Convex Optimization and Optimal Control

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Reminders on Real Analysis

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Chapter 2

Convex Sets

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2.1 Definition and first properties

2.1.1 Definition and first algebraic properties

Definition 1. *Let X be a (real) vector space. A set $A \subset X$ is said to be convex if*

$$\forall (x, y) \in A^2, \forall \theta \in [0, 1], qq\theta x + (1 - \theta)y \in A..$$

Proposition 1. *An (arbitrary) intersection of convex subsets of X is convex.*

Proposition 2. *The sum of two convex sets is convex.*

2.1.2 First topological properties

Proposition 3. *If X is a normed vector space then all balls (open or closed) of X are convex.*

Proof. □

Proposition 4. *If A is convex subset of a normed vector space then $\overline{A}$ and $\int A$ are also convex.*

Proof. □

2.2 Convex combinations and convex hull

2.2.1 Convex combinations

Definition 2. *Let X be a vector space. A convex combination of m elements $x_1, \ldots, x_m \in X$ is an element $y \in X$ that can be decomposed as*

$$y = \sum_{i=1}^{m} \theta_i x_i \quad \text{with} \quad (\theta_1, \ldots, \theta_m) \in (\mathbb{R}_+)^m, \quad \sum_{i=1}^{m} \theta_i = 1.$$

Proposition 5. *If A is convex then any convex combination of elements of A belong to A.*

Proof. □

Theorem 1 (Carathéodory)**.** *In a vector space of dimension n, all convex combination of m elements, $m > n+1$, can be written as a convex combination of at most $n+1$ of these elements.*

Proof. □

2.2.2 Convex hull

Definition 3. *Let X be a vector space and A be a subset of X. The convex hull of A, denoted by* $\operatorname{conv} A$ *is the smallest (inclusion) convex subset of X containing A. It is the intersection of all convex sets containing A.*

Proposition 6. *The convex hull of A is the set of all convex combinations of elements of A.*

Proposition 7. *Let X, Y be two vector spaces, $A \subset X, B \subset Y$. We have*

$$\operatorname{conv}(A \times B) = \operatorname{conv} A \times \operatorname{conv} B..$$

Proof. □

2.3 Projection and separation

2.3.1 Projection of a point onto a closed convex set

Theorem 2. *Let X be a Hilbert space and A be a nonempty closed convex subset of X. For all $x \in X$ there exists a unique $y \in A$ such that*

$$\|x - y\| = \min_{z \in A} \|x - z\|.$$

This element is characterized by the variational inequality (see in **??***)*

$$\langle x - y, z - y \rangle \leq 0 \quad \forall z \in A.$$

It is called the projection of x onto A, denoted by $y = P_A(x)$.

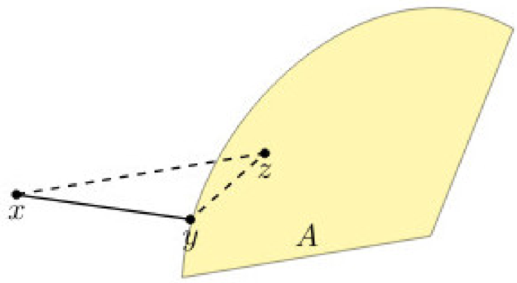

Figure 2.9: Projection onto a convex set

Proof. □

Proposition 8. *Under the assumptions of the above theorem, we have*

$$\|P_A(x) - P_A(y)\| \leq \|x - y\| \quad \forall x, y \in X, .$$

i.e. the projection operator is 1*-Lipschitz continuous.*

Proof. □

Proposition 9. *Let X be a Hilbert space and M be a closed linear subspace of X. The projection $y = P_M(x) \in M$ is characterized by*

$$\langle x - y, v \rangle = 0 \quad \forall v \in M..$$

Moreover P_M is a linear map. It is called orthogonal projection.

Proof. □

2.3.2 Representation of linear functionals and applications: gradient, adjoint

Theorem 3 (Riesz)**.** *Let X be a Hilbert space and φ be a continuous linear functional on X. There exists a unique $w \in X$ such that*

$$\langle w, x \rangle = \varphi(x) \quad \forall x \in X.$$

The vector w is called the Riesz representation of φ.

