Tensorial Formalism

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Contents

We give here a presentation a bit more precise on the notion of tensors. The domain of application of tensors go far beyond relativity since it is used in euclidian geometry, in particular in continuum mechanics as well as in the framework of General Relativity.

1 Vectors

We consider a vector space E of finite dimension n. If we introduce a vectorial basis, made of n independent vectors $\underline{e}_\mu$, all vectors $\underline{u}$ of E can be decomposed in the form of

$$\underline{u} = u^\mu \underline{e}_\mu \tag{1}$$

These vectors in the new basis, begin linear combinations of old basis vectors, the new contravariant components u'^{μ} are linear combinations of old components u^{ν}, which we can write in the form

$$u'^{\mu} = \Lambda^{\mu}_{\nu} u^{\nu} \tag{2}$$

The coefficients Λ^{μ}_{ν} do not depend on the vector $\underline{u}$ considered here, but only on the basis vectors $\underline{e}_{\mu}$ and $\underline{e}'_{\mu}$.

2 Covectors

We call **covector** all linear forms defined on E, that is to say all linear applications which to all vectors associate a real number,

$$\lambda : E \to \mathbb{R} \quad \text{such that} \quad \underline{u} \mapsto \lambda(\underline{u}) \tag{3}$$

Given a vectorial basis $\{\underline{e}_{\mu}\}$, the components, said **covariants**, of a linear form λ are defined by

$$\lambda_{\mu} \equiv \lambda(\underline{e}_{\mu}) \tag{4}$$

The action of a linear form on a vector can be written in components

$$\lambda(\underline{u}) = \lambda(u^{\mu}\underline{e}_{\mu}) = u^{\mu}\lambda(\underline{e}_{\mu}) = u^{\mu}\lambda_{\mu}, \tag{5}$$

where we used the linearity of λ. In another basis $\{\underline{e}'_{\mu}\}$, the covariant components of λ are given by

$$\lambda'_{\mu} = \lambda(\underline{e}'_{\mu}) \tag{6}$$

The action of λ on an arbitrary vector $\underline{u}$ writes in components $\lambda(\underline{u}) = u'^{\mu}\lambda'_{\mu}$ in this new basis. Comparing to (5), and knowing that the components of $\underline{u}$ are transformed following (2), we can deduce that the old and new components of λ are related by

$$\lambda_{\nu} = \Lambda^{\mu}_{\nu}\lambda'_{\mu}, \tag{7}$$

or, by inversing,

$$\lambda_{\mu} = \tilde{\Lambda}^{\nu}_{\mu}\lambda_{\nu}, \quad \tilde{\boldsymbol{\Lambda}} = (\boldsymbol{\Lambda}^{-1})^{T}. \tag{8}$$

The covectors can also be seen as vectors belonging to a vectorial subspace called the dual of E which we will denote as E^{*}. The dual space is also of dimension n, and for each choice of a basis $\{\underline{e}_{\mu}\}$ in E, corresponds a dual basis in E^{*}, which we will denote as $\underline{e}^{\mu}_{*}$, defined by[1],

$$\underline{e}^{\mu}_{*}(\underline{e}_{\nu}) = \delta^{\mu}_{\nu} \tag{9}$$

[1] The superscript position of the index μ will allow us to use the Einstein notation in the dual space

We can then decompose all elements of E^* on the dual basis and write

$$\lambda = \lambda_\mu \underline{e}_*^\mu \tag{10}$$

The linear forms on E^*, that is to say acting on the covectors, can be identified with vectors of E, i.e[2]

$$(E^*)^* \equiv E \tag{11}$$

In fact, all vectors $\underline{v}$ define a linear form on E^*, which associates all covectors λ with the number $\lambda(\underline{v})$. Reciprocally, in finite dimensions, to each linear form ζ defined on E^*, one can associate a vector $\underline{v}$ such that, for all element λ of E^*, we have,

$$\zeta(\lambda) = \lambda(\underline{v}). \tag{12}$$

It suffices to define $\zeta(\underline{e}_*^\mu) = v^\mu$; this definition being independent of the choice of basis.

