Minkowski Spacetime

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Contents

In this chapter, we will see that special relativity can be reformulated in a framework of *quadri-dimensional*, where time associates itself to the three ordinary spatial coordinates. This quadri-dimensional presentation of Einstein's relativity is in principal due to the mathematician Hermann Minkowski

1 Euclidian geometry

Let's put ourselves in a Euclidean space, combined with a cartesian coordinate system with a bsis $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ and an origin O. Any point M of the space is distinguished by it's coordinates x, y and z, which means that the position vector $\vec{x}$ that connects O to M can be decomposed in the form,

$$O\vec{M} = \vec{x} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \tag{1}$$

We will mostly use the identification $x^1 \equiv x$, $x^2 \equiv y$ and $x^3 = z$. The decomposition of position-vector is written as,

$$\vec{x} = \sum_{i=1}^{3} x^i \vec{e}_i. \tag{2}$$

We can furthermore simplify this expression if we adopt the convention of the **Einstein summation**, in which the summation Σ is not written explicitely; the summation is indicated implicitely by the *repetition* of the symbol i in superscript position (in x^i) and in subscript position, which lets us write

$$\vec{x} = x^i \vec{e}_i \tag{3}$$

Even though it is not *a priori* necessary, we use in general an orthonormal basis with respect to the euclidean scalar product. That is to say,

$$\vec{e}_i \cdot \vec{e}_j \equiv g(\vec{e}_i, \vec{e}_j) = \delta_{ij} \tag{4}$$

where δ_{ij} is the Kronecker symbol. The scalar square of $\vec{x}$ is thus,

$$\vec{x}^2 \equiv g(x^i\vec{e}_i, x^j\vec{e}_j) = x^i X^j g(\vec{e}_i, \vec{e}_j) = \delta_{ij} x^i x^j = x^2 + y^2 + z^2 \tag{5}$$

where we used the Einstein summation convention.

By choosing another orthonormal basis $\{\vec{e}'_i\} = \{\vec{e}'_x, \vec{e}'_y, \vec{e}'_z\}$, while still keeping the same origin, the position vector $\vec{x}$ of the point M admits a new decomposition

$$\vec{x} = x'^i \vec{e}'_i \tag{6}$$

The components/coordinates ${x'}^i$ in the new basis are different than the old x^i. However, the scalar square $\vec{x}^2$ does not depend on the orthonormal basis chosen and we then have,

$$\vec{x}^2 = x^2 + y^2 + z^2 = {x'}^2 + {y'}^2 + {z'}^2 \tag{7}$$

As we will see later, it is natural, in the context of special relativity, to extend the notion of tri-dimentional euclidean geometry to a new geometry, this time quadri-dimensional, by associating the temporal dimension to the three spacial dimensions.

2 Quadri-dimensional geometry

If to the three spatial coordinates x, y, z, we add the temporal coordinate

$$x^o \equiv ct, \tag{8}$$

we can define a quadri-dimensional space, called the **spacetime**, where points corresponds to events characterised by their 4 coordinates that we will collectively note as x^μ:

$$x^\mu = \{ct, x, y, z,\}, \quad \mu = 0, 1, 2, 3. \tag{9}$$

We have seen that a same event $\mathcal{E}$, observed in two different referentials $\mathcal{R}$ and $\mathcal{R}'$, is distinguished by different coordinates, respectively (t, x, y, z) and (t', x', y', z'), but related by the equality

$$c^2t^2 - x^2 - y^2 - z^2 = c^2{t'}^2 - {x'}^2 - {y'}^2 - {z'}^2 \tag{10}$$

The similarity between this equation and the scalar square in euclidean geometry suggests that we should think of a change in inertial referential in Special Relativity as a change in basis in the 4 dimensional spacetime , moving from the coordinates x^μ to the coordinates ${x'}^\mu$.