Proof. □

Definition 4. *Let X be a Hilbert space and $f : X \to \mathbb{R}$ be a function. Let $x \in X$ be such that d is Fréchet differentiable at x. The gradient of f at x, denoted by $\nabla f(x)$ is the Riesz representation of the linear functional $df(x)$.*

Theorem 4. *Let X, Y be two Hilbert spaces and $L \in \mathcal{L}(X,Y)$. There exists a unique map $L^* \in \mathcal{L}(Y,X)$, called the adjoint of L such that*

$$\langle Lx, y\rangle = \langle x, L^*y\rangle \quad \forall (x,y) \in X \times Y..$$

2.3.3 Separation of convex sets

Theorem 5. *Lat A be a closed convex subset of a Hilbert space X and let $X \in X \backslash A$. There exists $r \in X\backslash\{0\}$ such that*

$$\sup_{z\in A}\langle r, z\rangle < \langle r, x\rangle.$$

Proof. □

Corollary 1. *For all subset A of a Hilbert space, $\overline{\operatorname{conv} A}$ is equal to the intersection of all affine half-spaces containing A. It is called the closed convex hull of A.*

Proof. □

Theorem 6. *Let A and B be two disjoint convex subsets of a Hilbert space X with A closed and B compact. There exists $r \in X\backslash\{0\}$ such that*

$$\sup_{x\in A}\langle r, x\rangle < \inf_{y\in B}\langle r, y\rangle.$$

Proof. □

Theorem 7. *Let A be a nonempty open convex subset of a finite dimensional Hilbert space X and let $x \in X\backslash A$. There exists $r \in X\backslash\{0\}$ such that*

$$\langle r, z\rangle \leq \langle r, x\rangle \quad \forall z \in A..$$

Proof. □

2.4 Cones

2.4.1 Definitions

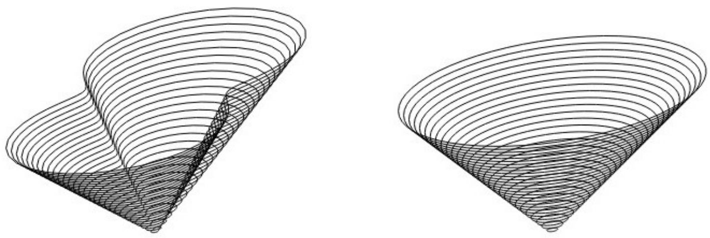

Figure 2.15: Non-convex cone (left) and convex cone (right)

Definition 5. *A subset C of a vector space X is said to be a cone if*

$$\forall(\alpha, x) \in \mathbb{R}_+ \times C, \alpha x \in C.$$

Cones can be convex or not as we can see in **??**.

Proposition 10. *If C is a cone of a normed vector space X then $\overline{C}$ is also a cone.*

Proof. □

2.4.2 Polar cones

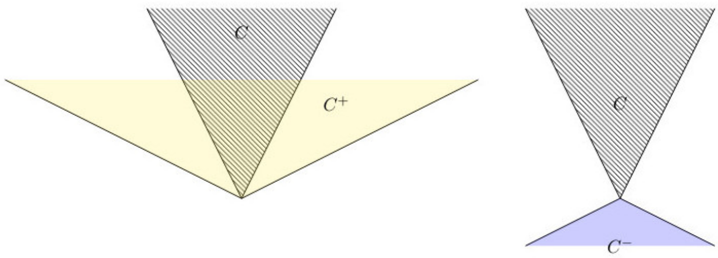

Figure 2.16: Positive and negative polar cones

Definition 6. *Let C be a cone of a Hilbert space X. The positive and negative polar (or dual) cones of C are respectively defined by (see* **??***)*

$$C^+ = \{x \in X \ s.t \ \langle x, y\rangle \geq 0, \forall y \in C\}$$
$$C^- = \{x \in X \ s.t \ \langle x, y\rangle \leq 0, \quad \forall y \in C\}.$$

Example 1. *For the positive orthant it is easily checked that*

$$(\mathbb{R}^n_+)^+ = \mathbb{R}^n_+, \qquad (\mathbb{R}^n_+)^- = \mathbb{R}^n_-..$$

Indeed, the negative polar cone is

$$(\mathbb{R}^n_+)^- = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \ s.t \ \sum_{i=1}^n x_i y_i \leq 0 \quad \forall (y_1, \ldots, y_n) \in \mathbb{R}^n_+ \right\}.$$

Theorem 8. *If C is a convex cone of a Hilbert space X then*

$$C^{--} = \overline{C}.$$

Proof. □

2.4.3 Normal cone and tangent cone of a convex set

Definition 7. *Let A be a convex set of a Hilbert space X and let $a \in A$. The normal cone of A at point a is the set*

$$N_A(a) = \{x \in X \ s.t \ \langle x, y - a \rangle \leq 0 \quad \forall y \in A\}.$$

Definition 8. *Let A be a convex set of a Hilbert space X and let $a \in A$. The tangent cone of A at point a is the smallest (inclusion) closed convex cone containing $A - a$. It is the intersection of all closed convex cones containing $A - a$. It is denoted by $T_A(x)$.*

