PHIL3110 - Exam

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Part A

Problem 1

1. Let $\beta = \{\varphi \mid \varphi \in \Gamma \land \varphi \in \Delta\}$.

   Say there's some φ such that $\beta \models \varphi$ but $\varphi \notin \beta$. If $\varphi \notin \beta$, then $\varphi \notin \Gamma$ and $\varphi \notin \Delta$.

   ABORTATIVE I looked for a counter-example φ that could both be true in two seperate sets, but when stripped of its environment (all the ψs in which $\psi \in \Gamma$ but $\psi \notin \Delta$, and vice-versa) allowed a derivation to something that was not in Γ and Δ. I found this very difficult.

2. Let Ξ be $\{\varphi\}$ such that for any ψ not equal to φ, $\varphi \not\vdash \psi$. In other words, φ only derives itself.

   Ξ is a theory. By completeness, $\Xi \models \varphi$, and $\varphi \in \Xi$. Furthermore, also by completeness, there is no ψ such that $\Xi \models \psi$ and $\psi \notin \Xi$.

   Choose some arbitrary χ. Let $\Gamma = \Xi \cup \{\chi\} \cup \{$ sentences required to make Γ a theory $\}$.

   Ξ is a proper subset of Γ. Γ is a theory. Yet Ξ is a theory. This serves as a counter-example.

Problem 2

$$(P \to Q) \to P$$

Problem 3

Let A be the range of the increasing total recursive function $f : \omega \to \omega$. We interpret "range" to mean the image of the function, not its co-domain.

Thus,

$$A = im(f) = \{f(a) \mid a \in dom(f)\}$$

Recalling DEFINITION 190[1] and DEFINTION 191[2], we'll construct a function that for any n will verify whether or not it is in A, and halts in finite time.

We'll reason informally about this function.

First, we'll assume that $im(f)$ is finite. As f is increasing, it is injective. As it is injective, $dom(f)$ is finite. As f is a total recursive function, it halts in finite time for all inputs in its domain. Therefore, collecting all $f(n)$ for every $n \in dom(f)$ halts in finite time. Checking membership of a finite set halts in finite time. If a is in this set, $a \in A$, otherwise $a \notin A$.

Secondly, we'll assume that $im(f)$ is infinite. For every $n \in dom(f)$, we compute $f(n)$ (in finite time). If $f(n) = a$, we halt, and confirm that $a \in A$. If $f(n) > a$, we halt, and confirm that $a \notin A$. As f is increasing, we can assume that some successive n will never produce something that is equal to a - e.g. we are allowed to halt when $f(n) > a$. As $a \in A$ describes some finite point in the ordered ω, checking that $a \in A$ will halt in finite time.

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Problem 4

Given:

- for $x \in M$, x is regular $\Leftrightarrow x \in R^{\mathcal{M}}$
- for $x \in N$, x is perfect $\Leftrightarrow x \in P^{\mathcal{N}}$
- Every derivation in M is regular
- Every derivation in N is perfect

We'll assume for both $\mathcal{M}$ and $\mathcal{N}$, that for each constant symbol a, $a^{\mathcal{M}} = a$ and $a^{\mathcal{N}} = a$
We can thus conclude that $M = R^{\mathcal{M}}$ and $N = P^{\mathcal{N}}$.
We can say that $\mathcal{M} \models \forall x Rx$ and $\mathcal{N} \models \forall x Px$.

1. $\mathcal{N}$ has a non-empty quantifiable domain, N. If $\mathcal{N} \models \forall x Px$, then $\mathcal{N} \models \exists x Px$. Therefore, $\mathcal{N} \models \exists x Px$.
2. Fix $\mathcal{M}^+$ such that there is some new $m \in \mathcal{L}(C)$, such that $m^{\mathcal{M}^+} = m$, where $m \notin P^{\mathcal{M}^+}$. $\mathcal{M}^+$ is a valid substitute for $\mathcal{M}$, as it can be constructed using the rules specified prior.

   Thus, $\mathcal{M} \not\models \forall x Px$.
3. Note that $\forall x Rx$ is a Π_1 sentence (a $\forall$-sentence). By Theorem 139(2)[3], if φ is Π_1, and $N \subseteq M$, and $\mathcal{M} \models \varphi$, then $\mathcal{N} \models \varphi$. As specified prior $\mathcal{M} \models \forall x Rx$. Therefore $\mathcal{N} \models \forall x Rx$.
4. $\mathcal{N}$ has a non-empty quantifiable domain, N. If $\mathcal{N} \models \forall x Px$, then $\mathcal{N} \models \exists x Px$. For $N \subseteq M$ to be true, if $m \in P^{\mathcal{N}}$, then $m \in P^{\mathcal{M}}$. Therefore $m \in P^{\mathcal{M}}$. Therefore $\mathcal{M} \models \exists x Px$.

