Behavioral Game Models for Real-World Games

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Abstract

Behavioral game theory has mostly relied on lab experiments using "simple games" developed by economists like beauty contest and centipede games. This is problematic because such games are complex for participants due to unfamiliarity and unrelated to real-world decision making. In this paper, I suggest a rethinking of the ideology and show that it is possible to use real-world games in behavioral game experiments. Using mancala heuristics—sowing games commonly played in Africa, I show that standard behavioral game theory models (level k and cognitive hierarchy) are inadequate to explain the more realistic multi-dimensional reasoning processes required in mancala.

Keywords: Behavioral game theory, Heuristics, Stochastic choice, Tensor.

1.0. Introduction

Behavioral game theory has over the past few decades offered explanations to non-equilibrium behaviors of economic agents—behaviors that are inconsistent with conventional game-theoretic predictions (e.g., Nash equilibrium). Experimental games in lab settings have been the springboard of new theories that have enriched the predictions of agents under strategic situations (see a review by Crawford et al 2013). The models commonly used by economists include level-*k* (Nagel 1995, Costa-Gomes and Crawford 2006) and cognitive hierarchy (CH) model (Camerer et al 2004). These models have led to several analyses in tax policy (Farhi and Gabaix 2020), monetary policy (Farhi and Werning 2019), auction theory (Camerer, Nunnari and Palfrey 2016) and managerial ability (Goldfarb and Xiao 2011; Goldfarb and Yang 2009). Motivated by actual games, economists have created games that are far removed from the games people play in their daily lives and have used the new models to explain what is observed in such games. This has the advantage of showcasing only the simple aspects of the games that matter to the game theoretic predictions. That is, "economics games" are not muddled with emotional, social and other complexities of game rules.

There are two downsides to this practice. First, though "economics games" have simple rules and have direct relationships to economic theory predictions; most research participants are not familiar with these games as such a lot of mistakes are made that are difficult to disentangle from the game choices (Mkondiwa

2020). Games that people play in culture[1] in contrast have many rules and are very complex in mathematical terms but are very simple to the people who play them. Second, it is difficult to imagine how play in these artificial "economics games" relates to how people play in actual games and even more to the purported reason of the experiments—actual decision making. The goal of this paper is to change how economists and other social scientists approach behavioral games by developing the foundational theory and review literature on how to use games played in real-world or culture (e.g., mancala) in developing behavioral models that are closely related to people's decision making.

Resolving the issue of how to use games in culture to recover economic parameters is useful because of the ongoing debates that when "economics games" are taken out of the lab to another lab in a different context or in the field, the unfamiliarity with any related games in those contexts delivers mixed results (see for example a debate between Levitt and List (2007, 2015) and Camerer (2015)). A well-known study by Henrich et al (2010) demonstrates this puzzle. Henrich and coauthors characterized the conventional experimental subjects in ultimatum games as WEIRD (Western, Educated, Industrialized, Rich, and Democratic) because their responses are different from others. However, it is difficult to know whether this is an artifact of the game itself (ultimatum games). Certainly, some societies have had games like ultimatum games. Perhaps, ultimatum games are "weird" games for the "weird" economists. While debates in the economics literature have focused on lab-field experiments, only few studies have used games that people play in culture (e.g., rock-paper-scissors game by Batzilis et al. 2019 and hide-and-seek game by Crawford and Iriberri 2007; Li and Camerer 2020). These examples clearly demonstrate that the apathy towards real-world games by behavioral game researchers is unwarranted. It is important to ask: who plays ultimatum games or beauty context games at home with their friends and family? Unlike "economics games" which are often highly contrived[2], exceptional and artificial; games played in culture are the norm and thus represent well the decision-making heuristics[3] in that society.

The key contribution of the paper is in showing that behavioral game theory can be conducted using any games relevant to the cultures. The three main results that allow the use of mancala in behavioral game

[1] Culture in the "games in culture" phrase as commonly used in game anthropology is being used loosely to refer to real-world games.
[2] Borrowing from Aumann's definition (Aumann, 2019), these games are contrived because the experimenter builds a situation (or game) to get the effect he/she wants to get and gets it.
[3] Heuristics are criteria, methods, or principles for deciding which among several alternative courses of action promises to be the most effective in order to achieve some goal (Pearl 1984, p.3).

theory are: (i) stochastic choice functions[1] expressed as mixture choice data[2] are rationalizable by a logit quantal CH model (proposition 1), (ii) mancala game presents such mixture choice data (proposition 2), and therefore (iii) mancala can be used in quantal CH experiments (proposition 3) that take into account the network structure of choices. These results also apply to other games in culture like Go. For instance, a recent paper by Igami (2020) has shown how discrete choice modeling in structural estimation is closely linked to AI of games played in culture like Go. In terms of practical implementation of the approach proposed in this paper through either experimentation or observation, I relate the approach to two strands of literature. The first is on game experiments to measure salience either in location games using quantal CH models (Li and Camerer 2020) or in hide and seek games using level-*k* models (Crawford and Iriberri 2007). The second is on consumer choice in product networks (Masatlioglu and Suleymanov 2019; Ruiz et al 2020).

The rest of this section discusses the details on mancala and related games, and the related literature. I present the link between AI models of mancala (especially probabilistic opponent model search) and standard behavioral game theory models in section 2. In section 3, I provide modifications of a CH model to accommodate mancala complexity and the nature of experiments that can be conducted. In section 4, I provide a catalogue of mancala rules, winning strategies, and heuristics that are consistent with extensions of quantal CH predictions. I finally conclude in section 5.

1.1. Mancala and related games

Mancala is a generic term for hundreds of zero-sum games of perfect information characterized by sowing and counting (see a picture of the board for a variant of mancala, *Bao* in figure 1) that are played in all parts of Africa and around the world. In the case of Africa, Mkondiwa (2020b) compiled a database of mancala games in Africa comprising of 102 uniquely matched ethnic group-mancala game combinations. This database has the details on geo-coded location of the ethnic group and mancala game, rules of the game, complexity classifications, number of seeds used in the game, whether played by two or more players, allowable directions of play and dimensions of the board. In this paper, I use *Bao*, a complex mancala variant played in Malawi and Tanzania to demonstrate how the game can be used in behavioral game

[1] A stochastic choice function is defined as follows: Let X be an arbitrary finite set and A for the collection of all nonempty subsets of X (menus). A stochastic choice function is defined as a function $\rho: X \times A \rightarrow [0,1]$ such that $\sum \rho(x, A) = 1$ and $\rho(y, A) = 0$ for every $A \in X$ and $y \in X \backslash A$ (Ok and Tserenjigmid 2020). Interested readers are referred to Chambers and Echenique (2016) for details on how stochastic choice functions are used in economics.

[2] According to Dardanoni et al (2020a), mixture choice data consist of the joint distribution of choices of a population of agents from a collection of menus. If menus are independent, then mixture choice data are identifiable by a stochastic choice function.

experiments. The rules for the game are discussed in Mkondiwa (2020a, pp. 152-154) and adapted for the simple version in appendix A.

Using the parlance of game theory, I summarize some of the features of *Bao* (including some of its cousins). The game is played by two players (top and bottom). In other variants of mancala, the game is played by more than two players. It is a zero-sum game of perfect information, with moves that are sequential and asymmetric. The goal is to capture seeds of the opponent so that he/she does not have any seed in inner row or to immobilize the opponent so that he/she cannot make a legal move. Therefore, one can lose the game even while having more seeds. A move consists of picking seeds from a pocket with at least two seeds, drop one in each pocket in a direction chosen towards a pocket in which the corresponding inner pocket of the opponent has seeds to be captured (see appendix B for example of game moves). The captured seeds are the dropped one seed for each pocket from the far left or right pockets in the inner row. A move ends when the last seed is dropped into an empty pocket[1].

Figure 1: Mancala (*Bao*) board game

1.2. Related literature

This paper draws from and contributes to three literatures: behavioral game theory, artificial intelligence, and game anthropology. This is not a first attempt to connect these fields. There are two connections that have been suggested in the literature. First, the links between behavioral game theory and artificial intelligence were introduced by Wright and Leyton-Brown (2019) (see a review by Camerer 2017). Following Wright and Leyton-Brown, hybrid models of machine learning and behavioral models have been used by Fudenberg and Liang (2019) and Li and Camerer (2020). For complex and large games (those with

[1] Be warned: These rules may be confusing for anyone who has never played a variant of mancala games but perhaps just mildly confused as compared to research participants with any of the "economic games".

large game trees), the motivations of the opponent model search algorithm in computer game playing and the cognitive hierarchy models in human decision making are the same: computational and human inability to use game theoretic procedures like backward induction to play the games (Donkers et al 2001 and Camerer et al 2004). A recent review of recursive modeling of opponent's decision making by Doshi, Gmytrasiewicz, and Durfee (2020) also recognized the connection to cognitive hierarchy models commonly used in economics.

