BACHELOR THESIS: SPECTRAL THEORY

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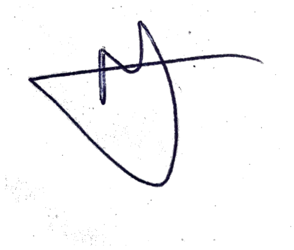

ABSTRACT

This thesis follows a two month internship at the Center of Mathematics Laurent Schwartz of École Polytechnique under the supervision of M. Julien Sabin. This report is directed towards readers with knowledge in Quantum Mechanics and knowledge in Hilbert Spaces and Analysis.
Different topics of Spectral Theory will be treated here; however, this does not constitute a full image of Spectral Theory. The concepts treated here are some introductory concepts of Sobolev spaces in the Appendix as well as theoretical notion of spectrums in the case of symmetric and auto-adjoint operators.
In the last part, we will be looking at the applications of such theoretical notions on the momentum operator, both in $\mathbb{R}$ and on an interval, as well as the Laplacian; however due to time-constraints, the Laplacian will be introduced in a less rigorous manner.

I would like to thank M. Julien Sabin for all his help during my internship and even after to make sure that I have understood the concepts seen.

NOTATIONS

- $L^2(\mathbb{R})$ denotes the Lebesgue space of square integrable functions on $\mathbb{R}$;
- $\ell^2(\mathbb{R})$ denotes the space of square summable sequences on $\mathbb{R}$;
- $\mathcal{C}^\infty(\mathbb{R})$ denotes the space of smooth functions;
- $\|\cdot\|_X$ denotes the norm on the space X. If this norm is not explicitely defined, it is taken to be the inherited norm on X;
- $|||\cdot|||$ denotes the operator norm;
- ${}^{\mathrm{cl}}A$ denotes the closure of A;
- $\overline{z}$ denotes the complex conjuguate of z;
- $\langle\cdot,\cdot\rangle$ denotes the inner product associated to the space under study;
- $D^\perp$ denotes the orthogonal complement of D;
- $\mathcal{F}^{-1}$ denotes the inverse Fourier transform;
- Im denotes the image of an application;
- $\Im$ denotes the imaginary part;
- $\partial A = {}^{\mathrm{cl}}A \backslash \mathrm{int}(A)$ denotes the boundary of a set A;
- $(v_n) \subset \mathfrak{H}$ denotes a sequence with parameter $n \in \mathbb{N}$ and elements in $\mathfrak{H}$;
- $\mathbb{I}$ denotes the indicator function;
- $\widehat{f}$ denotes the Fourier transform of a function f;
- $\|\cdot\|_{L^2}$ denotes the L^2 norm given by $\|f\|_{L^2} = \left(\int |f|^2\right)^{1/2}$.

INTRODUCTION

Spectral theory is a theory in Mathematics developped in the field of Operator Theory. This theory based on multiple fields such as Functional Analysis and Complex Analysis on Hilbert Spaces emerged due to a need of it in Quantum Mechanics. The name Spectral Theory became known thanks to *David Hilbert* in his original formulation of Hilbert Space Theory, combining it with the notion of quadratic forms in infinitely many variables. Further developpment of this theory came with the works of *John Von Neumann* in the conditions of the Laplacian operator.
It gained popularity when Quantum Mechanics was formulated in terms of the *Schrödinger equation*. It was then a link between atomic spectra and spectral theory was made.
Since then, Spectral theory has been useful in the mathematical formulation of Quantum Mechanics as some physical examples of Quantum Mechanics lead to contradiction. Indeed, in the works of [Gie00], counterintuitive examples of spectral theory are given related to the Dirac's formalism in Quantum Mechanics.
Such an example is the case of commutators in relation to the momentum P operator and position Q operator for a particle in one dimension.
The Heisenberg's canonical commutator relation give us,

$$[P,Q] = \frac{\hbar}{i}\mathbf{1} \tag{1}$$

However, taking the trace of this relationship, one finds a vanishing term on the left-hand side whereas $\mathrm{Tr}\left(\frac{\hbar}{i}\mathbf{1}\right) \neq 0$ leading to a contradiction.
A solution to this problem can be found in [Gie00] and goes as follows. Suppose the commutation relation $[P,Q] = \frac{\hbar}{i}\mathbf{1}$ is satisfied by operators P and Q defined on a Hilbert space $\mathfrak{H}$ of finite dimension $n \times n$. In this case, P and Q can be represented by $n \times n$ matrices; the trace is then well-defined and we obtain the result,

$$0 = \mathrm{Tr}[P,Q] = \mathrm{Tr}\left(\frac{\hbar}{i}\mathbf{1}_n\right) = \frac{\hbar}{i}n \tag{2}$$

From this result, one concludes that the Heisenberg's relation cannot be realised on a *finite* dimension Hilbert space. Thus, quantum mechanics has to be formulated on an infinite dimensional Hilbert Space.

CHAPTER I - OPERATORS AND SPECTRUM

In this chapter, we will introduce operators in infinite dimensions as well as the spectrum of operators. We will talk about symmetry and adjoints of an operator all in an arbitrary number of dimensions, which can even be applied to infinite dimensions. Let $\mathfrak{H}$ be a Hilbert space on the complex plane.

Operators, graphs and extensions

Definition 1 (Densely defined operators)**:** *An operator on the space $\mathfrak{H}$ is a couple $(A, D(A))$ given by a dense subspace $D(A) \subset \mathfrak{H}$ and a linear application $A : D(A) \to \mathfrak{H}$.*

Remark.
The biggest difference between infinite dimensions operators and finite dimension operators is the domain of the operator.
Indeed, in finite dimensions, a vectorial subspace is automatically closed. Hence, knowing that $D(A) \subset \mathfrak{H}$ is dense in $\mathfrak{H}$, in finite dimensions, $D(A) = \mathfrak{H}$. This is not always the case in infinite dimensions.

For the sake of simplicity, we will work on the notion of graphs for the study of closure of an operator.

Definition 2 (Graph of an operator)**:** *We call the graph of an operator $(A, D(A))$ the vectorial subspace of $\mathfrak{H} \times \mathfrak{H}$ defined by,*

$$G(A) = \{(v, Av),\ s.t.\ v \in D(A)\} \tag{3}$$

The following lemma introduces the necessary conditions to determing whether a subset of $\mathfrak{H} \times \mathfrak{H}$ is the graph of an operator.

Lemma 1 (Graph)**:** *A set $G \subset \mathfrak{H} \times \mathfrak{H}$ is the graph of an operator, if and only if,*

1. *G is vectorial subspace of $\mathfrak{H} \times \mathfrak{H}$,*
2. *$(0, y) \in G$ implies that $y = 0$,*
3. *the projection $D = \{x \in \mathfrak{H} : \exists y \in \mathfrak{H}, (x, y) \in G\}$ is dense in $\mathfrak{H}$.*

Proof. The first condition implies that the operator associated to the set $G \subset \mathfrak{H} \times \mathfrak{H}$ is a linear operator. The second condition implies the unicity of the image of a point. Lastly, the third condition imply the densely defined domain of the operator. □

Definition 3 (Extension of an operator)**:** *Let $(A, D(A))$. We call the operator $(B, D(B))$ an extension of $(A, D(A))$, writing $A \subset B$, when $D(A) \subset D(B)$ and B coincides with A on $D(A)$, that is to say,*

$$Ax = Bx \quad \forall x \in D(A). \tag{4}$$

Remark.
It is interesting to note that an equivalent definition can be given using the notion of graphs.
Indeed, an operator $(B, D(B))$ is an extension of $(A, D(A))$ if $G(A) \subset G(B)$.

Spectrum

In this section, we will define the notion of spectrum of an operator. Let us do a little reminder on the notion of spectrum of an operator in finite dimensions.
In finite dimensions, the spectrum of a square matrix A is the set of complex numbers $\lambda \in \mathbb{C}$ such that $\det(A - \lambda) = 0$, that is to say that $A - \lambda$ is not invertible. This notion of spectrum is linked to non-injectivity or non-surjectivity of the operator $A - \lambda$. In finite dimensions, non-injectivity and non-surjective are equivalent. Non-injectivity is the definition of eigenvector.
In infinite dimensions, the surjectivity does not necessarily imply injectivity and vice-versa.
Furthermore, the inverse may exist without being continuous since a linear application is not necessarily continuous in infinite dimensions.

Definition 4 (Spectrum)**:** *Let $(A, D(A))$ be an operator. We call <u>resolvant set</u> of A the subset of $\mathbb{C}$ defined by,*

$$\rho(A) \equiv \left\{ \lambda \in \mathbb{C} \text{ s.t. } A - \lambda : D(A) \to \mathfrak{H} \text{ is invertible, of } \underline{\text{bounded}} \text{ inverse } (A - \lambda)^{-1} : \mathfrak{H} \to D(A) \right\}$$

The spectrum of A is given by the complement of the resolvant set in $\mathbb{C}$, i.e. $\sigma(A) \equiv \mathbb{C} \backslash \rho(A)$.
It should be worth noting that to the dense set $D(A) \subset \mathfrak{H}$, we associate the topology given on $\mathfrak{H}$.

Indeed, in infinite dimensions, the notion of spectrum does not only reduce to eigenvalues. In fact, a complex number can also be in the spectrum if $A - \lambda$ is not-surjective or the inverse is not bounded. The following example demonstrates this fact.

Example 1.
On the Hilbert space $\mathfrak{H} = \ell^2(\mathbb{N})$, let us define the operator $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ of the arithmetic right shift by,

$$S : x = (x_0, x_1, \dots) \mapsto (0, x_0, x_1, \dots)$$

Then, S is in fact injective but not surjective therefore $\lambda = 0 \in \sigma(S)$. Moreover,

$$Sx_\lambda = \lambda x_\lambda \iff Sx_\lambda = 0 \iff (0, (x_\lambda)_0, (x_\lambda)_1, \dots) = 0 \iff x_\lambda = 0$$

Therefore, there are no non-zero solutions to the eigenvalue problem with $\lambda = 0$, thus $\lambda = 0$ is not an eigenvalue of S.

Lemma 2 ($\sigma(A)$ is closed)**:** *Let $(A, D(A))$ be an operator and let $z \in \rho(A)$. Then,*

$$B\left(z, \frac{1}{\|(A - z)^{-1}\|}\right) \subset \rho(A) \tag{5}$$

In particular, $\sigma(A)$ is closed in $\mathbb{C}$.

Let us now proceed with the proof.

