

Fundamental Limits for Sensor-Based Robot Control

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Abstract—Our goal is to develop theory and algorithms for establishing *fundamental limits* on performance imposed by a robot’s sensors for a given task. In order to achieve this, we define a quantity that captures the amount of *task-relevant information* provided by a sensor. Using a novel version of the generalized Fano inequality from information theory, we demonstrate that this quantity provides an upper bound on the highest achievable expected reward for one-step decision making tasks. We then extend this bound to multi-step problems via a dynamic programming approach. We present algorithms for numerically computing the resulting bounds, and demonstrate our approach on three examples: (i) the lava problem from the literature on partially observable Markov decision processes, (ii) an example with continuous state and observation spaces corresponding to a robot catching a freely-falling object, and (iii) obstacle avoidance using a depth sensor with non-Gaussian noise. We demonstrate the ability of our approach to establish strong limits on achievable performance for these problems by comparing our upper bounds with achievable lower bounds (computed by synthesizing or learning concrete control policies).

I. INTRODUCTION

Robotics is often characterized as the problem of transforming “pixels to torques” [1]: how can an embodied agent convert raw sensor inputs into actions in order to accomplish a given task? In this paper, we seek to understand the *fundamental limits* of this process by studying the following question: is there an *upper bound* on performance imposed by the sensors that a robot is equipped with?

As a motivating example, consider the recent debate around the “camera-only” approach to autonomous driving favored by Tesla versus the “sensor-rich” philosophy pursued by Waymo [2]. Is an autonomous vehicle equipped only with cameras *fundamentally limited* in terms of the collision rate it can achieve? By “fundamental limit”, we mean a bound on performance or safety on a given task that holds *regardless* of the form of control policy one utilizes (e.g., a neural network with billions of parameters or a nonlinear model predictive control scheme combined with a particle filter), how the policy is synthesized (e.g., via model-free reinforcement learning or model-based control), or how much computation is available to the robot or software designer.

While there have been tremendous algorithmic advancements in robotics over decades, we currently lack a “science” for understanding such fundamental limits [3]. Current practice in robotics is often *empirical* in nature. For example, practitioners today often implement a process of trial and error with different

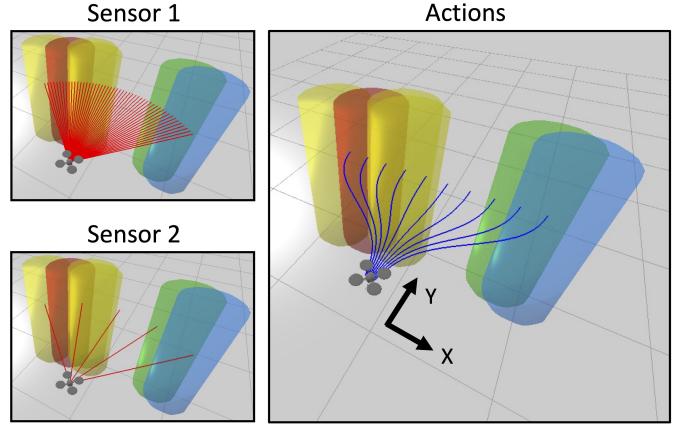


Fig. 1. Our goal is to establish fundamental limits on performance for a given task imposed by a robot’s sensors. We propose a quantity that captures the amount of *task-relevant information* provided by a sensor and use it to bound the highest achievable performance (expected reward) using the sensor. We demonstrate our approach on examples that include obstacle avoidance with a noisy depth sensor (left figure) using motion primitives (right figure). Our framework also allows us to establish the superiority of one sensor (Sensor 1: a dense depth sensor) over another (Sensor 2: a sparse depth sensor) for a given task.

perception and control architectures that use neural networks of varying sizes and architectures. Techniques for establishing fundamental limits imposed by a sensor would potentially allow us to glean important design insights such as realizing that a particular sensor is not sufficient for a task and must be replaced. Further, such techniques could allow us to establish the superiority of one suite of sensors over another from the perspective of a given task; this may be achieved by synthesizing a control policy for one sensor suite that achieves better performance than the fundamental bound for another suite.

In this paper, we take a step towards these goals. We consider settings where the robot’s task performance or safety may be quantified via a reward function. We then observe that any technique for establishing fundamental bounds on performance imposed by a given sensor must take into account two factors: (i) the *quality* of the sensor as measured by the amount of information about the state of the robot and its environment provided by the sensor, and, importantly, (ii) the *task* that the robot is meant to accomplish. As an example, consider a drone equipped with a (noisy) depth sensor (Figure 1). Depending on the nature of the task, the robot may need more

or less information from its sensors. For example, suppose that the obstacle locations are highly constrained such that a particular sequence of actions always succeeds in avoiding them (i.e., there is a purely open-loop policy that achieves good performance on the task); in this case, even an extremely noisy or sparse depth sensor allows the robot to perform well. However, if the distribution of obstacles is such that there is no pre-defined gap in the obstacles, then a noisy or sparse depth sensor may fundamentally limit the achievable performance on the task. The achievable performance is thus intuitively influenced by the amount of *task-relevant information* provided by the robot’s sensors.

Statement of contributions. Our primary contribution is to develop theory and algorithms for establishing fundamental bounds on performance imposed by a robot’s sensors for a given task. Our key insight is to define a quantity that captures the *task-relevant information* provided by the robot’s sensors. Using a novel version of the *generalized Fano’s inequality* from information theory, we demonstrate that this quantity provides a fundamental upper bound on expected reward for one-step decision making problems. We then extend this bound to multi-step settings via a dynamic programming approach and propose algorithms for computing the resulting bounds for systems with potentially continuous state and observation spaces, nonlinear and stochastic dynamics, and non-Gaussian sensor models (but with discretized action spaces). We demonstrate our approach on three examples: (i) the lava problem from the literature on partially observable Markov decision processes (POMDPs), (ii) a robot catching a freely-falling object, and (iii) obstacle avoidance using a depth sensor (Figure 1). We demonstrate the strength of our upper bounds on performance by comparing them against *lower bounds* on the best achievable performance obtained from concrete control policies: the optimal POMDP solution for the lava problem, a model-predictive control (MPC) scheme for the catching example, and a learned neural network policy and heuristic planner for the obstacle avoidance problem. We also present applications of our approach for establishing the superiority of one sensor over another from the perspective of a given task. To our knowledge, the results in this paper are the first to provide general-purpose techniques for establishing fundamental bounds on performance for sensor-based control of robots.

A preliminary version of this work was published in the proceedings of the Robotics: Science and Systems (RSS) conference [4]. In this significantly extended and revised version, we additionally present: (i) the extension of the definition of the task-relevant information potential (TRIP) from using KL-divergence only to the more general f -divergence (Section IV), (ii) the generalization of the single-step performance bound with the extended TRIP definition (Theorem 1), (iii) the generalization of the upper bound of performance for multi-step problems (Theorem 2), (iv) the application of the generalized bounds to the multi-step lava problem (Section VII-A), (v) a method for optimizing the upper bound by varying the function used to define the f -divergence in the multi-step lava problem (Section VII-A), and (vi) results demonstrating that

our novel version of the generalized Fano’s inequality results in tighter bounds as compared to the original generalized Fano’s inequality [5, 6] (Section VII-A).

A. Related Work

Domain-specific performance bounds. Prior work in robotics has established fundamental bounds on performance for particular problems. For example, [7] and [8] consider high-speed navigation through an ergodic forest consisting of randomly-placed obstacles. Results from percolation theory [9] are used to establish a critical speed beyond which there does not exist (with probability one) an infinite collision-free trajectory. The work by [10] establishes limits on the speed at which a robot can navigate through unknown environments in terms of perceptual latency. Classical techniques from robust control [11] have also been utilized to establish fundamental limits on performance for control tasks (e.g., pole balancing) involving linear output-feedback control and sensor noise or delays [12]. The results obtained by [13] demonstrate empirical correlation of the complexity metrics presented in the work by [12] with sample efficiency and performance of learned perception-based controllers on a pole-balancing task. The approaches mentioned above consider specific tasks such as navigation in ergodic forests, or relatively narrow classes of problems such as linear output-feedback control. In contrast, our goal is to develop a general and broadly applicable theoretical and algorithmic framework for establishing fundamental bounds on performance imposed by a sensor for a given task.