Lemma 1. *The tangent cone $T_A(a)$ is the closure of the convex convex (called radial cone)*

$$T^\circ_A(a) = \{\alpha(x - a), \quad \alpha \geq 0, \quad x \in A\}.$$

Proof. □

Theorem 9. *Let A be a convex set of a Hilbert space X and let $a \in A$. We have*

$$N_A(a) = (T_A(a))^-.$$

Proof. □

Chapter 3

Convex functions

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3.1 Definitions and first properties

3.1.1 Extended real valued functions

3.1.2 Convex functions

Definition 9. *Let X be a vector space and $f : X \to \overline{\mathbb{R}}$. The epigraph of f is the set*

$$\operatorname{epi} f = \{(x, \zeta) \in X \times \mathbb{R} \text{ s.t. } \zeta \geq f(x)\}..$$

Proposition 11. *Let X be a vector space and $f, g : X \to \overline{\mathbb{R}}$ be two functions. Then*

$$f = g \Leftrightarrow \operatorname{epi} f = \operatorname{epi} g..$$

Definition 10. *Let X be a vector space and $f : X \to \overline{\mathbb{R}}$. The function f is said to be convex if its epigraph is convex.*

Proposition 12. *A function $f : X \to \overline{\mathbb{R}}$ is convex iff*

$$\forall x, y \in \operatorname{dom} f, \forall \theta \in (0,1), \qquad f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y).$$

Proof. □

Proposition 13. *If $f : X \to \overline{\mathbb{R}}$ is convex then we have that for all $(x_1, \ldots, x_m) \in$*

$\operatorname{dom} f$ *and all* $(\theta_1, \dots, \theta_m) \in (0,1)^m$ *with* $\sum_{i=1}^m \theta_i = 1$,

$$f\left(\sum_{i=1}^m \theta_i x_i\right) \le \sum_{i=1}^m \theta_i f(x_i).$$

Proof. □

Definition 11. *A function* $f : X \to \overline{\mathbb{R}}$ *is strictly convex iff*

$$\forall x, y \in \operatorname{dom} f, x \neq y, \forall \theta \in (0,1), \qquad f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta) f(y).$$

A function $f : X \to \overline{\mathbb{R}}$ *is said to be (strictly) concave if* $-f$ *is (strictly) convex.*

3.1.3 Level sets

Definition 12. *Let* X *be a vector space,* $f : X \to \overline{\mathbb{R}}$ *and* $\gamma \in \overline{\mathbb{R}}$*. The* γ*-level set of* f *is the set*

$$\operatorname{lev}_\gamma f = \{x \in X \text{ s.t. } f(x) \le \gamma\}.$$

Proposition 14. *Let* X *be a vector space,* $f : X \to \overline{\mathbb{R}}$ *and* $\gamma \in \overline{\mathbb{R}}$*. If* f *is convex then* $\operatorname{lev}_\gamma f$ *is convex.*

3.2 Particular cases

3.2.1 Marginal functions

Definition 13. *Let* X *be a vector space,* I *be an arbitrary set and* $(f_i)_{i \in I}$ *be a family of functions* $f_i : X \to \overline{\mathbb{R}}$*. We define the upper marginal function*

$$\begin{aligned} \sup_{i \in I} f_i : X &\to \overline{\mathbb{R}} \\ x &\mapsto \sup_{i \in I} f_i(x). \end{aligned}$$

Lemma 2. *Let* X *be a vector space,* I *be a set and* $(f_i)_{i \in I}$ *be a family of functions* $f_i : X \to \overline{\mathbb{R}}$*. We have*

$$\operatorname{epi} \sup_{i \in I} f_i = \bigcap_{i \in I} \operatorname{epi} f_i..$$

Corollary 2. *Let* X *be a vector space,* I *be an arbitrary set and* $(f_i)_{i \in I}$ *be a family of convex functions* $f_i : X \to \overline{\mathbb{R}}$*. Then,* $\sup_{i \in I} f_i$ *is convex.*

Proposition 15. *Let X, Y be vector spaces and $f : X \times Y \to \overline{\mathbb{R}}$ be a convex function. The function $g : X \to \overline{\mathbb{R}}$ defined by*

$$g(x) = \inf_{y \in Y} f(x, y).$$

is convex.