Proof. Let $z \in \rho(A)$ and $\eta \in \mathbb{R}$. Let us now find for which values of η, $A - z - \eta$ is invertible with bounded inverse. We can write,

$$A - z - \eta = \left(1 - \eta(A - z)^{-1}\right)(A - z) \tag{6}$$

By hypothesis, $z \in \rho(A)$, therefore $A - z$ is invertible with bounded inverse. Furthermore, knowing that $(A-z)^{-1}$ is bounded, $1 - \eta(A-z)^{-1}$ is bounded for a fixed η. With the help of Taylor and Maclaurins series, we know that an operator B such that $\|B\| < 1$, the operator $(1-B)^{-1}$ is given by,

$$(1-B)^{-1} = \sum_{n\geq 0} B^n \tag{7}$$

Therefore, for all $\eta \in \mathbb{R}$ such that $\eta\|(A-z)^{-1}\| < 1$, $1 - \eta(A-z)^{-1}$ is invertible of inverse given by,

$$(A-z-\eta)^{-1} = (A-z)^{-1}\sum_{n\geq 0} \eta^n (A-z)^{-n} \tag{8}$$

It is worth noting that for all $n \geq 0$, $(A-z)^{-n}$ is bounded. Knowing now that $A - z - \eta$ is invertible with bounded inverse for all $z \in \rho(A)$ and for all $\eta \in \mathbb{R}$ such that $\eta\|(A-z)^{-1}\| < 1$. We can therefore conclude that,

$$B\left(z, \frac{1}{\|(A-z)^{-1}\|}\right) \subset \rho(A), \forall z \in \rho(A) \tag{9}$$

Seeing that the above equation implies the openness of $\rho(A)$, by using the complement, we find that $\sigma(A)$ is closed. □

CLOSURE OF AN OPERATOR

Definition 5 (Closed operator)**:** *We say that an operator $(A, D(A))$ is closed when one of the following conditions is satisfied:*

- *the graph of A, $G(A)$, is closed in $\mathfrak{H} \times \mathfrak{H}$ by the induced product topology.*
- $\forall (x_n) \in D(A)$ *such that,* $\exists x, y \in \mathfrak{H}$

$$x_n \to x, \ \textit{in } \mathfrak{H} \tag{10}$$
$$Ax_n \to y, \ \textit{in } \mathfrak{H} \tag{11}$$

we have that $x \in D(A)$ and $Ax = y$

Remark.
These two conditions ultimately are equivalent, however in most cases, the first condition can be easier to prove.

Proposition 1 (Spectrum of a non-closed operator)**:** *If $(A, D(A))$ is a non-closed operator, then we have*

$$\sigma(A) = \mathbb{C} \tag{12}$$

Proof. The proof can be found in [Lew21].
The idea of the proof is as follows. We prove this by contrapositive. In other words, we prove that if $\sigma(A) \neq \mathbb{C}$, then A is necessarily closed. We assume there exists $z \notin \sigma(A)$. Thus, $z \in \rho(A)$. Now, take a sequence $(x_n) \subset D(A)$ such that $x_n \to x$ and $Ax_n \to y$.
Since $(A - z)$ is invertible with bounded inverse coupled with the fact that $x_n \in D(A) \subset \mathfrak{H}$, we can write

$$(A-z)^{-1}Ax_n = x_n + z(A-z)^{-1}x_n. \tag{13}$$

Passing to the limit, we get that,

$$(A-z)^{-1}y = x + z(A-z)^{-1}x \tag{14}$$

Letting us conclude that $(x,y) \in G(A)$ and therefore A is closed. $\square$

Definition 6 (Closure of an operator)**:** *Let $(A, D(A))$ be an operator. We say that A is closable if it admits at least one closed extension.*
In this case, ${}^{\mathrm{cl}}G(A)$ is the graph of the operator noted ${}^{\mathrm{cl}}A$ of domain $D({}^{\mathrm{cl}}A)$ and is called the closure of A.
It is worth to note that it is the smalled closed extension of A.

Theorem 1 (Closure of ∂_x and of ∂_x^2 in $\mathbb{R}$)**:** *In $\mathfrak{H} = L^2(\mathbb{R})$, let $P^{\min}$ be the operator defined on the domain $D(P^{\min}) = \mathcal{C}_c{}^{\infty}(\mathbb{R})$ by $P^{\min}f = -if'$.*
Then, $P^{\min}$ is closable and it's closure is the operator $P \equiv {}^{\mathrm{cl}}P^{\min}$ given by,

$$Pf = -if' \quad \text{on the domain} \quad D(P) = \{f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R})\} = H^1(\mathbb{R}) \tag{15}$$

where f' is the derivative in the sence of distributions. This is indeed the Sobolev space H^1 as defined in the Appendix A.

In a similar way, let $A^{\min}$ be the operator defined by $A^{\min}f = -f''$ on the domain $D(A^{\min}) = \mathcal{C}_c^{\infty}(\mathbb{R})$.
Then, $A^{\min}$ is closable and it's closure is the operator $A \equiv {}^{\mathrm{cl}}A^{\min}$ given by,

$$Af = -f'' \quad \text{on the domain} \quad D(A) = \{f \in L^2(\mathbb{R}) : f'' \in L^2(\mathbb{R})\} = H^2(\mathbb{R}) \tag{16}$$

where ∂^2 is also in the sense of distributions.

Proof. For the proof, we will proceed with the impulsion operator and the argument is similar for the Laplacian operator.
Let us first show that the operator P given by $Pf = -i\partial_x f$ on $D(P) = H^1(\mathbb{R})$ is indeed closed.
Let us take a sequence of functions $(f_n)_{n\geq 0} \subset H^1(\mathbb{R})$ such that there exists $f, g \in L^2(\mathbb{R})$ such that,

$$\begin{cases} f_n & \to f, \\ -if_n' & \to g \end{cases} \quad \text{in } L^2(\mathbb{R}) \tag{17}$$

We will prove that $f \in H^1(\mathbb{R})$ and that $-if' = g$. It is interesting to note that the derivatives in these cases are derivatives in the sense of distributions.
We have that,

$$\left| i \int_{\mathbb{R}} \varphi(x) g(x) \,\mathrm{d}x - \int_{\mathbb{R}} \varphi(x) f_n'(x) \,\mathrm{d}x \right| = \left| \int_{\mathbb{R}} \varphi(x)(ig(x) - f_n') \,\mathrm{d}x \right| \tag{18}$$

$$\leq \left(\int_{\mathbb{R}} |\varphi(x)|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}} |ig(x) - f_n'(x)|^2 \right)^{1/2} \tag{19}$$

$$\leq \|\varphi(x)\|_{\infty} \|ig(x) - f_n'(x)\|_{L^2} \tag{20}$$

Moreover, as $-if_n' \to g$ in $L^2(\mathbb{R})$,

$$\|ig(x) - f_n'(x)\|_{L^2} = \|-if_n'(x) - g(x)\|_{L^2} \to 0 \tag{21}$$

Thus,

$$i\int_{\mathbb{R}}\varphi(x)g(x)\,\mathrm{d}x=\lim_{n\to\infty}\int_{\mathbb{R}}\varphi(x)f_n'(x)\,\mathrm{d}x \tag{22}$$

Now, as $f_n\in H^1(\mathbb{R})$ for all $n\geq 0$, by definition of the weak derivative, we get that,

$$\int_{\mathbb{R}}\varphi(x)f_n'(x)\,\mathrm{d}x=-\int_{\mathbb{R}}\varphi'(x)f_n(x)\,\mathrm{d}x\quad\forall n\geq 0 \tag{23}$$

Therefore,

$$i\int_{\mathbb{R}}\varphi(x)g(x)\,\mathrm{d}x=-\lim_{n\to\infty}\int_{\mathbb{R}}\varphi'(x)f_n(x)\,\mathrm{d}x\quad\forall n\geq 0 \tag{24}$$

In the same way,

$$\left|\int_{\mathbb{R}}\varphi'(x)f(x)\,\mathrm{d}x-\int_{\mathbb{R}}\varphi'(x)f_n(x)\,\mathrm{d}x\right|=\left|\int_{\mathbb{R}}\varphi'(x)(f(x)-f_n(x))\,\mathrm{d}x\right| \tag{25}$$

$$\leq\left(\int_{\mathbb{R}}|f_n(x)-f(x)|^2\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}}|\varphi'(x)|^2\right)^{1/2} \tag{26}$$

$$=\|f(x)-f_n(x)\|_{L^2}\|\varphi'\|_\infty \tag{27}$$

Now, since $f_n\to f$ in $L^2(\mathbb{R})$, the above inequality tends to 0 and we can write,

$$\int_{\mathbb{R}}\varphi'(x)f(x)\,\mathrm{d}x=\lim_{n\to\infty}\int_{\mathbb{R}}\varphi'(x)f_n(x)\,\mathrm{d}x \tag{28}$$

And therefore, we get that,

$$i\int_{\mathbb{R}}\varphi(x)g(x)\,\mathrm{d}x=\int_{\mathbb{R}}\varphi'(x)f(x)\,\mathrm{d}x \tag{29}$$

Proving then that $f\in H^1(\mathbb{R})$ and that $-if'=g$ and therefore that $(P,D(P))$ is closed.
Let us now prove that ${}^{\mathrm{cl}}(P^{\min})=P$ on the domain $D({}^{\mathrm{cl}}(P^{\min}))=D(P)$ by means of graphs. We wish to show that ${}^{\mathrm{cl}}(G(P^{\min}))=G(P)$.
Knowing that $(P,D(P))$ is closed, we already know that ${}^{\mathrm{cl}}(G(P^{\min}))\subset G(P)$.
Take $(f,-if')\in G(P)$. By density of $\mathcal{C}_c^\infty(\mathbb{R})$ spaces in $H^1(\mathbb{R})$ as proven in the appendix, we can find a sequence $(f_n)_{n\geq 0}\subset\mathcal{C}_c^\infty(\mathbb{R})$ such that, $f_n\to f$ in $L^2(\mathbb{R})$ and $f_n'\to f'$ in $L^2(\mathbb{R})$.
Therefore, we have that $(f,-if')\in{}^{\mathrm{cl}}(G(P^{\min}))$ and finally we get that the closure of $P^{\min}$ is the operator P. □

Adjoint

Let us define now the notion of adjoint of an operator. In a similar manner to the case of finite dimensions, knowing an operator $(A,D(A))$, we are looking for an operator $(A^*,D(A^*))$ such that for all $u\in D(A)$ and for all $v\in D(A^*)$,

$$\langle v,Au\rangle=\langle A^*v,u\rangle \tag{30}$$

As we can see we wish to find the "biggest" domain $D(A^*)$ such that this relation is valid. Rewriting this condition, we find that,

$$\langle v,Au\rangle=\langle A^*v,u\rangle\iff\langle v,Au\rangle+\langle A^*v,-u\rangle=0\quad\forall u\in D(A),v\in D(A^*) \tag{31}$$

Thus, defining the inner product on the space $\mathfrak{H} \times \mathfrak{H}$ as,

$$\langle (u_1, u_2), (v_1, v_2) \rangle_{\mathfrak{H}\times\mathfrak{H}} \equiv \langle u_1, v_1 \rangle_{\mathfrak{H}} + \langle u_2, v_2 \rangle_{\mathfrak{H}}, \tag{32}$$

we can rewrite the adjoint condition as,

$$\langle (v, A^*v), (Au, -u) \rangle_{\mathfrak{H}\times\mathfrak{H}} = 0 \tag{33}$$

This tells us, informaly, that $G(A^*) = \{(Au, -u), u \in D(A)\}^{\perp}$ is the adjoint of A. Let us now do it properly.

Theorem 2: *Let $(A, D(A))$ be an operator. Let*

$$G = \{(Au, -u), u \in D(A)\}^{\perp} \tag{34}$$

Then, $(A, D(A))$ admits an adjoint operator $(A^*, D(A^*))$ with $G(A^*) = G$ if and only if A is closable.

To prove this Theorem, we will make use of the following Lemmas.