Comparing sensors. The notion of a *sensor lattice* was introduced by [14, 15] for comparing the power of different sensors. The works by [16] and [17] present similar approaches for comparing robots, sensors, and actuators. The sensor lattice provides a partial ordering on different sensors based on the ability of one sensor to simulate another. However, most pairs of sensors are *incomparable* using such a scheme. Moreover, the sensor lattice does not establish the superiority of one sensor over another from the perspective of a given task; instead, the partial ordering is based on the ability of one sensor to perform as well as another in terms of filtering (i.e., state estimation). In this paper, we also demonstrate the applicability of our approach for comparing different sensors. However, this comparison is *task-driven*; we demonstrate how one sensor can be proved to be fundamentally better than another from the perspective of a given task, without needing to estimate states irrelevant to the task.

Fano’s inequality and its extensions. In its original form, *Fano’s inequality* [18] relates the lowest achievable error of estimating a signal x from an observation y in terms of the noise in the channel that produces observations from signals. In recent years, Fano’s inequality has been significantly extended and applied for establishing fundamental limits for various statistical estimation problems, e.g., lower bounding the Bayes and minimax risks for different learning problems [5, 6, 19]. In this paper, we build on generalized versions of Fano’s inequality [5, 6] in order to obtain fundamental bounds on performance for robotic systems with noisy sensors. On the technical front,

we contribute by deriving a stronger version of the generalized Fano's inequalities presented by [5, 6] by utilizing the *inverse of the f -divergence* (Section III) and computing it using *convex programming* [20, Ch. 4]. The resulting inequality, which may be of independent interest, allows us to derive fundamental upper bounds on performance for one-step decision making problems. We then develop a dynamic programming approach for recursively applying the generalized Fano inequality in order to derive bounds on performance for multi-step problems.

II. PROBLEM FORMULATION

A. Notation

We denote sequences by $x_{i:j} := (x_k)_{k=i}^j$ for $i \leq j$. We use abbreviations \inf (infimum) and \sup (supremum), and also use the abbreviations LHS (left hand side) and RHS (right hand side) for inequalities. Conditional distributions are denoted as $p(x|y)$. Expectations are denoted as $\mathbb{E}[\cdot]$ with the variable of integration or its measure appearing below it for contextual emphasis, e.g.: $\mathbb{E}_x[\cdot]$, $\mathbb{E}_{p(x)}[\cdot]$. Expectations with multiple random variables are denoted as $\mathbb{E}_{x,y}[\cdot]$ or $\mathbb{E}_{p(x),p(y)}[\cdot]$, while conditional expectations are denoted as $\mathbb{E}_{x|y}[\cdot]$ or $\mathbb{E}_{p(x|y)}[\cdot]$.

B. Problem Statement

We denote the state of the robot and its environment at time-step t by $s_t \in \mathcal{S}$. Let p_0 denote the initial state distribution. Let the robot's sensor observation and control action at time-step t be denoted by $o_t \in \mathcal{O}$ and $a_t \in \mathcal{A}$ respectively. Denote the stochastic dynamics of the state by $p_t(s_t|s_{t-1}, a_{t-1})$ and suppose that the robot's sensor is described by $\sigma_t(o_t|s_t)$. We note that this formulation handles multiple sensors by concatenating the outputs of different sensors into o_t . In this work, we assume that the robot's task is prescribed using reward functions $r_0, r_1, \dots, r_{T-1} : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$ at each time-step (up to a finite horizon). We use R to represent cumulative expected reward over a time horizon denoted with a subscript. In subsequent sections, we use superscript \star to denote optimality, and use superscript \perp to represent reward achieved by open-loop policies.

Assumption 1 (Bounded rewards). *We assume that rewards are bounded, and without further loss of generality we assume that $r_t(s_t, a_t) \in [0, 1]$, $\forall s_t \in \mathcal{S}, a_t \in \mathcal{A}, t \in \{0, \dots, T-1\}$.*

The robot's goal is to find a potentially time-varying and history-dependent control policy $\pi_t : \mathcal{O}^{t+1} \rightarrow \mathcal{A}$ that maps observations $o_{0:t}$ to actions in order to maximize the total expected reward:

$$R_{0 \rightarrow T}^* := \sup_{\pi_{0:T-1}} \mathbb{E}_{\substack{s_{0:T-1} \\ o_{0:T-1}}} \left[\sum_{t=0}^{T-1} r_t(s_t, \pi_t(o_{0:t})) \right]. \quad (1)$$

Goal: Our goal is to *upper bound* the best achievable expected reward $R_{0 \rightarrow T}^*$ for a given sensor $\sigma_{0:T-1}$. We note that we are allowing for completely general policies that are arbitrary time-varying functions of the entire history of observations received up to time t (as long as the functions satisfy measurability

conditions that ensure the existence of the expectation in (1)). An upper bound on $R_{0 \rightarrow T}^*$ thus provides a fundamental bound on achievable performance that holds regardless of how the policy is parameterized (e.g., via neural networks or receding-horizon control architectures) or synthesized (e.g., via reinforcement learning or optimal control techniques).

III. BACKGROUND

In this section, we briefly introduce some background material that will be useful throughout the paper.

A. KL Divergence, f -Divergence, and f -informativity

The Kullback-Leibler (KL) divergence between two distributions, $p(x)$ and $q(x)$, is defined as:

$$\mathbb{D}(p(x) \parallel q(x)) := \mathbb{E}_{p(x)} \left[\log \frac{p(x)}{q(x)} \right]. \quad (2)$$

This definition can be extended to the more general notion of f -divergence:

$$\mathbb{D}_f(p(x) \parallel q(x)) := \mathbb{E}_{q(x)} \left[f \left(\frac{p(x)}{q(x)} \right) \right], \quad (3)$$

where f is a convex function on \mathbb{R}^+ , and $f(1) = 0$. The KL divergence is a special case of the f -divergence with $f(x) = x \log x$.

The f -informativity [21] between two random variables is defined as:

$$\mathbb{I}_f(x; y) := \inf_{q(y) p(x)} \mathbb{E} \left[\mathbb{D}_f(p(y|x) \parallel q(y)) \right], \quad (4)$$

where $p(y|x)$ is the conditional distribution of y on x , $q(y)$ is any probability distribution on the random variable y , $p(x, y)$ is the joint distribution, and $p(x)$ and $p(y)$ are the resulting marginal distributions. When the subscript f is dropped, \mathbb{I} is simply the Shannon mutual information (i.e., $f(x) = x \log x$ is assumed). The f -informativity captures the amount of information obtained about a random variable (e.g., the state s_t) by observing another random variable (e.g., sensor observations o_t).

B. Inverting Bounds on the f -divergence

Let \mathcal{B}_p and \mathcal{B}_q be Bernoulli distributions on $\{0, 1\}$ with mean p and q respectively. For $p, q \in [0, 1]$, we define:

$$\mathbb{D}_{f,\mathcal{B}}(p \parallel q) := \mathbb{D}_f(\mathcal{B}_p \parallel \mathcal{B}_q) = qf\left(\frac{p}{q}\right) + (1-q)f\left(\frac{1-p}{1-q}\right).$$

In Section IV, we will obtain bounds on the single-step best achievable expected reward $R_{0 \rightarrow 1}^* \in [0, 1]$ through bounds that take the form: $\mathbb{D}_{f,\mathcal{B}}(R_{0 \rightarrow 1}^* \parallel q) \leq c$ for some $q \in [0, 1]$ and an upper bound $c \geq 0$ on $\mathbb{D}_{f,\mathcal{B}}(R_{0 \rightarrow 1}^* \parallel q)$. In order to upper bound $R_{0 \rightarrow 1}^*$, we will use the *f -divergence inverse* (f -inverse for short):

$$\mathbb{D}_f^{-1}(q|c) := \sup \{p \in [0, 1] \mid \mathbb{D}_{f,\mathcal{B}}(p \parallel q) \leq c\}. \quad (5)$$

It is then easy to see that $R_{0 \rightarrow 1}^* \leq \mathbb{D}_f^{-1}(q|c)$.

Since $\mathbb{D}_{f,B}(\cdot\|\cdot)$ is jointly convex in both arguments [22], the optimization problem in (5) is a convex problem. One can thus compute the f -inverse efficiently using a *convex program* [20, Ch. 4] with a single decision variable p .

IV. PERFORMANCE BOUND FOR SINGLE-STEP PROBLEMS

In this section, we will derive an upper bound on the best achievable reward $R_{0 \rightarrow 1}^*$ in the single time-step decision-making setting. This bound will then be extended to the multi-step setting in Section V.

