Other Regarding Preferences - CES

December 8, 2016

1 Introduction

1.1 Budget

$p = \frac{p_x}{p_y} = \frac{m_y}{m_x}; m_y = \frac{M}{p_y}; m_x = \frac{M}{p_x}$ (if $m_y = m_x$ we denote both by m).

1.2 CES Preferences

$u(x, y) = [\alpha \times x^{\rho} + (1 - \alpha) \times y^{\rho}]^{\frac{1}{\rho}}$ where $\alpha \in [0, 1]$.[1]

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[1] The definition of elasticity of substitution is $\sigma_{xy} = -\frac{\frac{\Delta \frac{x}{y}}{\frac{x}{y}}}{\frac{\Delta \frac{p_x}{p_y}}{\frac{p_x}{p_y}}}$. If the DM maximizes utility and the chosen bundle is interior then at this bundle the price ratio equals the MRS. Therefore, $\sigma_{xy} = -\frac{\frac{\Delta \frac{x}{y}}{\frac{x}{y}}}{\frac{\Delta \frac{u_x}{u_y}}{\frac{u_x}{u_y}}}$. Next, we take advantage of the equality $d\ln(f(x)) = \frac{df(x)}{f(x)}$ and we write $\sigma_{xy} = -\frac{d\ln(\frac{x}{y})}{d\ln(\frac{u_x}{u_y})} = -\frac{d\ln(\frac{x}{y})}{d\ln(MRS_{xy})}$. For an interior optimal choice in a standard case ($\alpha \in (0,1)$, $x > 0$, $y > 0$, $\rho \notin \{-\infty, 0, 1, \infty\}$): $MRS_{xy} = \frac{\alpha}{1-\alpha}\left(\frac{x}{y}\right)^{\rho-1}$. Now, denote $MRS_{xy} = \theta$. Then, $\theta = \frac{\alpha}{1-\alpha}\left(\frac{x}{y}\right)^{\rho-1}$ and therefore $\frac{x}{y} = (\theta\frac{1-\alpha}{\alpha})^{\frac{1}{\rho-1}}$. By the expression we found above, $\sigma_{xy} = -\frac{d\ln((\theta\frac{1-\alpha}{\alpha})^{\frac{1}{\rho-1}})}{d\ln(\theta)}$. Hence, we get, $\sigma_{xy} = -\frac{\frac{d((\theta\frac{1-\alpha}{\alpha})^{\frac{1}{\rho-1}})}{(\theta\frac{1-\alpha}{\alpha})^{\frac{1}{\rho-1}}}}{\frac{d\theta}{\theta}} = -\frac{d((\theta\frac{1-\alpha}{\alpha})^{\frac{1}{\rho-1}})}{d\theta} \times \frac{\theta}{(\theta\frac{1-\alpha}{\alpha})^{\frac{1}{\rho-1}}} = -\frac{1}{\rho-1} \times (\theta\frac{1-\alpha}{\alpha})^{\frac{1}{\rho-1}-1} \times \frac{1-\alpha}{\alpha} \times \frac{\theta}{(\theta\frac{1-\alpha}{\alpha})^{\frac{1}{\rho-1}}} = \frac{1}{1-\rho}$. Hence, the elasticity of substitution is constant since it does not depend on the chosen bundle. As ρ increases ($\rho \leqslant 1$) the elasticity of substitution increases, meaning that the DM is more sensitive to changes in the relative price.

2 Utility Maximization Problem

Case 1 (Extreme Altruism). *$\alpha = 0$ means that $u(x, y) = y$ and the chosen bundle is $(0, m_y)$.*

Case 2 (Extreme Selfishness). *$\alpha = 1$ means that $u(x, y) = x$ and the chosen bundle is $(m_x, 0)$.*

Case 3 ($\sigma_{xy} = 0$). *If $\alpha \in (0, 1)$ then $\rho \to -\infty$ means that $u(x, y) = \min\{x, y\}$ and the chosen bundle is $(\frac{m_y}{1+p}, \frac{m_y}{1+p})$.*[2]

Case 4 ($\sigma_{xy} = 1$). *If $\alpha \in (0, 1)$ then $\rho = 0$ means that the DM has Cobb-Douglas preferences, $u(x, y) = x^{\alpha} \times y^{1-\alpha}$ and the chosen bundle is $(\alpha m_x, (1-\alpha) m_y)$.*[3]