Lemma 3: *Let $\mathfrak{H}$ be a Hilbert space and $D \subset \mathfrak{H}$ be a vectorial subspace. We have,*

$$D \subset \mathfrak{H} \textit{ is dense} \iff D^{\perp} = \{0\} \tag{35}$$

Remark.
The left to right implication works for any subset of the Hilbert space. However, the reverse implication only work for vector subspaces. A good counterexample of this is the unit sphere $S = \{u \in \mathfrak{H}, \|u\| = 1\}$.
First, $0 \in S^{\perp}$.
Now for any $v \in \mathfrak{H}, v \neq 0$, taking $u = \frac{v}{\|v\|} \in S$, we see that

$$\langle v, u \rangle = \left\langle v, \frac{v}{\|v\|} \right\rangle = 1 \tag{36}$$

Therefore $v \notin S^{\perp}$ and $S^{\perp} = \{0\}$ but we can clearly see that S is not dense in $\mathfrak{H}$.

Proof of Lemma 3.
$\boxed{\Rightarrow}$
Take $a \in D^{\perp}$. Then, for all $x \in D$, $\langle a, x \rangle = 0$. As $D \subset \mathfrak{H}$ is dense, there exists a sequence $(y_n) \subset D$ such that $y_n \to a$ as $n \to \infty$.
For any $n \in \mathbb{N}$, $y_n \in D$ therefore $\langle a, y_n \rangle = 0$. Passing to the limit, we get that,

$$\langle a, y_n \rangle = 0 \xrightarrow[n\to\infty]{} \langle a, a \rangle = \|a\|^2 = 0 \implies a = 0 \tag{37}$$

Therefore $D^{\perp} = \{0\}$.
$\boxed{\Leftarrow}$
$D \subset \mathfrak{H}$ is a vectorial subspace such that $D^{\perp} = \{0\}$. Let us remark that $({}^{\mathrm{cl}}D)^{\perp} = D^{\perp} = \{0\}$.
Using the Hilbert projection theorem and using the fact that ${}^{\mathrm{cl}}D$ is closed, we have that $\mathfrak{H} = {}^{\mathrm{cl}}D \oplus ({}^{\mathrm{cl}}D)^{\perp}$.
Finally, as $({}^{\mathrm{cl}}D)^{\perp} = \{0\}$, we get that $\mathfrak{H} = {}^{\mathrm{cl}}D$ and therefore D is dense in $\mathfrak{H}$. $\square$

Lemma 4 (Double-adjoint of an operator)**:** *Let $(A, D(A))$ be an operator.*
Assume that $(A^*, D(A^*))$ defined by $G(A^*) = \{(Au, -u), u \in D(A)\}^\perp$ is the adjoint of A.
Then we have that

$$(A^*)^* = {}^{\mathrm{cl}}A \tag{38}$$

giving us,

$$((A^*)^*)^* = ({}^{\mathrm{cl}}A)^* = A^* \tag{39}$$

Proof of Lemma 4. Let us first prove that $(A^*)^* = {}^{\mathrm{cl}}A$.
Define

$$J : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{H} \times \mathfrak{H} \quad \text{by} \quad (\xi, \eta) \mapsto J(\xi, \eta) = (\eta, -\xi) \tag{40}$$

We will show that $J^* = -J$ or more specifically that $\langle Jx, y\rangle_{\mathfrak{H}\times\mathfrak{H}} = \langle x, -Jy\rangle_{\mathfrak{H}\times\mathfrak{H}}$ for all $x, y \in \mathfrak{H} \times \mathfrak{H}$.
Indeed for any $(\xi, \eta), (\xi', \eta') \in \mathfrak{H} \times \mathfrak{H}$, we have that,

$$\begin{aligned}
\langle J(\xi, \eta), (\xi', \eta')\rangle_{\mathfrak{H}\times\mathfrak{H}} &= \langle (\eta, -\xi), (\xi', \eta')\rangle_{\mathfrak{H}\times\mathfrak{H}} \\
&= \langle \eta, \xi'\rangle + \langle -\xi, \eta'\rangle \\
&= -\left(\langle \xi, \eta'\rangle + \langle \eta, -\xi'\rangle\right) \\
&= -\langle (\xi, \eta), (\eta', -\xi')\rangle_{\mathfrak{H}\times\mathfrak{H}} \\
&= -\langle (\xi, \eta), J(\xi', \eta')\rangle_{\mathfrak{H}\times\mathfrak{H}} \\
&= \langle (\xi, \eta), -J(\xi', \eta')\rangle_{\mathfrak{H}\times\mathfrak{H}}
\end{aligned}$$

Now, we can see that the operator $(J, \mathfrak{H} \times \mathfrak{H})$ is bijective.
Moreover, for any $y \in \mathfrak{H} \times \mathfrak{H}$, $V \subset \mathfrak{H} \times \mathfrak{H}$, $y \in (JV)^\perp$ if and only if $Jy \in V^\perp$.
Indeed, for any $x \in JV$, there exists $x' \in V$ such that $Jx' \in JV$. We get,

$$\begin{aligned}
y \in (JV)^\perp &\iff \langle x, y\rangle_{\mathfrak{H}\times\mathfrak{H}} = 0 \quad \forall x \in JV \\
&\iff \langle Jx', y\rangle_{\mathfrak{H}\times\mathfrak{H}} = 0 \quad \forall x' \in V \\
&\iff \langle x', -Jy\rangle_{\mathfrak{H}\times\mathfrak{H}} = 0 \quad \forall x' \in V \\
&\iff \langle x', Jy\rangle_{\mathfrak{H}\times\mathfrak{H}} = 0 \quad \forall x' \in V \\
&\iff Jy \in V^\perp.
\end{aligned}$$

This can thus be rewritten as $J[(JV)^\perp] = V^\perp$ giving us,

$$\left(J[(JV)^\perp]\right)^\perp = {}^{\mathrm{cl}}V \tag{41}$$

Furthermore, as $G(A^*) = \{(Au, -u), u \in D(A)\}^\perp$, we can see that $G(A^*) = (J[G(A)])^\perp$.
Similarly, $G((A^*)^*) = (J[G(A^*)])^\perp$. Therefore putting $V = G(A)$ in (41), we get the desired equation,

$$G((A^*)^*) = {}^{\mathrm{cl}}G(A) = G({}^{\mathrm{cl}}A) \tag{42}$$

As for the second statement in the Lemma, we can remark that $({}^{\mathrm{cl}}V)^\perp = V^\perp$ and therefore by (41), we have that $((A^*)^*)^* = ({}^{\mathrm{cl}}A)^* = A^*$ □

Remark.
The function $J : \mathfrak{H}\times\mathfrak{H} \to \mathfrak{H}\times\mathfrak{H}$ defined by $(\xi, \eta) \mapsto J(\xi, \eta) = (\eta, -\xi)$ is called the symplectic operator.

Let us now proceed with the proof of the Theorem.

Proof of the Theorem. We will show that $G(A^*)$ as defined in the theorem does in fact concurr with the graph of an operator.
We will thus check the conditions of Lemma 1.
The orthogonal of a set is a vector space therefore, $G = \{(Au, -u), u \in D(A)\}^\perp$ is in fact a vector subspace. Now, let us check the second condition.

$$\begin{aligned}(0, w) \in G &\iff \langle 0, Au\rangle + \langle w, -u\rangle = 0, \quad \forall u \in D(A) &(43)\\ &\iff \langle w, u\rangle = 0 \quad \forall u \in D(A) &(44)\\ &\iff w \in D(A)^\perp &(45)\end{aligned}$$

Therefore we get,

$$D(A)^\perp = \{w \in \mathfrak{H} : (0, w) \in G\} \tag{46}$$

However, since $D(A)$ is dense in $\mathfrak{H}$, by Lemma 3, $D(A)^\perp = \{0\}$. Hence, the second condition is satisfied.

Finally, we will check the third condition. Let $D = \{x \in \mathfrak{H}, \exists y \in \mathfrak{H} : (x, y) \in G\}$. We have that,

$$w \in D^\perp \iff \langle u, w\rangle = 0 \quad \forall u \in D, w \in \mathfrak{H} \tag{47}$$

Since $u \in D$, there exists a unique $v \in \mathfrak{H}$ such that, $(u, v) \in G$. Furthermore, as $\langle v, 0\rangle = 0$, we have that

$$\begin{aligned}w \in D^\perp &\iff \langle u, w\rangle + \langle v, 0\rangle = 0 \quad \forall u \in D, w \in \mathfrak{H} &(48)\\ &\iff \langle (u, v), (w, 0)\rangle_{\mathfrak{H}\times\mathfrak{H}} = 0 \quad \forall u \in D, w \in \mathfrak{H} &(49)\\ &\iff (w, 0) \in G^\perp &(50)\end{aligned}$$

We can observe that,

$$G = \{(Au, -u), u \in D(A)\}^\perp = (J(G(A)))^\perp \tag{51}$$

Hence,

$$G^\perp = {}^{\mathrm{cl}}JG(A) = J^{\mathrm{cl}}G(A) \tag{52}$$

Therefore,

$$(w, 0) \in G^\perp \iff (0, -w) \in {}^{\mathrm{cl}}G(A) \tag{53}$$

The equivalence comes from the fact that since conditions (1) and (3) of Lemma 1 are satisfied, condition (2) is the only condition needed to be satisfied for equivalence.
Now, we can see that if $(A, D(A))$ is closable, then $w = 0$ and this means that D is dense in $\mathfrak{H}$, by using the fact that D is a vector subspace. This proves that G is indeed the graph of an operator. Furthermore, if A is not closable then w is not necessarily 0 and D is not necessarily dense in $\mathfrak{H}$.
Finally, we just have to prove that $G \equiv G(A^*)$ is indeed the graph of the adjoint of the operator A which we will call A^*.
For any $(v, A^*v) \in G$, the following equality is satisfied,

$$\begin{aligned}&\langle (v, A^*v), (Au, -u)\rangle_{\mathfrak{H}\times\mathfrak{H}} = 0 &(54)\\ \iff &\langle v, Au\rangle + \langle A^*v, -u\rangle = 0 &(55)\\ \iff &\langle v, Au\rangle - \langle A^*, u\rangle = 0 &(56)\\ \iff &\langle v, Au\rangle = \langle A^*, u\rangle &(57)\end{aligned}$$

We do indeed get the equality of the adjoint operator.
We have therefore proved that $(A, D(A))$ admits an adjoint $(A^*, D(A^*))$ if and only if A is closable. Moreover, we have also proved the equation for the graph of the adjoint. □

Remark.
Now that we have found in which cases we can define the adjoint of an operator, we can replace the assumption in the lemma 4 to the case of a closable operator.

Lemma 5: *Let $(A, D(A))$ be a closable operator. Then, we have the following equality,*

$$\ker(A^* - \overline{z}) = \operatorname{Im}(A - z)^\perp = \operatorname{Im}({}^{\mathrm{cl}}A - z)^\perp \tag{58}$$

for all $z \in \mathbb{C}$.

