2-states Disappointment Aversion

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1 Introduction

1.1 Preferences

$u(x,y) = \gamma w(\max\{x,y\}) + (1-\gamma) w(\min\{x,y\})$
$\gamma = \frac{1}{2+\beta} \qquad -1 < \beta < \infty$

Remark 1. *Note that $\beta < 0$ implies $1+\beta < \frac{1}{1+\beta}$.*

1.2 Budget

$p = \frac{p_x}{p_y} = \frac{m_y}{m_x}; m_y = \frac{M}{p_y}; m_x = \frac{M}{p_x}$ (if $m_y = m_x$ we denote both by m).

2 CRRA

$$w(x) = \begin{cases} \frac{x^{1-\rho}}{1-\rho} & \rho \geqslant 0 \quad (\rho \neq 1) \\ ln(x) & \rho = 1 \end{cases}$$

Remark 2. *Assume $\rho \geqslant 0$ since $\rho < 0$ is not identifiable (the non-concavity of the VNM index cannot be disentangled from the non-convexity of the elation loving preferences).*

$$MRS_{xy} = \begin{cases} \frac{1}{1+\beta}\left(\frac{y}{x}\right)^{\rho} & x > y \\ \left[\frac{1}{1+\beta}, 1+\beta\right] & x = y \\ (1+\beta)\left(\frac{y}{x}\right)^{\rho} & x < y \end{cases}$$

2.1 Utility Maximization Problem

Case 1. $\rho > 0, \beta \geqslant 0:$

$$(x,y)^d = \begin{cases} \left(\frac{m_x}{\left[1+\frac{[p(1+\beta)]^{1/\rho}}{p}\right]}, \frac{m_y}{1+\frac{p}{[p(1+\beta)]^{1/\rho}}} \right) & p < \frac{1}{1+\beta} \\ \left(\frac{m_y}{p+1}, \frac{m_y}{p+1} \right) & \frac{1}{1+\beta} \leqslant p \leqslant 1+\beta \\ \left(\frac{m_x}{1+\frac{1}{p}\left(\frac{p}{1+\beta}\right)^{1/\rho}}, \frac{m_y}{1+\frac{p}{\left(\frac{p}{1+\beta}\right)^{1/\rho}}} \right) & 1+\beta < p \end{cases}$$

Case 2. $\rho > 0, -1 < \beta < 0:$

$$(x,y)^d = \begin{cases} \left(\frac{m_x}{\left[1+\frac{[p(1+\beta)]^{1/\rho}}{p}\right]}, \frac{m_y}{1+\frac{p}{[p(1+\beta)]^{1/\rho}}} \right) & p < 1 \\ \left\{ \left(\frac{m}{1+(1+\beta)^{1/\rho}}, \frac{m}{1+(1+\beta)^{-1/\rho}} \right), \left(\frac{m}{1+(1+\beta)^{-1/\rho}}, \frac{m}{1+(1+\beta)^{1/\rho}} \right) \right\} & p = 1 \\ \left(\frac{m_x}{1+\frac{1}{p}\left(\frac{p}{1+\beta}\right)^{1/\rho}}, \frac{m_y}{1+\frac{p}{\left(\frac{p}{1+\beta}\right)^{1/\rho}}} \right) & 1 < p \end{cases}$$

Case 3. $\rho \geqslant 0, \beta = -1$ *or* $\rho = 0, -1 \leqslant \beta < 0$*:*

$$(x,y)^d = \begin{cases} (m_x, 0) & p < 1 \\ \{(m,0),(0,m)\} & p = 1 \\ (0, m_y) & 1 < p \end{cases}$$

Case 4. $\rho = 0, \beta > 0:$

$$(x,y)^d = \begin{cases} (m_x, 0) & p < \frac{1}{1+\beta} \\ \{x \geqslant y, px + y = m_y\} & p = \frac{1}{1+\beta} \\ \left(\frac{m_y}{p+1}, \frac{m_y}{p+1} \right) & \frac{1}{1+\beta} < p < 1+\beta \\ \{\ x \leqslant y, px + y = m_y\} & p = 1+\beta \\ (0, m_y) & 1+\beta < p \end{cases}$$