Case 5 ($\sigma_{xy} \to \infty$). *If $\alpha \in (0, 1)$ then $\rho = 1$ means that $u(x, y) = \alpha \times x + (1 - \alpha) \times y$. Therefore, the indifference curves are linear with the slope $\frac{\alpha}{1-\alpha}$. Thus, the chosen bundle is,*

$$\begin{array}{cl} (0, m_y) & \textit{if } \frac{\alpha}{1-\alpha} < p \\ \{(x, y) | p_x x + p_y y = M\} & \textit{if } \frac{\alpha}{1-\alpha} = p \\ (m_x, 0) & \textit{if } \frac{\alpha}{1-\alpha} > p \end{array}$$

Case 6. *If $\alpha \in (0, 1)$, $\rho \in (-\infty, 1)$ and $\rho \neq 0$, the CES utility function*

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[2] The Leontief function is due to the fact that the lower quantity is the dominant quantity when raised to the power of minus infinity.

[3] When $\rho = 0$ the value of the utility cannot be determined. To recover the utility function consider,

$$\begin{aligned} \lim_{\rho \to 0} u(x, y) &= \lim_{\rho \to 0} [\alpha \times x^{\rho} + (1 - \alpha) \times y^{\rho}]^{\frac{1}{\rho}} &&= \lim_{\rho \to 0} e^{\ln[\alpha \times x^{\rho} + (1-\alpha) \times y^{\rho}]^{\frac{1}{\rho}}} \\ &= e^{\lim_{\rho \to 0} \ln[\alpha \times x^{\rho} + (1-\alpha) \times y^{\rho}]^{\frac{1}{\rho}}} &&= e^{\lim_{\rho \to 0} \frac{\ln[\alpha \times x^{\rho} + (1-\alpha) \times y^{\rho}]}{\rho}} \end{aligned}$$

By L'Hopital rule,

$$\begin{aligned} \lim_{\rho \to 0} \frac{\ln[\alpha \times x^{\rho} + (1-\alpha) \times y^{\rho}]}{\rho} &= \lim_{\rho \to 0} \frac{\frac{\alpha \times x^{\rho} \times \ln x + (1-\alpha) \times y^{\rho} \times \ln y}{\alpha \times x^{\rho} + (1-\alpha) \times y^{\rho}}}{1} \\ &= \lim_{\rho \to 0} \frac{\alpha \times x^{\rho} \times \ln x + (1-\alpha) \times y^{\rho} \times \ln y}{\alpha \times x^{\rho} + (1-\alpha) \times y^{\rho}} \\ &= \alpha \times \ln x + (1 - \alpha) \times \ln y \end{aligned}$$

Hence, $\lim_{\rho \to 0} u(x, y) = e^{\alpha \times \ln x + (1-\alpha) \times \ln y} = x^{\alpha} \times y^{1-\alpha}$.

represents monotonic[4] *and convex*[5] *preferences. Therefore, the chosen bundle is interior and it satisfies* $MRS_{xy} = p$ *and* $p_x x + p_y y = M$*. Hence, the chosen bundle is* $(\frac{m_y}{p+[p\frac{1-\alpha}{\alpha}]^{\frac{1}{1-\rho}}}, \frac{m_y}{1+\frac{p}{[p\frac{1-\alpha}{\alpha}]^{\frac{1}{1-\rho}}}})$.

Case 7. *If* $\alpha \in (0,1)$*,* $\rho \in (1,\infty)$*, the CES utility function represents monotonic and concave preferences. Therefore, the chosen bundle is*

$$\begin{array}{cl} (0, m_y) & \text{if } \frac{\alpha}{1-\alpha} < p \\ \{(0, m_y), (m_x, 0)\} & \text{if } \frac{\alpha}{1-\alpha} = p \\ (m_x, 0) & \text{if } \frac{\alpha}{1-\alpha} > p \end{array}$$

Remark 1. *Cases 6 and 7 demonstrate that when* $\rho \geqslant 1$*,* ρ *cannot be identified. Cases 1 and 2 demonstrate that when* $\alpha \in \{0,1\}$*,* ρ *also cannot be identified. Therefore,* ρ *is identified only when* $\alpha \in (0,1)$ *and* $\rho < 1$

3 Expenditure Minimization Problem

Case 8 (Extreme Altruism)**.** $\alpha = 0 : e\left(p_x, p_y, (x_0, y_0)\right) = p_y \times y_0$.