Proof. We will prove this by double inclusion. To be more specific, we will prove that,

$$\operatorname{Im}({}^{\mathrm{cl}}A - z)^\perp \subseteq_1 \operatorname{Im}(A - z)^\perp \subseteq_2 \ker(A^* - \overline{z}) \subseteq_3 \operatorname{Im}({}^{\mathrm{cl}}A - z)^\perp \tag{59}$$

$\boxed{\subseteq_1}$
We can easily see that since $G(A) \subseteq G({}^{\mathrm{cl}}A)$, $\operatorname{Im}(A - z) \subseteq \operatorname{Im}({}^{\mathrm{cl}}A - z)$ and therefore,

$$\operatorname{Im}({}^{\mathrm{cl}}A - z)^\perp \subseteq \operatorname{Im}(A - z)^\perp \tag{60}$$

$\boxed{\subseteq_2}$
Take $y \in \operatorname{Im}(A - z)^\perp$, then,

$$\langle (A - z)x, y\rangle = 0 \quad \forall x \in D(A) \tag{61}$$

Now, using the fact that $y \in \mathfrak{H}$ and that $D(A^*)$ is dense in $\mathfrak{H}$, we know that there exists a sequence $(y_n)_n \subseteq D(A^*)$ such that $y_n \to y$ in $\mathfrak{H}$.
We have that,

$$\Big|\langle (A - z)x, y_n\rangle\Big| = \Big|\langle (A - z)x, y_n - y + y\rangle\Big| \tag{62}$$

$$\leq \Big|\langle (A - z)x, y_n - y\rangle\Big| \tag{63}$$

$$\leq \|(A - z)x\|_{\mathfrak{H}} \|y_n - y\|_{\mathfrak{H}} \xrightarrow[n\to\infty]{} 0 \tag{64}$$

since for all fixed $x \in D(A)$, $\|(A - z)x\|_{\mathfrak{H}} \leq C$ for some constant $C > 0$ independent of $n \in \mathbb{N}$.
So we can see that,

$$\Big|\langle x, (A^* - \overline{z})y_n\rangle\Big| \xrightarrow[n\to\infty]{} 0 \tag{65}$$

Therefore, to prove that the limit $y_n \to y$ is in the $\ker(A^* - \overline{z})$, we just have to show that the kernel is a topologically closed space.
As we can see,

$$\ker(A^* - \overline{z}) = \{w \in \mathfrak{H}, \text{ such that } (w, 0) \in G(A^* - \overline{z})\} \tag{66}$$

Moreover,

$$(w, 0) \in G(A^* - \overline{z}) \iff \langle (w, 0), ((A - z)u, -u)\rangle = 0 \quad \forall u \in D(A) \tag{67}$$

$$\iff \langle (w, 0), J(u, (A - z)u)\rangle = 0 \tag{68}$$

$$\iff \langle -J(w, 0), (u, (A - z)u)\rangle = 0 \tag{69}$$

$$\iff \langle J(w, 0), (u, (A - z)u)\rangle = 0 \tag{70}$$

$$\iff (0, -w) \in G(A - z)^\perp \tag{71}$$

Therefore,

$$\ker(A^* - \overline{z}) = \{w \in \mathfrak{H}, \text{ such that } (0, -w) \in G(A - z)^\perp\} \tag{72}$$

And finally since $G(A - z)^\perp$ is a topologically closed space, it follows that $\ker(A^* - \overline{z})$ is also a topologically closed space.
We can thus conclude that $y = \lim_{n\to\infty} y_n \in \ker(A^* - \overline{z})$
$\boxed{\subseteq_3}$
For all $y \in \ker(A^* - \overline{z})$,

$$\langle (A - z)x, y\rangle = \langle x, (A^* - \overline{z})y\rangle = 0 \quad \forall x \in D(A) \tag{73}$$

Furthermore, using the fact that $A^* = ({}^{\text{cl}}A)^*$, we have that,

$$0 = \langle x, (A^* - \overline{z})y\rangle = \langle x, (({}^{\text{cl}}A)^* - \overline{z})y\rangle \quad \forall x \in D(A) \tag{74}$$

Hence,

$$\langle ({}^{\text{cl}}A - z)x, y\rangle = 0 \tag{75}$$

So, $y \in \operatorname{Im}({}^{\text{cl}}A - z)^\perp$. □

Proposition 2: *Let $(A, D(A))$ and $(B, D(B))$ be two closable operators. If $A \subset B$, then $B^* \subset A^*$*

Proof. Let $(A, D(A))$ and $(B, D(B))$ be two closable operators. Then the adjoints are well-defined. Now, since $G(A) \subset G(B)$, by continuity of J, $J(G(A)) \subset J(G(B))$. Moreover, since $G(A^*) = (J(G(A))^\perp$ and $G(B^*) = (J(G(B))^\perp$,

$$G(B^*) = (J(G(B)))^\perp \subset (J(G(A)))^\perp = G(A^*) \tag{76}$$

giving us that $B^* \subset A^*$ □

Symmetry

From now on, to make sure that we always have a densely defined adjoint to an operator, we will exclusively work with closable operators.

Definition 7 (Symmetry)**:** *Let $(A, D(A))$ be an operator. We say that A is symmetric when,*

$$\langle Av, w\rangle = \langle v, Aw\rangle \quad \forall v, w \in D(A) \tag{77}$$

Equivalently, $A \subset A^*$ or $G(A) \subset G(A^*)$.

Proposition 3: *Let $(A, D(A))$ be a closable operator. If A is symmetric, then it's closure ${}^{\text{cl}}A$ is also symmetric.*

Proof. Let $(A, D(A))$ be a closable operator. Let $(A, D(A))$ be symmetric. Then, $A \subset A^*$. Making use of Proposition 2, we get that $(A^*)^* \subset A^*$. Moreover, we know that ${}^{\text{cl}}A = (A^*)^*$ and $A^* = ({}^{\text{cl}}A)^*$, therefore we get that ${}^{\text{cl}}A \subset ({}^{\text{cl}}A)^*$. □

We will now introduce the spectrum of symmetric operators and prove it. To do so, we will first emphasize on some points.

In the following theorem, we will state the different possibilities for the spectrum of a symmetric operator.
In [Lew21], the theorem is stated for the closure of $(A, D(A))$. We, on the other hand, will state it for the operator itself, for the following reason.
If $(A, D(A))$ is a non-closed operator, the spectrum of A is given by $\sigma(A) = \mathbb{C}$. Therefore $\sigma(A)$ falls in one of the four cases.
If now $(A, D(A))$ is closed, then the four cases apply to the spectrum of A.

Theorem 3 (Spectrum of symmetric operators)**:** *Let $(A, D(A))$ be a symmetric operator. The spectrum of A is*

- *either equal to the whole complex plane:*

$$\sigma(A) = \mathbb{C}; \tag{78}$$

- *either equal to the closed complex upper half-plane:*

$$\sigma(A) = \mathbb{C}_+ = \{z \in \mathbb{C} : \Im z \geq 0\}; \tag{79}$$

- *either equal to the closed complex lower half-plane:*

$$\sigma(A) = \mathbb{C}_- = \{z \in \mathbb{C} : \Im z \leq 0\}; \tag{80}$$

- *either included in $\mathbb{R}$:*

$$\sigma(A) \subset \mathbb{R} \tag{81}$$

In any of the cases, the spectrum never contains eigenvalues in $\mathbb{C}\backslash\mathbb{R}$. That is to say that , $\ker({}^{\mathrm{cl}}A - z) = \{0\}$ *if* $\Im(z) \neq 0$.
Moreover, if there exist $z \in \mathbb{C}$ with $\Im(z) \neq 0$ such that $\mathrm{Im}(A - z) = \mathfrak{H}$, then $A = {}^{\mathrm{cl}}A$ and $z \in \rho(A)$, such that $\sigma(A)$ is included in the half-plane not containing z.

The proof will be left to the reader, however we will remind the reader of a few facts to help with the proof.

For some operator $(A, D(A))$ we have that for all $u \in D(A)$ and $a, b, \in \mathbb{R}$,

$$\begin{aligned} \|(A - a - ib)u\|^2 &= \langle (A - a - ib)u, (A - a - ib)u\rangle && (82)\\ &= \|(A - a)u\|^2 + b^2\|u\|^2 - 2b\Im\langle (A - a)u, u\rangle && (83)\\ &= \|(A - a)u\|^2 + b^2\|u\|^2, && (84) \end{aligned}$$

where we used that for any symmetric operator, $\langle (A - a)u, u\rangle = \langle u, (A - a)u\rangle = \overline{\langle (A - a)u, u\rangle}$.
The relationship

$$\|(A - a - ib)u\|^2 = \|(A - a)u\|^2 + b^2\|u\|^2 \geq b^2\|u\|^2 \tag{85}$$

directly imply that,

$$(A - a - ib)u = 0, b \neq 0 \implies u = 0. \tag{86}$$

Making use of the fact that $z \in \mathbb{C}$ can be rewritten as $z = a + ib$ with $a, b \in \mathbb{R}$, we can immediately conclude that there cannot be any imaginary eigenvalues.
Furthermore, if we assume that $(A - a - ib)^{-1}$ exists, we have that,

$$\frac{1}{|b|}\|(A - a - ib)u\| \geq \|u\| = \left\|(A - a - ib)(A - a - ib)^{-1}u\right\| \tag{87}$$

giving us that if the inverse exists, then it is bounded by,

$$\left|\left|\left|(A - a - ib)^{-1}\right|\right|\right| \leq \frac{1}{|b|} \tag{88}$$

From all these facts, we can conclude that $A - a - ib$ is necessarily injective and if it is surjective too, then the inverse is bounded.
Therefore, the only case for which $a + ib$ with $a, b \in \mathbb{R}$ is not in the resolvent set is if $A - a - ib$ is not surjective.

Proof. The proof can be found in [Lew21], in the proof of Theorem 2.21. □

Self-adjoint

Definition 8 (Self-adjoint)**:** *An operator $(A, D(A))$ is said to be self-adjoint when $A = A^*$, or in other words, $(A, D(A))$ is symmetric and $D(A^*) = D(A)$. We say that it is essentially self-adjoint if it is symmetric and ${}^{\mathrm{cl}}A$ is self-adjoint.*

Remark.
It is worth to note that since ${}^{\mathrm{cl}}A = (A^*)^* = A^* = A$ for any self-adjoint operator, we conclude that all self-adjoint operators are closed.

Proposition 4 (Bounded self-adjoint operators)**:** *If $(A, D(A))$ is a symmetric operator such that $D(A) = \mathfrak{H}$, then A is self-adjoint and bounded.*

Proof. First of all, since A is a symmetric operator, $D(A) \subset D(A^*)$. Therefore, using $D(A) = \mathfrak{H}$, we get that $D(A) = D(A^*)$ and A is self-adjoint. As for the boundness, since A is closed and linear between two Banach spaces, by the closed graph theorem, A is continuous and therefore bounded. □

Theorem 4 (Characterisation of self-adjoint operators)**:** *Let $(A, D(A))$ be a symmetric operator. The following assertions are equivalent:*

1. *A is self-adjoint, that is to say that it verifies $D(A^*) = D(A)$;*
2. *the spectrum of A is real: $\sigma(A) \subset \mathbb{R}$;*
3. *there exists $\lambda \in \mathbb{C}$ such that $A - \lambda$ and $A - \overline{\lambda}$ are surjective from $D(A)$ to $\mathfrak{H}$.*

Proof.