3 Tensors

A *multilinear* form, which at p vectors E and at q covectors associates a real number

$$\begin{aligned} \mathbf{T} : E \times \cdots \times E \times E^* \times \cdots \times E^* \to \mathbb{R}, \\ (\underline{u}_1, \ldots, \underline{u}_p, \lambda_1, \ldots, \lambda_q) \mapsto \mathbf{T}(\underline{u}_1, \ldots, \underline{u}_p, \lambda_1, \ldots, \lambda_q) \end{aligned}$$

is called a **tensor** p times **covariant** and q times **contravariant**. The components of the tensor $\mathbf{T}$ are defined as,

$$T^{\nu_1 \cdots \nu_p}_{\mu_1 \cdots \mu_p} = \mathbf{T}(\underline{e}_{\mu_1}, \ldots, \underline{e}_{\mu_p}, \underline{e}_*^{\nu_1}, \cdots, \underline{e}_*^{\nu_q}) \tag{13}$$

and are characterised by p subscript indices and q superscript indices. Let us give a few particular cases of tensors. A vector of E, seen as a linear form defined on E^*, is a tensor one time contravariant. A covector of E^*, seen as a linear form defined on E, is a tensor one time covariant. In a change of basis, the relationship between new and old components of a tensor is derived from the linear relationship between the new and old bases of E and E^* and of the multilinearity of the tensor. We thus obtain,

$$(T')^{\mu_1 \cdots \nu_q}_{\mu_1 \cdots \mu_p} = \tilde{\mathbf{\Lambda}}^{\rho_1}_{\mu_1} \ldots \tilde{\mathbf{\Lambda}}^{\rho_p}_{\mu_p} \mathbf{\Lambda}^{\nu_1}_{\sigma_1} \ldots \mathbf{\Lambda}^{\nu_q}_{\sigma_q} T^{\sigma_1 \cdots \sigma_q}_{\rho_1 \cdots \rho_p} \tag{14}$$

This formula generalizes to the analogous formulas obtained for the vectors and covectors.

[2]This is true only in finite dimensions!

4 Scalar product

A scalar product (or scalar pseudo-product) is a non-degenerate symmetric bilinear form,

$$\begin{aligned} \mathbf{g} : \quad & E \times E \to \mathbb{R} \\ & (\underline{u}, \underline{v}) \mapsto \mathbf{g}(\underline{u}, \underline{v}) = \mathbf{g}(\underline{v}, \underline{u}), \end{aligned}$$

and constitutes an example of tensor twice covariant. In a vectorial basis $\underline{e}_\mu$, the components of the scalar product g are,

$$g_{\mu\nu} \equiv \mathbf{g}(\underline{e}_\mu, \underline{e}_\nu), \tag{15}$$

and the scalar product of two vectors writes, in components,

$$\mathbf{g}(\underline{u}, \underline{v}) = g_{\mu\nu} u^\mu v^\nu \tag{16}$$

A scalar product let's us create a natural correspondance between vectors and covectors, and thus to identify E with it's dual E^*. Indeed, using the scalar product $\mathbf{g}$. we can associate to any vector $\underline{u}$ a linear form λ_u for which the action on all vectors $\underline{v}$ is the scalar product of $\underline{u}$ and $\underline{v}$,

$$\begin{aligned} \lambda_u : \quad & E \to \mathbb{R} \\ & \underline{v} \mapsto \lambda_u(\underline{v}) = \mathbf{g}(\underline{u}, \underline{v}) \end{aligned}$$

Reciprocally, the action of all linear form can be expressed as a scalar product with a vector $\underline{u}$. In practice, we will use the same symbol to write the components of these two mathematical objects, $\underline{u}$ and $\underline{\lambda}_u$, the position of the index helping us to distinguish them; the components of vectors will be noted u^μ and the ones of covectors associated u_μ. They are related by the expression

$$u_\mu = g_{\mu\nu} u^\nu. \tag{17}$$

The scalar product of $\underline{u}$ and $\underline{v}$ can be written in different ways:

$$\mathbf{g}(\underline{u}, \underline{v}) = \lambda_u(\underline{v}) = \lambda_v(\underline{u}) \tag{18}$$

which corresponds, in components, to

$$g_{\mu\nu} u^\mu v^\nu = u_\mu v^\nu = u_\mu v^\mu = v_\mu u^\mu \tag{19}$$

5 Levi-Civita Tensor

We call **p-form** an *antisymmetric* tensor p times covariant, that is to say,

$$\omega(\underline{u}_{\sigma(1)}, \ldots, \underline{u}_{\sigma(p)}) = \varepsilon(\sigma)\omega(\underline{u}_1, \ldots, \underline{u}_p), \tag{20}$$

where $\varepsilon(\sigma)$ is the signature of the permutation σ.