When $T = 1$, our goal is to upper bound the following quantity:

$$R_{0 \rightarrow 1}^*(\sigma_0; r_0) := \sup_{\pi_0} \mathbb{E}_{s_0, o_0} [r_0(s_0, \pi_0(o_0))] \quad (6)$$

$$= \sup_{\pi_0} \mathbb{E}_{p_0(s_0)} \mathbb{E}_{\sigma_0(o_0|s_0)} [r_0(s_0, \pi_0(o_0))]. \quad (7)$$

The notation $R_{0 \rightarrow 1}^*(\sigma_0; r_0)$ highlights the dependence of the best achievable reward in terms of the robot's sensor and task (as specified by the reward function). As highlighted in Section I, the amount of information that the robot requires from its sensors in order to obtain high expected reward depends on its task; certain tasks may admit purely open-loop policies that obtain high rewards, while other tasks may require high-precision sensing of the state. We formally define a quantity that captures this intuition and quantifies the *task-relevant information* provided by the robot's sensors. We then demonstrate that this quantity provides an upper bound on $R_{0 \rightarrow 1}^*(\sigma_0; r_0)$.

Definition 1 (Task-relevant information potential). *Let $\mathbb{I}_f(o_0; s_0)$ be the f -informativity between the robot's sensor observation and state. Define:*

$$R_{0 \rightarrow 1}^\perp := \sup_{a_0} \mathbb{E}_{s_0} [r_0(s_0, a_0)] \quad (8)$$

as the highest achievable reward using an open-loop policy. Then define the task-relevant information potential (TRIP) of a sensor σ_0 for a task specified by reward function r_0 as:

$$\tau(\sigma_0; r_0) := \mathbb{D}_f^{-1}(R_{0 \rightarrow 1}^\perp | \mathbb{I}_f(o_0; s_0)). \quad (9)$$

Remark 1. The TRIP depends on the specific choice of f one uses. In Section VII, we will empirically compare the usefulness of different choices of commonly used functions f (Table I) from the perspective of establishing fundamental limits.

In order to interpret the TRIP, we state two useful properties of the f -inverse.

Proposition 1 (Monotonicity of f -inverse). *The f -inverse $\mathbb{D}_f^{-1}(q|c)$ is:*

- 1) *monotonically non-decreasing in $c \geq 0$ for fixed $q \in [0, 1]$,*
- 2) *monotonically non-decreasing in $q \in [0, 1]$ for fixed $c \geq 0$.*

Proof: The first property follows from the fact that increasing c loosens the f -divergence constraint in the optimization

problem in (5). The proof of the second property is provided in Appendix A (Proposition1). ■

The TRIP $\tau(\sigma_0; r_0)$ depends on two factors: the f -informativity $\mathbb{I}_f(o_0; s_0)$ (which depends on the robot's sensor) and the best reward $R_{0 \rightarrow 1}^\perp$ achievable by an open-loop policy (which depends on the robot's task). Using Proposition 1, we see that as the sensor provides more information about the state (i.e., as $\mathbb{I}_f(o_0; s_0)$ increases for fixed $R_{0 \rightarrow 1}^\perp$), the TRIP is monotonically non-decreasing. Moreover, the TRIP is a monotonically non-decreasing function of $R_{0 \rightarrow 1}^\perp$ for fixed $\mathbb{I}_f(o_0; s_0)$. This qualitative dependence is intuitively appealing: if there is a good open-loop policy (i.e., one that achieves high reward), then the robot's sensor can provide a small amount of information about the state and still lead to good overall performance. The specific form of the definition of TRIP is motivated by the result below, which demonstrates that the TRIP upper bounds the best achievable expected reward $R_{0 \rightarrow 1}^*(\sigma_0; r_0)$ in Equation (6).

Theorem 1 (Single-step performance bound). *The best achievable reward is upper bounded by the task-relevant information potential (TRIP) of a sensor:*

$$\tau(\sigma_0; r_0) \geq R_{0 \rightarrow 1}^*(\sigma_0; r_0) = \sup_{\pi_0} \mathbb{E}_{s_0, o_0} [r_0(s_0, \pi_0(o_0))]. \quad (10)$$

Proof: The proof is provided in Appendix A and is inspired by the proof of the generalized Fano inequality presented in Proposition 14 in the work by [6]. The bound (10) tightens the generalized Fano inequality [5, 6] by utilizing the f -inverse (in contrast to the methods presented by [5] and [6], which may be interpreted as indirectly bounding the f -inverse). The result presented here may thus be of independent interest. ■

Theorem 1 provides a *fundamental bound* on performance (in the sense of Section I) imposed by the sensor for a given single-step task. This bound holds for *any* policy, independent of its complexity or how it is synthesized or learned. Since the TRIP depends on the choice of f -divergence, the bound may be tightened by judiciously choosing f ; we investigate this empirically in Section VII.

V. PERFORMANCE BOUND FOR MULTI-STEP PROBLEMS: FANO'S INEQUALITY WITH FEEDBACK

In this section, we derive an upper bound on the best achievable reward $R_{0 \rightarrow T}^*$ defined in (1) for the general multi time-step setting. The key idea is to extend the single-step bound from Theorem 1 using a dynamic programming argument.

Let $\pi_t^k: \mathcal{O}^{k-t+1} \rightarrow \mathcal{A}$ denote a policy that takes as input the sequence of observations $o_{t:k}$ from time-step t to k (for $k \geq t$). Thus, a policy π_0^k at time-step k utilizes all observations received up to time-step k . Given an initial state distribution p_0 and an open-loop action sequence $a_{0:t-1}$, define the reward-to-go from time $t \in \{0, \dots, T-1\}$ given $a_{0:t-1}$ as:

$$R_{t \rightarrow T} := \mathbb{E}_{s_{t:T-1}, o_{t:T-1} | a_{0:t-1}} \left[\sum_{k=t}^{T-1} r_k(s_k, \pi_t^k(o_{t:k})) \right], \quad (11)$$

where the expectation,

$$\mathbb{E}_{\substack{s_{t:T-1}, o_{t:T-1} \\ a_{0:t-1}}} [\cdot] \quad (12)$$

is taken with respect to the distribution of states $s_{t:T-1}$ and observations $o_{t:T-1}$ one receives if one propagates p_0 using the open-loop sequence of actions from time-steps 0 to $t-1$, and then applies the closed-loop policies $\pi_t^t, \pi_{t+1}^{t+1}, \dots, \pi_{T-1}^{T-1}$ from time-steps t to $T-1$. We further define $R_{T \rightarrow T} := 0$.

Now, for $t \in \{0, \dots, T-1\}$, define:

$$R_{t \rightarrow T}^\perp := \sup_{a_t} \left[\mathbb{E}_{s_t | a_{0:t-1}} [r_t(s_t, a_t)] + R_{t+1} \right], \quad (14)$$

and

$$R_{t \rightarrow T}^{\perp \star} := \sup_{\pi_{t+1}^{t+1}, \dots, \pi_{T-1}^{T-1}} R_{t \rightarrow T}^\perp. \quad (15)$$

The following result then leads to a recursive structure for computing an upper bound on $R_{0 \rightarrow T}^*$.

Proposition 2 (Recursive bound). *For any $t = 0, \dots, T-1$, the following inequality holds for any open-loop sequence of actions $a_{0:t-1}$:*

$$\sup_{\pi_t^t, \dots, \pi_{T-1}^{T-1}} R_{t \rightarrow T} \leq \underbrace{(T-t) \cdot \mathbb{D}_f^{-1} \left(\frac{R_{t \rightarrow T}^{\perp \star}}{T-t} \mid \mathbb{I}_f(o_t; s_t) \right)}_{=: \tau_t(\sigma_{t:T-1}; r_{t:T-1})}. \quad (16)$$

Proof: The proof follows a similar structure to Theorem 1 and is presented in Appendix A. \blacksquare

To see how we can use Proposition 2, we first use (1) and (11) to note that the LHS of (16) for $t = 0$ is equal to $R_{0 \rightarrow T}^*$:

$$R_{0 \rightarrow T}^* = \sup_{\pi_0^0, \dots, \pi_{T-1}^{T-1}} R_{0 \rightarrow T} \leq \tau_0(\sigma_{0:T-1}; r_{0:T-1}). \quad (17)$$