Case 9 (Extreme Selfishness)**.** $\alpha = 1 : e\left(p_x, p_y, (x_0, y_0)\right) = p_x \times x_0$.

Case 10 ($\sigma_{xy} = 0$)**.** $\rho \to -\infty : e\left(p_x, p_y, (x_0, y_0)\right) = (p_x + p_y) \times \min\{x_0, y_0\}$.

Case 11 ($\sigma_{xy} = 1$)**.** $\rho = 0 : e\left(p_x, p_y, (x_0, y_0)\right) = [\frac{p_x x_0}{\alpha}]^{\alpha} \times [\frac{p_y y_0}{1-\alpha}]^{1-\alpha}$.

Case 12 ($\sigma_{xy} \to \infty$)**.** $\rho = 1 : e\left(p_x, p_y, (x_0, y_0)\right) =$

$$\begin{array}{cl} p_y \times (y_0 + \frac{\alpha}{1-\alpha} x_0) & \text{if } \frac{\alpha}{1-\alpha} < p \\ p_x \times x_0 + p_y \times y_0 & \text{if } \frac{\alpha}{1-\alpha} = p \\ p_x \times (x_0 + \frac{1-\alpha}{\alpha} y_0) & \text{if } \frac{\alpha}{1-\alpha} > p \end{array}$$

[4]Consider the case where $\alpha \in (0,1)$ and $\rho \in (-\infty, \infty)$ and $\rho \notin \{0,1\}$. Then, $u_x = \alpha[\alpha \times x^{\rho} + (1-\alpha) \times y^{\rho}]^{\frac{1}{\rho}-1} x^{\rho-1} \geqslant 0$ and $u_y = (1-\alpha)[\alpha \times x^{\rho} + (1-\alpha) \times y^{\rho}]^{\frac{1}{\rho}-1} y^{\rho-1} \geqslant 0$. Hence, the CES preferences are monotonic.

[5]Consider the case where $\alpha \in (0,1)$ and $\rho \in (-\infty, \infty)$ and $\rho \notin \{0,1\}$. Then, $MRS_{xy} = \frac{\alpha}{1-\alpha}\left(\frac{x}{y}\right)^{\rho-1}$. If $\rho < 1$ then when $\frac{x}{y}$ increases the MRS decreases while if $\rho > 1$ then when $\frac{x}{y}$ increases the MRS increases. Since the the CES preferences are monotonic, when $\rho < 1$ the CES preferences are convex while when $\rho > 1$ the CES preferences are concave.

Case 13. $\rho < 1, \rho \neq 0, \alpha \in (0,1) : e\left(p_x, p_y, (x_0, y_0)\right) =$

$$u(x_0, y_0) \times \left[\frac{p_x}{\left[\alpha + \frac{\alpha^{\frac{\rho}{\rho-1}}}{(1-\alpha)^{\frac{1}{\rho-1}} p^{\frac{\rho}{\rho-1}}}\right]^{\frac{1}{\rho}}} + \frac{p_y}{\left[(1-\alpha) + \frac{(1-\alpha)^{\frac{\rho}{\rho-1}} p^{\frac{\rho}{\rho-1}}}{\alpha^{\frac{1}{\rho-1}}}\right]^{\frac{1}{\rho}}} \right]$$

Case 14. *If* $\alpha \in (0,1), \rho \in (1, \infty) : e\left(p_x, p_y, (x_0, y_0)\right) =$

$$\begin{array}{ll} \frac{p_y}{1-\alpha}\left[\alpha \times x_0^\rho + (1-\alpha) \times y_0^\rho\right]^{\frac{1}{\rho}} & \textit{if} \quad \frac{\alpha}{1-\alpha} \leqslant p \\ \frac{p_x}{\alpha}\left[\alpha \times x_0^\rho + (1-\alpha) \times y_0^\rho\right]^{\frac{1}{\rho}} & \textit{if} \quad \frac{\alpha}{1-\alpha} > p \end{array}$$