In the spacetime, a basis is made of 4 quadri-dimensional vectors, or **four-vectors**, $\underline{e}_0$, $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$. To each event $\mathcal{E}$, of coordinate (ct, x, y, z), we can associate a **position four-vector**, noted by $\underline{X}$,

$$\underline{X} = \sum_{\mu=0}^{3} x^\mu \underline{e}_\mu, \tag{11}$$

which characterize the separation, in *time* and *space* of the event $\mathcal{E}$ with the origin-event characterized by $t = x = y = z = 0$.

Similarly to the case of euclidean geometry, we will now use the Einstein notation: the repetition of the symbol μ in superscript position (in x^μ) and in subscript position (in $\underline{e}_\mu$) will be synonym to the summation on that index and we will write the position four-vector in the simplified form

$$\underline{X} = x^\mu \underline{e}_\mu \tag{12}$$

To distinguish between three and four dimensional vectors we will use the convention of greek letters for the indices of 4 dimensional vectors and the latin letters for the indices of 3 dimensional vectors.

2.1 Minkowski scalar (or metric) product

As we have seen in euclidean geometry, the quadratic expression is invariant under a change of basis but depends on the scalar product taken. The signs in (10) forbids us to choose a 4 dimensional euclidean scalar product.

We will thus introduce the following scalar product, called the **Minkowski scalar product**[1], defined by it's action on the four basis vectors:

$$\underline{e}_0 \cdot \underline{e}_0 = 1, \quad \underline{e}_0 \cdot \underline{e}_i = \underline{e}_i \cdot \underline{e}_0 = 0, \quad \underline{e}_i \cdot \underline{e}_j = -\delta_{ij}, \tag{13}$$

which we can rewrite as,

$$\underline{e}_\mu \cdot \underline{e}_\nu = \eta_{\mu\nu}, \quad [\eta_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = G. \tag{14}$$

We can then see that the scalar square of $\underline{X}$ writes as,

$$\underline{X}^2 \equiv \underline{X} \cdot \underline{X} = c^2t^2 - x^2 - y^2 - z^2 \tag{15}$$

this is exactly what we found in (10). The scalar square is automatically independent of the referential in which we place ourselves, in the same way that the norm of a vector is independent of the referential in euclidean geometry.

We talk also about **Minkowski metric**, since the scalar product allow us to define, as in euclidean geometry, a *distance* between two points of the space-time, located by the four-vectors $\underline{X}_1$ and $\underline{X}_2$, form the scalar square $(\underline{X}_1 - \underline{X}_2)^2$ of their four-vector separation.

[1] Rigorously, we should talk about pseudo-scalar product as the associated symmetric bilinear form is not positive-definite

Having defined the scalar product by it's action on the basis vectors, we can easily deduce the scalar product of any two four-vectors, thanks to it's bilinearity. For the four-vectors,

$$\underline{X} = x^\mu \underline{e}_\mu, \quad \underline{Y} = y^\mu \underline{e}_\mu, \tag{16}$$

we find, using the Einstein summation,

$$\underline{X} \cdot \underline{Y} = \left(x^\mu \underline{e}_\mu\right) \cdot \left(y^\nu \underline{e}_\nu\right) = x^\mu y^\nu \underline{e}_\mu \underline{e}_\nu = \eta_{\mu\nu} x^\mu y^\nu \tag{17}$$

It is sometimes useful to decompose the four-vectors, as well as the Minkowski scalar product, in a spatial part and a temporal part:

$$\underline{X}: \quad \{x^0, \vec{x}\}, \quad \underline{Y}: \quad \{y^0, \vec{y}\}, \quad \underline{X} \cdot \underline{Y} = x^0 y^0 - \vec{x} \cdot \vec{y} \tag{18}$$

2.2 Change of basis

Similarly to the case of euclidean geometry, we can change basis and consider the new decomposition

$$\underline{X} = x'^\mu \underline{e}'_\mu. \tag{19}$$

Analogous to a change of orthonormal basis in euclidean geometry, we will only be interested with the new basis $\{\underline{e}'_\mu\}$ that are, like $\{\underline{e}_\mu\}$, pseudo-orthonormal, that is to say that verify

$$\underline{e}'_\mu \cdot \underline{e}'_\nu = \eta_{\mu\nu} \tag{20}$$

The new basis vectors, being linear combinations of the old ones, we can see that the new components of the four-vector $\underline{X}$ is as well a linear combination of the old components, if the origin of the spacetime is left unchanged, For an arbitrary four-vector $\underline{X}$, we can thus write

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \tag{21}$$

where the coefficients Λ^μ_ν does not depend on the basis $\{\underline{e}_\mu\}$ and $\{\underline{e}'_\mu\}$.