The quantity $\tau_t(\sigma_{t:T-1}; r_{t:T-1})$ may be interpreted as a multi-step version of the TRIP from Definition 1 (which again depends on the specific choice of f one uses). This quantity depends on the f -informativity $\mathbb{I}_f(o_t; s_t)$, which is computed using the distribution $p_t(s_t | a_{0:t-1})$ over s_t that one obtains by propagating p_0 using the open-loop sequence of actions $a_{0:t-1}$:

$$\mathbb{I}_f(o_t; s_t) = \inf_q \mathbb{E}_{\substack{s_t \\ a_{0:t-1}}} \mathbb{D}_f(\sigma_t(o_t | s_t) \| q(o_t)). \quad (18)$$

In addition, $\tau_t(\sigma_{t:T-1}; r_{t:T-1})$ depends on $R_{t \rightarrow T}^{\perp \star}$, which is then divided by $(T-t)$ to ensure boundedness between $[0, 1]$ (see Assumption 1). The quantity $R_{t \rightarrow T}^{\perp \star}$ can itself be upper bounded using (16) with $t+1$, as we demonstrate below. Such an upper bound on $R_{t \rightarrow T}^{\perp \star}$ for $t = 0$ leads to an upper bound on $R_{0 \rightarrow T}^*$ using (17) and the monotonicity of the f -inverse (Proposition 1). Applying this argument recursively leads to

¹For $t = 0$, we use the convention that $a_{0:-1}$ is the empty sequence and:

$$\mathbb{E}_{\substack{s_{0:T-1}, o_{0:T-1} \\ a_{0:-1}}} [\cdot] := \mathbb{E}_{s_{0:T-1}, o_{0:T-1}} [\cdot]. \quad (13)$$

Algorithm 1, which computes an upper bound on $R_{0 \rightarrow T}^*$. In Algorithm 1, we use \bar{R} to denote recursively-computed upper bounds on the RHS of (16).

Algorithm 1 Multi-Step Performance Bound

- 1: Initialize $\bar{R}_{T \rightarrow T}(a_{0:T-1}) = 0, \forall a_{0:T-1}$.
 - 2: **for** $t = T-1, T-2, \dots, 0$ **do**
 - 3: $\forall a_{0:t-1}$, compute:
 $\bar{R}_{t \rightarrow T}(a_{0:t-1}) := (T-t) \cdot \mathbb{D}_f^{-1} \left(\frac{\bar{R}_{t \rightarrow T}^{\perp \star}}{T-t} \mid \mathbb{I}_f(o_t; s_t) \right)$,
where:
 $\bar{R}_{t \rightarrow T}^{\perp \star} := \sup_{a_t} \mathbb{E}_{s_t | a_{0:t-1}} [r_t(s_t, a_t)] + \bar{R}_{t+1 \rightarrow T}(a_{0:t})$.
 - 4: **end for**
 - 5: **return** $\bar{R}_{0 \rightarrow T}$ (bound on achievable expected reward).
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Theorem 2 (Multi-step performance bound). *Algorithm 1 returns an upper bound on the best achievable reward $R_{0 \rightarrow T}^*$.*

Proof: We provide a sketch of the proof here, which uses backwards induction. In particular, Proposition 2 leads to the inductive step. See Appendix A for the complete proof.

We prove that for all $t = T-1, \dots, 0$,

$$\sup_{\pi_t^t, \dots, \pi_{T-1}^{T-1}} R_{t \rightarrow T} \leq \bar{R}_{t \rightarrow T}(a_{0:t-1}), \quad \forall a_{0:t-1}. \quad (19)$$

Thus, in particular,

$$R_{0 \rightarrow T}^* = \sup_{\pi_0^0, \dots, \pi_{T-1}^{T-1}} R_{0 \rightarrow T} \leq \bar{R}_{0 \rightarrow T}. \quad (20)$$

We prove (19) by backwards induction starting from $t = T-1$. We first prove the base step. Using (16), we obtain:

$$\sup_{\pi_{T-1}^{T-1}} R_{T-1 \rightarrow T} \leq \mathbb{D}_f^{-1} \left(R_{T-1 \rightarrow T}^{\perp \star} \mid \mathbb{I}_f(o_{T-1}; s_{T-1}) \right). \quad (21)$$

Using the fact that $R_{T \rightarrow T} = 0$, we can show that $R_{T-1 \rightarrow T}^{\perp \star} = R_{T-1 \rightarrow T}^{\perp \star}$. Combining this with (21) and the monotonicity of the f -inverse (Proposition 1), we see:

$$\sup_{\pi_{T-1}^{T-1}} R_{T-1 \rightarrow T} \leq \mathbb{D}_f^{-1} \left(\bar{R}_{T-1 \rightarrow T}^{\perp \star} \mid \mathbb{I}_f(o_{T-1}; s_{T-1}) \right) \quad (22)$$

$$= \bar{R}_{T-1 \rightarrow T}(a_{0:T-2}). \quad (23)$$

In order to prove the induction step, suppose that for $t \in \{0, \dots, T-2\}$, we have

$$\sup_{\pi_{t+1}^{t+1}, \dots, \pi_{T-1}^{T-1}} R_{t+1 \rightarrow T} \leq \bar{R}_{t+1 \rightarrow T}(a_{0:t}). \quad (24)$$

We then need to show that

$$\sup_{\pi_t^t, \dots, \pi_{T-1}^{T-1}} R_{t \rightarrow T} \leq \bar{R}_{t \rightarrow T}(a_{0:t-1}). \quad (25)$$

We can use the induction hypothesis (24) to show that $R_{t \rightarrow T}^{\perp \star} \leq \bar{R}_{t \rightarrow T}^{\perp \star}$. Combining this with (16) and the monotonicity of the f -inverse (Proposition 1), we obtain the desired result (25):

$$\sup_{\pi_t^t, \dots, \pi_{T-1}^{T-1}} R_{t \rightarrow T} \leq (T-t) \cdot \mathbb{D}_f^{-1} \left(\frac{\bar{R}_{t \rightarrow T}^{\perp \star}}{T-t} \mid \mathbb{I}_f(o_t; s_t) \right) \quad (26)$$

$$= \bar{R}_{t \rightarrow T}(a_{0:t-1}). \quad (27)$$

\blacksquare

VI. NUMERICAL IMPLEMENTATION

In order to compute the single-step bound using Theorem 1 or the multi-step bound using Algorithm 1, we require the ability to compute (or bound) three quantities: (i) the f -inverse, (ii) the f -informativity $\mathbb{I}_f(o_t; s_t)$, and (iii) the quantity $\bar{R}_{t \rightarrow T}^{\perp \star}$. As described in Section III-B, we can compute the f -inverse efficiently using a convex program [20, Ch. 4] with a single decision variable. There are a multitude of solvers for convex programs including Mosek [23] and the open-source solver SCS [24]. Next, we describe the computation of $\mathbb{I}_f(o_t; s_t)$ and $\bar{R}_{t \rightarrow T}^{\perp \star}$ in different settings.

A. Analytic Computation

In certain settings, one can compute $\mathbb{I}_f(o_t; s_t)$ and $\bar{R}_{t \rightarrow T}^{\perp \star}$ exactly. We discuss two such settings of interest below.

Discrete POMDPs. In cases where the state space \mathcal{S} , action space \mathcal{A} , and observation space \mathcal{O} are finite, one can compute $\mathbb{I}_f(o_t; s_t)$ exactly by propagating the initial state distribution p_0 forward using open-loop action sequences $a_{0:t-1}$ and using the expression (18) (which can be evaluated exactly since we have discrete probability distributions). The expectation term in $\bar{R}_{t \rightarrow T}^{\perp \star}$ can be computed similarly. In addition, the supremum over actions can be evaluated exactly via enumeration.

Linear-Gaussian systems with finite action spaces. One can also perform exact computations in cases where (i) the state space \mathcal{S} is continuous and the dynamics $p_t(s_t|s_{t-1}, a_{t-1})$ are given by a linear dynamical system with additive Gaussian uncertainty, (ii) the observation space \mathcal{O} is continuous and the sensor model $\sigma_t(o_t|s_t)$ is such that the observations are linear (in the state) with additive Gaussian uncertainty, (iii) the initial state distribution p_0 is Gaussian, and (iv) the action space \mathcal{A} is finite. In such settings, one can analytically propagate p_0 forward through open-loop action sequences $a_{0:t-1}$ using the fact that Gaussian distributions are preserved when propagated through linear-Gaussian systems (similar to Kalman filtering [25]). One can then compute $\mathbb{I}_f(o_t; s_t)$ using (18) by leveraging the fact that all the distributions involved are Gaussian, for which KL divergences (and some f -divergences) can be computed in closed form [26]. One can also compute $\bar{R}_{t \rightarrow T}^{\perp \star}$ exactly for any reward function that permits the analytic computation of the expectation term using a Gaussian (e.g., quadratic reward functions); the supremum over actions can be evaluated exactly since \mathcal{A} is finite.

