COMPSCI 611 Advanced Algorithm

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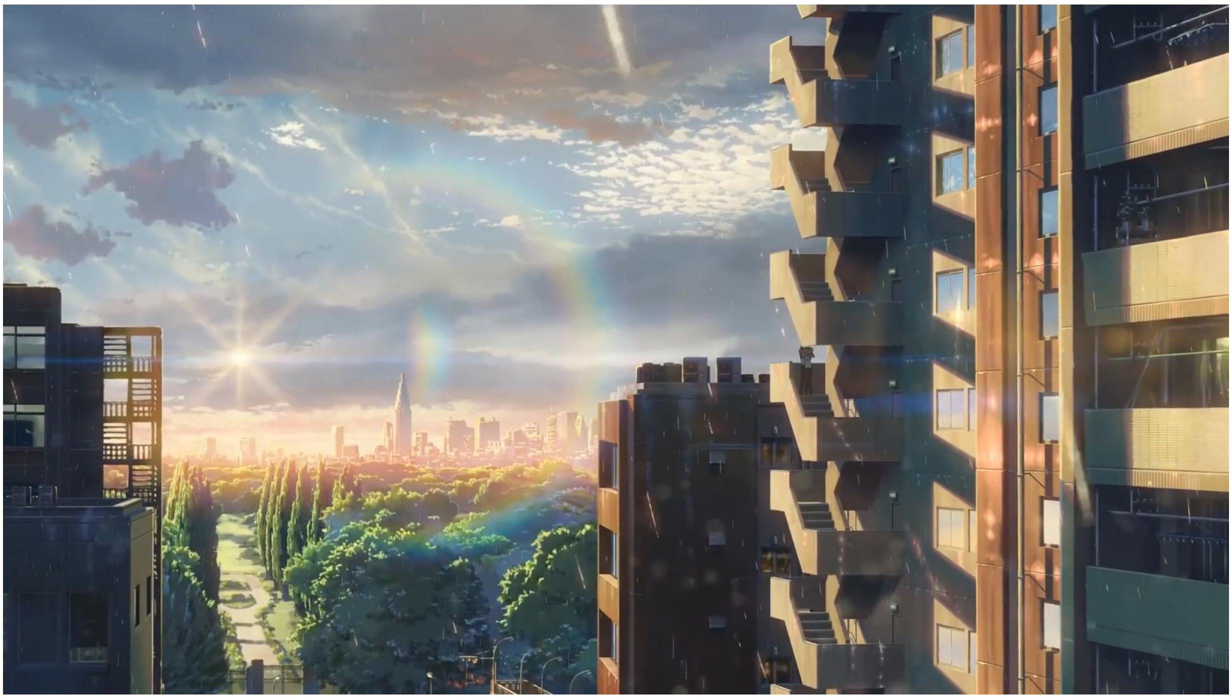

Honest or Liar

There is a group of n people in which more than $n/2$ people are honest (and others are liars). You want to identify honest people in the group by paring them up. For each pair (A, B), they will talk and assess each other. If A says that B is honest and B says that A is honest, then you know that either both of them are honest or both of them are liars. For other combinations of the answers, you know that at least one of them is a liar.

a) Show that to identify an honest person, $\lfloor n/2 \rfloor$ pairing tests are sufficient to reduce the problem size by at least $\lfloor n/2 \rfloor$.

Let P be the set of n people to be tested. First, we make b $\lfloor n/2 \rfloor$ pairs of people and let people in each pair assess each other. (There will be one person leftover if n is odd; for now, we just focus on the pairs.) We then throw out every pair whose answer is not (honest, honest); at least one person in this kind of pair is a liar. Let X be the number of remaining pairs. We form a new set Q by picking exactly one (arbitrary) person per pair of X. At this point, we know that Q has at least $|Q|/2$ honest people, but we need more than $|Q|/2$ honest people.

Observe that if Q has exactly $|Q|/2$ honest people, then n is odd and the leftover person is honest. This is because initially there are more than $n/2$ honest people, and that we threw out as many honest people as liars. (But we cannot immediately add the leftover person, if any, to Q.) We consider two cases:

- $|X|$ is odd. There must be strictly more honest pairs than liar pairs. Thus, Q has more than honest people. We do not need to add the leftover person. (Indeed, we could not add, since otherwise, we could possibly increase the number of liars in Q.)

- $|X|$ is even. We add the leftover person to Q. There cannot be more liar pairs than honest pairs in X, since otherwise, even if the leftover person is honest, there are more liars than honest people initially. If the number of liar pairs is strictly less than the number of honest pairs, then the number of honest pairs is at least 2 more than the number of liar pairs since $|X|$ is even. Thus, even if the leftover person is a liar, the number of honest people is still strictly more than the number of liars in Q. Finally, if the number of liar pairs is equal to the number of honest pairs, then the leftover person must be honest. Thus, the number of honest people is still strictly more than the number of liars in Q.

Clearly, the number of tests is $\lfloor n/2 \rfloor$, and for every pair, we throw out at least 1 person. Thus, $|Q| \leq |P| - \lfloor n/2 \rfloor$ as desired.

b) Show that to identify all honest people, it suffices to use O(n) pairing tests.

We apply the strategy in (a) to reduce the problem size until $n \leq 2$. Then pick any person among (at most 2) remaining people, this person is guaranteed to be honest. The number of test we make is at most:

$$\frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \ldots \leq n$$

We pair the honest person to each person in the original set P, which costs $n - 1$ tests. Based on the assessments of the honest person, we can identify all honest people. Thus, the total number of tests is $n + n - 1 = O(n)$.