Second, Allis et al (1994), Donkers et al (2001), Akinyemi et al (2009) and others have showcased the use of artificial intelligence in mancala games e.g., *Bao*, using insights from game anthropology (especially heuristics used by players to win games) to build the algorithms. Such algorithms include greedy, minimax, alpha-beta pruning, and Monte-Carlo Tree Search (MCTS). The algorithms that have been found to be better are hybrid algorithms that embed expert heuristics to the traditional AI algorithms. This paper uses mancala games to illustrate the advantages of getting insights from these fields. Throughout the paper, I use one key insight related to complexity: complexity of mancala rules, strategic sophistication, and computational complexity. The game anthropology literature suggests that it takes many years for one to learn to play mancala games well due to the many rules. For this reason, computer scientists and artificial intelligence experts of games have relied on stripping most of the mancala rules and creating simple games that are amenable to computational analysis. For example, Allis (1994) created *Kalah*, a game related to *Aware* for computational reasons. According to Rovaris (2017), *Awari* was also adapted from *Wari* rules by Allis et al (1991) for computational purposes. Such simplifications have led to several insights in artificial intelligence and have provided avenues of increasing the scholarship on these games and their computerization.

Nonetheless, the level of computational complexity that the algorithms computer scientists use on these simple versions of mancala is extremely high and far much removed from the strategies used by mancala experts. This is where insights from behavioral game theory can be useful in that they bring forth simple non-equilibrium algorithms like level-k and cognitive hierarchy models which can demonstrate how AI and computational algorithms are an upper bound in the level-k complexity but too simplistic on rules as compared to how humans play the games. Level-k and CH models can also be extended using insights from AI of mancala games because the complexity of mancala games implies AI algorithms for these games use heuristics to approximate payoffs which is closer to human decision making than the assumption of known payoffs in CH models. Unlike games used in CH models (e.g., centipede and beauty contest games) which assume that under unlimited levels of thinking the results should be the same as the Nash equilibrium; most mancala games do not have a solution. A CH model of mancala therefore must take into account that the payoffs in each subgame or move as at best approximations just as one would expect in real life decision making.

Figure 2 illustrates these important relationships. This study seeks to propose a rethinking in behavioral game theory that relies on game anthropology especially of games that are played in cultures rather than economic games that economists have developed. Such a rethinking in the case of mancala games can start with learning the tools in AI of using game heuristics or strategies from game anthropology for complex games (e.g., Divilly et al. 2013) and game theoretic features (e.g., maximin) within behavioral game theoretic models (e.g., Wright and Leyton-Brown 2019; Fudenberg and Liang 2019). Game anthropology therefore adds decision making rules unrelated to the cognitive costs imposed by the game features. Such additional factors including emotions, moral standards, and social norms (Heller and Winter 2016) may matter more both in real-world economic decisions and mancala play yet these are abstracted away in most economic games. I show that such a rethinking is possible through a key insight that the underlying connection between behavioral game models, AI of games, game theory, and anthropology of games is a stochastic choice function. In the parlance of game theory, this is called stochastic game theory (Goeree et al 2016, 2020).

Two sets of papers that are in the spirit of the approach suggested in this paper are: (i) on design of experiments for complex games by Arad and Rubinstein (2012), and Li and Camerer (2020), and (ii) data analysis of data from complex games by Chen et al (2015) and Igami (2020). The first set of papers deals with how to design experiments or observational studies using complex games. Arad and Rubinstein (2012) like this paper also argues for using complex games in behavioral game experiments such that it is possible to develop new decision procedures. They used Colonel Blotto game—a complex game with a large and unordered strategy space—to showcase how a new multi-dimensional iterative reasoning procedure is more appropriate than standard level *k* and cognitive hierarchy models. Li and Camerer (2020) uses eye tracking and choice data from a salience algorithm of locations in hider-seeker games. Data from mancala can be collected using similar approaches. The second set of papers deals with the analytics of data from complex games. Chen et al (2015) suggest a general modeling framework, a sequential Bayesian network approach to analyzing human behavior in real-world games while Igami (2020) derives the equivalence between AI tools like reinforcement learning and econometric models of dynamic discrete choices. This paper is complementary to this literature in that this paper proposes use of heuristics-derived stochastic choice response functions when explaining human behavior while Chen et al (2015) and Igami (2020) focus on developing frameworks for predicting human behavior but may fail at explaining the behaviors observed.

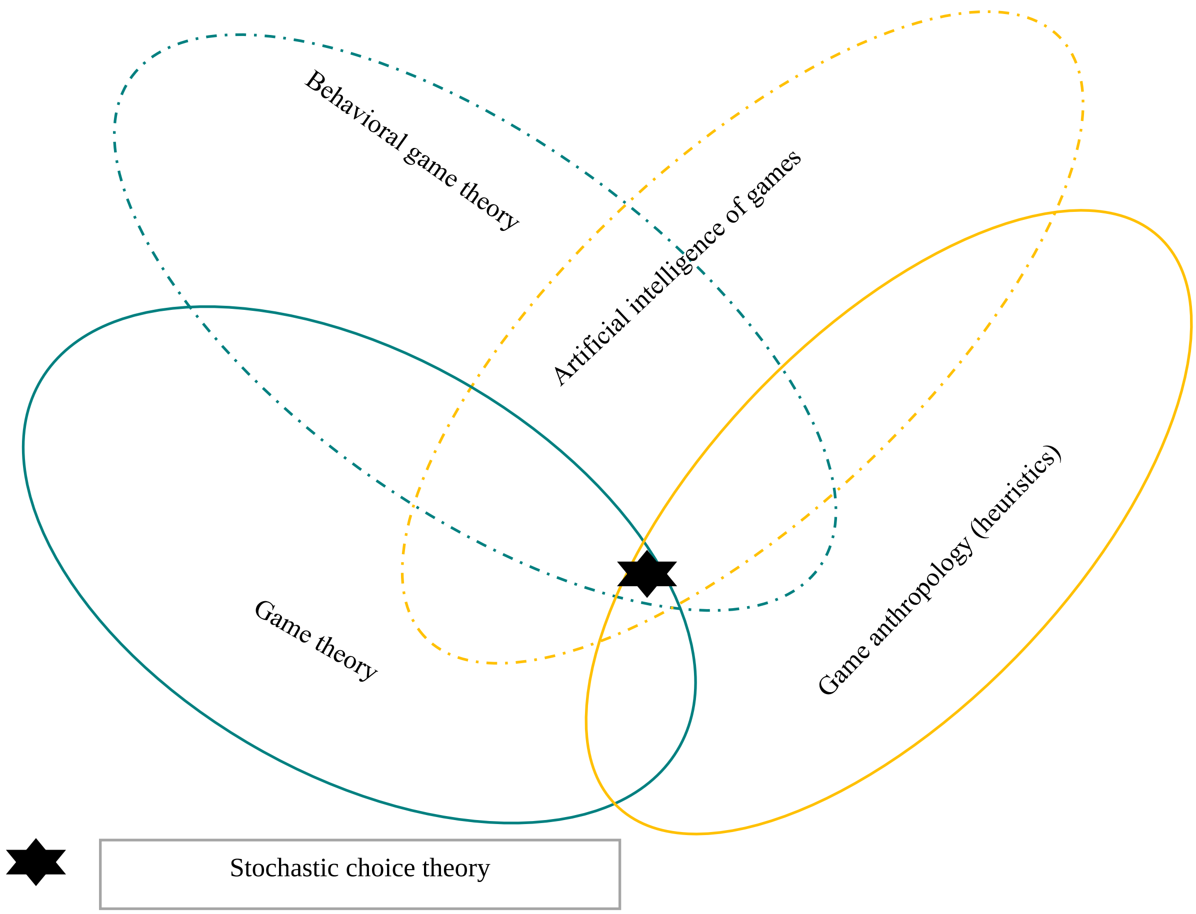

Figure 2: Schematic view of using games to understand economic behavior. Note: the solid lines show active areas of interdisciplinary research while the dotted line shows a neglected link.

2.0. AI of Mancala and Cognitive Hierarchy Model

In this section I provide a brief review of opponent-model search algorithms in artificial intelligence of mancala games and the cognitive hierarchy model in behavioral game theory. I specifically focus on the model assumptions, strategies, algorithm, solutions and extensions. According to Donkers et al (2004), opponent-model search is a game-tree search method that explicitly uses the knowledge of the opponent in order to exploit weak points in the opponent's search strategy. The key extension to the opponent-model search that we use in this study is the probability search model defined by the assumptions in definition 1 (Donkers et al 2001).

Definition 1: *Probabilistic search model is defined by the following assumptions (Donkers et al 2001),*

1) A player has knowledge of n *different opponent types* $\omega_0, \ldots, \omega_{n-1}$

2) All opponent types are assumed to use the same search-tree depth and the same move ordering as the player

3) A Player has subjective probabilities $Pr(\omega_i)$ *on the range of opponents, such that* $\sum_i Pr(\omega_i) = 1$.