Case 5. $\rho = 0, \beta = 0:$

$$(x,y)^d = \begin{cases} (m_x, 0) & p < 1 \\ \{(x,y), px + y = m\} & p = 1 \\ (0, m_y) & 1 < p \end{cases}$$

Remark 3. *Comparing case 3 to case 5 shows that when $\beta = -1$ or β is negative when $\rho = 0$, the choices can be rationalized by $\rho = 0$ and $\beta = 0$ but not the other way around.*

2.2 Expenditure Minimization Problem

Remark 4. *If $x_0 = 0$ or $y_0 = 0$ then $0 \leqslant \rho < 1$ ($\rho \geqslant 1$ implies $u(x_0, y_0) = -\infty$).*

Case 6. $\rho > 0, \rho \neq 1, \beta \geqslant 0 : e\left(p_x, p_y, (x_0, y_0)\right) =$

$$\begin{cases} p_x\left[\frac{(1-\rho)(2+\beta)u(x_0,y_0)}{1+(1+\beta)^{\frac{1}{\rho}}p^{\frac{1-\rho}{\rho}}}\right]^{\frac{1}{1-\rho}} + p_y\left[\frac{(1-\rho)(2+\beta)u(x_0,y_0)}{1+(1+\beta)^{\frac{1}{\rho}}p^{\frac{1-\rho}{\rho}}}\right]^{\frac{1}{1-\rho}}\left((1+\beta)\,p\right)^{\frac{1}{\rho}} & p < \frac{1}{1+\beta} \\ (p_x + p_y)\left[(1-\rho)u(x_0,y_0)\right]^{\frac{1}{1-\rho}} & \frac{1}{1+\beta} \leqslant p \leqslant 1+\beta \\ p_x\left[\frac{(1-\rho)(2+\beta)u(x_0,y_0)}{(1+\beta)+\left(\frac{p}{1+\beta}\right)^{\frac{1-\rho}{\rho}}}\right]^{\frac{1}{1-\rho}} + p_y\left[\frac{(1-\rho)(2+\beta)u(x_0,y_0)}{(1+\beta)+\left(\frac{p}{1+\beta}\right)^{\frac{1-\rho}{\rho}}}\right]^{\frac{1}{1-\rho}}\left(\frac{p}{1+\beta}\right)^{\frac{1}{\rho}} & 1+\beta < p \end{cases}$$

Case 7. $\rho = 1, \beta \geqslant 0 : e\left(p_x, p_y, (x_0, y_0)\right) =$

$$\begin{cases} (2+\beta)\left[\frac{p_x^{\frac{1}{1+\beta}}p_y}{1+\beta}\right]^{\frac{1+\beta}{2+\beta}}e^{u(x_0,y_0)} & p < \frac{1}{1+\beta} \\ (p_x + p_y)\,e^{u(x_0,y_0)} & \frac{1}{1+\beta} \leqslant p \leqslant 1+\beta \\ (2+\beta)\left[\frac{p_y^{\frac{1}{1+\beta}}p_x}{1+\beta}\right]^{\frac{1+\beta}{2+\beta}}e^{u(x_0,y_0)} & 1+\beta < p \end{cases}$$

Case 8. $\rho > 0, \rho \neq 1, -1 < \beta < 0 : e\left(p_x, p_y, (x_0, y_0)\right) =$