B. Computation via Sampling and Concentration Inequalities

General settings. Next, we consider more general settings with: (i) continuous state and observation spaces, (ii) arbitrary (e.g., non-Gaussian/nonlinear) dynamics $p_t(s_t|s_{t-1}, a_{t-1})$, which are potentially not known analytically, but can be sampled from (e.g., as in a simulator), (iii) arbitrary (e.g., non-Gaussian/nonlinear) sensor $\sigma_t(o_t|s_t)$, but with a probability density function that can be numerically evaluated given any particular state-observation pair, (iv) an arbitrary initial state distribution p_0 that can be sampled from, and (v) a finite action space. Our bound is thus broadly applicable, with the primary

restriction being the finiteness of \mathcal{A} ; we leave extensions to continuous action spaces for future work (see Section VIII).

We first discuss the computation of $\bar{R}_{t \rightarrow T}^{\perp \star}$. Since the supremization over actions can be performed exactly (due to the finiteness of \mathcal{A}), the primary challenge here is to evaluate the expectation:

$$\mathbb{E}_{s_t|a_{0:t-1}} [r_t(s_t, a_t)]. \quad (28)$$

We note that any upper bound on this expectation leads to an upper bound on $\bar{R}_{t \rightarrow T}^{\perp \star}$, and thus a valid upper bound on $R_{0 \rightarrow T}^{\star}$ (due to the monotonicity of the f -inverse; Proposition 1). One can thus obtain a high-confidence upper bound on (28) by sampling states $s_t|a_{0:t-1}$ and using any concentration inequality [27]. In particular, since we assume boundedness of rewards (Assumption 1), we can use Hoeffding's inequality.

Theorem 3 (Hoeffding's inequality [27]). *Let z be a random variable bounded within $[0, 1]$, and let z_1, \dots, z_n denote i.i.d. samples. Then, with probability at least $1 - \delta$ (over the sampling of z_1, \dots, z_n), the following bound holds:*

$$\mathbb{E}[z] \leq \frac{1}{n} \sum_{i=1}^n z_i + \sqrt{\frac{\log(1/\delta)}{2n}}. \quad (29)$$

In our numerical examples (Section VII), we utilize a slightly tighter version of Hoeffding's inequality (see Appendix B).

Next, we discuss the computation of $\mathbb{I}_f(o_t; s_t)$. Again, we note that any upper bound on $\mathbb{I}_f(o_t; s_t)$ yields a valid upper bound on $R_{0 \rightarrow T}^{\star}$ due to the monotonicity of the f -inverse. In general, since f -informativity is the infimum among all distributions $q(o_t)$ over the observations at time t , the marginal distribution $\sigma_t(o_t)$ provides a valid upper bound:

$$\mathbb{I}_f(o_t; s_t) \leq \mathbb{D}_f \left(p_t(s_t|a_{0:t-1}) \sigma_t(o_t|s_t) \| p_t(s_t|a_{0:t-1}) \sigma_t(o_t) \right). \quad (30)$$

For KL divergence and mutual information specifically, we utilize *variational bounds*; in particular, we use the “leave-one-out bound” [28]:

$$\mathbb{I}(o_t; s_t) \leq \mathbb{E} \left[\frac{1}{K} \sum_{i=1}^K \left[\log \frac{\sigma_t(o_t^{[i]}|s_t^{[i]})}{\frac{1}{K-1} \sum_{j \neq i} \sigma_t(o_t^{[j]}|s_t^{[j]})} \right] \right], \quad (31)$$

where the expectation is over size- K batches $\{(s_t^{[i]}, o_t^{[i]})\}_{i=1}^K$ of sampled states $s_t|a_{0:t-1}$ and observations sampled using $\sigma_t(o_t|s_t)$. The quantity $\sigma_t(o_t^{[i]}|s_t^{[i]})$ denotes (with slight abuse of notation) the evaluation of the density function corresponding to the sensor model. Since the bound (31) is in terms of an expectation, one can again obtain a high-confidence upper bound by sampling state-observation batches and applying a concentration inequality (e.g., Hoeffding's inequality if the quantity inside the expectation is bounded).

We note that the overall implementation of Algorithm 1 may involve the application of multiple concentration inequalities (each of which holds with some confidence $1 - \delta_i$). One can obtain the overall confidence of the upper bound on $R_{0 \rightarrow T}^{\star}$ by using a union bound: $1 - \delta = 1 - \sum_i \delta_i$.

C. Tightening the Bound with Multiple Horizons

We end this section by discussing a final implementation detail. Let T denote the time horizon of interest (as in Section II). For any $H \in \{1, \dots, T\}$, one can define

$$R_{0 \rightarrow H}^* := \sup_{\pi_{0:H-1}} \mathbb{E}_{s_{0:H-1} \sim \pi_{0:H-1}} \left[\sum_{t=0}^{H-1} r_t(s_t, \pi_t(o_{0:t})) \right] \quad (32)$$

as the best achievable reward for a problem with horizon H (instead of T). One can then apply Algorithm 1 to compute an upper bound on $R_{0 \rightarrow H}^*$. Since rewards are assumed to be bounded within $[0, 1]$ (Assumption 1), we can observe that $R_{0 \rightarrow T}^* \leq R_{0 \rightarrow H}^* + (T - H)$; this is equivalent to computing the bound using a horizon H and then adding a reward of 1 for times beyond this horizon. In practice, we sometimes find that this bound provides a tighter bound on $R_{0 \rightarrow T}^*$ for some $H < T$ (as compared to directly applying Algorithm 1 with a horizon of T). For our numerical examples, we thus sweep through different values for the horizon H and report the lowest upper bound $R_{0 \rightarrow H}^* + (T - H)$.

VII. EXAMPLES

We demonstrate our approach on three examples: (i) the lava problem from the POMDP literature, (ii) an example with continuous state and observation spaces corresponding to a robot catching a freely-falling object, and (iii) obstacle avoidance using a depth sensor with non-Gaussian noise. We illustrate the strength of our upper bounds on these examples by comparing them against the performance achieved by concrete control policies (i.e., lower bounds on achievable performance). We also demonstrate the applicability of our approach for establishing the superiority of one sensor over another (from the perspective of a given task). Code for all examples can be found at: <https://github.com/irom-lab/performance-limits>.

A. Lava Problem

The first example we consider is the lava problem (Figure 2) [29–31] from the POMDP literature.

Dynamics. The setting consists of five discrete states (Figure 2) and two actions (`left` and `right`). If the robot falls into the lava state, it remains there (i.e., the lava state is absorbing). If the robot attempts to go left from state 1, it remains at state 1. The initial state distribution p_0 is chosen to be uniform over the non-lava states.

Sensor. The robot is equipped with a sensor that provides a noisy state estimate. The sensor reports the correct state (i.e., $o_t = s_t$) with probability p_{correct} , and a uniformly randomly chosen incorrect state with probability $1 - p_{\text{correct}}$.

Rewards. The robot’s objective is to navigate to the goal state (which is an absorbing state), within a time horizon of $T = 5$. This objective is encoded via a reward function $r_t(s_t, a_t)$, which is purely state-dependent. The reward associated with being in the lava is 0; the reward associated with being at the goal is 1; the reward at all other states is 0.1.

Results. An interesting feature of this problem is that it admits a purely *open-loop* policy that achieves a high expected

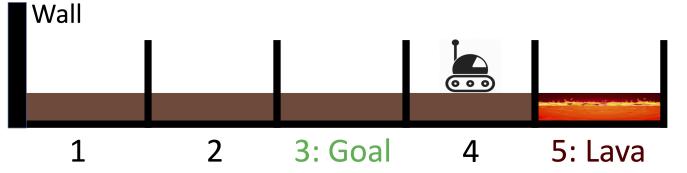


Fig. 2. An illustration of the lava problem. The robot needs to navigate to a goal without falling into the lava (using a noisy sensor).

reward. In particular, consider the following sequence of actions: `left`, `right`, `right`. No matter which state the robot starts from, this sequence of actions will steer the robot to the goal state (recall that the goal is absorbing). Given the initial distribution and rewards above, this open-loop policy achieves an expected reward of 3.425. Suppose we set $p_{\text{correct}} = 1/5$ (i.e., the sensor just uniformly randomly returns a state estimate and thus provides no information regarding the state). In this case, Algorithm 1 returns an upper bound: $3.5 \geq R_{0 \rightarrow T}^*$.