$\boxed{(1) \Rightarrow (2)}$

Let A be a self-adjoint operator. For some $z \in \mathbb{C}\backslash\mathbb{R}$, we have that $\ker({}^{\mathrm{cl}}A - z) = \ker({}^{\mathrm{cl}}A - \overline{z}) = \{0\}$, which implies that $A - z$ is injective.
As $A - z$ is injective, let us now check that it is surjective. By Lemma 5, we have that,

$$\ker(A - \overline{z}) = \operatorname{Im}(A - z)^{\perp} = \{0\} \tag{89}$$

Therefore, $\operatorname{Im}(A - z)$ is dense in $\mathfrak{H}$. If we now show that $\operatorname{Im}(A - z)$ is closed, then $\operatorname{Im}(A - z) = \mathfrak{H}$.
Take $(v_n) \subset D(A)$ such that $(A - z)v_n \to w$. By the inequality $\|(A - z)u\|^2 \geq b^2\|u\|^2$,

$$0 \leftarrow \|(A - z)(v_n - v_p)\| \geq |\operatorname{Im}(z)|\|v_n - v_p\| \tag{90}$$

where $\operatorname{Im}(z) \neq 0$ since $z \in \mathbb{C}\backslash\mathbb{R}$. We can thus say that $(v_n)_n$ is a Cauchy sequence in $\mathfrak{H}$. Now, using the fact that A and therefore $G(A)$ are closed, we get that $v \in D(A)$ and therefore $w \in \operatorname{Im}(A - z)$ concluding that $\operatorname{Im}(A - z) = \mathfrak{H}$.
By Theorem 3, this gives us that $z \in \rho(A)$ and therefore, since $z \in \mathbb{C}\backslash\mathbb{R}$ was chosen arbitrarily, $\sigma(A) \subset \mathbb{R}$.

$\boxed{(2) \Rightarrow (3)}$

If $\sigma(A) \subset \mathbb{R}$, $\rho(A) \subset \mathbb{C}\backslash\mathbb{R}$. Therefore, there exists $\lambda \in \mathbb{C}\backslash\mathbb{R}$ such that $A - \lambda$ and $A - \overline{\lambda}$ are surjective.

$\boxed{(3) \Rightarrow (1)}$

Suppose there exists $\lambda \in \mathbb{C}\backslash\mathbb{R}$ such that $A - \lambda$ and $A - \overline{\lambda}$ are surjective.
Let $(v, w) \in G(A^*)$. Then,

$$\langle v, Au\rangle = \langle w, u\rangle \quad \forall u \in D(A) \tag{91}$$

$$\langle v, (A - z)u\rangle = \langle w - \overline{z}v, u\rangle \quad \forall u \in D(A) \tag{92}$$

Knowing that $A - \overline{z}$ is surjective, there exists $y \in D(A)$ such that $w - \overline{z}v = (A - \overline{z})y$.
Therefore,

$$\langle v, (A - z)u\rangle = \langle (A - \overline{z})y, u\rangle \quad \forall u \in D(A) \tag{93}$$

$$= \langle y, (A - z)u\rangle \quad \forall u \in D(A) \tag{94}$$

Implying that,

$$\langle v - y, (A - z)u\rangle = 0 \quad \forall u \in D(A) \implies v - y \in \operatorname{Im}(A - z)^{\perp}. \tag{95}$$

However, since $\operatorname{Im}(A - z)^{\perp} = \{0\}$ since $A - z$ is surjective, we get that $v = y$ showing that $w = Av$ and $G(A) = G(A^*)$. $\square$

Theorem 5 (Spectrum of self-adjoint operators)**:** *Let $(A, D(A))$ be an self-adjoint operator and $\lambda \in \mathbb{R}$. The following assertions are equivalent:*

1. *$\lambda \in \sigma(A)$;*
2. *$\inf\limits_{\substack{v \in D(A) \\ \|v\|=1}} \|(A - \lambda)v\| = 0$;*
3. *there exists a sequence $(v_n) \subset D(A)$ such that $\|v_n\| = 1$ and $\|(A - \lambda)v_n\| \to 0$.*

Proof. The proof can be found in [Lew21], in the proof of Theorem 2.28. □

Definition 9 (Suite de Weyl)**:** *A sequence (v_n) verifying the properties in point (3) of Theorem 5 is called a Weyl sequence.*

We will make a small aparté on the notion of weak convergence and Weyl sequences.

Definition 10: *A sequence $(v_n) \subset \mathfrak{H}$ is said to converge weakly in $\mathfrak{H}$ to a point $v \in \mathfrak{H}$ if $\langle v_n, y\rangle \to \langle v, y\rangle$ for all $y \in \mathfrak{H}$.*

Lemma 6 (Weak convergence of Weyl sequence)**:** *Let $(v_n) \subset \mathfrak{H}$ is a Weyl sequence, then it is bounded and admits a subsequence (v_{n_k}) that convergences weakly to a vector $v \in \mathfrak{H}$.*

Proof. Let $(v_n) \subset \mathfrak{H}$ be a Weyl sequence. Since $\|v_n\| = 1$ for all $n \in \mathbb{N}$, it is bounded. Now every bounded sequence admits a convergent subsequence. Let us now show that if (v_{n_k}) converges usually, it converges weakly. Let $v \in \mathfrak{H}$ such that $v_{n_k} \to v$ as $n_k \to \infty$. Then,

$$\Big|\langle v_{n_k}, u\rangle - \langle v, u\rangle\Big| = \Big|\langle v_{n_k} - v, u\rangle\Big| \leq \|v_{n_k} - v\|\|u\| \to 0 \quad \forall u \in \mathfrak{H} \tag{96}$$

Hence, (v_{n_k}) converges weakly. □

Corollary 1 (Spectrum localization)**:** *Let $(A, D(A))$ be an self-adjoint operator. We let,*

$$m = \inf_{\substack{v \in D(A) \\ \|v\|=1}} \langle v, Av\rangle \quad \textit{and} \quad M = \sup_{\substack{v \in D(A) \\ \|v\|=1}} \langle v, Av\rangle \tag{97}$$

Then, $\sigma(A) \subset [m, M]$.

Remark.
If furthermore, A is a bounded operator, then $m \in \sigma(A), M \in \sigma(A)$.

Proof. The proof can be found in [Bre05], in the proof of Proposition VI.9. □

CHAPTER II - APPLICATIONS OF SPECTRAL THEORY

Now that we have studied the notions of spectrum on symmetric and auto-adjoint operators let's put it to good use with well-known and well-seeked operators in physics: the Momentum and the Laplacian operator on different domains of study.

Momentum on $\mathbb{R}$

We have seen the following 2 differential operators:

$$P^{\min} f = -if', \quad D(P^{\min}) = \mathcal{C}_c^{\infty}(\mathbb{R}) \tag{98}$$

As we have seen, this operators is not closed and it's closure is given by, respectively,

$$Pf = -if', \quad D(P) = \{f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R})\} = H^1(\mathbb{R}) \tag{99}$$

In this part, we will show that P is well-defined and self-adjoint.
$P^{\min}$ and $A^{\min}$ are essentially self-adjoint as they admit only one self-adjoint extension which is their closure.

Theorem 6 (Momentum and Laplacian on $\mathbb{R}$)**:** *The operator $P = -i\partial$ on $D(P) = H^1(\mathbb{R}) \subset \mathcal{H} = L^2(\mathbb{R})$ is self-adjoint and it's spectrum is given by*

$$\boxed{\sigma(P) = \mathbb{R}} \tag{100}$$

The operator P does not exhibit any eigenvalues.

Proof. Let us first show that P is symmetric on $H^1(\mathbb{R})$. Using the integration by parts formulae, we get that

$$\langle g, Pf\rangle = -i \int_{\mathbb{R}} \overline{g(x)} f'(x) \,\mathrm{d}x = i \int_{\mathbb{R}} \left(\overline{g(x)}\right)' f(x) \,\mathrm{d}x \tag{101}$$

$$= \int_{\mathbb{R}} \overline{-ig'(x)} f(x) \,\mathrm{d}x = \langle Pg, f\rangle \quad \forall f, g \in H^1(\mathbb{R}) \tag{102}$$

Now that we know P is symmetric, we will make use of Theorem 4 with $\lambda = i$ to show that P is auto-adjoint.
We wish to prove that $P - i$ and $P + i$ are both surjective. Therefore, given $g \in L^2(\mathbb{R})$ we wish to find $f \in H^1(\mathbb{R})$ and $h \in H^1(\mathbb{R})$ such that $g = (P - i)f$ and $g = (P + i)h$. We will thus move in Fourier space to find such a function.
Let us start with the operator $P - i$. We wish to find $f \in H^1(\mathbb{R})$ such that $-i(f + f') = g$. Moving on to Fourier space, this relation can be rewritten as,

$$-if' - if = g \iff -i\widehat{f'(x)} - i\widehat{f(x)} = \widehat{g(x)} \tag{103}$$

Using properties of the Fourier transform, we can simplify this equation

$$-i\widehat{f'(x)} - i\widehat{f(x)} = k\widehat{f}(k) - i\widehat{f}(k) = (k - i)\widehat{f}(k) = \widehat{g}(k). \tag{104}$$

Since $k \in \mathbb{R}$, $k - i \neq 0$ and we get that,

$$-if' - if = g \iff \widehat{f}(k) = \frac{\widehat{g}(k)}{k-i}. \tag{105}$$

We are then done if we can prove that,

$$f(x) = \mathcal{F}^{-1}\left(\frac{\widehat{g}(k)}{k-i}\right) \in H^1(\mathbb{R}) \tag{106}$$

We will thus use the Fourier characterisation to prove that $f(x) \in H^1(\mathbb{R})$, i.e. we will prove that $(1+|k|)\widehat{f}(k) \in L^2(\mathbb{R})$.
By the Fourier transform, $\widehat{g}(k) \in L^2(\mathbb{R})$ and $\frac{1+|k|}{k-i} \in L^\infty(\mathbb{R})$, therefore,

$$(1+|k|)\widehat{f}(k) = \frac{1+|k|}{k-i}\widehat{g}(k) \in L^2(\mathbb{R}) \iff f(x) \in H^1(\mathbb{R}) \tag{107}$$

Thus, $P - i$ is surjective.
Using the same argument, we conclude that,

$$(1+|k|)\widehat{h}(k) = \frac{1+|k|}{k+i}\widehat{g}(k) \in L^2(\mathbb{R}) \iff h(x) \in H^1(\mathbb{R}) \tag{108}$$

Hence, $P + i$ is surjective.
Finally, by Theorem 4, $(P, D(P))$ is self-adjoint.
As for the proof of the spectrum, the proof can be found in [Lew21] in the proof of Theorem 2.33. The idea, however, is to create a Weyl sequence to find out the spectrum of P. □

Momentum on an interval

In this section, we will study more in details the operators,

$$f \mapsto -if' \qquad \text{on } \mathfrak{H} = L^2(I), \text{ with } I =]0,1[\text{ or } I =]0,\infty[. \tag{109}$$
$$\tag{110}$$

- ### Momentum $P = -i\mathrm{d}/\mathrm{d}x$ on $]0,1[$

We wish to define the operator $f \mapsto -if'$ on an appropriate domain in the Hilbert space $\mathfrak{H} = L^2(]0,1[)$. The two natural domain we think of are $\mathcal{C}_c^\infty(]0,1[)$ and $H^1(]0,1[)$. Therefore, we define,

$$P^{\min} f = -if', \quad D(P^{\min}) = \mathcal{C}_c^\infty(]0,1[) \tag{111}$$
$$P^{\max} f = -f', \quad D(P^{\max}) = H^1(]0,1[) \tag{112}$$