Problem 6

1. Let $\mathcal{M}$ be a model with domain $M = \{\}$.

   $\nexists x(x = x)$ is a consequence.

   Furthermore, we run into some strangeness when we find that $\forall x(x \neq x)$ is a consequence.

   $\forall x \varphi(x) \nRightarrow \exists x \varphi(x)$ no longer holds either.

   In other words, free logic admits existentially quantified formulas always being false in the empty domain, and universally quantified formulas always being true.

   Furthermore, the notion that $\neg\exists x \varphi(x) \nRightarrow \forall x \neg\varphi(x)$ ceases to be meaningful.

2. **Rules**

   We're also going to consider $\exists$ in the set logical vocabulary, because, why not?

   $\varphi \land \psi (\land)$
   |
   $\varphi\ \psi$

   $\neg(\varphi \land \psi)(\neg\land)$
   / \
   φ ψ

   $\neg\neg\varphi(\neg\neg)$
   |
   φ

   $\forall x \varphi(x)(\forall)$
   / \
   $\neg E!a$ $\varphi(a)$
   |
   where a is any name

   $\neg\forall x \varphi(x)(\neg\forall)$
   |
   $E!a\ \varphi(a)$
   |
   where a is new

   $\exists x \varphi(x)(\exists)$
   |
   $E!a\ \varphi(a)$
   |
   where a is new

   $\neg\exists x \varphi(x)(\neg\exists)$
   |
   ???

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where $E!c$ expands to $\exists x(x = c)$[4].

By new we mean that the name a has not occured anywhere on the branches above.

3. Our goal is to show that if $\models_f \varphi$, then $\vdash_{Tab} \varphi$. In other words, if every model $\mathcal{M}$ (of the language φ) is such that if $\mathcal{M} \models \varphi$ then the tableau commencing with $\neg\varphi$ is closed.

   By contraposition, if $\nvdash_{Tab} \varphi$ then $\not\models_f \varphi$.

   In other words, if the tableau commencing with $\neg\varphi$ does not close, then there is some $\mathcal{M}$ such that $M \models \neg\varphi$.

   Rather than choosing some open branch $\mathcal{B}$ of a tableau commencing with $\neg\varphi$, and defining a $\mathcal{M}^{\mathcal{B}}$, and reasoning with $+$-complexity, we'll use a Δ that is maximally consistent[5] and existentially witnessed[6] set of of sentences.

   Let $\mathcal{M}^{\Delta}$ be such that:

   - M^{Δ} is the set of constant symbols a occurring in the stentences in Δ.
   - for each constant symbol a, let $a^{\mathcal{M}} = a$
   - for each n-ary relation symbol R of $\mathcal{L}(C)$ let $R^{\mathcal{M}}$ be the set of n-tuples $\langle a_1, ..., a_n\rangle$ such that $Ra_1, ..., a_n$ is in Δ.

   Reiterating

   Claim 120 *Suppose Δ is maximal consistent. Then $\varphi \in \Delta \Leftrightarrow \mathcal{M}^{\Delta} \models \varphi$*

   but eliding Proof 120[7].

   We will show $\varphi \in \Delta \Rightarrow \mathcal{M}^{\Delta} \models \varphi$

   We proceed by induction on the complexity of the forumula.

   Base

   Suppose $\psi := \chi$. Then

   $$\chi \in \Delta \Leftrightarrow \mathcal{M}^{\Delta} \models \chi$$

   This flows trivially from Claim 120.

   Induction Step

   - Suppose $\psi := \chi \wedge \delta \in \Delta$.
     We claim that $\chi \wedge \delta \in \Delta$ iff both $\chi \in \Delta$ and $\delta \in \Delta$.
     If $\chi \wedge \delta \in \Delta$, then by Claim 120 and tableau rule ($\wedge$), we have both χ and δ in Δ.

     $$\begin{aligned} \chi \wedge \delta \in \Delta &\Rightarrow \chi \in \Delta \text{ and } \delta \in \Delta \\ &\Leftrightarrow \mathcal{M}^{\Delta} \models \chi \text{ and } \mathcal{M}^{\Delta} \models \delta \\ &\Leftrightarrow \mathcal{M}^{\Delta} \models \chi \wedge \delta \end{aligned}$$

     The first $\Rightarrow$ is via our claim; the second $\Leftrightarrow$ is by induction hypothesis; and the last $\Leftrightarrow$ is from the $\models$ definition.