3.2.2 Indication function

Definition 14. *Let A be a subset of a vector space X. The indicator function of A is the function $I_A : X \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$I_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{if } x \neq A \end{cases}.$$

Proposition 16. *A subset A of a vector space X is convex iff its indicator function I_A is convex.*

3.2.3 Support function

Definition 15. *Let A be a nonempty subset of a Hilbert space X. The support function of A is the function $\sigma_A : X \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$\sigma_A(p) = \sup_{x \in A} \langle p, x \rangle.$$

Theorem 10. *Let A be a nonempty subset of a Hilbert space X. Then*

$$\overline{\operatorname{conv} A} = \{x \in X \text{ s.t. } \langle p, x \rangle \leq \sigma_A(p), \quad \forall p \in X\}.$$

3.3 Continuity and lower semicontinuity

3.3.1 Continuity

Proposition 17. *Let X be a normed vector space and $f : X \to \overline{\mathbb{R}}$ be a convex function. If $f(x_0) \in \mathbb{R}$ and f is bounded from above in a neighborhood of x_0, then f is Lipschitz continuous in a (possibly smaller) neighborhood of x_0.*

Proposition 18. *Let $f : X \to \overline{\mathbb{R}}$ be a convex function, where X is a normed vector space of finite dimension. Then f is continuous at each $x_0 \in \int \operatorname{dom} f$ such that $f(x_0) \in \mathbb{R}$.*

3.3.2 Lower semicontinuity

Definition 16. *Let X be a normed vector space and $f : X \to \overline{\mathbb{R}}$. We say that f is lower semicontinuous (or closed) if its epigraph is a closed subset of $X \times \mathbb{R}$.*

Proposition 19. *Let X be a normed vector space and $f : X \to \overline{\mathbb{R}}$. The following assertions are equivalent.*

- *f is lower semicontinuous;*
- *all level sets $\operatorname{lev}_\gamma f$, $\gamma \in \mathbb{R}$ are closed;*
- *for all sequence (x_n) of elements of X such that*

$$\lim_{n\to+\infty} x_n = x \in X \quad \text{and} \quad \lim_{n\to+\infty} f(x_n) = y \in \overline{\mathbb{R}}.$$

  it holds $f(x) \leq y$.

Proposition 20. *Let X be a normed vector space, I be a set and $(f_i)_{i\in I}$ be a famiily of lower semicontinuous functions $f_i : X \to \overline{\mathbb{R}}$. Then $\sup_{i\in I} f_i$ is lower semicontinuous.*

3.4 Closed convex hull of an extended real valued function

3.4.1 Affine minorant

Definition 17. *Let X be a Hilbert space and $f : X \to \overline{\mathbb{R}}$. We say that f admits an affine minorant if there exists $(a,) \in X \times \mathbb{R}$ such that*

$$f(x) \geq \langle a, x\rangle + \alpha \quad \forall x \in X..$$

Proposition 21. *Let X be a Hilbert space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinous convex function. Then f admits an affine minorant.*

3.4.2 Closed convex hull

Proposition 22. *Let X be a Hilbert space and $f : X \to \overline{\mathbb{R}}$ be a function with affine minorant. Let G be the set of convex lower semicontinuous functions $g : X \to \mathbb{R} \cup \{+\infty\}$ such that $g(x) \leq f(x)$ for all $x \in X$. Then*

- *$G \neq \emptyset$;*

- *test*
- *$\cap_{g\in G} \operatorname{epi} g$ is the epigraph of a convex lower semicontinuous function called the closed convex hull of f, denoted by $\overline{\operatorname{conv}}\, f$;*
  $\overline{\operatorname{conv}}\, f$ is the greatest convex lower semicontinuous function minorizing f.

3.5 Legendre-Fenchel transform

3.5.1 Definition

Definition 18. *Let X be a Hilbert space and $f : X \to \overline{\mathbb{R}}$ be an arbitrary function. The Legendre-Fenchel transform of f (or conjuguate of f) is the function $f^* : X \to \overline{\mathbb{R}}$ defined by*

$$f^*(p) = \sigma_{\mathrm{epi}\, f}(p, -1) = \sup_{x \in X} \langle p, x \rangle - f(x).$$

Remark. *The definition immediately shows , if A is an arbitrary nonempty subset of X, then*

$$I_A^* = \sigma_A.$$

Lemma 3. *Let $f : X \to \overline{\mathbb{R}}$ be proper. If f admits an affine minorant then f^* is proper.*

3.5.2 Fenchel-Moreau-Rockafellar theorem

Theorem 11 (Fenchel-Moreau-Rockafellar)**.** *Let X be a Hilbert space and $f : X \to \overline{\mathbb{R}}$ be a proper function with affine minorant. Then*

$$f^{**} = \sup\{g, g \text{ affine minorant of } f\} = \overline{\mathrm{conv}}\, f.$$

Remark. *We have shown in the proof the equality*

$$\mathrm{epi}\,\overline{\mathrm{conv}}\, f = \overline{\mathrm{conv}\,\mathrm{epi}\, f}, .$$

i.e. the epigraph of the closed convex hull of a proper function with affine minorant is the closed convex hull of its epigraph.

Corollary 3. *If f is convex, proper and lower semicontinuous then $f^{**} = f$.*