Since the new basis is pseudo-orthonormal, the scalar product of any two vectors $\underline{X}$ and $\underline{Y}$ writes as,

$$\underline{X} \cdot \underline{Y} = \eta_{\mu\nu} x'^\mu y'^\nu = \eta_{\mu\nu} x^\mu y^\nu, \tag{22}$$

respectively in the new basis $\{\underline{e}'_\mu\}$ and in the old basis $\{\underline{e}_\mu\}$.

By changing the new coordinates x'^μ and y'^ν by their expression, we can deduce, $\underline{X}$ and $\underline{Y}$ being arbitrary, that

$$\eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = \eta_{\rho\sigma} \tag{23}$$

This conditions on the coefficients Λ^μ_ν, equivalent to the invariance in the quadratic form (See Chapter 2), characterize the Lorentz transformations. These last equations play an analogous role, in special relativity, to the one of isometries in euclidian geometry.

3 Types of quadri-vectors

The Minkowski (pseudo-) scalar product not being positive-definite, the scalar square of non-zero vectors can be negative ($\underline{X}^2 < 0$) or zero ($\underline{X}^2 = 0$), contrarily to euclidean geometry.

We can thus classify the spacetime four-vectors in 3 categories.

3.1 Temporal quadri-vectors

The four-vector $\underline{X}$ is said to be of type **temporal** if $\underline{X}^2 > 0$. Since

$$\underline{X}^2 = (x^0)^2 - \|\vec{x}\|^2, \tag{24}$$

the components of $\underline{X}$ need to verify the condition

$$|x^0| > \|\vec{x}\|, \tag{25}$$

in other words, the temporal component of $\underline{X}$ dominates over it's spatial component.

Even if the components are modified through a change of basis, this property stays true in all referential.

Moreover, it is possible to find a particular referential in which the spatial component is null, $\vec{x}' = \vec{0}$. Let us prove it for a vector $x^\mu == \{x^0, x^1, 0, 0\}$, a case we can always come to by performing a rotation of the spatial referential defined in the initial referential.

In a Lorentz transformation along the x-axis, the components of $\underline{X}$ become

$$x'^0 = \gamma(x^0 - \beta x^1), \qquad x'^1 = \gamma(x^1 - \beta x^0), \tag{26}$$

the two other spatial components stay null. In consequence, by choosing

$$\beta = \frac{x^1}{x^0}, \tag{27}$$

which is possible here since $|x^1| < |x^0|$ (making sure that $|\beta| < 1$), we obtain a new referential in which $\vec{x}' = \vec{0}$. Therefore, by choosing carefully the referential (in fact by choosing $\underline{e}'_0$ colinear with $\underline{X}$), we can eliminate the spatial components of all four-vector of temporal type, however not it's temporal component.

If two events in spacetime, $\mathcal{E}_1$ and $\mathcal{E}_2$, are separated by a four-vector,

$$\underline{S} = \underline{X}_2 - \underline{X}_1 \tag{28}$$

of *temporal type*, then the same reasoning leads to a referential in which these two events have the *same spatial coordinates*.

3.2 Spatial quadri-vectors

3.3 Light quadri-vectors

3.4 Causality

4 Trajectory of space-time

4.1 Proper time and speed of quadri-vectors

4.2 Change of referential

4.3 Instantaneous inertial referential

4.4 Proper time mesured by different observers

4.5 Acceleration of quadri-vectors

4.6 Accelerated uniform movement

5 Wave quadri-vectors

5.1 Electromagnetic waves