$$\begin{cases} p_x\left[\frac{(1-\rho)(2+\beta)u(x_0,y_0)}{1+(1+\beta)^{\frac{1}{\rho}}p^{\frac{1-\rho}{\rho}}}\right]^{\frac{1}{1-\rho}} + p_y\left[\frac{(1-\rho)(2+\beta)u(x_0,y_0)}{1+(1+\beta)^{\frac{1}{\rho}}p^{\frac{1-\rho}{\rho}}}\right]^{\frac{1}{1-\rho}}\left((1+\beta)\,p\right)^{\frac{1}{\rho}} & p < 1 \\ p_x\left[\frac{(1-\rho)(2+\beta)u(x_0,y_0)}{1+(1+\beta)^{\frac{1}{\rho}}}\right]^{\frac{1}{1-\rho}} + p_y\left[\frac{(1-\rho)(2+\beta)u(x_0,y_0)}{1+(1+\beta)^{\frac{1}{\rho}}}\right]^{\frac{1}{1-\rho}}(1+\beta)^{\frac{1}{\rho}} & p = 1 \\ p_x\left[\frac{(1-\rho)(2+\beta)u(x_0,y_0)}{(1+\beta)+\left(\frac{p}{1+\beta}\right)^{\frac{1-\rho}{\rho}}}\right]^{\frac{1}{1-\rho}} + p_y\left[\frac{(1-\rho)(2+\beta)u(x_0,y_0)}{(1+\beta)+\left(\frac{p}{1+\beta}\right)^{\frac{1-\rho}{\rho}}}\right]^{\frac{1}{1-\rho}}\left(\frac{p}{1+\beta}\right)^{\frac{1}{\rho}} & p > 1 \end{cases}$$

Case 9. $\rho = 1, -1 < \beta < 0 : e\left(p_x, p_y, (x_0, y_0)\right) =$

$$\begin{cases} (2+\beta)\left[\frac{p_x^{\frac{1}{1+\beta}}p_y}{1+\beta}\right]^{\frac{1+\beta}{2+\beta}}e^{u(x_0,y_0)} & p < 1 \\ (2+\beta)p_x\left(\frac{1}{1+\beta}\right)^{\frac{1+\beta}{2+\beta}}e^{u(x_0,y_0)} & p = 1 \\ (2+\beta)\left[\frac{p_y^{\frac{1}{1+\beta}}p_x}{1+\beta}\right]^{\frac{1+\beta}{2+\beta}}e^{u(x_0,y_0)} & 1 < p \end{cases}$$

Case 10. $\rho > 0, \rho \neq 1, \beta = -1 :$

$$e(p_x, p_y, (x_0, y_0)) = \min\{p_x, p_y\}\left[(1-\rho)u(x_0, y_0)\right]^{\frac{1}{1-\rho}}$$

Case 11. $\rho = 1, \beta = -1:$

$$e(p_x, p_y, (x_0, y_0)) = \min\{p_x, p_y\} e^{u(x_0,y_0)}$$

Case 12. $\rho = 0, \beta \geqslant 0:$

$$e\left(p_x, p_y, (x_0, y_0)\right) = \begin{cases} p_x(2+\beta)u(x_0, y_0) & p < \frac{1}{1+\beta} \\ (p_x + p_y)u(x_0, y_0) & \frac{1}{1+\beta} \leqslant p \leqslant 1+\beta \\ p_y(2+\beta)u(x_0, y_0) & 1+\beta < p \end{cases}$$

Case 13. $\rho = 0, -1 \leqslant \beta < 0:$

$$e(p_x, p_y, (x_0, y_0)) = \min\{p_x, p_y\}(2+\beta)u(x_0, y_0)$$

3 CARA

$w\left(x\right) = -e^{-ax}$

Remark 5. *Assume $a \geqslant 0$, since $a < 0$ implies a decreasing VNM index.*

$$MRS_{xy} = \begin{cases} \frac{1}{1+\beta} e^{-a(x-y)} & x > y \\ \left[\frac{1}{1+\beta}, 1+\beta\right] & x = y \\ \left(1+\beta\right) e^{-a(x-y)} & x < y \end{cases}$$