[4] I had to do this because the forest package wouldn't typeset leaves over a certain number of characters

[5] for an arbitrary φ either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$

[6] if $\exists x\varphi(x) \in \Delta$, then for some constant symbol a, $\varphi(a) \in \Delta$

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- Suppose $\psi := \neg\chi$.
  If $\chi \in \delta$, then by Claim 120 and tableau rule $(\neg)$, $\chi \notin \Delta$.

$$\begin{aligned}\neg\chi \in \Delta &\Leftrightarrow \chi \notin \Delta \\ &\Leftrightarrow \mathcal{M}^\Delta \not\models \chi \\ &\Leftrightarrow \mathcal{M}^\Delta \models \neg\chi\end{aligned}$$

  The first $\Leftrightarrow$ is via the completeness of Δ; the second $\Leftrightarrow$ is by the induction hypothesis; and the last is via the $\models$ definition.
- Suppose $\psi := \forall x\chi(x)$.
  We claim $\forall x\chi(x) \in \Delta$ for every constant symbol $a \in \mathcal{L}(C)$ such that $\chi(a) \in \Delta$.
  Suppose $\forall x\chi(x) \in \Delta$.
  Suppose $\mathcal{L}(C) \neq \{\}$, which given our definiton of $\mathcal{M}^\Delta$, occurs when $M^\Delta \neq \{\}$.
  Then by claim Claim 120 and tableau rule $(\forall)$, for all $a \in \mathcal{L}(C)$, $\chi(a) \in \Delta$.
  Suppose $\mathcal{L}(C) = \{\}$. We assert that there must be something in Δ that says there is no $a \in \mathcal{L}(C)$.
  Then by Claim 120 and tableau rule $(\forall)$, there is some φ of the form $\neg\exists y(y = a)$ such that $\varphi \in \Delta$.
  On the other hand, if for some some $a \in \mathcal{L}(C)$, $\chi(a) \in \Delta$, then by Claim 120 and $(\forall)$, $\forall x\chi(x) \in \Delta$.
  Then we have

$$\begin{aligned}\forall x\chi(x) \in \Delta &\Rightarrow \text{for all } a \in M^\Delta,\ \chi(a) \in \Delta \text{ or } \neg\exists y(y = a) \in \Delta \\ &\Leftrightarrow \text{for all } a \in M^\Delta,\ \mathcal{M}^\Delta \models \chi(a) \in \Delta \text{ or } \mathcal{M}^\Delta \models \neg\exists y(y = a) \\ &\Leftrightarrow \mathcal{M}^\Delta \models \forall x\chi(x)\end{aligned}$$

  The first $\Rightarrow$ is via our claim; the second $\Leftrightarrow$ is by induction hypothesis; and the final $\Leftrightarrow$ is was via the $\models$ definition.
- Suppose $\psi := \exists x\chi(x)$.
  Then we claim $\exists x\chi(x) \in \Delta$ iff there is some constant symbol $a \in \mathcal{L}(C)$ such that $\chi(a) \in \Delta$.
  Suppose $\exists x\chi(x) \in \Delta$.
  By the construction of Δ[8], Claim 120 and tableau rule $(\exists)$, there is some $a \in \mathcal{L}(C)$ such that $\chi(a) \in \Delta$.
  Furthermore we may also say that there exists some φ of the form $\exists y(y = a)$, such that $\varphi \in \Delta$.
  We would not be able to say this if Δ were not existentially witnessed.
  Then we have

$$\begin{aligned}\exists x\chi(x) \in \Delta &\Leftrightarrow \text{there is some } a \in M^\Delta \text{ such that } \chi(a) \in \Delta \text{ and } \exists y(y = a) \in \Delta \\ &\Leftrightarrow \text{there is some } a \in M^\Delta \text{ such that } \mathcal{M}^\Delta \models \chi(a) \text{ and } \mathcal{M}^\Delta \models \exists y(y = a) \\ &\Leftrightarrow \mathcal{M}^\Delta \models \exists x\chi(x)\end{aligned}$$

  The first $\Rightarrow$ is via our claim; the second $\Leftrightarrow$ is by induction hypothesis; and the final $\Leftrightarrow$ is was via the $\models$ definition.
  The other cases are left as exercises for the marker.

[8] it is existentially witnessed