \in \Delta \\ &\Leftrightarrow \mathcal{M}^{\Delta} \models \chi \text{ and } \mathcal{M}^{\Delta} \models \delta \\ &\Leftrightarrow \mathcal{M}^{\Delta} \models \chi \wedge \delta \end{aligned}$$

     The first $\Rightarrow$ is via our claim; the second $\Leftrightarrow$ is by induction hypothesis; and the last $\Leftrightarrow$ is from the $\models$ definition.

[1] I had to do this because the forest package wouldn't typeset leaves over a certain number of characters
[2] for an arbitrary φ either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$
[3] if $\exists x\varphi(x) \in \Delta$, then for some constant symbol a, $\varphi(a) \in \Delta$
[4] Page 77 of course notes

- Suppose $\psi := \neg\chi$.
  If $\chi \in \delta$, then by Claim 120 and tableau rule $(\neg)$, $\chi \notin \Delta$.

$$\begin{aligned}\neg\chi \in \Delta &\Leftrightarrow \chi \notin \Delta \\ &\Leftrightarrow \mathcal{M}^\Delta \not\models \chi \\ &\Leftrightarrow \mathcal{M}^\Delta \models \neg\chi\end{aligned}$$

  The first $\Leftrightarrow$ is via the completeness of Δ; the second $\Leftrightarrow$ is by the induction hypothesis; and the last is via the $\models$ definition.
- Suppose $\psi := \forall x\chi(x)$.
  We claim $\forall x\chi(x) \in \Delta$ for every constant symbol $a \in \mathcal{L}(C)$ such that $\chi(a) \in \Delta$.
  Suppose $\forall x\chi(x) \in \Delta$.
  Suppose $\mathcal{L}(C) \neq \{\}$, which given our definiton of $\mathcal{M}^\Delta$, occurs when $M^\Delta \neq \{\}$.
  Then by claim Claim 120 and tableau rule $(\forall)$, for all $a \in \mathcal{L}(C)$, $\chi(a) \in \Delta$.
  Suppose $\mathcal{L}(C) = \{\}$. We assert that there must be something in Δ that says there is no $a \in \mathcal{L}(C)$.
  Then by Claim 120 and tableau rule $(\forall)$, there is some φ of the form $\neg\exists y(y = a)$ such that $\varphi \in \Delta$.
  On the other hand, if for some some $a \in \mathcal{L}(C)$, $\chi(a) \in \Delta$, then by Claim 120 and $(\forall)$, $\forall x\chi(x) \in \Delta$.
  Then we have

$$\begin{aligned}\forall x\chi(x) \in \Delta &\Rightarrow \text{for all } a \in M^\Delta,\ \chi(a) \in \Delta \text{ or } \neg\exists y(y = a) \in \Delta \\ &\Leftrightarrow \text{for all } a \in M^\Delta,\ \mathcal{M}^\Delta \models \chi(a) \in \Delta \text{ or } \mathcal{M}^\Delta \models \neg\exists y(y = a) \\ &\Leftrightarrow \mathcal{M}^\Delta \models \forall x\chi(x)\end{aligned}$$

  The first $\Rightarrow$ is via our claim; the second $\Leftrightarrow$ is by induction hypothesis; and the final $\Leftrightarrow$ is was via the $\models$ definition.
- Suppose $\psi := \exists x\chi(x)$.
  Then we claim $\exists x\chi(x) \in \Delta$ iff there is some constant symbol $a \in \mathcal{L}(C)$ such that $\chi(a) \in \Delta$.
  Suppose $\exists x\chi(x) \in \Delta$.
  By the construction of Δ[5], Claim 120 and tableau rule $(\exists)$, there is some $a \in \mathcal{L}(C)$ such that $\chi(a) \in \Delta$.
  Furthermore we may also say that there exists some φ of the form $\exists y(y = a)$, such that $\varphi \in \Delta$.
  We would not be able to say this if Δ were not existentially witnessed.
  Then we have

$$\begin{aligned}\exists x\chi(x) \in \Delta &\Leftrightarrow \text{there is some } a \in M^\Delta \text{ such that } \chi(a) \in \Delta \text{ and } \exists y(y = a) \in \Delta \\ &\Leftrightarrow \text{there is some } a \in M^\Delta \text{ such that } \mathcal{M}^\Delta \models \chi(a) \text{ and } \mathcal{M}^\Delta \models \exists y(y = a) \\ &\Leftrightarrow \mathcal{M}^\Delta \models \exists x\chi(x)\end{aligned}$$

  The first $\Rightarrow$ is via our claim; the second $\Leftrightarrow$ is by induction hypothesis; and the final $\Leftrightarrow$ is was via the $\models$ definition.
  The other cases are left as exercises for the marker.

[5] it is existentially witnessed