Next, we plot the upper bounds provided by Algorithm 1 for different values of sensor noise by varying p_{correct} . With the freedom to choose the exact f -divergence to use in Algorithm 1, we evaluate a series of common f -divergences shown in Table I.

TABLE I
 f -DIVERGENCES USED TO COMPUTE UPPER BOUNDS.

Divergence	Corresponding f
Kullback-Leibler	$x \log(x)$
Negative log	$-\log(x)$
Total variation	$\frac{1}{2} x - 1 $
Pearson χ^2	$(x - 1)^2$
Jensen-Shannon	$-(x + 1) \ln(\frac{x+1}{2}) + x \ln x$
Squared Hellinger	$(\sqrt{x} - 1)^2$
Neyman χ^2	$\frac{1}{t} - 1$

Three of the resulting bounds are shown in Figure 3. The three bounds are chosen to be plotted because: 1) KL divergence is one of the best-known divergences, 2) Total variation distance provides the tightest bounds among those computed when sensor accuracy is low, and 3) Neyman χ^2 -divergence provides the tightest bounds among computed when sensor accuracy is higher. Since the results for different sensor noise levels are independent of each other, we can always choose the particular f -divergence that returns the tightest bound.

Since the lava problem is a finite POMDP, one can compute $R_{0 \rightarrow T}^*$ exactly using a POMDP solver. Figure 3 compares the upper bounds on $R_{0 \rightarrow T}^*$ returned by Algorithm 1 with $R_{0 \rightarrow T}^*$ computed using the `pomdp_py` package [32]. The figure illustrates that our approach provides strong bounds on the best achievable reward for this problem. We also note that computation of the POMDP solution (for each value of p_{correct}) takes ~ 20 s, while the computation of the bound takes ~ 0.2 s (i.e., $\sim 100 \times$ faster).

Tightening the upper bound. In Figure 3, we note that different f -divergences yield different upper bound values.

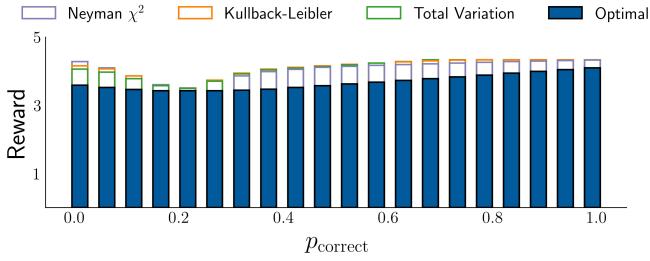


Fig. 3. Results for the lava problem. We compare the upper bounds on achievable expected rewards computed by our approach using three different f -divergences with the optimal POMDP solution for different values of sensor noise.

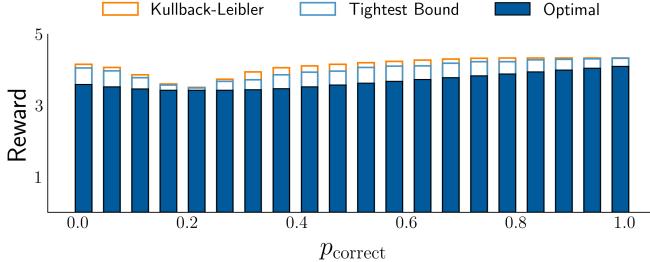


Fig. 4. Tightest upper bounds for the lava problem. We compare the optimized upper bounds on achievable expected rewards with the bounds computed by Algorithm 1 using KL divergence and the optimal POMDP solution.

While all bounds are valid, we are motivated to search for the tightest bound by optimizing the function f . To do so, we define a family of generic piecewise-linear functions that are convex and satisfy $f(1) = 0$; then for a given sensor noise level, we minimize the upper bounds over the defined family of piecewise-linear functions using `scipy.optimize`. To define such a family of functions, we divide the interval $(0, 2]$ into n steps of equal length, each step with a corresponding slope s_i . Since f is a function on \mathbb{R}^+ , we extend the last step interval to positive infinity. The n slopes, along with the constraints that f is convex (therefore continuous) and $f(1) = 0$, complete the definition of f . The results obtained by setting $n = 10$ are shown in Figure 4, and the resulting functions f that give the tightest bounds for the first 15 sensor noise level are shown in Figure 5. We can see that the functions tightening the bounds follow a general trend from more parabolic to more log-like as p_{correct} increases.

Comparing with generalized Fano's inequality. In the work by [5], the generalized Fano's inequality gives a lower bound for the Bayesian risk, which translates to an upper bound for the expected reward of:

$$R_{0 \rightarrow 1}^*(\sigma_0, r_0) \leq \frac{\mathbb{I}(o_0; s_0) + \log(1 + R_{0 \rightarrow 1}^\perp)}{\log(1/(1 - R_{0 \rightarrow 1}^\perp)} \quad (33)$$

We compare the upper bounds obtained via our approach (Theorem 1) using KL divergence with the ones obtained by equation (33) for the single-step lava problem. The results shown in Figure 6 demonstrate that our novel bound in Theorem 1 is significantly tighter, especially for larger values of p_{correct} . Indeed, the bound from Theorem 1 is *always* at least as tight as inequality (33) (not just for this specific example).

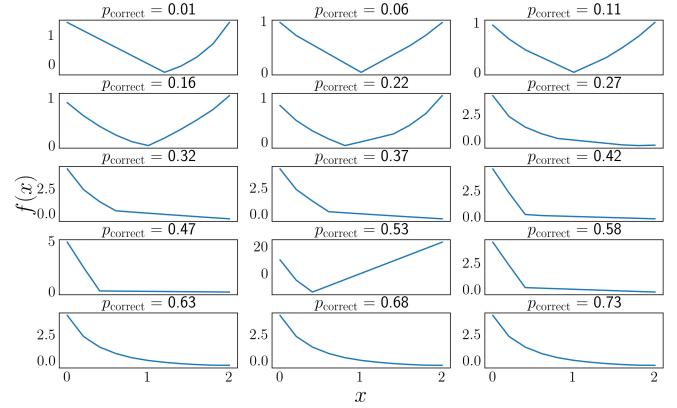


Fig. 5. Functions f that provide the tightest bounds for corresponding sensor noise level.

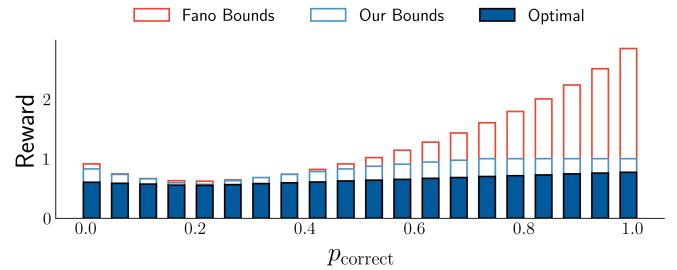


Fig. 6. Results for the one-step lava problem. We compare the upper bounds on achievable expected rewards computed via our approach (using KL divergence) with the bounds computed via generalized Fano's inequality.

This is because the RHS of (10) is a lower bound on the RHS of (33). Our bound may thus be of independent interest for applications considered by [5] such as establishing sample complexity bounds for various learning problems.

B. Catching a Falling Object

Next, we consider a problem with continuous state and observation spaces. The goal of the robot is to catch a freely-falling object such as a ball (Figure 7).

Dynamics. We describe the four-dimensional state of the robot-ball system by $s_t := [x_t^{\text{rel}}, y_t^{\text{rel}}, v_t^{\text{x,ball}}, v_t^{\text{y,ball}}] \in \mathbb{R}^4$, where $[x_t^{\text{rel}}, y_t^{\text{rel}}]$ is the relative position of the ball with respect to the robot, and $[v_t^{\text{x,ball}}, v_t^{\text{y,ball}}]$ corresponds to the ball's velocity. The action a_t is the horizontal speed of the robot and can be chosen within the range $[-0.4, 0.4]$ m/s (discretized in increments of 0.1m/s). The dynamics of the system are given by:

$$s_{t+1} = \begin{bmatrix} x_{t+1}^{\text{rel}} \\ y_{t+1}^{\text{rel}} \\ v_{t+1}^{\text{x,ball}} \\ v_{t+1}^{\text{y,ball}} \end{bmatrix} = \begin{bmatrix} x_t^{\text{rel}} + (v_t^{\text{x,ball}} - a_t)\Delta t \\ y_t^{\text{rel}} + \Delta t v_t^{\text{y,ball}} \\ v_t^{\text{x,ball}} \\ v_t^{\text{y,ball}} - g\Delta t \end{bmatrix}, \quad (34)$$

where $\Delta t = 1$ is the time-step and $g = 0.1$ m/s² is chosen such that the ball reaches the ground within a time horizon of $T = 5$. The initial state distribution p_0 is chosen to be a Gaussian with mean $[0.0\text{m}, 1.05\text{m}, 0.0\text{m/s}, 0.05\text{m/s}]$ and diagonal covariance matrix $\text{diag}([0.01^2, 0.1^2, 0.2^2, 0.1^2])$.