3.1 Utility Maximization Problem

Case 14. $a > 0, \beta \geqslant 0:$

$$(x, y)^d = \begin{cases} (m_x, 0) & p < \frac{1}{1+\beta} e^{-am_x} \\ \left(\begin{array}{c} \frac{1}{p+1}\left[m_y - \frac{1}{a}\ln\left(p\left(1+\beta\right)\right)\right], \\ \frac{1}{p+1}\left[m_y + \frac{p}{a}\ln\left(p\left(1+\beta\right)\right)\right] \end{array} \right) & \frac{1}{1+\beta} e^{-am_x} \leqslant p < \frac{1}{1+\beta} \\ \left(\frac{m_y}{p+1}, \frac{m_y}{p+1}\right) & \frac{1}{1+\beta} \leqslant p \leqslant 1+\beta \\ \left(\begin{array}{c} \frac{1}{p+1}\left[m_y - \frac{1}{a}\ln\left(\frac{p}{1+\beta}\right)\right], \\ \frac{1}{p+1}\left[m_y + \frac{p}{a}\ln\left(\frac{p}{1+\beta}\right)\right] \end{array} \right) & 1+\beta < p \leqslant \left(1+\beta\right) e^{am_y} \\ (0, m_y) & \left(1+\beta\right) e^{am_y} < p \end{cases}$$

Case 15. $a > 0, -1 < \beta < 0$:

$$(x,y)^d = \begin{cases} (m_x, 0) & p < \frac{1}{1+\beta} e^{-am_x} \\ \left(\left(\begin{array}{c} \frac{1}{p+1}\left[m_y - \frac{1}{a}\ln\left(p\left(1+\beta\right)\right)\right], \\ \frac{1}{p+1}\left[m_y + \frac{p}{a}\ln\left(p\left(1+\beta\right)\right)\right] \end{array} \right)\right) & \frac{1}{1+\beta} e^{-am_x} \leqslant p < 1 \\ \left\{ \begin{array}{c} \left(\frac{1}{2}\left[m_y - \frac{1}{a}\ln\left(1+\beta\right)\right], \frac{1}{2}\left[m_y + \frac{1}{a}\ln\left(1+\beta\right)\right]\right), \\ \left(\frac{1}{2}\left[m_y + \frac{1}{a}\ln\left(1+\beta\right)\right], \frac{1}{2}\left[m_y - \frac{1}{a}\ln\left(1+\beta\right)\right]\right) \end{array} \right\} & p = 1 \\ \left(\begin{array}{c} \frac{1}{p+1}\left[m_y - \frac{1}{a}\ln\left(\frac{p}{1+\beta}\right)\right], \\ \frac{1}{p+1}\left[m_y + \frac{p}{a}\ln\left(\frac{p}{1+\beta}\right)\right] \end{array} \right) & 1 < p \leqslant \left(1+\beta\right) e^{am_y} \\ (0, m_y) & \left(1+\beta\right) e^{am_y} < p \end{cases}$$

Case 16. $a > 0, \beta = -1$:

$$(x,y)^d = \begin{cases} (m_x, 0) & p < 1 \\ \left\{(m_x, 0), (0, m_y)\right\} & p = 1 \\ (0, m_y) & 1 < p \end{cases}$$

Remark 6. *The Taylor series first approximation of $w(x) = -e^{-ax}$ around $a = 0$ is $w(x) = ax - 1$. Thus, when a goes to zero, we consider $w(x) = ax - 1$ rather than $w(x) = -e^{-ax}$. At exactly $a = 0$ we get $w(x) = -1$.*

Case 17. $a \to 0, \beta > 0$:

$$(x,y)^d = \begin{cases} (m_x, 0) & p < \frac{1}{1+\beta} \\ \left\{x \geqslant y, px + y = m_y\right\} & p = \frac{1}{1+\beta} \\ \left(\frac{m_y}{p+1}, \frac{m_y}{p+1}\right) & \frac{1}{1+\beta} < p < 1+\beta \\ \left\{ x \leqslant y, px + y = m_y\right\} & p = 1+\beta \\ (0, m_y) & 1+\beta < p \end{cases}$$

Case 18. $a \to 0, \beta = 0$:

$$(x,y)^d = \begin{cases} (m_x, 0) & p < 1 \\ \left\{(x,y), px + y = m\right\} & p = 1 \\ (0, m_y) & 1 < p \end{cases}$$

Case 19. $a \to 0, -1 \leqslant \beta < 0$*:*

$$(x,y)^d = \begin{cases} (m_x, 0) & p < 1 \\ \left\{(m, 0), (0, m)\right\} & p = 1 \\ (0, m_y) & 1 < p \end{cases}$$