4) A player uses a mixed strategy which consists of the n *opponent-type strategies. At every move, a player is supposed to pick randomly one strategy according to the opponent-type probabilities* $Pr(\omega_i)$.

These assumptions are closely related to cognitive hierarchy model assumptions in definition 2.

Definition 2: *Cognitive hierarchy model is defined by the following elements (Camerer et al 2004; Camerer 2015),*

1) A distribution of the frequency of level types, $f(k)$*;*

2) Specified actions of level-0 players (who act heuristically and do not have beliefs about other players);

3) Beliefs of level-k players (for $k = 1,2, \ldots$*) about what other players choose and how common they are;*

4) Assessment of expected payoffs for each level based on beliefs in (3);

5) A stochastic choice response function for each level-k based on the expected payoffs in (4).

In addition to these elements, the game in CH model consists of iterative decision rules for players doing k steps of thinking and the frequency distribution $f(k)$ is assumed to be Poisson of step k players. Step k thinkers assume that their opponents are distributed, according to a normalized Poisson distribution. It is clear from these CH elements that they are closely related to the elements in the probabilistic opponent search model used in AI of mancala. A review of search models in games by Albrecht and Stone (2018) showed that the cognitive hierarchy model (Camerer et al 2004) and opponent-model search are just different types of autonomous agents modelling of other agents. Table 1 shows the relationship between the probabilistic opponent-model search and cognitive hierarchy model using five model features.

First, both models require a payoff function as commonly known in game theory or as an evaluation function as commonly known in AI. Second, a player has beliefs on the probability types of the opponent. In the case of CH models, this corresponds to beliefs on the levels of thinking while in probabilistic opponent model it is based on arbitrary belief types. Third, the stochastic choice response function is similar. Fourth, the strategic choice probability is defined by a best response in both models. Fifth, CH model relies on iterative strategic reasoning (hierarchy of beliefs on the opponent) while probabilistic opponent search model relies on type-based search. In order to effectively use cognitive hierarchy model in mancala games and use the existing knowledge from AI especially probabilistic-model search, a hybrid model containing important features in each of the models is needed. Two important distinguishing features are: assumption on the behavior of level 0 player and parametrization of the opponent type probability. First, as shown in table 1, a CH model assumes a uniform distribution of play for level 0 players. The probabilistic-opponent search model to the contrary uses a set of maximin payoff equations for both parent and child nodes in the game tree. According to Wright and Leyton-Brown (2019), using a linear combination of binary features like maximin improves on performance of Poisson CH that assumes uniform distribution of payoffs.

Second, using the hierarchy of beliefs, the probability of the opponent types is parametrized to a relative proportion of Poisson distributions in CH while in probabilistic opponent search model one must rely on type-based probability space. Strzalecki (2014, p. 5: property 1) proved the equivalence between hierarchy of beliefs approach (like level-*k* and cognitive hierarchy) and the probability type space using earlier Harsanyi's (1967) characterization of these in the case of games of incomplete information. In the probability type space approach, he christened level-*k* as immediate-predecessor type space and cognitive hierarchy as common conditions type space. The parametrization using the Poisson distribution in CH model is favourable because it introduces recursion that allows the solution to be derived from recursive methods which are much easier, intuitive and closely related to human decision making than type- based search. Camerer and Smith (2012, p.348) claimed that recursive methods are easy to program and solve numerically. In AI, recursive modelling methods are extensively used also to solve games (see a review by Doshi et al 2019).

Table 1: Probabilistic opponent-model search and cognitive hierarchy model

Model feature	**Cognitive hierarchy model (Camerer et al 2004)**	**Probabilistic opponent model (Donkers et al 2001)**
Payoff/evaluation function value	*Uniform distribution* for level 0 player, and opponent best response given beliefs about steps of thinking *Maximize payoffs*, $\pi_i(s_i^j, s_{-i}^{j\prime})$	*Maximin* Player i *maximizes evaluation function value* $V_i(P_j)$ Where P_j is player j's move P
Opponent type probability	*Hierarchy of beliefs: k*-step players have accurate beliefs of relative proportions of players doing less thinking, $$g_k(h) = \frac{f(h)}{\sum_{l=0}^{k-1} f(l)}$$ Assumes $f(h)$ follows a Poisson distribution, where $$f(k) = \frac{e^{-\tau}\tau^k}{k!}$$	*Opponent probability types:* Subjective probability of opponent types, $\Pr(\omega_i)$. The opponent is assumed to use a weaker evaluation function
Stochastic choice response function	Expected payoff to a k-step thinker from choosing strategy s_i^j is $$E_k\left(\pi_i(s_i^j)\right) = \sum_j^m \pi_i(s_i^j, s_{-i}^{j\prime}) \left\{\sum_h^k g_k(h) \times P_h(s_i^j)\right\}$$	The search value is the expected value of child nodes $$E(P) = \sum_j v_i(P_j)\{\sum_i \Pr(\omega_i) \times W_i(P_j)$$
Strategy choice probability	*Best response* For optimal strategy, s_i^* $$P(s_i^*) = 1$$ If and only if $s_i^* = argmax\, s_i^j E_k(\pi_i(s_i^j)))$	*Best response* $$W_i(P_j) = \begin{cases} 1 & v_{\omega_i}(P_j) = \min_k v_{\omega_i}(P_k) \\ 0 & otherwise \end{cases}$$
Strategic reasoning algorithm	*Iterative strategic reasoning* Recursive methods (Assume Poisson distribution, then use recursion starting with step 0 player).	*Type-based search* (e.g., alpha-beta pruning search)

Recent extensions of the CH model embed quantal response (McKelvey and Palfrey 1995) instead of best response assumed by CH (strategy choice probability row in the table 1). By quantal response, it means that the probability of an agent selecting a strategy choice follows a logit quantal best response function (QBR) and can thus take any number between 0 and 1 unlike in the standard CH and probabilistic opponent search models. Note that the probability in probabilistic opponent search model is in the probability in beliefs over opponent types not in the responses which are also assumed to be best responses. Using notation by Wright and Leyton-Brown (2019), a quantal best response $QBR_i(s_i, \lambda)$ is defined by

$$s_i(a_i) = \frac{exp[\lambda \times u_i(a_i, s_{-i})]}{\sum_{a_i'}^m exp\left[\lambda \times u_i\left(a_i', s_{-i}\right)\right]} \qquad (1)$$

Where $u_i(a_i, s_{-i})$ is the expected utility of player i when playing strategy $a_i \in A_i$ against mixed strategy profile s_{-i}, λ is a precision or rationality parameter such that as λ approaches zero, the strategy choice is as random and if λ approaches infinity, the strategy choice follows Nash equilibrium or rationality. A quantal CH model combines quantal best response and iterative decision process of CH as defined in definition 3.

Definition 3 [Quantal CH, definition 2 in Wright and Leyton-Brown (2019)]: *Let $\pi_{i,m} \in \Delta(A_i)$ be the distribution over actions predicted for an agent i with level k. Let the truncated distribution over actions predicted for an agent conditional on that agent's having level $0 \leq l \leq k$ be $\pi_{i,0:k}$ and for agents other than i be $\pi_{-i,0:k}$.*

$$\pi_{i,0:k} = \sum_{l=0}^{k} \frac{Poisson\ (l;\tau)}{\sum_{l'=0}^{k} Poisson\ (l';\tau)} \pi_{i,l} \tag{2}$$

Starting with level-0, the predicted $\pi_{i,k}$ is

$$\pi_{i,0}(a_i) = |A_i|^{-1} \tag{3}$$

$$\pi_{i,k}\ (a_i) = QBR_i\ (\pi_{-i}, 0{:}\,k-1; \lambda) \tag{4}$$

Where QBR is as in equation (1). The overall predicted distribution of actions is a weighted sum of the distributions for each level:

$$Pr(a_i|\tau, \lambda) = \sum_{l=0}^{\infty} Poisson(l;\tau) \times \pi_{i,l}\ (a_i) \tag{5}$$

3.0. Towards Cognitive Hierarchy Models of Mancala Games

3.1. Quantal CH and mixture choice data

The insight I follow in this paper is to start with a general framework in which quantal response is one of the approaches of characterizing strategy choices. This insight has been suggested by Dardanoni et al (2020b) where one starts with mixture choice data—joint distribution of choices of a population of agents from a collection of menus. In the context of games and iterated reasoning, it implies that it is possible to observe strategies chosen from several alternatives under different menus of alternatives and also deduce the share of players who make particular combination of choices across a collection of moves (Dardanoni et al 2020b). This is one of the two interpretations of stochastic choice data in revealed preference theory[1]. It rests on the assumption of availability of aggregate data on the distribution of choices under different menus in the population. The second interpretation according to Chambers and Echenique (2016, p. 95) assumes observing an individual "*who randomizes among different alternatives overtime, observed enough*

[1] Filiz-Ozbay and Masatlioglu (2020, p. 3) succinctly called the interpretation with choices of different individuals in the same environment as "interpersonal" while the interpretation with choices of a single individual in different situations as "intrapersonal".