It can easily be seen that as $H^1(]0,1[)$ is the space containing all $L^2(]0,1[)$ such that their derivative (in the sence of distributions) is also in $L^2(]0,1[)$, it is the biggest domain we can imagine. Hence, $D(P^{\max}) = H^1(]0,1[)$ is the maximal domain.
It is also trivial that,

$$P^{\min} \subset P^{\max} \tag{113}$$

Furthermore, using the integration by parts formulae, we have that,

$$\begin{aligned}
\langle f, P^{\min} g\rangle &= -i\int_0^1 \overline{f(t)}g'(t)\,\mathrm{d}t && (114)\\
&= i\int_0^1 \overline{f'(t)}g(t)\,\mathrm{d}t && (115)\\
&= \int_0^1 \overline{-if'(t)}g(t)\,\mathrm{d}t && (116)\\
&= \langle P^{\min} f, g\rangle \quad \forall f, g && (117)
\end{aligned}$$

Therefore, $P^{\min}$ is a symmetric operator.
let us now check if $P^{\max}$ is a symmetric operator. If it is, I could be a potential self-adjoint extension of $P^{\min}$. Using the integration by parts formulae on $H^1(]0,1[)$, we get that

$$\begin{aligned}
\langle f, P^{\max} g\rangle &= -i\int_0^1 \overline{f(t)}g'(t)\,\mathrm{d}t && (118)\\
&= i\int_0^1 \overline{f'(t)}g(t)\,\mathrm{d}t - i\left(\overline{f(1)}g(1) - \overline{f(0)}g(0)\right) && (119)\\
&= \langle P^{\max} f, g\rangle - i\left(\overline{f(1)}g(1) - \overline{f(0)}g(0)\right) \quad \forall f, g \in H^1(]0,1[) && (120)
\end{aligned}$$

In particular, for $f(x) = 1, g(x) = x$,

$$\langle f, P^{\max} g\rangle - \langle P^{\max} f, g\rangle = -i \tag{121}$$

Therefore, $P^{\max}$ is not a symmetric extension of $P^{\min}$.

Lemma 7 (Closure and adjoints)**:** *The operator $P^{\max}$ is closed. On the other hand, the operator $P^{\min}$ is not closed and it's closure is given by the operator*

$$P_0 : f \mapsto -if' \qquad D(P_0) = H_0^1(]0,1[) \tag{122}$$

We have

$$(P^{\min})^* = (P_0)^* = P^{\max} \quad \textit{and} \quad (P^{\max})^* = P_0 \tag{123}$$

such that P_0 is not self-adjoint. The spectrums are given by,

$$\sigma(P^{\min}) = \sigma(P^{\max}) = \sigma(P_0) = \mathbb{C} \tag{124}$$

The spectrum of $P^{\max}$ is only composed of eigenvalues, whereas the spectrums of $P^{\min}$ and of P_0 do not exhibit any.

Proof. The proof of this Lemma will not be given as it is a very similar idea to the previous case. It can however be found in [Lew21] in the proof of Lemma 2.35. □

As we have demonstrated from our calculations, none of the three operators $P^{\min} \subset P_0 \subset P^{\max}$ are self-adjoint.
On the other hand, since $P^{\max} = (P^{\min})^*$, we know that all self-adjoint extensions verify

$$P^{\min} \subsetneq {}^{\mathrm{cl}}(P^{\min}) = P_0 \subsetneq P = P^* \subsetneq P^{\max}. \tag{125}$$

Since functions in the domain of the self-adjoint extension (if it exists) neither verify $f(0) = f(1) = 0$ nor the fact that they do not relate, the next natural thought would be to relay them through some periodic relation. This brings us to the next Theorem.

Theorem 7 (Momentum on $]0,1[$: self-adjoint and spectrum)**:** *The strict symmetric extensions of P_0 are the operators $P_{\text{per},\theta}$ defined by,*

$$P_{\text{per},\theta} : f \mapsto -if' \quad \textit{with} \quad D(P_{\text{per},\theta}) = H^1_{\text{per},\theta}(]0,1[) \equiv \{f \in H^1(]0,1[) : f(1) = e^{i\theta} f(0)\} \tag{126}$$

where $\theta \in [0, 2\pi]$. These operators are all self-adjoint and their spectrum is given by

$$\sigma(P_{\text{per},\theta}) = \{k + \theta, \quad k \in 2\pi\mathbb{Z}\}. \tag{127}$$

Each element $k+\theta$ of the spectrum is a simple eigenvalue with associated eigenfunction $x \mapsto e^{i(k+\theta)}x$

The conditions $f(1) = e^{i\theta} f(0)$ are known as the *Born-von Karman conditions*. This theorem is extremely powerful as it states that the only self-adjoint extensions of the Momentum operator on a bounded interval corresponds to the operator satisfying the Born-von Karman conditions.
The intuition behind this is that since the Momentum operator is a differential operator of order one, only one condition needs to be fulfilled at the boundary. Therefore, the two Dirichlet conditions $f(0) = f(1) = 0$ are too restrictive for the Momentum operator to be self-adjoint.

Proof. The full proof of this Theorem will not be given and the reader can refer to the proof of Theorem 2.36 in [Lew21] for the detailed proof. □

- **Momentum $P = -i\mathrm{d}/\mathrm{d}x$ on $]0,\infty[$**

Let us examine now the same operator on the interval $]0,\infty[$. We will show that in this case, contrarily to the case of the bounded interval $]0,1[$, the Momentum operator does not admit any auto-adjoint extension. Let us summarize what we know.
Similarly to the previous part, we have three operators:

$$\begin{cases} P^{\min} : f \mapsto -if', \quad \text{with} \quad D(P^{\min}) = \mathcal{C}^\infty_c(]0,\infty[), \\ P_0 : f \mapsto -if', \quad \text{with} \quad D(P_0) = H^1_0(]0,\infty[) = \{f \in H^1(]0,\infty[) : f(0^+) = 0\} \\ P^{\max} : f \mapsto -if', \quad \text{with} \quad D(P^{\max}) = H^1(]0,\infty[) \end{cases}$$

Following the same arguments as before we can show that,

- P_0 and $P^{\max}$ are closed whereas $P^{\min}$ is not;
- $P^{\min}$ and P_0 are symmetric whereas $P^{\max}$ is not;
- ${}^{\text{cl}}(P^{\min}) = P_0$,
- $(P^{\min})^* = (P_0)^* = P_{\max}$,
- $(P^{\max})^* = P_0$.

This leads us to the following Theorem.

Theorem 8 (Momentum on the positive half-line)**:** *The operator P_0 does not admit any strict symmetric extension. In particular, the operator $P^{\min}$ does not admit any self-adjoint extension. Furthermore,*

$$\sigma(P_0) = \mathbb{C}_-, \tag{128}$$

a spectrum exhibitting no eigenvalues.

Proof. We will not provide a full detailed proof to this theorem and if the reader wishes to get the full proof, it can be found in the proof of Theorem 2.37 in [Lew21]. Let us prove that P_0 does not admit any strict symmetric extension by contradiction. Suppose P is such an extension.
Then, we have $P \subset P^* \subset (P_0)^* = P^{\max}$. The symmetry of P implies, after integration by parts on $H^1(]0,\infty[)$ that,

$$\overline{f(0)}g(0) = 0 \qquad \forall f, g \in D(P) \tag{129}$$

Since P is a strict symmetric extension of P_0, there exists $0 \neq f \in D(P) \backslash D(P_0)$ that is to say $f \in H^1(]0,\infty[)$ such that $f(0) \neq 0$.
We deduce then that $g(0) = 0$ for all $g \in D(P)$, this means that $P \subset P_0$ which is a contradiction since $P \neq P_0$. □

Laplacian on an interval

- $A = -\mathrm{d}^2/\mathrm{d}x^2$ on $]0,1[$

Out of interest in the application of spectral theory on common operators, we will introduce the Laplacian on the interval $]0,1[$ in a less rigorous manner as to understand the idea behind it.

Let us find out the self-adjoint extensions to the Laplician defined as,

$$A^{\min} f = -f'', \quad \text{with} \quad D(A^{\min}) = \mathcal{C}_c^\infty(]0,1[) \tag{130}$$

in the space $\mathfrak{H} = L^2(]0,1[)$. As done before, we can easily see that $A^{\min}$ is symmetric; however it is not closed. It's closure is given by the operator $A_0 f = -f''$ defined on the domain

$$D(A_0) = H_0^2(]0,1[) = \{f \in H^2(]0,1[) : f(0) = f(1) = f'(0) = f'(1) = 0\} \tag{131}$$

Naturally, we define the operator $A^{\max} f = -f''$ on the maximal domain

$$D(A^{\max}) = H^2(]0,1[). \tag{132}$$

Now this last operator $A^{\max}$ is closed; however, since the border terms do not cancel each other out, it is not symmetric.
Similarly to the previous part, we have

$$(A^{\min})^* = (A_0)^* = A^{\max}, \quad (A^{\max})^* = A_0, \tag{133}$$

such that now of the three operators, none are self-adjoint.
In the same way as the spectrum of the momentum, we introduce the spectrum of the Laplacian operators.

Lemma 8 (Spectrum of Laplacian operators on $]0,1[$)**:** *We have that,*

$$\sigma(A^{\min}) = \sigma(A^{\max}) = \sigma(A_0) = \mathbb{C} \tag{134}$$

The spectrum of $A^{\max}$ is only composed of eigenvalues whereas the spectrum of $A^{\min}$ and A_0 don't exhibit any eigenvalues.

Proof. The proof of this Lemma can be found in the proof of Lemma 2.40 of [Lew21]. □

Let us now find the self-adjoint extensions A of $A^{\min}$ which necessarily verify

$$A^{\min} \subsetneq {}^{\mathrm{cl}}(A^{\min}) = A_0 \subsetneq A = A^* \subsetneq A^{\max}. \tag{135}$$

As in the previous sections, after integration by parts on $H^2(]0,1[)$, we deduce that all self-adjoint extensions A of A_0 verify

$$\overline{g'(1)}f(1) - \overline{g(1)}f'(1) + \overline{g(0)}f'(0) - \overline{g'(0)}f(0) = 0, \quad \forall f, g \in D(A) \tag{136}$$

Rewriting this condition in matrix form, we get

$$\left\langle \begin{pmatrix} g(0) \\ g'(0) \\ g(1) \\ g'(1) \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f(0) \\ f'(0) \\ f(1) \\ f'(1) \end{pmatrix} \right\rangle_{\mathbb{C}^4} = 0. \tag{137}$$

In an algebraic interpretation of this, this means that

$$V \equiv \{(f(0), f'(0), f(1), f'(1)) \in \mathbb{C}^4 \quad : \quad f \in D(A)\} \tag{138}$$

is an isotropic subspace of the matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{139}$$

Remark.
As a small reminder. An isotropic subspace V of a matrix M is a subspace of the considered vector space such that $\langle v, Mv \rangle = 0$ for all $v \in V$.
Now, by polarisation, this can be extended to $\langle w, Mv \rangle = 0$ for all $v, w \in V$.

Naturally, this tells us to define the operator,

$$A_V : f \mapsto -f'', \quad \text{with} \quad D(A_V) = \{f \in H^2(]0,1[) : (f(0), f'(0), f(1), f'(1)) \in V\} \tag{140}$$

for all isotropic subspace V of the matrix M.

Remark.
It may be worth noting that when we talk about the isotropic subspace of a matrix, we mean the isotropic subspace of the symplectic form associated to the matrix.
However, for the sake of simplicity we say the isotropic subspace of a matrix.

