Objective: produce a numerical scheme for wave propagation in spacetime-varying media.

In one spatial dimension wave propagation is described by[1],

$$\bar{\kappa}(x,t)\cdot q(x,t)_t + f_x(q,x) = \psi(q,x,t) \tag{1}$$

The numerical scheme is given by,

$$\frac{\partial Q_i}{\partial t} = \frac{1}{K_i \Delta x}\left(A^+\Delta q_{i-1/2} + \mathcal{A}^-\Delta q_{i+1/2} + \mathcal{A}\Delta q_i\right) \tag{2}$$

where

$$\mathcal{A}^-\Delta q = \sum_{p,s^p\leq 0} \mathcal{Z}^p, \tag{3a}$$

$$\mathcal{A}^+\Delta q = \sum_{p,s^p\geq 0} \mathcal{Z}^p, \tag{3b}$$

where $\mathcal{Z}$ is commonly known as the f–wave.

And the total fluctuation,

$$f(q^L_{i+1/2}) - f(q^R_{i-1/2}) - \Delta x\psi(q_l,q_r) \tag{4}$$

We use f–wave Riemann solvers to decompose the flux difference in waves,

$$f(q_r) - f(q_l) = \sum_p \beta^p r^p = \sum_p \mathcal{Z}^p, \tag{5}$$

1 Riemann solver

For Maxwell's equation in 1D we have:

$$q = \begin{pmatrix} q_e \\ q_h \end{pmatrix}, \qquad f = \begin{pmatrix} \frac{q_h}{\eta_e^0} \\ \frac{q_h}{\eta_h^0} \end{pmatrix} \tag{6}$$

with,

$$\bar{\kappa} = \begin{pmatrix} \eta_e(x,t) & 0 \\ 0 & \eta_h(x,t) \end{pmatrix} \tag{7}$$

and, in the case of linear materials $\psi = \partial_t\bar{\kappa}\cdot q$.

The the flux Jacobian, f_q, becomes:

$$f_q = A = \begin{pmatrix} 0 & 1/\eta_e^0 \\ 1/\eta_h^0 & 0 \end{pmatrix}. \tag{8}$$

[1]Note that in the problems we are interested in the flux f does not depend explicitly on *time*.

With resulting eigenvectors, r^p.and eigenvalues, s^p:

$$r^1 = \begin{pmatrix} -Z \\ 1 \end{pmatrix}, \qquad s^1 = -c(x,t), \tag{9a}$$

$$r^2 = \begin{pmatrix} +Z \\ 1 \end{pmatrix}, \qquad s^2 = c(x,t), \tag{9b}$$

where Z is the impedance,

$$Z = \sqrt{\frac{\eta_e^0}{\eta_h^0}}. \tag{10}$$

and eigenvalues, $s^{1,2}$:

$$s^{1,2} = \pm c(x,t) = \frac{\pm 1}{\sqrt{\eta_e^0 \eta_h^0}}. \tag{11}$$

For the Riemann problem between the cells $i-1$ and i, the f–waves, $\mathcal{Z}^{1,2}$, are:

$$\mathcal{Z}^1 = \beta^1 r_{i-1}^1, \qquad \mathcal{Z}^2 = \beta^2 r_i^2. \tag{12}$$

Let $R = [r^1\ r^2]$, then we can determine β^i by solving the system:

$$R\beta = f_i(Q_i) - f_{i.1}(Q_{i-1}) = \Delta f. \tag{13}$$

To obtain

$$\beta^1 = \frac{-\Delta f^1 + \Delta f^2 Z_i}{Z_i + Z_{i-1}}, \tag{14a}$$

$$\beta^2 = \frac{\Delta f^1 + \Delta f^2 Z_{i-1}}{Z_i + Z_{i-1}}. \tag{14b}$$

2 Algorithm

A succinct description of the numerical algorithm is,

1. Set the initial cell averages of $\bar{\eta}_l(x_i, t=0)$, $\bar{\eta}_l(x_i, t=0)_t$ and $q(x_i, t=0) \equiv Q_i$
2. Reconstruction
   (a) Solve the Riemann problem at $x_{i-1/2}$ using the cell averages Q_{i-1}, Q_i
   (b) Using WENO compute reconstructed piecewise function
   (c) Get states $q_{i-1/2}^R, q_{i+1/2}^L$
3. Compute the fluctuations, $\mathcal{A}^+\Delta q_{i-1/2}, \mathcal{A}^-\Delta q_{i+1/2}$. We do this by solving the Riemann problem with initial states $(q_{i-1/2}^L, q_{i+1/2}^R)$. *We do not incorporate the effect of the source on this step, rather we will subtract it in the total fluctuation*

4. Calculate the total fluctuation $\mathcal{A}\Delta q_i$, use states $q^L_{i+1/2}, q^R_{i-1/2}$
5. Add the net effect of all waves going to the right, left and the total fluctuation
6. Integrate in time and get Q^{n+1}