Case 20. $a = 0$:

$$(x,y)^d = \left\{(x,y), p_x x + p_y y \leqslant M\right\}$$

3.2 Expenditure Minimization Problem

Remark 7. *The indifference curve through* $u(x_0, y_0)$ *intersects the axes, when* $\exists x$ *such that* $u(x_0, y_0) = \frac{1}{2+\beta}[-e^{-ax} - (1+\beta)]$. *Since* $-e^{-ax}$ *is negative and bounded by zero, if* $-u(x_0, y_0) > \frac{1+\beta}{2+\beta}$ *the indifference curve through* $u(x_0, y_0)$ *intersects the axes.*

Case 21. $a > 0, \beta \geqslant 0$ *and* $-u(x_0, y_0) > \frac{1+\beta}{2+\beta} : e(p_x, p_y, (x_0, y_0)) =$

$$\begin{cases} -\frac{p_x}{a}\ln\left(-\left((2+\beta)u(x_0,y_0)+(1+\beta)\right)\right) & p < -\frac{(2+\beta)u(x_0,y_0)+(1+\beta)}{1+\beta} \\ p_x\left[\frac{1}{a}\ln\left(-\frac{(1+p)}{p(2+\beta)u(x_0,y_0)}\right)\right]+p_y\left[\frac{1}{a}\ln\left(-\frac{(1+p)(1+\beta)}{(2+\beta)u(x_0,y_0)}\right)\right] & -\frac{(2+\beta)u(x_0,y_0)+(1+\beta)}{1+\beta} \leqslant p < \frac{1}{1+\beta} \\ (p_x+p_y)\left[\frac{1}{a}\ln\left(-\frac{1}{u(x_0,y_0)}\right)\right] & \frac{1}{1+\beta} \leqslant p \leqslant 1+\beta \\ p_x\left[\frac{1}{a}\ln\left(-\frac{(1+\beta)(1+p)}{p(2+\beta)u(x_0,y_0)}\right)\right]+p_y\left[\frac{1}{a}\ln\left(-\frac{(1+p)}{(2+\beta)u(x_0,y_0)}\right)\right] & 1+\beta < p \leqslant -\frac{1+\beta}{(2+\beta)u(x_0,y_0)+(1+\beta)} \\ -\frac{p_y}{a}\ln\left(-\left((2+\beta)u(x_0,y_0)+(1+\beta)\right)\right) & -\frac{1+\beta}{(2+\beta)u(x_0,y_0)+(1+\beta)} < p \end{cases}$$

Case 22. $a > 0, \beta \geqslant 0$ *and* $-u(x_0, y_0) \leqslant \frac{1+\beta}{2+\beta} : e(p_x, p_y, (x_0, y_0)) =$

$$\begin{cases} p_x\left[\frac{1}{a}\ln\left(-\frac{(1+p)}{p(2+\beta)u(x_0,y_0)}\right)\right]+p_y\left[\frac{1}{a}\ln\left(-\frac{(1+p)(1+\beta)}{(2+\beta)u(x_0,y_0)}\right)\right] & p < \frac{1}{1+\beta} \\ (p_x+p_y)\left[\frac{1}{a}\ln\left(-\frac{1}{u(x_0,y_0)}\right)\right] & \frac{1}{1+\beta} \leqslant p \leqslant 1+\beta \\ p_x\left[\frac{1}{a}\ln\left(-\frac{(1+\beta)(1+p)}{p(2+\beta)u(x_0,y_0)}\right)\right]+p_y\left[\frac{1}{a}\ln\left(-\frac{(1+p)}{(2+\beta)u(x_0,y_0)}\right)\right] & 1+\beta < p \end{cases}$$

Case 23. $a > o, -1 < \beta < 0$ *and* $-u(x_0, y_0) \geqslant \frac{2+2\beta}{2+\beta} : e(p_x, p_y, (x_0, y_0)) =$

$$\begin{cases} -\frac{p_x}{a}\ln\left(-\left((2+\beta)u(x_0,y_0)+(1+\beta)\right)\right) & p \leqslant 1 \\ -\frac{p_y}{a}\ln\left(-\left((2+\beta)u(x_0,y_0)+(1+\beta)\right)\right) & p > 1 \end{cases}$$