to infer a stochastic rule that they use to select an element at random when faced with a given set of available choices." The first interpretation has been adopted in behavioral game theory to test the models (e.g., level-*k* and cognitive hierarchy) in the field. For example, Ostling et al (2011) uses field data from the Swedish lowest unique positive integer (LUPI) game called Limbo to test cognitive hierarchy models in which they did not have access to individual level data. They only had access to aggregate daily frequencies of the integers between 1 and 99,999 chosen by players. Using the second interpretation, it is also possible to collect experimental choice data from games played in culture if it is possible within the game to observe enough moves to determine the stochastic choice rule being used by the player. Such mixture choice data provides conditions for analyzing both preferences and rationality using a quantal or multinomial logit CH model as stated in proposition 1:

Proposition 1 [Dardanoni et al., 2020b, proposition 7]: *If A contains all binary menus and menu*[1] X*, then both the utility functions and the rationality parameters of the multinomial logit model are revealed by the type-conditional stochastic choice functions.*

Proposition 1 is a restatement of the well-known conditions for the stochastic choice function to be represented by the Luce model or multinomial logit (Luce 1959; Kovach and Tserenjigmid 2018, definition 3, p. 4). The proof of proposition 1 is provided by Dardanoni et al (2020b, p. 26). In the context of behavioral games, Rogers, Palfrey and Camerer (2009) showed how to combine quantal response and CH models by assuming that they are all cases of subjective quantal response models. Their definition of subjective expected payoff is the same as the definition of the mixture choice function in Dardanoni et al (2020b).

While Dardanoni et al (2020b) stated that mixture data can allow one to identify primitive preference (e.g., risk and time preferences) and cognition parameters; Rogers, Palfrey and Camerer (2009) argued that "*subjective quantal response equilibrium allows an intuitive link between two models (quantal and CH) developed from different perspectives about behavior—one with stochastic choice, and the other based on heterogeneous cognitive limitations."*

The idea of mixture choice data is also closely related to M equilibrium proposed by Goeree and Louis (2018) which uses algebraic sets to develop game theoretic predictions. M equilibrium relies on insights from semi-algebraic sets and algebraic geometry while Dardanoni et al (2020b) relies on related field of

[1] A menu is a nonempty set of alternatives contained in a finite universal set of alternatives. Dardanoni et al (2020a, p.1273) provides several examples of menus. Some these are: retailers selling a product and banks offering fixed deposits.

algebraic statistics particularly identifiability using Kruskal rank decomposition theorem; a concept that is used for decomposition of tensors[1] (Seigal 2019).

3.2. Mancala moves as mixture choice data

Given the current knowledge, it is difficult to implement the CH models of mancala. Nonetheless, I present theoretical results that allows linking of predictions from CH and level-*k* models to the heuristics in several mancala games. To do this, I rely on a theoretical result in algebraic statistics especially Kruskal rank decomposition theorem as proposed by Dardanoni et al (2020b) and as explained in proposition 1. The theoretical result can be stated succinctly that that quantal CH type models can be conceived as mixture choice data which embed preferences and cognition. In this section, I show that a mancala game presents mixture choice data and that the formulation of AI tools in mancala is equivalent to quantal CH type models.

A mancala game can be conceived as a perfect example of mixture choice data in which different menus (board arrangements as the game progresses) are presented to each player to choose a move from. In aggregate, one can observe the proportion of players of different preference types (e.g., attack or defend) or can observe a stochastic choice rule that a player is using in playing the game. If proposition 2 holds, then choice or moves data from a mancala game ensure identifiability using a quantal CH model.

Proposition 2 [Mancala game produces a tensor or mixture choice data]: *Mancala board game with multiple moves by each player generates a tensor or mixture choice data that is decomposable in the sense of Kruskal decomposition theorem to characterize preference and cognitive types.*

To satisfy proposition 2 and therefore use mancala game data in quantal CH experiments; it is necessary and sufficient to show conditions under which data from mancala can achieve identifiability in the context of Kruskal (1977) three-way array rank decomposition (Allman et al. 2009; Dardanoni et al. 2020b). This can be illustrated in two steps as shown in the example below. First, I show that mancala board and moves represent a tensor –a three-dimensional array of data. Second, I show how to convert this tensor into an array of conditional probabilities that satisfy Kruskal identifiability conditions.

Example 1: Following Mkondiwa (2020a), let the mancala (*Bao*) board (see figures of the board in the appendices) be a matrix of four rows, $i = 1, \ldots, 4$ and eight columns, $j = 1, \ldots, 8$. Given the matrix and assuming each hole represents a payoff to each player with the top player having rows 1 and 2 while bottom player has rows 3 and 4. The key question in behavioral game theory is whether we can build a model that

[1] A tensor is a multidimensional array that generalizes the concept of a matrix to allow for an arbitrary number of indices—this number being the order of the tensor (Dardanoni et al 2020a). A primer on tensor decomposition for identifiability is presented in the appendix B of Dardanoni et al (2020a). Note that mixture choice data are a specific example of a tensor—a probabilistic graphical representation of a tensor.

predicts which hole will be frequently chosen by a player when it's her turn to play. The *Bao* boards in the appendices can be represented by the four matrices: M, N, O and P.

$$M = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \leftarrow \mathbf{2} \\ \mathbf{2} & \mathbf{2} & 2 & 2 & 2 & 2 & \mathbf{2} & \mathbf{2} \end{bmatrix}, \quad N = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & \mathbf{2} & 2 \\ 0 & 0 & \mathbf{2} \rightarrow & 2 & \mathbf{2} & 0 & 0 & 2 \\ 5 & 1 & 4 & 4 & 4 & 0 & 1 & 3 \\ 3 & 3 & 0 & 3 & 3 & 0 & 3 & 3 \end{bmatrix}$$

$$O = \begin{bmatrix} 4 & 1 & 0 & 4 & 1 & 4 & 0 & 1 \\ 6 & 1 & 5 & 3 & 2 & 0 & 4 & 6 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{3} \uparrow & 3 & 0 & 3 & 3 & 0 & 3 & 3 \end{bmatrix}, \quad P = \begin{bmatrix} \mathbf{4} & 1 & 0 & \mathbf{4} & 1 & 4 & 0 & 1 \\ 6 & 1 & 0 & \mathbf{3} & 2 & 0 & 4 & \mathbf{6} \\ 2 & 2 & 6 & 1 & 1 & 0 & 0 & 0 \\ \mathbf{0} & 3 & 0 & 3 & 3 & 0 & 3 & 3 \end{bmatrix}$$

Using the rules in appendix A, if bottom player plays cell (3,8) in M towards the left and makes all legal captures, the resultant matrix is as shown in N. After the top player moves, the matrix is as shown in O and if bottom player moves again, the resultant matrix is shown in P. Assuming one wants to evaluate after these two moves for bottom player, the four matrices represent a tensor of order three, 4x8x4. The question is whether one can use such tensors to make behavioral predictions in terms of levels of thinking.

To do that, consider first that each possible action has a probability distribution. In M, bottom player has as many as 4 actions that can go clockwise or anti-clockwise, and 8 that can go either clockwise or anti-clockwise. Thus, there are a total of 16 legal moves. In N, there are 3 possible actions for top player. These are: (1,7), (2,3) and (2,5). In O, bottom player has only 1 possible action (4,1). In matrix P, top player has 4 possible actions: (1,1), (1,4), (2,4) and (2,8). If a player makes a move, we do not need to see each sowing or seeds captured. We need to observe the payoffs before the play and payoffs after play. The same for the opponent. The nature of the moves is therefore considered as a hidden layer. To use the analytical apparatus of discrete choice modeling, the 4×8 matrix is vectorized or flattened into a 32×1 vector. The choice probabilities from a player's entire game are then linked to mancala heuristics (explained in section 4.2).The key question in identifiability[1] concerns how many legal moves of each menu would allow one to isolate the deep parameters on preferences and rationality. This is trivial in mancala (e.g., *Bao*) because a common game consists of more than twenty moves by each player therefore there is enough variation to identify the parameters.

