2-States Disappointment Aversion - Implementation

MMI CRRA – Quantities transformation (HPZ_CRRA_Quantities_Transformation)

The Expenditure Minimization Problem (EMP) is the following problem:

$$\min_{x,y}\{p_x x + p_y y\}$$

$$\text{s.t.}\quad u(x,y) = u(x_0, y_0)$$

Definition: The Transformed Expenditure Minimization Problem (TEMP-c, c>0) is the following problem:

$$\min_{x^*,y^*}\{p_x^* x^* + p_y^* y^*\}$$

$$\text{s.t.}\quad u(x^*,y^*) = u(x_0^*, y_0^*)$$

where $x^* = cx$, $y^* = cy$, $x_0^* = cx_0$, $y_0^* = cy_0$, $p_x^* = \frac{p_x}{c}$, $p_y^* = \frac{p_y}{c}$

Claim:

1. $\forall p_x, p_y\colon \min_{x^*,y^*}\{p_x^* x^* + p_y^* y^*\} = \min_{x,y}\{p_x x + p_y y\}$.
2. $\forall p_x, p_y\colon arg\min_{x^*,y^*}\{p_x^* x^* + p_y^* y^*\} = \{cz \mid z \in arg\min_{x,y}\{p_x x + p_y y\}\}$.
3. For the DA2-CRRA utility function:
   $\forall x, y, x_0, y_0\colon u(x,y) = u(x_0,y_0) \Leftrightarrow u(x^*,y^*) = u(x_0^*, y_0^*)$.

Proof:

The first part:

$$\min_{x^*,y^*}\{p_x^* x^* + p_y^* y^*\} = \min_{cx,cy}\{\tfrac{p_x}{c}cx + \tfrac{p_y}{c}cy\} = \min_{x,y}\{p_x x + p_y y\}$$

The second part:

Let $z \in arg\min_{x,y}\{p_x x + p_y y\}$. Then, $\min_{x,y}\{p_x x + p_y y\} = p_x z_1 + p_y z_2$. By Claim 1, $\min_{x^*,y^*}\{p_x^* x^* + p_y^* y^*\} = p_x z_1 + p_y z_2$. Therefore, $\min_{x^*,y^*}\{p_x^* x^* + p_y^* y^*\} = p_x^* c z_1 + p_y^* c z_2$. Therefore, cz $\in arg\min_{x^*,y^*}\{p_x^* x^* + p_y^* y^*\}$. The other direction is similar using $\frac{1}{c}$.

The third part:

The DA2-CRRA utility function for $\rho \neq 1$ is:

$u(x,y) = \frac{1}{1-\rho}(\gamma\,(\max(x,y))^{1-\rho} + (1-\gamma)\,(\min(x,y))^{1-\rho})$.

$$\boldsymbol{u(x^*,y^*) = u(x_0^*, y_0^*)} \quad \Leftrightarrow$$

$$\frac{1}{1-\rho}(\gamma\,(\max(x^*,y^*))^{1-\rho} + (1-\gamma)(\min(x^*,y^*))^{1-\rho}) = \frac{1}{1-\rho}(\gamma\,(\max(x_0^*,y_0^*))^{1-\rho} + (1-\gamma)(\min(x_0^*,y_0^*))^{1-\rho}) \quad \Leftrightarrow$$

$$\frac{1}{1-\rho}(\gamma\,(\max(cx,cy))^{1-\rho}+(1-\gamma)(\min(cx,cy))^{1-\rho})$$
$$=\ \frac{1}{1-\rho}(\gamma\,(\max(cx_0,cy_0))^{1-\rho}+(1-\gamma)(\min(cx_0,cy_0))^{1-\rho})\quad\Leftrightarrow$$

$$\frac{c^{1-\rho}}{1-\rho}(\gamma\,(\max(x,y))^{1-\rho}+(1-\gamma)(\min(x,y))^{1-\rho})$$
$$=\ \frac{c^{1-\rho}}{1-\rho}(\gamma\,(\max(x_0,y_0))^{1-\rho}+(1-\gamma)(\min(x_0,y_0))^{1-\rho})\quad\Leftrightarrow$$

$$\frac{1}{1-\rho}(\gamma\,(\max(x,y))^{1-\rho}+(1-\gamma)(\min(x,y))^{1-\rho})$$
$$=\ \frac{1}{1-\rho}(\gamma\,(\max(x_0,y_0))^{1-\rho}+(1-\gamma)(\min(x_0,y_0))^{1-\rho})\quad\Leftrightarrow$$

$$\boldsymbol{u(x,y)=u(x_0,y_0)}$$

For $\rho=1$: $\quad u(x,y)=\ \gamma\ ln(\max(x,y))+(1-\gamma)\ ln(\min\,(x,y))$

$$\boldsymbol{u(x^*,y^*)=u(x_0^*,y_0^*)}\quad\Leftrightarrow$$

$$\gamma\ ln(\max(x^*,y^*))+(1-\gamma)ln(\min(x^*,y^*))$$
$$=\ \gamma\ ln\big(\max(x_0^*,y_0^*)\big)+(1-\gamma)ln(\min(x_0^*,y_0^*))\quad\Leftrightarrow$$

$$\gamma\ ln(\max(cx,cy))+(1-\gamma)ln(\min(cx,cy))$$
$$=\ \gamma\ ln\big(\max(cx_0,cy_0)\big)+(1-\gamma)ln(\min(cx_0,cy_0))\quad\Leftrightarrow$$

$$\gamma\ ln(\max(x,y))+(1-\gamma)ln(\min(x,y))\ +\ ln(\mathrm{c})$$
$$=\ \gamma\ ln\big(\max(x_0,y_0)\big)+(1-\gamma)ln(\min(x_0,y_0))\ +\ ln(\mathrm{c})\quad\Leftrightarrow$$

$$\gamma\ ln(\max(x,y))+(1-\gamma)ln(\min(x,y))=\ \gamma\ ln\big(\max(x_0,y_0)\big)+(1-\gamma)ln(\min(x_0,y_0))$$
$$\Leftrightarrow$$

$$\boldsymbol{u(x,y)=u(x_0,y_0)}$$

Conclusion:

For the DA2-CRRA utility function the EMP problem and the TEMP-c problem are equivalent.