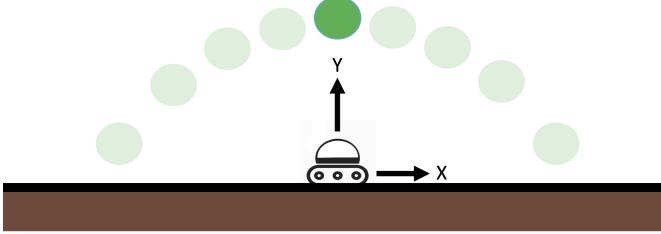


Fig. 7. An illustration of the ball-catching example with continuous state and observation spaces. The robot is constrained to move horizontally along the ground and can control its speed. Its goal is to track the position of the falling ball using a noisy estimate of the ball’s state.

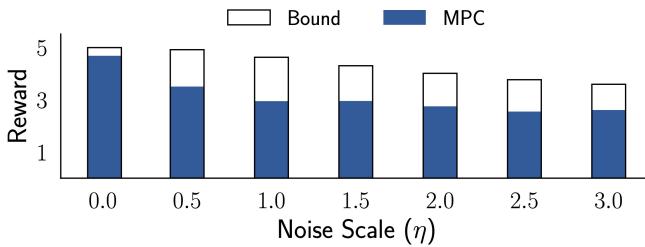


Fig. 8. Results for the ball-catching example. We compare the upper bounds on achievable expected rewards with the expected rewards using MPC combined with Kalman filtering for different values of sensor noise (results for MPC are averaged across five evaluation seeds; the std. dev. across seeds is too small to visualize).

Sensor. The robot’s sensor provides a noisy state estimate $o_t = s_t + \epsilon_t$, where ϵ_t is drawn from a Gaussian distribution with zero mean and diagonal covariance matrix $\eta \cdot \text{diag}([0.5^2, 1.0^2, 0.75^2, 1.0^2])$. Here, η is a noise scale that we will vary in our experiments.

Rewards. The reward function $r_t(s_t, a_t)$ is chosen to encourage the robot to track the ball’s motion. In particular, we choose $r_t(s_t, a_t) = \max(1 - 2|x_t^{\text{rel}}|, 0)$. The reward is large when x_{rel} is close to 0 (with a maximum reward of 1 when $x_t^{\text{rel}} = 0$); the robot receives no reward if $|x_t^{\text{rel}}| \geq 0.5$. The robot’s goal is to maximize the expected cumulative reward over a time horizon of $T = 5$.

Results. Unlike the lava problem, this problem does not admit a good open-loop policy since the initial distribution on $v_0^{\text{x,ball}}$ is symmetric about 0; thus, the robot does not have *a priori* knowledge of the ball’s x-velocity direction (as illustrated in Figure 7). In this example, we choose KL divergence as the specific f -divergence to compute bounds with. Figure 8 plots the upper bound on the expected cumulative reward obtained using Algorithm 1 for different settings of the observation noise scale η . Since the dynamics (34) are affine, the sensor model is Gaussian, and the initial state distribution is also Gaussian, we can apply the techniques described in Section VI-A for *analytically* computing the quantities of interest in Algorithm 1.

Figure 8 also compares the upper bounds on the highest achievable expected reward $R_{0 \rightarrow T}^*$ with *lower bounds* on this quantity. To do this, we note that the expected reward achieved by any *particular* control policy provides a lower

bound on $R_{0 \rightarrow T}^*$. In this example, we compute lower bounds by computing the expected reward achieved by a model-predictive control (MPC) scheme combined with a Kalman filter for state estimation. We estimate this expected reward using 100 initial conditions sampled from p_0 . Figure 8 plots the average of these expected rewards across five random seeds (the resulting standard deviation is too small to visualize). As the figure illustrates, the MPC controller obeys the fundamental bound on reward computed by our approach. Moreover, the performance of the controller qualitatively tracks the degradation of achievable performance predicted by the bound as η is increased. Finally, we observe that sensors with noise scales $\eta = 1$ and higher are *fundamentally limited* as compared to a noiseless sensor. This is demonstrated by the fact that the MPC controller for $\eta = 0$ achieves higher performance than the fundamental limit on performance for $\eta = 1$.

C. Obstacle Avoidance with a Depth Sensor

For our final example, we consider the problem of obstacle avoidance using a depth sensor (Figure 1). This is a more challenging problem with higher-dimensional (continuous) state and observation spaces, and non-Gaussian sensor models.

State and action spaces. The robot is initialized at the origin with six cylindrical obstacles of fixed radius placed randomly in front of it. The state $s_t \in \mathbb{R}^{12}$ of this system describes the locations of these obstacles in the environment. In addition, we also place “walls” enclosing a workspace $[-1, 1]\text{m} \times [-0.1, 1.2]\text{m}$ (these are not visualized in the figure to avoid clutter). The initial state distribution p_0 corresponds to uniformly randomly choosing the x-y locations of the cylindrical obstacles from the set $[-1, 1]\text{m} \times [0.9, 1.1]\text{m}$. The robot’s goal is to navigate to the end of the workspace by choosing a motion primitive to execute (based on a noisy depth sensor described below). Figure 1 illustrates the set of ten motion primitives the robot can choose from; this set corresponds to the action space.

Rewards. We treat this problem as a one-step decision making problem (Section IV). Once the robot chooses a motion primitive to execute based on its sensor measurements, it receives a reward of 0 if the motion primitive results in a collision with an obstacle; if the motion primitive results in collision-free motion, the robot receives a reward of 1. The expected reward for this problem thus corresponds to the probability of safe (i.e., collision-free) motion.

Sensor. The robot is equipped with a depth sensor which provides distances along $n_{\text{rays}} = 10$ rays. The sensor has a field of view of 90° and a maximum range of 1.5m. We use the noise model for range finders described in Ch. 6.3 in the work by [25] and consider two kinds of measurement errors: (i) errors due to failures to detect obstacles, and (ii) random noise in the reported distances. For each ray, there is a probability $p_{\text{miss}} = 0.1$ that the sensor misses an obstacle and reports the maximum range (1.5m) instead. In the case that an obstacle is not missed, the distance reported along a given ray is sampled from a Gaussian with mean equal to the true distance along that ray and std. dev. equal to η . The noise for each ray is

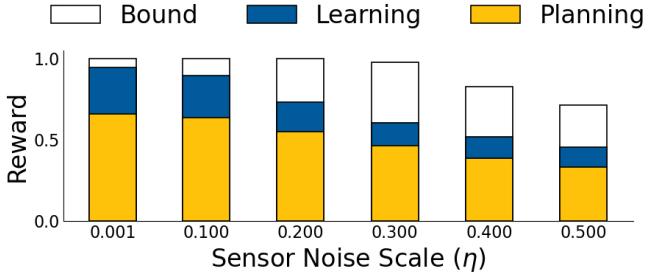


Fig. 9. Comparing sensors with varying levels of noise for the obstacle avoidance problem. Upper bounds on achievable expected rewards are compared with the expected rewards using (i) a learned neural network policy and (ii) a heuristic planner.

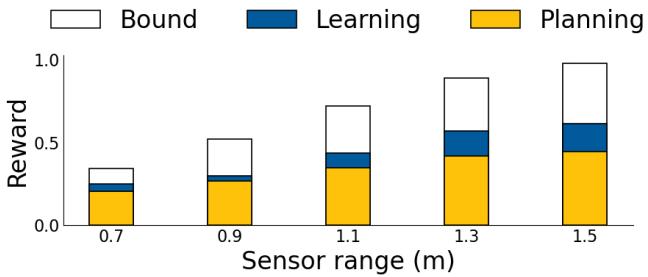


Fig. 10. Comparing sensors with varying distance ranges for the obstacle avoidance problem. Upper bounds on achievable expected rewards are compared with the expected rewards using (i) a learned neural network policy and (ii) a heuristic planner.

sampled independently. Overall, this is a non-Gaussian sensor model due to the combination of the two kinds of errors.