Since $\langle v, Mv \rangle_{\mathbb{C}^4}$ for all $v \in V$, we can see that $V \subset (MV)^{\perp}$.
Furthermore, since M is invertible,

$$\dim((MV)^{\perp}) = 4 - \dim(MV) = 4 - \dim(V) \geq \dim(V) \iff \dim(V) \leq 2. \tag{141}$$

From this, we can get the famous examples of conditions at the border,

Dirichlet	$f(0) = f(1) = 0$	$V = \text{Vect}\{e_2, e_4\}$
Neumann	$f'(0) = f'(1) = 0$	$V = \text{Vect}\{e_1, e_3\}$
Periodic	$f(0) = f(1)$ and $f'(0) = f'(1)$	$V = \text{Vect}\{e_1 + e_3, e_2 + e_4\}$
Robin	$af(0) - bf'(0) = af(1) + bf'(1) = 0$, with $a, b \in \mathbb{R}^2 \backslash (0,0)$	$V = \text{Vect}\{be_1 + ae_2, be_3 - ae_4\}$
Born-von Karman	$e^{i\theta} f(0) - f(1) = e^{i\theta} f'(0) - f'(1) = 0$ with $\theta \in [0, 2\pi[$	$V = \text{Vect}\left\{e_1 + e^{i\theta} e_3, e_2 + e^{i\theta} e_4\right\}$

We will now state the self-adjoint extensions of A.

Theorem 9 (Self-adjoint extensions of the Laplacian on $]0,1[$)**:** *The symmetric extensions of A_0 are exactly the operators A_V as introduced above, as V sweeps all isotropic subspaces of M. The domain of A_V can be written as,*

$$D(A_V) = H_0^2(]0,1[) + \text{Vect}\{f_i\}, \tag{142}$$

where $f_i \in H^2(]0,1[)$ are arbitrary functions chosen such that $(f_i(0), f_i'(0), f_i(1), f_i'(1))$ generates a basis on V.
Then we have,

$$A_V \textit{ is self-adjoint} \iff \dim(V) = 2 \tag{143}$$

When $\dim(V) \in \{0,1\}$, we have that,

$$\sigma(A_V) = \mathbb{C} \tag{144}$$

Proof. The proof to this theorem can be found in the proof of Theorem 2.41 in [Lew21]. □

Interesting operators studied in Spectral Theory

Spectral theory creates an extensive theory to study the spectrums of operators.
In other examples of operators that have an interest in physics, we have the periodic and Born-von Karman Laplacians in dimensions $d \geq 2$, showing interesting boundary conditions,

$$f(x + \ell) = f(x) e^{i\xi\cdot\ell}, \qquad \xi \in (0, 2\pi)^d.$$

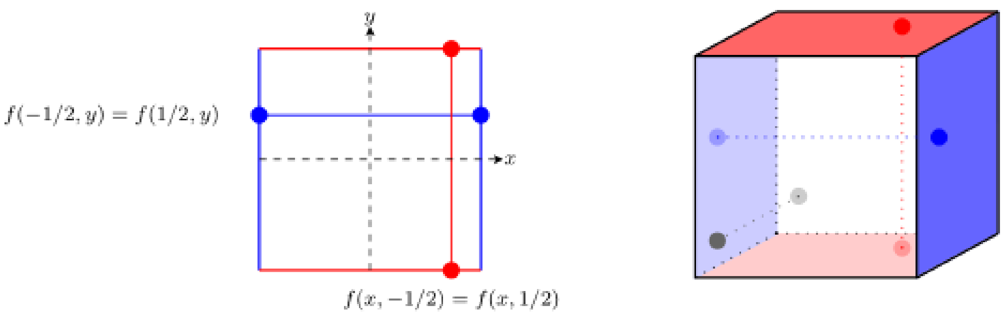

Figure 1: Periodic boundary conditions in dimensions 2 and 3 (Picture from [Lew21]).

CONCLUSION

Spectral Theory creates for a mathematical theory that becomes inevitable when considering the mathematical aspects of operators in Quantum mechanics (see [Gie00]). Indeed, whenever studying position operators or momentum operators, one must keep in mind that the $\mathfrak{H}$ space is infinite dimensional and might lead to contradiction if not considered carefully.
One needs to step carefully when dealing with complicated mathematical objects and conclusions might be more counterintuitive than one might think.
Let us once again look at one of the examples in [Gie00].
Consider the operators $P = \frac{\hbar}{i}\frac{d}{dx}$ and $Q =$ "multiplication by x" acting on wave functions in $\mathbb{R}$. Since P and Q are Hermitian operators, the operator $A = PQ^3 + Q^3P$ has this same property since its adjoint is given by

$$A^\dagger = (PQ^3 + Q^3P)^\dagger = Q^3P + PQ^3 = A. \tag{145}$$

It follows that all eigenvalues of A are real. Nevertheless, one easily verifies that,

$$Af = \frac{\hbar}{i}f \quad \text{with} \quad f(x) = \begin{cases} \frac{1}{\sqrt{2}}|x|^{-3/2}\exp\left(-\frac{1}{4x^2}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}, \tag{146}$$

meaning that A admits the complex eigenvalue $\frac{\hbar}{i}$.
Note that the function f is infinitely differentiable on $\mathbb{R}$ and that it is square integrable.
One might ask where the error arises from.
In fact, from Spectral Theory, we know that the maximal domain for the Momentum operator is $D(P^{\max}) = H^1(\mathbb{R})$.
Moreover, the chosen function $f(x)$ is $L^2(\mathbb{R})$ however it's derivative isn't.

Spectral Theory even let's use study complicated operators such as the Shrödinger operator $-\Delta+V$ with some potential V, giving us a pretty point of view,

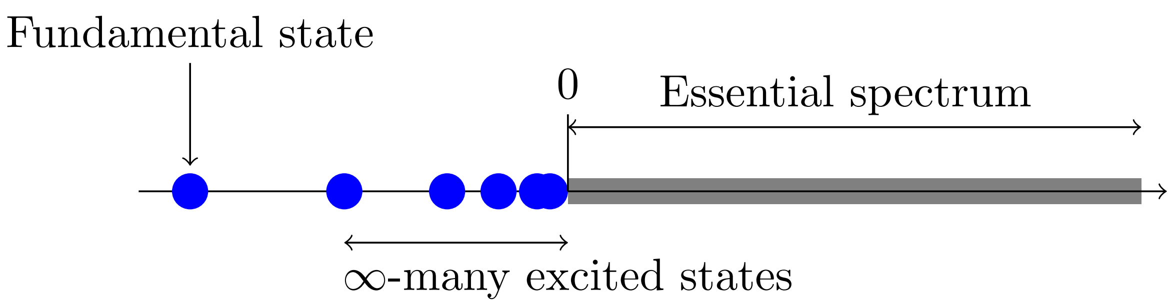

Figure 2: Spectrum of the Friedrichs extension of the Schrödinger operator $-\Delta + V$

APPENDIX A - SOBOLEV SPACES

Sobolev spaces play a major role in analysis, especially in the study of partial differential equation. However, in this case, we use Sobolev spaces as domains for the study of differential operators.

Sobolev spaces on $\mathbb{R}$

Before introducing the notion of Sobolev spaces, we will introduce a notion called "weak-derivatives" which is a generalisation of usual derivatives. As a simplification, we will work with functions taking one dimensional parameters.

Definition 11 (Weak derivative)**:** *Let $f \in L^2(\mathbb{R}, \mu)$. We say that f admits a weak derivative if,*

$$\exists g \in L^2(\mathbb{R}, \mu) \quad \text{such that} \quad \forall \varphi \in \mathcal{C}^1(\mathbb{R}), \quad \int_{\mathbb{R}} f\varphi' \, \mathrm{d}\mu = -\int_{\mathbb{R}} g\varphi \, \mathrm{d}\mu \tag{147}$$

Remark (Existence of such derivative)**.**
In the above definition, all functions are either $\mathcal{C}^1(\mathbb{R})$ or $L^2(\mathbb{R})$ and therefore are integrable.
Hence, the existence of such derivative follows naturally from the integration by parts formulae.
It is worth to note that when $f \in L^2(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$ admits a usual derivative that is $L^2(\mathbb{R})$, it's usual and weak derivative coincide and $f \in H^1(\mathbb{R})$.

Lastly, from the definition we can see that the weak derivative is only defined up to sets of measure zero.

Such a derivative is actually unique.

Lemma 9 (Unicity of the weak derivative)**:** *In the above case where $f \in L^2(\mathbb{R})$ admits a weak derivative, the function g is unique.*
In that case, we denote $f' = g$ the derivative of f.

Remark.
We admit the fact that $\mathcal{C}_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.

Proof. The proof can be found in [Kin21], in the proof of Lemma 1.4.
Let $f \in L^2(\mathbb{R})$. Now, let $g_1, g_2 \in L^2(\mathbb{R})$ be two functions such that,

$$\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}), \quad -\int_{\mathbb{R}} g_1\varphi = \int_{\mathbb{R}} f\varphi' = -\int_{\mathbb{R}} g_2\varphi \tag{148}$$

Then it is given that,

$$\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}), \quad \int_{\mathbb{R}} (g_1 - g_2)\varphi = 0 \tag{149}$$

Now take $g = g_1 - g_2$ and $\varphi \ in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \varphi(x) \, \mathrm{d}x = 1$. The above condition can thus be rewriting as,

$$n \int_{\mathbb{R}} g(x)\varphi(n(y-x)) \, \mathrm{d}x = 0 \qquad \forall y \in \mathbb{R}, \forall n \in \mathbb{N}^* \tag{150}$$

Hence, for $\varphi_n : x \mapsto n\varphi(nx)$, we have that $g * \varphi_n = 0$ for all $n \in \mathbb{N}$.
We will show that $0 = g * \varphi_n \xrightarrow[n\to\infty]{} g$ in $L^2(\mathbb{R})$.