3.3. Quantal CH model of mancala

Using the equivalence results between mancala tensors and mixture choice data, I propose that quantal CH of mancala be modelled as follows:

[1] Note that it is widely known that the identifiability proofs of Allman et al (2009) may not be constructive for estimation purposes and it is advisable to rely on extensions presented in Compiani and Kitamura (2016, C98)

Proposition 3 [Quantal CH of mancala]: *Given mixture choice data/tensor from mancala game (proposition 1 and 2), a quantal cognitive hierarchy model of mancala games is the same as the traditional quantal CH model described in definition 3 with added layers from a neural, tensor or Bayesian network.*

The traditional quantal CH model is usually analyzed using a multinomial logit choice model which can be seamlessly used to analyze mancala data where each pocket in a mancala board is considered an alternative. However, such a model would not capture well the relationships between the position of these pockets and their contents to type preferences of the player. To capture these dependencies, I rely on theoretical results demonstrating that neural networks are extensions of multinomial logit models (e.g., Bentz and Merunka 2000) and that tensor networks graphically represent complex tensors including such dependencies[1]. The definition of quantal CH in proposition 3 borrows from the neural network-based approach to include game theoretic features in behavioral models as proposed by Hartford, Wright and Leyton-Brown (2016). The standard quantal CH model (definition 3) is equivalent to the following specification when a neural network is written without a hidden layer

$$ar_i = softmax\left(\lambda_i\left(\sum_{j=0}^{i-1} v_{i,j}\ U \times ar_j\right)\right) \tag{6}$$

Where ar_i is the action response layer, v_{ij} is as in the standard CH model, $v_{i,j} = \frac{Poisson\ (l;\tau)}{\sum_{l'=0}^{k} Poisson\ (l';\tau)}$. $softmax$ is a common function used in machine learning to refer to the normalizing exponential function equivalent to equation (1).

To ensure that game features are taken into account in a quantal CH, a feed forward neural network with one or more hidden layers $\left(\sum_{l=1}^{k} w_{i,l}\ H_{i,l}\right)$ was suggested by Hartford, Wright and Leyton-Brown (2016),

$$ar_i = softmax\left(\lambda_i\left(\sum_{j=0}^{i-1} v_{i,j}\left(\sum_{l=1}^{k} w_{i,l}\ H_{i,l}\right) \times ar_j\right)\right) \tag{7}$$

For mancala, this is a natural modeling strategy because recent efforts in AI of mancala use neural networks[2] and other network models like normalized Gaussian Network (NGN) (Donkers et al 2004).

[1] Mixture choice function representation of Dardanoni et al (2020b) already allows for correlations of choices across menus but not within the same menu as in a mancala move.

[2] There are several variations of these given the multiplicity of the games including: https://mnebuerquo.github.io/mancalai/#1 for Awari, and Conradie and Engelbrecht (2006)

Example 2 [Mancala as an asymmetric tensor or Bayesian network game]: Beyond embedding level-0 features as suggested by Hartford, Wright, and Leyton-Brown (2016), neural network models have been used in structural discrete choice modeling in economics to analyze choice interactions (e.g., Ruiz et al 2020). The foundational theory of using neural networks for these problems can be well articulated using mancala as an example. Borrowing from literature that links tensor decomposition to neural networks (e.g., Levine et al 2018) it is natural to generalize for quantal CH of mancala. First, standard tensor decomposition is equivalent to shallow neural networks. For example, a multinomial logit model is equivalent to a neural network without a hidden layer. Second, deep neural networks as in Hartford, Wright and Leyton-Brown (2016) are equivalent to hierarchical Tucker tensor decomposition. Finally, the network structure of game play in mancala implies that a more complex decomposition is required to understand the interactions of the choices. This can potentially be analyzed using tensor networks which are simpler graphical representations of complex tensors.

3.4. Extensions

Beyond considering mancala as mixture choice data or a tensor, one can utilize the rules of mancala to define a tensor network and therefore consider more complex behavioral models. The mathematical apparatus is as explained in Robeva and Seigal (2019) who provide duality results between tensor networks and graphical models[1]. The treatment of mancala as a collection of type-dependent menus also rests of recursion or iteration of plays. This implies that the payoffs are changed during each iteration. One can use this feature to extend CH models by using modular arithmetic to define mancala matrices based on board and move vectors (Taalman et al 2013) which has the same mathematical structure of category theory as in compositional game theory. The network dependence characterization of mancala implies that the game is different from normal form game matrices usually analyzed in behavioral game theories. However, it is closer in style to location games that Li and Camerer (2020) demonstrated can be analyzed using behavioral models. The game features of mancala form a complex network guided by the game rules. This implies that if one wants to use the neural network apparatus proposed by Hartford, Wright and Leyton-Brown (2016), one needs some form of interpretable deep neural network which maintains the interpretability of the multinomial logit model that maps to discrete choices.

The network structure of mancala pockets and moves point to several methodological extensions for the future research. First, mancala games would require generative models like sigmoid belief networks or Bayesian network learning—as proposed by Neal 1992 and expounded by Gan et al 2015—to accurately capture human decision-making processes in the game. The identifiability results for these models are discussed in Allman et al. (2015). The hand-crafted features from mancala can also be evaluated on other

[1] Other equivalence results are presented by Tucci (1995) using quantum Bayesian networks and Glasser et al (2019)

economic prediction problems that are closely linked to decision making in mancala. Second, the idea of using AI of games within a discrete choice framework has also been discussed for Go and Bonanza games by Igami (2020) who emphasized the difficulty of AI for Go because of its network structure, that is, *"local positions of stones seamlessly interact with the global one"* Igami (2020, p.S8). He however, demonstrated how dynamic structural econometrics uses the same statistical tools as AI of Go. Thus, the methodological advancements in structural econometrics would be directly applicable to analyzing data from mancala games. A similar suggestion is by Chen et al (2015) who proposed that sequential Bayesian networks represent a general framework for learning in real-world games. The two studies extend the empirics from normal form games as studied in conventional CH models to sequential game environments like in mancala.

The practical side of behavioral game theory is to relate game play to economic decision making in real life. The anthropological and philosophical expositions regarding the similarities between game language and rules, and everyday language and reasoning in societies that play the games were discussed in Mkondiwa (2020a, 2020b). In this paper, I suggest two applications for which the network structure of action choices as in mancala is of importance. First, the network structure of different pockets in a mancala game can be related to product networks and therefore be used in analyzing decisions within a product network (Masatlioglu and Suleymanov 2019). Mancala can be used for experiments to elicit decision making processes in scenarios where alternatives or products are related as complements or substitutes (Ruiz et al 2020). Ruiz et al (2020) developed a probabilistic model of consumer choice (they called it SHOPPER) that allowed for item interactions. Second and relatedly, mancala can be used in game experiments to measure salience just as in location games using quantal CH models (Li and Camerer 2020), and hide and seek games using level-*k* (Crawford and Iriberri 2007).

4.0. Stochastic Revealed Preference and Mancala Heuristics

Because this study is theoretical and exploratory, I do not implement the recursive quantal CH model of mancala as proposed because that requires a careful design of an experimental study and knowledge yet to be discovered. Nonetheless, I provide heuristics from cognitive hierarchy experiments, revealed preference theories, and make a link to heuristics that have been suggested in mancala by game experts, game anthropologists, and AI experts. I use the connections to explain heuristic choices just as recent efforts in AI focus on interpretation (also called explainable AI) (Miller 2019). I differentiate heuristics that are artifacts of the strategic environment from those that pertain to commonly used strategies by players to win the game. The former, I call game heuristics while the later I call play heuristics. For two games, with similar rules and complexity; one can expect that game heuristics will be similar but play heuristics may differ because play heuristics depend on norms of play. These heuristics are related and may be difficult to isolate in practice.

4.1. Heuristics from mancala game for behavioral game theory

What insights can be learned from mancala game environment that can enrich behavioral game theory predictions? The games can teach us the game theoretic features needed when analyzing economic decisions. For example, for most games one randomizes in low stakes play but thinks deeper for high stakes plays. Beyond this common feature of game play, mancala games present five important features that need to be considered as part of behavioral analysis of economic decisions.

Heuristic 1: *Large strategy space and complexity forces players to choose strategy categories first rather than evaluating each strategy.*

Most mancala games are complex and have a very large strategy space such that players mostly choose categories of the strategies rather than strategies themselves. For example, mancala (e.g., for *Bao*) players would mostly choose to play the inner rows and those with more seeds. This feature of mancala places it within many concepts in game theory and revealed preference. First, the multiplicity of characteristics that must be considered places it in games that Arad and Rubinstein (2012, 2019) claimed require multi-dimensional reasoning. Though their multi-dimensional reasoning in Arad and Rubinstein (2019) is within an equilibrium concept, they suggested the framework could also be adapted to non-equilibrium approaches like quantal CH. Second, this insight is also consistent with a concept of categorizing then choosing proposed by Manzini and Mariotti (2012) in which consumers choose categories in the first stage, then within the chosen category they make the choice. Finally, the choice of categories other than strategies is the same as the concept of rule rationality in which agents are assumed to choose rules of behavior that do well in usual or naturally occurring situations rather than acts (Aumann 2019). This has been found to be the case in game experiments that measure salience in location games (Li and Camerer 2020).

Another related insight is to avoid the worst plays and choose whichever doesn't have an obvious win or loss. This is the same as to say players eliminate the dominated strategies. According to Leyton-Brown, such strategies have not yet been incorporated in CH and level-*k* models. In section 4.2, I discuss some of the heuristics that mancala players use when selecting actions.

