Lorentz Group

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We call Lorentz transformation all linear transformation

$$x'^{\mu} = \sum_{\nu}^{3} \Lambda^{\mu}_{\nu} x^{\nu} \tag{1}$$

connecting the coordinates $x'^{\mu} = \{ct', x', y', z'\}$ with $\mu = 0, 1, 2, 3$ to the coordinates $x^{\mu} = \{ct, x, y, z\}$ verifying the relation

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 \tag{2}$$

This last expression can be rewritten as

$$\sum_{\mu,\nu=0}^{3} x'^{\mu} \eta_{\mu\nu} x'^{\nu} = \sum_{\mu,\nu=0}^{3} x^{\mu} \eta_{\mu\nu} x^{\nu} \tag{3}$$

with $\eta_{00} - 1, \eta_{11} = \eta_{22} = \eta_{33} = -1$ and $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$. In matrix form, the equation (1) becomes

$$\mathbf{X}' = \mathbf{\Lambda} \mathbf{X} \tag{4}$$

with

$$\mathbf{X} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \qquad \mathbf{X}' = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \tag{5}$$

and $\mathbf{\Lambda}$ is the square matrix of the components Λ^{μ}_{ν}.
By defining the matrix G of the components $\eta_{\mu\nu}$, that is to say,

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6}$$

the equation (2) take the form

$$\mathbf{X'}^T\mathbf{G}\mathbf{X'} = \mathbf{X}^T\mathbf{G}\mathbf{X} \iff \mathbf{X}^T\mathbf{\Lambda}^T\mathbf{G}\mathbf{\Lambda}\mathbf{X} = \mathbf{X}^T\mathbf{G}\mathbf{X} \tag{7}$$

where $\mathbf{X}^T$ is the transposed of the matrix X. This equation being satisfied for all $\mathbf{X}$, we can deduce that $\mathbf{\Lambda}$ needs to satisfy the condition

$$\mathbf{\Lambda}^T\mathbf{G}\mathbf{\Lambda} = \mathbf{G} \tag{8}$$

It is easy to verify that the matrices satisfying this condition create a group, in which the neutral element is the identity matrix. The inverse of all element $\mathbf{\Lambda}$ is given by

$$\mathbf{\Lambda}^{-1} = \mathbf{G}\mathbf{\Lambda}^T\mathbf{G} \tag{9}$$

since $\mathbf{G}^2 = I$. The matrix relation that characterise Lorentz transformation contains 16 equations; however, this relation being symmetric, only 10 equations are independent. In consequence, the Lorentz transformations is characterized by $4 \times 4 - 10 = 6$ continuous independent parameters. Let us also remark, taking the determinant of (8) that

$$\det(\mathbf{\Lambda})^2 = 1 \implies \det(\mathbf{\Lambda}) = \pm 1 \tag{10}$$

which let us distinguish the transformations said proper ($\det(\mathbf{\Lambda}) = 1$) and the transformations said improper ($\det(\mathbf{\Lambda}) = -1$). Moreover, the 00 component of the matrix in (8) writes as,

$$(\mathbf{\Lambda}^0_0)^2 - \sum_{i=1}^{3}(\mathbf{\Lambda}^i_0)^2 = 1, \tag{11}$$

where we deduce that $|\mathbf{\Lambda}^0_0| \geq 1$. This let's us distinguish the transformations said orthochronous ($\mathbf{\Lambda}^0_0 \geq 1$) and transformations said anti-orthochronous ($\mathbf{\Lambda}^0_0 \leq -1$). In the first case, the direction of flow of time is unchanged (t' is increasing if t is), whereas it is inverted in the case of anti-orthochronous transformations. As we can easily verify, the composition of two orthochronous transformations, or anti-orthochronous, give a orthochronous transformation whereas the composition of an orthochronous and anti-orthochronous transformation give an anti-orthochronous transformation. The signs of $\det(\mathbf{\Lambda})$ and of $\mathbf{\Lambda}_0^0$ let's us distinguish 4 sub-groups of the Lorentz group. The proper and orthochronous transformations, in which the identity is part of, create a subgroup of the Lorentz group called the restricted Lorentz group.

1 Sub-groups of rotations

The rotations in space are particular cases of Lorentz transformations, which leave the time coordinate unchanged. They are of the form

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Omega} \end{pmatrix}, \tag{12}$$

where $\mathbf{\Omega}$ is an orthogonal matrix. The rotations constitute a sub-group of the restricted Lorentz group.

2 Generators of the Lorentz group

The Lorentz transformation corresponding to infinitesimal devistions with respect to the identity can be written as

$$\mathbf{\Lambda} \simeq \mathbf{1} + \sum_a \varepsilon_a \mathbf{G}_a, \tag{13}$$

where the ε_a are six infinitesimal parameters (because of the six continuous parameters). The six matrices G_a can for example be constructed from infinitesimal rotations of the axes Ox, Oy and Oz in one part and of infinitesimal boosts along Ox, Oy and Oz in another part. Hence, the rotation of axis Ox and of angle $\varepsilon << 1$ give

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\varepsilon) & \sin(\varepsilon) \\ 0 & 0 & -\sin(\varepsilon) & \cos(\varepsilon) \end{pmatrix} \simeq \mathbf{1} - \varepsilon \mathbf{S}_x, \quad \text{with} \quad \mathbf{S}_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{14}$$

In a same manner, from the infinitesimal roations of axis Oy and Oz, we obtain the matrices

$$\mathbf{S}_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{S}_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

An infinitesimal boost along Ox writes as,

$$
\mathbf{\Lambda} = \begin{pmatrix} \mathrm{ch}(\varepsilon) & -\mathrm{sh}(\varepsilon) & 0 & 0 \\ -\mathrm{sh}(\varepsilon) & \mathrm{ch}(\varepsilon) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \simeq \mathbf{1} - \varepsilon \mathbf{K}_x, \quad \text{with} \quad \mathbf{K}_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{16}
$$

and the infinitesimal boosts along the axes Oy and Oz give, in a similar manner, the matrices

$$
\mathbf{K}_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{with} \quad \mathbf{K}_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{17}
$$

The six matrices S_i and K_i are called **infinitesimal generators** of the (restricted) Lorentz group and we can write an element of this group in the exponential form

$$
\mathbf{\Lambda} = \exp\Big(-\vec{\theta} \cdot \vec{\mathbf{S}} - \vec{\alpha} \cdot \vec{\mathbf{K}}\Big), \tag{18}
$$

where the angles θ^i and the rapidities $\$\alpha^{\mathrm{i}\,\hat{}}$ constitute the six continuous parameters. We can easily verify the following commutation rules satisfied by the infinitesimal generators:

$$
[S_x, S_y] = S_z, \quad [S_y, S_z] = S_x, \quad [S_z, S_x] = S_y \tag{19}
$$

$$
[S_x, K_y] = K_z \quad \text{(and permutations)} \tag{20}
$$

$$
[K_x, K_y] = -S_z \quad \text{(and permutations)} \tag{21}
$$

where $[A, B] = AB - BA$ represents the commutator for the matrices A and B. In particular, the commutators (21) express the fact that the composition of two boosts contain, in general, a rotation.