We define

$$g_R = g\,\mathbb{I}_{\mathcal{B}(0,R)}\,\mathbb{I}_{|g|\leq R}$$

for all $R>0$, where $\mathbb{I}$ is the indicator function. Let us prove that $g_R \xrightarrow[R\to\infty]{} g$ in $L^2(\mathbb{R})$.
We can easily see that $\lim_{R\to\infty} g_R(x) = g(x)$ for all $x\in\mathbb{R}$. Furthermore, for any $R>0$ and for any $x\in\mathbb{R}$,

$$|g_R - g|^2 = \left|g\,\mathbb{I}_{\mathcal{B}(0,R)}\,\mathbb{I}_{|g|\leq R} - g\right|^2 \leq 4|g|^2 \tag{151}$$

which is integrable. Therefore, the Dominated Convergence Theorem can be used and,

$$\lim_{R\to\infty} \|g_R - g\|_{L^2}^2 = \lim_{R\to\infty}\int_{\mathbb{R}} |g_R - g|^2 = \int_{\mathbb{R}} \lim_{R\to\infty}\left|g\,\mathbb{I}_{\mathcal{B}(0,R)}\,\mathbb{I}_{|g|\leq R} - g\right|^2 = 0 \tag{152}$$

Thus, $g_R \xrightarrow[R\to\infty]{} g$ in L^2. We can write that, for any $R>0$,

$$\|g*\varphi_n - g\|_{L^2} \leq \|(g-g_R)*\varphi_n\|_{L^2} + \|g_R*\varphi_n - g_R\|_{L^2} + \|g_R - g\|_{L^2} \tag{153}$$

Let us examine the first term on the RHS. Using Young's convolution inequality, we have that,

$$\|(g-g_R)*\varphi_n\|_{L^2} \leq \|g-g_R\|_{L^2}\|\varphi_n\|_{L^1} = \|g-g_R\|_{L^2}\|\varphi\|_{L^1} = \|g-g_R\|_{L^2} \tag{154}$$

Now onto the second term. For all $R>0$, we write

$$\begin{aligned}
\|g_R*\varphi_n - g_R\|_{L^2}^2 &= \int_{\mathbb{R}} \left|(g_R*\varphi_n)(y) - g_R(y)\right|^2 \mathrm{d}y && (155)\\
&= \int_{\mathbb{R}} \left|\int_{\mathbb{R}} g_R(y)\varphi_n(x)\,\mathrm{d}x - g_R(y)\right|^2 \mathrm{d}y && (156)\\
&= \int_{\mathbb{R}} \left|g_R(y)\left(\int_{\mathbb{R}}\varphi_n(x)\,\mathrm{d}x - 1\right)\right|^2 \mathrm{d}y && (157)
\end{aligned}$$

Now, we can easily see that,

$$\int_{\mathbb{R}} \varphi_n(x)\,\mathrm{d}x = \int_{\mathbb{R}} n\varphi(x(x-y))\,\mathrm{d}x = \int_{\mathbb{R}}\varphi(x)\,\mathrm{d}x = 1 \tag{158}$$

independently of y and n. Therefore,

$$\|g_R*\varphi_n - g_R\|_{L^2}^2 = 0 \quad \forall n\geq 0, \quad \forall R\geq 0 \tag{159}$$

To conclude, coming back to (153), we can see that the LHS does not depend on R and taking the limit as $R\to\infty$ on the RHS, we get that,

$$\|g*\varphi_n - g\|_{L^2} = 0 \quad \text{and} \quad g = 0 \iff g_1 = g_2 \tag{160}$$

□

Remark.
This proof did not only prove unicity of the weak derivative but also proved the density of $\mathcal{C}_c^\infty$ in L^2.

Let us define now Sobolev spaces.

Definition 12 (Sobolev spaces)**:** *Let $k \geq 1$. The Sobolev spaces $H^k(\mathbb{R})$ is defined by,*

$$H^k(\mathbb{R}) \equiv \{f \in L^2(\mathbb{R}) \text{ such that } f^{(\alpha)} \in L^2(\mathbb{R}), \quad \forall 0 < \alpha \leq k\} \tag{161}$$

It should be noted that the derivatives in this case are weak derivatives.

Remark.
It can easily be seen that we can equivalently define the Sobolev spaces in the following manner, $\forall k \geq 1$,

$$H^k(\mathbb{R}) \equiv \{f \in H^{k-1}(\mathbb{R}) \text{ such that } f^{(k)} \in L^2(\mathbb{R})\} \tag{162}$$

In this case, the derivative is also to be considered as a weak derivative.

Let us see some properties of such spaces.
As we will see below, the Sobolev spaces have a Hilbert structure.

Theorem 10: *Let us define the inner product on $H^k(\mathbb{R})$,*

$$\langle f, g\rangle_{H^k} \equiv \sum_{\alpha \leq k} \langle f^{(\alpha)}, g^{(\alpha)}\rangle_{L^2} \tag{163}$$

Then, $(H^k(\mathbb{R}), \langle\cdot,\cdot\rangle)$ is a Hilbert space.

Proof. The proof of this Theorem can be found in [Bre05] in the proof of Proposition VIII.1. □

The space of $\mathcal{C}_c^\infty(\mathbb{R})$ functions is dense in Sobolev spaces for the norm induced by the inner product.

Theorem 11 (Meyers-Serrin)**:** *The space $\mathcal{C}_c^\infty(\mathbb{R})$ is dense in $H^k(\mathbb{R})$ for the norm induced by the inner product.*

Proof. We will not prove this theorem and we will consider it as admitted. The proof however, can be found in [Fou03] □

Sobolev spaces on an interval I

Let $I \subset \mathbb{R}$ be an interval. Compared to the definition of Sobolev spaces, we have no conflicts of definition. Hence,

Definition 13 (Sobolev spaces on an interval of $\mathbb{R}$)**:** *Let $k \geq 1$. The Sobolev spaces $H^k(I)$ is defined by,*

$$H^k(I) \equiv \{f \in L^2(I) \text{ such that } \partial^\alpha f \in L^2(I), \quad \forall 0 < \alpha \leq k\} \tag{164}$$

It should be noted that the derivatives in this case are weak derivatives.

Remark.
As a reminder, L^p spaces are defined by the means of equivalence classes, therefore pointwise evaluation of a H^1 function is a bit subtle.

With the help of the following Lemma, we will be able to perform pointwise evaluation of $H^k(I)$ functions.

Theorem 12 (Representation of H^1 functions)**:** *Let $f \in H^1(I)$. Then, there exists a function $\tilde{f} \in \mathcal{C}({}^{cl}I)$ such that $f = \tilde{f}$ almost everywhere on I and we have the relation*

$$\tilde{f}(x) - \tilde{f}(y) = \int_y^x f'(t)\,\mathrm{d}t \quad \forall x, y \in I \tag{165}$$

Proof. The proof can be found in [Bre05] in the proof of Theorem VIII.2. □

Lemma 10 (Elliptical regularity on $(0,1)$)**:** *Let $f \in L^2(]0,1[)$ such that $f'' \in L^2(]0,1[)$. Then, $f' \in \mathcal{C}^0(]0,1[)$ with*

$$\max_{[0,1]} |f'| \leq 6||f||_{L^2(]0,1[)} + 2||f''||_{L^2(]0,1[)} \tag{166}$$

In particular, $f' \in L^2(]0,1[)$ and $f \in H^2(]0,1[)$.

Remark.
With the help of this Lemma, in the case of $H^2(]0,1[)$ functions, we can tell things about f' even though $H^2(]0,1[)$ introduces only f and f''.

Proof. The proof for this Lemma will not be done here as it requires some knowledge of Distribution Theory. However, it can be found in the proof of Lemma A.4 in [Lew21]. □

Lemma 11 (Edge-restriction for $H^1(]0,1[)$)**:** *We have the continuous injection*

$$H^1(]0,1[) \subset \mathcal{C}([0,1]) \tag{167}$$

The edge-restricting application

$$f \in H^1(]0,1[) \mapsto (f(0^+), f(1^-)) \in \mathbb{C}^2 \tag{168}$$

is therefore continuous.

Remark.
This Lemma tells us that in one dimension, $H^1(]0,1[)$ functions are continuous on $\mathcal{C}([0,1])$.

Proof. The proof of this Lemma can be found in [Lew21] in the proof of Lemma A.2. □

With the following two Lemmas, we will introduce the formulae of integration by parts in Sobolev spaces.

Lemma 12: *Suppose I is not bounded and let $u \in H^k(I)$, $k \geq 1$. Then we have that,*

$$\lim_{\substack{x \in I \\ |x| \to \infty}} u(x) = 0 \tag{169}$$

Proof. The proof for this Lemma can be found in the proof of Corollary VIII.8 in [Bre05]. □

Lemma 13 (Derivative of a product)**:** *Let $u, v \in H^1(I)$. Then, $uv \in H^1(I)$ and,*

$$(uv)' = u'v + uv' \tag{170}$$

Moreover, we have the formulae of integration by parts

$$\int_y^x u'v = u(x)v(x) - u(y)v(y) - \int_y^x uv' \quad \forall x, y \in {}^{\mathrm{cl}}I \tag{171}$$

Remark.
First of all, it is worth noting that if I is not bounded, the terms evaluated at the diverging boundary of I cancel out because of Lemma 12.
Secondly, we can extend this integration by parts formulae to H^2 spaces since $f'' = (f')'$ for any function $f \in H^2$.

Proof. The proof of this Lemma can be found in the proof of Corollary VIII.9 in [Bre05]. □

- $H_0^1(I)$ Spaces

We define $H_0^1(I)$ spaces as follows.

Definition 14 ($H_0^1(I)$ Spaces)**:** *Let I be an interval. We define,*

$$H_0^1(I) = \{f \in H_0^1(I) : \quad f = 0 \text{ on } \partial I\} \tag{172}$$

Fourier characterisation of Sobolev spaces

Sobolev spaces can also be characterised in a Fourier basis as we will see below.

Remark.
As a note to the reader. It is worth noting that Fourier transforms are isometric between $L^2(\mathbb{R})$ and $L^2(\mathbb{R})$. When dealing with bounded intervals, the right choice is the Fourier series as they are an isometry from $L^2(\mathbb{R})$ to $L^2(I)$.

Since we have that,

$$\widehat{f^{(n)}(x)} = (ik)^n \widehat{f(x)}, \tag{173}$$

Using the definition of Sobolev spaces, we know that,

$$f \in H^n(\mathbb{R}) \iff f \in L^2(\mathbb{R}) \text{ and } f^{(n)} \in L^2(\mathbb{R}) \tag{174}$$

Now using the fact that $f \in L^2(\mathbb{R})$ if and only if it's Fourier transform $\widehat{f}(k) \in L^2(\mathbb{R})$.
Therefore,

$$f^{(n)} \in L^2(\mathbb{R}) \iff \widehat{f^{(n)}}(k) \in L^2(\mathbb{R}) \iff \int_{\mathbb{R}} \left(|k|^2\right)^n |\widehat{f}(k)|^2 \, \mathrm{d}k < \infty \tag{175}$$

Combining this with the fact that,

$$f \in L^2(\mathbb{R}) \iff \int_{\mathbb{R}} |\widehat{f}(k)|^2 \, \mathrm{d}k < \infty \tag{176}$$

We can introduce the following characterisation of Sobolev spaces.

Proposition 5 (Fourier characterisation of Sobolev spaces)**:** *The Sobolev spaces as defined in* (161) *can be redefined in the following way,*

$$H^n(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \quad \int_{\mathbb{R}} (1 + |k|^2)^n |\widehat{f}(k)|^2 \, \mathrm{d}k < \infty \right\} \tag{177}$$

Remark.
Using this equation, we can extend the definition of $H^k(\mathbb{R}^N)$ to the case where k is not an integer by setting,

$$H^s(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \quad \int_{\mathbb{R}} (1+|k|^2)^s |\hat{f}(k)|^2 \, \mathrm{d}k < \infty \right\} \tag{178}$$

for all $s > 0$.

Theorem 13 (Density of $\mathcal{C}_c^\infty(\mathbb{R})$)**:** *The space $\mathcal{C}_c^\infty(\mathbb{R})$ is dense in $H^s(\mathbb{R})$ for all $s > 0$.*

Proof. This proof can be found in [Lew21]. The idea of the proof is as follows.
We prove first that $\mathcal{C}^\infty(\mathbb{R})$ is dense in $H^s(\mathbb{R})$.
To do so, we construct the function $f_n = f * \chi$ for all $n \in \mathbb{N}$, for any $f \in H^s(\mathbb{R})$ and $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \chi = 1$ and prove that it converges to f in $H^s(\mathbb{R})$.
Finally, we prove the case of the $\mathcal{C}_c^\infty(\mathbb{R})$ functions by introducing $\tilde{f}_n(x) = f(x)\chi(x/n)$ which is $\mathcal{C}_c^\infty(\mathbb{R})$ and proving that it converges towards f. $\square$

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