Heuristic 2: *Players make opening and other strategies that are mostly theoretical or experiential.*

Mancala players usually use theoretical or experiential openings and strategies during the game (Mkondiwa 2020). Therefore, the choice of which hole is played at each point in the game depends on the characteristics of the player like patience, risk preferences, conformity, e.t.c. In addition, for most mancala, the game rules differ in complexity for the games played by kids and those played by adults. Thus, the distribution of strategy choices embeds the interactions of these characteristics of the players and the payoffs available to each player. In order to embed these features one can use the methods in psychometrics which rely on defining a Q-matrix—a binary undirected adjacency matrix—that links player characteristics to the choice set (Xu and Shang 2018).

Heuristic 3: *When you have an opportunity to eat (ntaji); you are required to play to eat thereby being forced to deal with a narrow consideration set.*

This reduces the number of strategies to those needed to eat. This task of eliminating strategies that are in the dominated class allows one to focus on the essential strategies. This also implies that though the board may have many pockets, only few are within the consideration set. This also applies when thinking of what the opponent plays next. This idea is related to consumer choice theories on stochastic choice and consideration sets (Manzini and Mariotti 2014).

Heuristic 4: *Ability of a player or agent in mancala is known from playing under no eating play (game ndikutakata).*

This saying obviously demonstrates the level of thinking that is required for agents when they are under pressure. They are required to think harder. One can also conclude that the number of levels of thinking is higher under *takata* play (no eating play) than during a *ntaji* play (eating play). And the player in a *takata* play thinks at higher levels than a player during a *ntaji* play. Indeed, a player is socially encouraged to make a move quickly in a *ntaji* play than in a *takata* play.

Heuristic 5: *There is an inverted U relationship between number of choices and costs in choice making. It is better to give excess of choices and very few choices than medium number of choices to an opponent.*

With limited capacity to think on the part of the opponent, a poor move will be picked if excess strategies are offered because this raises the consideration costs (Aguiar et al. 2018) and you may then end up with an arrangement that gives the best strategies on your part. Giving very few strategies to the opponent forces the opponent to play those strategies that are beneficial to your game. With medium number of strategies, however, it is possible for the opponent to see all strategies and think harder on how to turn the game in the next moves. For example, in the matrices M, N, O, P with 16, 3, 1 and 4 strategies available to players respectively, a player with 3 and 4 actions available can think carefully on the best choice than the other cases. This inverted U relationship between the number of strategies available to a player and demands of thinking in mancala games has not been incorporated in behavioral game models. This heuristic is however related to the concept of choice overload and single-option aversion (Maltz and Rachmilevitch 2020).

4.2. Heuristics from mancala play for behavioral game theory

The multiplicity of mancala games and the complexity of the rules means that it is impossible to list all heuristics and winning strategies. The literature on mancala games has however attempted to develop common heuristics (from game anthropology) that are used in hybrid algorithms (Divilly, Rovaris). According to game anthropology literature, there are several types of mancala. The common ones are two-row and four-row mancala. In terms of complexity, the game anthropology literature (Murray 1952 and Townshend 1979) suggests that type 2A and type 4A (e.g., *Bao* and *Omweso*) are the most complex. Mkondiwa (2020b) suggested that 2A and 4A are more complex because they are divergent, that is, seeds are not taken out of the game. Den Herik et al. (2002) suggests that convergence is an important aspect of any two-player game. They particularly define a convergent game (e.g., *Oware* and *Ayo*) as one in which the size of the state space decreases as the game progresses while for a divergent game the size of the state space increases. Table 2 shows heuristics/strategies for convergent mancala as suggested by Divilly et al (2013) and Rovalis (2015) while table 3 shows heuristics/strategies for divergent mancala as proposed by Mkondiwa (2020a).

Table 2: Heuristics/strategies for convergent mancala (e.g., *Awari*, *Oware*, *Vai Lung Thlan*, *Erherhe*)

Strategy number	**Convergent mancala heuristics (strategy characteristic)**[1]	**Heuristic category (player characteristics)**
C1	**Hoard** as many counters as possible in one pit	Hoarding
C2	**Keep** as many counters on the players **own side**	Hoarding
C3	Have as **many moves as possible** from which to choose	
C4	**Maximize** the amount of counters in a players **own store**	Attacking
C5	Move the seeds from the pit closest to the opponents side	Attacking
C6	Keep the **opponents score** to a **minimum**	Defensive

Notes: 1. Sources: Divilly et al (2013) and Rovaris (2017).

Divilly and Rovaris used a combination of the heuristics in table 2 in understanding play in several convergent mankala variants including *Kalah*, *Oware* and *Awari*. They both found that these hybrid models (heuristics plus alpha-beta search) are the most effective. Similarly, Donkers et al (2004) considers several algorithms for solving[1] *Bao* (a divergent mancala variant) one of which depends on the knowledge of strategies that expert players use in the game. The heuristic is stated as *"it is good to have more stones in your back row since this increases the mobility in the second stage of the game."* They also used the difference in the number of seeds available to each player as another heuristic. These two heuristics were compared to other algorithms based on machine learning.

[1] According to Allis (1994), "games can be solved if it is possible to determine strategies leading to the best possible result for both players."

Table 3: Heuristics/strategies for divergent mancala (e.g., *Bao*)

Strategy number	Divergent mancala heuristics (strategy characteristic)[1]	Heuristic category (player characteristics)	Description
D1	Spread your counters throughout the row closest to the opponent.	Defensive	That way, it's difficult for your opponent to eat them all in one move.
D2	Keep houses (piles of seeds) for key strategic moves outside	Risk averse	Donkers et al. (2004) found this heuristic to be better than the difference in the number of seeds for each player in *Bao*.
D3	Make the calculations very fast and make the move as quickly as possible.	Impatience	This has a social implication in that one is expected to move faster
D4	Play to accumulate enough on your house and strategize so you can move it away and spread the game.	Hoarding	
D5	Keep the game on your right side but make strategies to carefully put seeds on the left so you eat your opponent's house.	Risk averse if on left of the house and risk loving if on the right of the house	This heuristic mostly applies when sowing the game. It may also apply in a regular move because the hole for the house has a different shape and is used as a reference.

Notes: 1. Source: Mkondiwa (2020a).

5.0. Conclusion

This paper has introduced a general theoretical framework that allows strategy games played in culture like mancala to be used in behavioral game theory just as "economics" games (e.g., centipede games, beauty contest games, etc.). The theoretical framework is based on relating ideas in artificial intelligence of mancala games, game anthropology, and behavioral game theory. The key insight is that mancala games can be represented as mixture choice data, with an underlying stochastic choice function. Mixture choice data can be analyzed using a quantal CH model (a leading behavioral game model); a model that is similar to probabilistic opponent search model commonly used in AI of mancala. These relationships demonstrate that cognitive hierarchy models can be useful models for building a theory that explains rules and strategies of mancala. Such a theoretical understanding allows the use of mancala games not only in behavioral experiments but also in gamification of many economic activities like business management and savings.

It suffices to reiterate the enormous challenge that this paper tries to solve. As Camerer, Ho and Chong (2015) cautioned, "*applications of CH models in field setting are challenging because they require simplifying complicated choices and uncertain payoffs into a form that can be analyzed mathematically*." Berger et al (2016) also cautioned that the *"thinking steps approach (e.g., QCH and Level-k) to predicting behavior is useful when a game is 'pure' and 'simple', i.e., stripped of all complications introduced by social preferences, algebraic complexities and risk issues."* However, purity and simplicity of a game is in the eyes of the beholder, real-world games being pure and simple to those who play them daily but impure and complex to behavioral game theorists. The future of behavioral game theory in developing economies will not be in transferring pure and simple lab games that economists have developed to the field but rather harnessing AI algorithms, heuristics and algebraic statistics of real-world games to build theoretic predictions, then either by observation, intervention or anthropology relate the predictions to economic decisions people of different abilities in playing the games make in those societies. This approach, however, needs much more information on the games. That is, for each game, one must think harder on when, how, and why the game is played; how it got its currency in that society; and how it has been intertwined in peoples' strategic behavior either through proverbs, folk lore, or language. Then one should consider strategies and heuristics that are commonly known to be effective at winning the game. One can then take these to predictions of AI of mancala games or related behavioral game theories and ask the following questions: what would these models predict for such actions or strategies? Is it generalizable for someone with such theoretic behavior to make such decisions in their economic lives?

This approach inevitably broadens the number of games beyond mancala amenable to economic analysis thereby allowing a much deeper understanding of economic behavior especially in societies whose behavior can be misconstrued when using games not available in their culture. Strategic games played in culture should be the default option in behavioral development economics not "economics" games created by economists. The key advantage of the games in culture is that they can reveal other forms of reasoning consistent with the argument that level *k* model may be one of many possible decision processes players employ to select strategies in games (Georganas et al. 2015). Mancala for example requires spatial and network thinking which expands behavioral development economics into problems that demand people's multi-dimensional reasoning. Recent efforts in behavioral game theory to elicit the strategic reasoning processes (e.g., Arad and Penczynski 2020) are therefore important towards a new philosophy of doing behavioral game theory using real-world games.