If the dimension of the underlying vectorial subspace is n, it is easy to see that the \$n\$-forms constitute a vectorial subspace of dimension 1. In other words, all \$n\$-forms are proportional between them. By choosing an orthonormal basis $\{\underline{e}_\mu\}$, relatively to a scalar product defined on E, we can define the *normalised* \$n\$-form, called the **Levi-Civita tensor**.

Hence, in dimension 4, which interests us the most, the components of the Levi-Civita tensor are determined by,

$$\epsilon_{0123} \equiv \epsilon(\underline{e}_0, \underline{e}_1, \underline{e}_2, \underline{e}_3) = 1 \tag{21}$$

the other components deduced by antisymmetry. In an other orthonormal basis $\{\underline{e}'_\mu\}$, which are deduced firsst by the Lorentz transformation, the new components fo the Levi-Civita tensor are obtained using (14). We thus find,

$$\epsilon'_{0123} \equiv \epsilon(\underline{e}'_0, \underline{e}'_1, \underline{e}'_2, \underline{e}'_3) = \tilde{\Lambda}_0^\mu \tilde{\Lambda}_1^\nu \tilde{\Lambda}_2^\rho \tilde{\Lambda}_3^\sigma \epsilon_{\mu\nu\rho\sigma} = \det \tilde{\Lambda} \tag{22}$$

If we confine ourself to the restricted Lorentz group, which comes back to only consider orthochrone Lorentz transformations and which conserves the orientation of the spatial trihedron, we conclude that $\epsilon_{\mu\nu\rho\sigma}$ are unchanged since $\det \tilde{\Lambda} = \det \Lambda = 1$.

6 Tensorial product

We can construct tensors from tensors of inferior order using the notion of **tensorial product**, written as $\otimes$. Therefore, from two vectors $\underline{u}$ and $\underline{v}$ of E, we can construct a tensor twice contravariant, noted $\underline{u} \otimes \underline{v}$ and are defined by,

$$\underline{u} \otimes \underline{v}(\lambda, \mu) = \underline{u}(\lambda)\underline{v}(\mu) = \lambda(\underline{u})\mu(\underline{v}) \quad (\lambda, \mu \in E^*) \tag{23}$$

wehere the second equality is a consequence of (11). This definition of tensorial product can be generalised directly to any tensors.

We can now, given a basis $\underline{e}_\mu$ of the vectorial subspace E, generalise the vectors and covectors decompositions to any arbitrary tensor, p times covariant and q times contravariant:

$$\mathbf{T} = T^{\nu_1 \cdots \nu_q}_{\mu_1 \cdots \mu_p} \underline{e}_*^{\mu_1} \otimes \cdots \otimes \underline{e}_*^{\mu_p} \otimes \underline{e}_{\nu_1} \otimes \cdots \otimes \underline{e}_{\nu_q}. \tag{24}$$

The components $T^{\nu_1\cdots\nu_q}_{\mu_1\cdots\mu_p}$ are simply obtained by evaluating the tensor $\mathbf{T}$ on the basis vectors, which we can explicitely verify by using the decomposition (24) and to the definition of the dual basis.

Let us remark that the scalar product $\mathbf{g}$ has a decomposition

$$\mathbf{g} = g_{\mu\nu} \underline{e}_*^\mu \otimes \underline{e}_*^\nu \tag{25}$$

7 Contraction of tensors

We can construct from a tensor $\mathbf{T}$, p times covariant and q times contravariant, a tensor $(p-1)$ times covariant and $(q-1)$ times contravariant.

This operation, called **contraction**, is defined par,

$$C(\mathbf{T}) = \sum_\mu \mathbf{T}(\ldots, \underline{e}^\mu, \ldots; \ldots, \underline{e}^*_\mu, \ldots). \tag{26}$$

We can verify that this definition does not depend on the basis considered and is hence intrinsic.