Case 24. $a > 0, -1 < \beta < 0$ *and* $\frac{2+2\beta}{2+\beta} > -u(x_0, y_0) > \frac{1+\beta}{2+\beta} : e(p_x, p_y, (x_0, y_0)) =$

$$\begin{cases} -\frac{p_x}{a}\ln\left(-\left((2+\beta)u(x_0,y_0)+(1+\beta)\right)\right) & p < -\frac{(2+\beta)u(x_0,y_0)+(1+\beta)}{1+\beta} \\ p_x\left[\frac{1}{a}\ln\left(-\frac{(1+p)}{p(2+\beta)u(x_0,y_0)}\right)\right]+p_y\left[\frac{1}{a}\ln\left(-\frac{(1+p)(1+\beta)}{(2+\beta)u(x_0,y_0)}\right)\right] & -\frac{(2+\beta)u(x_0,y_0)+(1+\beta)}{1+\beta} \leqslant p \leqslant 1 \\ p_x\left[\frac{1}{a}\ln\left(-\frac{(1+\beta)(1+p)}{p(2+\beta)u(x_0,y_0)}\right)\right]+p_y\left[\frac{1}{a}\ln\left(-\frac{(1+p)}{(2+\beta)u(x_0,y_0)}\right)\right] & 1 < p \leqslant -\frac{1+\beta}{(2+\beta)u(x_0,y_0)+(1+\beta)} \\ -\frac{p_y}{a}\ln\left(-\left((2+\beta)u(x_0,y_0)+(1+\beta)\right)\right) & -\frac{1+\beta}{(2+\beta)u(x_0,y_0)+(1+\beta)} < p \end{cases}$$

Case 25. $a > 0, -1 < \beta < 0$ *and* $\frac{1+\beta}{2+\beta} \geqslant -u\left(x_0, y_0\right) : e\left(p_x, p_y, \left(x_0, y_0\right)\right) =$

$$\begin{cases} p_x\left[\frac{1}{a}\ln\left(-\frac{(1+p)}{p(2+\beta)u(x_0,y_0)}\right)\right] + p_y\left[\frac{1}{a}\ln\left(-\frac{(1+p)(1+\beta)}{(2+\beta)u(x_0,y_0)}\right)\right] & p \leqslant 1 \\ p_x\left[\frac{1}{a}\ln\left(-\frac{(1+\beta)(1+p)}{p(2+\beta)u(x_0,y_0)}\right)\right] + p_y\left[\frac{1}{a}\ln\left(-\frac{(1+p)}{(2+\beta)u(x_0,y_0)}\right)\right] & 1 < p \end{cases}$$

Case 26. $a > 0, \beta = -1$

$$e\left(p_x, p_y, \left(x_0, y_0\right)\right) = \begin{cases} p_x\left[\frac{1}{a}\ln\left(-\frac{1}{(2+\beta)u(x_0,y_0)}\right)\right] & p \leqslant 1 \\ p_y\left[\frac{1}{a}\ln\left(-\frac{1}{(2+\beta)u(x_0,y_0)}\right)\right] & 1 < p \end{cases}$$

Case 27. $a \to 0, \beta > 0 :$

$$e\left(p_x, p_y, \left(x_0, y_0\right)\right) = \begin{cases} \frac{1}{a}(2+\beta)p_x[1+u(x_0,y_0)] & p \leqslant \frac{1}{1+\beta} \\ \frac{1}{a}(p_x+p_y)[1+u(x_0,y_0)] & \frac{1}{1+\beta} \leqslant p \leqslant 1+\beta \\ \frac{1}{a}(2+\beta)p_y[1+u(x_0,y_0)] & 1+\beta \leqslant p \end{cases}$$

Case 28. $a \to 0, -1 \leqslant \beta \leqslant 0 :$

$$e\left(p_x, p_y, \left(x_0, y_0\right)\right) = \frac{1}{a}(2+\beta)\min\{p_x, p_y\}[1+u(x_0,y_0)]$$

Case 29. $a = 0 :$

$$e\left(p_x, p_y, \left(x_0, y_0\right)\right) = 0$$