References

Aguiar, V.H., Boccardi, M.J., Kashaev, N., and Kim, J. 2019. "Does Random Consideration Explain Behavior when Choice is Hard? Evidence from a Large-scale Experiment." Working paper.

Akinyemi, I.O., Adebiyi, E.F., and Longe, H.O.D. 2009. "Critical Analysis of Decision Making Experience with a Machine Learning Approach in Playing *Ayo* Game." World Academy of Science, Engineering and Technology 56. Url: http://eprints.covenantuniversity.edu.ng/1130/1/10.1.1.193.5469.pdf.

Albrecht, S.V., and Stone, P. 2018. "Autonomous Agents Modelling Other Agents: A Comprehensive Survey and Open Problems." *Artificial Intelligence*. Doi: https://doi.org/10.1016/j.artint.2018.01.002.

Allis, L.V. 1994. "Searching for Solutions in Games and Artificial Intelligence." Ph.D. Thesis, University of Limburg, Maastricht, Netherlands.

Allman, E.S., Matias, C., Rhodes, J.A. 2009. "Identifiability of Parameters in Latent Structure Models with Many Observed Variables." *The Annals of Statistics* 37(6A): 3099-3132. Doi: 10.1214/09-AOS689.

Allman, E.S., Rhodes, J.A., Stanghellini, E., and Valtorta, M. 2015. "Parameter Identifiability of Discrete Bayesian Networks with Hidden Variables." *Journal of Causal Inference* 3(2): 189-205.

Arad, A., and Penczynski, S. 2020. "Multi-Dimensional Reasoning in Competitive Resource Allocation Games: Evidence from Intra-Team Communication." Working paper. Url: https://www.tau.ac.il/~aradayal/MD_Reasoning_in_Resource_Allocation_Games.pdf.

Arad, A., and Rubinstein, A. 2012. "Multi-dimensional iterative reasoning in action: The case of the Colonel Blotto game." *Journal of Economic Behavior and Organization* 84: 571-585. Doi: http://dx.doi.org/10.1016/j.jebo.2012.09.004

Arad, A., and Rubinstein, A. 2019. "Multidimensional Reasoning in Games: Framework, Equilibrium, and Applications." *American Economic Journal: Microeconomics* 11(3): 285-318. Doi: https://doi.org/10.1257/mic.20170322.

Aumann, R.J. 2019. "A synthesis of behavioural and mainstream economics." *Nature human behavior* 3: 666-670. Doi: https://doi.org/10.1038/s41562-019-0617-3.

Batzilis, D., Jaffe, S., Levitt, S., List, J.A., and Picel, J. 2019. "Behavior in Strategic Settings: Evidence from a Million Rock-Paper-Scissors Games." *Games* 10,18. Doi: doi:10.3390/g10020018.

Bentz, Y., and Merunka, D. 2000. "Neural Networks and the Multinomial Logit for Brand Choice Modeling: A Hybrid Approach." *Journal of Forecasting* 19: 177-200.

Berger, U., Silva, De Silva, H., and Fellner-Rohling, G. 2016. “Cognitive Hierarchies in the Minimizer Game.” *Journal of Economic Behavior and Organization* 130: 337-348. Doi: https://doi.org/10.1016/j.jebo.2016.08.004.

Camerer, C.F. 2015. “The Promise and Success of Lab-Field Generalizability in Experimental Economics: A Critical Reply to Levitt and List. In: Handbook of Experimental Economic Methodology, edited by Frechette, G.R., and Schotter, A. Oxford University Press.

Camerer 2017. AI and Behavioural Economics. Conference Presentation Slides. Url: https://static1.squarespace.com/static/59c2a584be42d60a2772ba71/t/59c2c2f803596efeceb722ff/1505936127928/Camerer.pdf.

Camerer, C.F. 2015. “Cognitive Hierarchy Process Models of Strategic Thinking in Games.” *The Wiley Blackwell Handbook of Judgment and Decision Making* Ch 9: 543-567.

Camerer, C.F., Ho, T., and Chong, J. 2004. “A Cognitive Hierarchy Model of Games.” *Quarterly Journal of Economics* 119 (3): 861-898. Doi: https://doi.org/10.1162/0033553041502225.

Camerer, C.F., Ho, T., and Chong, J. 2015. “A Psychological Approach to Strategic Thinking in Games.” *Current Opinion in Behavioral Sciences* 3: 157-162.

Camerer, C., Nunnari, S., and Palfrey, T.R. 2016. “Quantal Response and Nonequilibrium Beliefs Explain Overbidding in Maximum-Value Auctions.” *Games and Economic Behavior* 98: 243-263. Doi: https://doi.org/10.1016/j.geb.2016.06.009.

Chambers, C.P., and Echenique, F. 2016. “Revealed Preference Theory.” Cambridge University Press.

Chen, W., Chen, Y., and Levine, D.K. 2015. “A Unifying Learning Framework for Building Artificial Game-Playing Agents.” *Annals of Mathematics and Artificial Intelligence* 73(3): 335-358. Doi: 10.1007/s10472-015-9450-1.

Compiani, G., and Kitamura, Y. 2016. “Using Mixtures in Econometric Models: A Brief Review and Some New Results.” *Econometrics Journal* 19: C95-C127. Doi: 10.1111/ectj.12068.

Conradie, J., and Engelbrecht, A.P. 2006. “Training Bao Game-Playing Agents using Coevolutionary Particle Swarm Optimization.” Proceedings of the 2006 IEEE Symposium on Computational Intelligence and Games 2: 67-74.

Costa-Gomes, M.A., and Crawford, V.P. 2006. "Cognition and behavior in two-person guessing games: An experimental study." *American Economic Review* 96 (5): 1737-1768. Doi: 10.1257/aer.96.5.1737.

Crawford, V.P., and Iriberri, N. 2007. "Fatal attraction: Salience, Naivete, and Sophistication in Experimental "Hide-and-Seek" Games." *American Economic Review* 97(5): 1731-1750. Doi: 10.1257/aer.97.5.1731.

Dardanoni, V., Manzini, P., Mariotti, M., Petri, H., Tyson, C.J. 2020a. "Inferring Cognitive Heterogeneity from Aggregate Choices." *Econometrica* 88 (3): 1269-1296. Doi: https://doi.org/10.3982/ECTA16382.

Dardanoni, V., Manzini, P., Mariotti, M., Petri, H., Tyson, C.J. 2020b. "Mixture Choice Data: Revealing Preferences and Cognition." Working paper.

Den Herik, H.J., Uiterwijk, J.W.H.M., Rijswijck, J. 2002. "Games solved: Now and in the future." *Artificial Intelligence* 134: 277-311. Doi: 10.1016/S0004-3702(01)00152-7.

Divilly, C., O'Riordan, C., and Hill, S. 2013. "Exploration and Analysis of the Evolution of Strategies for Mancala Variants." IEEE Conference on Computational Intelligence in Games

Donkers, H.H.L.M., van den Herik, H.J., and Uiterwijk, J.W.H.M. 2004. "Opponent-Model Search in Bao: Conditions for a Successful Application." In: Van Den Herik, H.J., Heinz, E.A. (eds) Advances in Computer Games. IFIP-The International Federation for Information Processing, 135. Springer, Boston, MA. Doi: https://doi.org/10.1007/978-0-387-35706-5_20.

Farhi, E., and Werning, I. 2019. "Monetary Policy, Bounded Rationality, and Incomplete Markets." *American Economic Review* 109 (11): 3887-3928. Doi: https://doi.org/10.1257/aer.20171400.

Farhi, E., and Gabaix, X. 2020. "Optimal Taxation with Behavioral Agents." *The American Economic Review* 110 (1): 298-336.

Filiz-Ozbay, E., and Masatlioglu, Y. 2020. "Progressive Random Choice." Working paper.

Fragiadakis, D.E., Kovaliukaite, A., and Arjona, D.R. 2017. "Testing Cognitive Hierarchy Assumptions." Working paper.

Fudenberg, D., and Liang, A. 2019. "Predicting and Understanding Initial Play." *American Economic Review* 109 (12): 4112-4141. Doi: https://doi.org/10.1257/aer.20180654.

Glasser, I., Pancotti, N., and Cirac, J.I. 2019. "From Probabilistic Graphical Models to Generalized Tensor Networks for Supervised Learning." Url: https://arxiv.org/abs/1806.05964.

Goeree, J.K., and Louis, P. 2018. M Equilibrium: A Dual Theory of Beliefs and Choices in Games. Working paper.

Goeree, J.K., Holt, C.A., and Palfrey, T.R. 2016. "Quantal Response Equilibrium: A Stochastic Theory of Games." Princeton, NJ: Princeton University Press.

Goeree, J.K., Holt, C.A., and Palfrey, T.R. 2020. "Stochastic Game Theory for Social Science: a Primer on Quantal Response Equilibrium." In: Handbook of Experimental Game Theory. Eds: Capra, C.M., Croson, R.T.A., Rigdon, M.L., and Rosenblat, T.S. Edward Elgar Publishing Limited.

Goldfarb, A., and Xiao, M. 2011. "Who Thinks about the Competition? Managerial Ability and Strategic Entry in US Local Telephone Markets." *American Economic Review* 101 (7): 3130-3161. Doi: DOI: 10.1257/aer.101.7.3130.

Goldfarb, A., and Yang, B. 2009. "Are All Managers Created Equal." *Journal of Marketing Research* 46 (5): 612-622.

Harsanyi, J.C. 1967. "Games with incomplete information played by "Bayesian" players, I-III. Part I. The basic model." *Management Science* 14(3): 159-182. Doi: http://dx.doi.org/10.1287/mnsc.14.3.159.

Hartford, J. 2016. "Deep Learning for Predicting Human Strategic Behavior." MSc Thesis: University of British Columbia.

Hartford, J. Wright, J.R., Leyton-Brown, K. 2016. "Deep Learning for Predicting Human Strategic Behavior." 29th Conference on Neural Information Processing Systems (NIPS 2016), Barcelona, Spain.

Heller, Y., and Winter, E. 2016. "Rule Rationality." *International Economic Review* 57 (3): 997-1026. Doi: https://doi.org/10.1111/iere.12183.

Henrich, J., Heine, S., and Norenzayan, A. 2010. "The weirdest people in the world?" *Behavioral and Brain Sciences* 33:61-135. Doi: 10.1017/S0140525X0999152X.

Igami, M. 2020. "Artificial intelligence as structural estimation: Deep Blue, Bonanza, and AlphaGo." *Econometrics Journal* 23: S1-S24. Doi:10.1093/ectj/utaa005.

Kovach, M., and Tserenjigmid, G. 2019. “Behavioral Foundations of Nested Stochastic Choice and Nested Logit.” Working paper.

Kruskal, J.B. 1977. “Three-Way Arrays: Rank and Uniqueness of Trilinear Decompositions, with Application to Arithmetic Complexity and Statistics.” *Linear Algebra and Its Applications* 18: 95-138.

Levine, Y., Yakira, D., Cohen, N., and Shashua, A. 2018. “Deep Learning and Quantum Entanglement: Fundamental Connections with Implications to Network Design.*” International Conference on Learning Representations (ICLR).*

Levitt, S.D., and List, J.A. 2007. “Viewpoint: On the Generalizability of Lab Behavior to the Field.” *Canadian Journal of Economics* 40(2): 347-370. Doi: https://doi.org/10.1111/j.1365-2966.2007.00412.x.

Levitt, S.D., and List, J.A. 2015. “What Do Laboratory Experiments Measuring Social Preferences Reveal About the Real World.” In: Handbook of Experimental Economic Methodology, edited by Frechette, G.R., and Schotter, A. Oxford University Press.

Li, X., and Camerer, C.F. 2020. “Predictable Effects of Bottom-up Visual Salience in Experimental Decisions and Games.” Working paper. Url: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3308886.

Luce, R.D. 1959. “Individual Choice Behavior: A Theoretical Analysis.” Dover Publications, Inc. Mineola, New York.

Maltz, A., and Rachmilevitch, S. 2020. “A Model of Menu-Dependent Evaluations and Comparison-Aversion.” *Journal of Behavioral and Experimental Economics* (In Press, Journal Pre-proof). Doi: https://doi.org/10.1016/j.socec.2020.101655.

Manzini, P., and Mariotti, M. 2012. “Categorize then Choose: Boundedly Rational Choice and Welfare.” *Journal of the European Economic Association* 10(5):1141-1165. Doi: https://doi.org/10.1111/j.1542-4774.2012.01078.x.

Manzini, P., and Mariotti, M. 2014. “Stochastic Choice and Consideration Sets.” *Econometrica* 82(3):1153-1176. Doi: https://doi.org/10.3982/ECTA10575.

Masatlioglu, Y., and Suleymanov, E. 2019. “Decision making within a product network.” *Economic Theory*. Doi: https://doi.org/10.1007/s00199-019-01238-z.

Miller, T. 2019. “Explanation in Artificial Intelligence: Insights from the Social Sciences.” *Artificial Intelligence* 267: 1-38. Doi: https://doi.org/10.1016/j.artint.2018.07.007.

Mkondiwa, M. 2020a. “Games of Strategy in Culture and Economics Research.” *Journal of Economic Methodology* 27(2): 146-163. Doi: 10.1080/1350178X.2019.1680858.

Mkondiwa, M. 2020b. “Mancala Board Games and Origins of Entrepreneurship in Africa.” *PLOS ONE* 15(10): e0240790. Doi: https://doi.org/10.1371/journal.pone.0240790

Murray, H.J.R. 1952. A History of Board Games Other than Chess. Oxford: Clarendon Press.

Nagel, R. 1995. “Unraveling in Guessing Games: An Experimental Study.” *American Economic Review* 85: 1313-1326. Url: https://www.jstor.org/stable/2950991.

Ok, E.A., and Tserenjimid, G. 2020. “Deterministic Rationality of Stochastic Choice Behavior.” Working paper.

Ostling, R., Wang, J.T., Chou, E.Y., and Camerer, C.F. 2011. “Testing Game Theory in the Field: Swedish LUPI Lottery Games.” *American Economic Journal: Microeconomics* 3: 1-33.

Pearl, J. 1984. Heuristics: Intelligent Search Strategies for Computer Problem Solving. Addison-Wesley Publishing Company.

Robeva, E., and Seigal, A. 2019. “Duality of graphical models and tensor networks.” *Information and Inference: A Journal of the IMA* 8: 273-288.

Rogers, B.W., Palfrey, T.R., and Camerer, C.F. 2009. “Heterogeneous Quantal Response Equilibrium and Cognitive Hierarchies.” *Journal of Economic Theory* 144: 1440-1467. Doi: 10.1016/j.jet.2008.11.010.

Rovaris, G. 2017. “Design of Artificial Intelligence for Mancala Games.” PhD Thesis.

Ruiz, F.J.R., Athey, S., and Blei, D.M. 2020. SHOPPER: A probabilistic model of consumer choice with substitutes and complements.” *Annals of Applied Statistics* 1-27. Url: https://projecteuclid.org/euclid.aoas/1587002662.

Seigal, A.L. 2019. “Structured Tensors and the Geometry of Data.” PhD Thesis, University of California, Berkeley.

Strzalecki, T. 2014. “Depth of reasoning and higher order beliefs.” *Journal of Economic Behavior and Organization* 108: 108-122. Doi: https://doi.org/10.1016/j.jebo.2014.09.002.

Townshend, P. 1979. Mankala in Eastern and Southern Africa: A Distributional Analysis. *Azania: Archaeological Research in Africa* 14 (1): 109–38. doi:10.1080/00672707909511266.

Tucci, R.R. 1995. “Quantum Bayesian Nets.” *International Journal of Modern Physics B* 9 (3):295-337. Doi: https://doi.org/10.1142/S0217979295000148.

Wright, J.R., and Leyton-Brown, K. 2019. “Level-0 Meta-Models for Predicting Human Behavior in Games.” *Journal of Artificial Intelligence Research* 64:357-383.

Xu, G., and Shang, Z. 2018. “Identifying Latent Structures in Restricted Latent Class Models.” *Journal of the American Statistical Association* 113(523):1284-1295.Doi: https://doi.org/10.1080/01621459.2017.1340889.

Online Appendix

Appendix A: *Bao* game rules (baby version) Mkondiwa (2020a)

In what I call the "baby" level or simple version, all the 64 seeds are planted in twos in the 32 pockets (see figures in appendix B). The players agree who is to start the game. If none wants to start, you agree to take out the seeds and start planting again, whoever finishes first will start the game. The five key rules for the baby-level game are (be warned that if you have never played the game before, the following rules may be confusing):

I. You eat/capture whenever there is a chance to. You eat when your move lands in a pocket where you have at least one seed in the first row and your opponent has also at least one seed in the corresponding pocket.

II. A player makes a move if and only if the pocket has at least two seeds. It is not allowed to move a single seed when starting a move. But one can use single seeded pocket when connecting a play.

III. If you are moving to the left and eat, you pick the seeds you have eaten and start from the first pocket from the side you came from. This applies only to the 6 pockets on the side you are coming from. If instead, you are on the 7th pocket (that is either {2,2} or {3,2} or {2,7} or {3,7}), you pick the seeds to the 8th pocket (that is either {2,1} or {3,1} or {2,8} or {3,8}) then start moving back inside the game.

IV. If you fail to eat but your move is still connecting, you continue moving in the same direction until you get an empty spot then you stop. The opponent is then supposed to make a play.

V. The game ends when the first row of the opponent is all eaten up. The game can also end if the opponent's seeds are all ones in the pockets. In that case, the opponent cannot make a move.

Appendix B: *Bao* game moves on a board

Figure B1: *Bao* board for matrix M

Figure B2: *Bao* board for matrix N

Figure B3: *Bao* board for matrix O

Figure B4: *Bao* board for matrix P