Results: varying sensor noise. We implement Algorithm 1 (with KL divergence) using the sampling-based techniques described in Section VI-B. We sample 20K obstacle environments for upper bounding the open-loop rewards associated with each action. We also utilize 20K batches (each of size $K = 1000$) for upper bounding the mutual information using (31). We utilize a version of Hoeffding’s inequality (Theorem 3) presented in Appendix B to obtain an upper bound on $R_{0 \rightarrow T}^*$ that holds with probability $1 - \delta = 0.95$.

Figure 9 shows the resulting upper bounds for different values of the observation noise standard deviation η . We compare these upper bounds with rewards achieved by neural network policies trained to perform this task; these rewards provide lower bounds on the highest achievable rewards for each η . For each η , we sample 5000 training environments and corresponding sensor observations. For each environment, we generate a ten-dimensional training label by recording the minimum distance to the obstacles achieved by each motion primitive and passing the vector of distances through an (element-wise) softmax transformation. We use a cross-entropy loss to train a neural network that predicts the label of distances for each primitive given a sensor observation as input. We use two fully-connected layers with a ReLU nonlinearity; the output is passed through a softmax layer. At test-time, a given sensor observation is passed as input to the trained neural network; the motion primitive corresponding to the highest predicted distance is then executed. We estimate the

expected reward achieved by the trained policy using 5000 test environments (unseen during training). Figure 9 plots the average of these expected rewards across five training seeds for each value of η (the std. dev. across seeds is too small to visualize). Figure 9 also presents rewards achieved by a heuristic planner that chooses the motion primitive with the largest estimated distance to the obstacles.

As expected, the rewards from both the learning- and planning-based approaches are lower than the fundamental limits computed using our approach. This highlights the fact that our upper bounds provide fundamental limits on performance that hold regardless of the size of the neural network, the network architecture, or algorithm used for synthesizing the policy. We also highlight the fact that the neural network policy for a sensor with noise $\eta = 0.1$ achieves higher performance than the fundamental limit for a sensor with noise $\eta = 0.4$ or 0.5 .

Results: varying sensor range. Next, we compare depth sensors with different distance ranges in Figure 10. For this comparison, we fix $\eta = 0.3$ and $p_{\text{miss}} = 0.05$. Consistent with intuition, the upper bounds on achievable performance increase as the range of the sensors increase. The performance of the learning- and heuristic planning-based approaches also improve with the range, while remaining below the fundamental limits. We highlight that the neural network policy for a sensor with range 1.5m surpasses the fundamental limit for a sensor with range 0.9m.

Results: varying sensor resolution. Finally, we apply our approach to compare two sensors with varying number n_{rays} of rays along which the depth sensor provides distance estimates (Figure 1). For this comparison, we fix $\eta = 0.3$ and $p_{\text{miss}} = 0.05$. We compare two sensors with $n_{\text{rays}} = 50$ (Sensor 1) and $n_{\text{rays}} = 5$ (Sensor 2) respectively. The upper bound on expected reward computed using our approach (with confidence $1 - \delta = 0.95$) for Sensor 2 is 0.79. A neural network policy for Sensor 1 achieves an expected reward of approximately 0.86, which surpasses the fundamental limit on performance for Sensor 2.

VIII. DISCUSSION AND CONCLUSIONS

We have presented an approach for establishing fundamental limits on performance for sensor-based robot control and policy learning. We defined a quantity that captures the amount of task-relevant information provided by a sensor; using a novel version of the generalized Fano inequality, we demonstrated that this quantity upper bounds the expected reward for one-step decision making problems. We developed a dynamic programming approach for extending this bound to multi-step problems. The resulting framework has potentially broad applicability to robotic systems with continuous state and observation spaces, nonlinear and stochastic dynamics, and non-Gaussian sensor models. Our numerical experiments demonstrate the ability of our approach to establish strong bounds on performance for such settings. In addition, we provided an application of our approach for comparing different sensors and establishing the superiority of one sensor over another for a given task.

Challenges and future work. There are a number of challenges and directions for future work associated with our approach. On the theoretical front, an interesting direction is to handle settings where the sensor model is inaccurate (in contrast to this paper, where we have focused on establishing fundamental limits given a particular sensor model). For example, one could potentially perform an adversarial search over a family of sensor models in order to find the model that results in the lowest bound on achievable performance.

On the algorithmic front, the primary challenges are: (i) efficient computation of bounds for longer time horizons, and (ii) extensions to continuous action spaces. As presented, Algorithm 1 requires an enumeration over action sequences. Finding more computationally efficient versions of Algorithm 1 is thus an important direction for future work. The primary bottleneck in extending our approach to continuous action spaces is the need to perform a supremization over actions when computing $\bar{R}_{t \rightarrow T}^{\perp\star}$ in Algorithm 1. However, we note that any upper bound on $R_{t \rightarrow T}^{\perp\star}$ also leads to a valid upper bound on $R_{0 \rightarrow T}^{\star}$. Thus, one possibility is to use a Lagrangian relaxation [20] to upper bound $\bar{R}_{t \rightarrow T}^{\perp\star}$ in settings with continuous action spaces. Another current limitation in computing the bounds for systems with continuous state and observation spaces (Section VI-B) is the need for a probability density function for the sensor model σ_t than can be numerically evaluated given any particular state-observation pair. This may be addressed by leveraging the broad range of numerical techniques that exist for mutual information estimation [28, 33].

Our work also opens up a number of exciting directions for longer-term research. While we have focused on establishing fundamental limits imposed by imperfect sensing in this work, one could envision a broader research agenda that seeks to establish bounds on performance due to other limited resources (e.g., onboard computation or memory). One concrete direction is to combine the techniques presented here with information-theoretic notions of bounded rationality [34–36]. Finally, another exciting direction is to turn the impossibility results provided by our approach into certificates of *robustness* against an adversary. Specifically, consider an adversary that can observe our robot’s actions; if one could establish fundamental limits on performance *for the adversary* due to its inability to infer the robot’s internal state (and hence its future behavior) using past observations, this provides a certificate of robustness against *any* adversary. This is reminiscent of a long tradition in cryptography of turning impossibility or hardness results into robust protocols for security [37, Ch. 9].

Overall, we believe that the ability to establish fundamental limits on performance for robotic systems is crucial for establishing a science of robotics. We hope that the work presented here along with the indicated directions for future work represent a step in this direction.

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APPENDIX A PROOFS

Proposition 1 (Monotonicity of f -inverse). *The f -inverse $\mathbb{D}_f^{-1}(q|c)$ is:*

- 1) *monotonically non-decreasing in $c \geq 0$ for fixed $q \in [0, 1]$,*
- 2) *monotonically non-decreasing in $q \in [0, 1]$ for fixed $c \geq 0$.*

Proof: Recall that the f -inverse is defined using the following optimization problem with decision variable p :

$$\mathbb{D}_f^{-1}(q|c) := \sup \{p \in [0, 1] \mid \mathbb{D}_{f,\mathcal{B}}(p||q) \leq c\}. \quad (35)$$

where

$$\mathbb{D}_{f,\mathcal{B}}(p||q) := \mathbb{D}_f(\mathcal{B}_p||\mathcal{B}_q) = qf\left(\frac{p}{q}\right) + (1-q)f\left(\frac{1-p}{1-q}\right).$$

The monotonicity condition consists of two parts, as stated above.

To prove 1), we see that increasing c loosens constraint to the optimization problem (35), thus the f -inverse can only take values larger than or equal to the ones corresponding to smaller c 's.

To prove 2), we first show that $\mathbb{D}_{f,\mathcal{B}}(p||q)$ is monotonically non-increasing in q for any fixed p . For notational simplicity, we drop the subscripts for f -divergence and denote $\mathbb{D}_{f,\mathcal{B}}(p||q)$ by $\mathbb{D}(p, q)$.

We note that $\mathbb{D}(p, q)$ is directionally differentiable, since f is a one-dimensional convex function.

For fixed p , the directional derivatives in the positive and negative q directions are:

$$\begin{aligned} \mathbb{D}'_{\pm}(p, q \pm \delta) &= \frac{\partial}{\partial q_{\pm}} \left[qf\left(\frac{p}{q}\right) + (1-q)f\left(\frac{1-p}{1-q}\right) \right] \\ &= f\left(\frac{p}{q}\right) - f\left(\frac{1-p}{1-q}\right) + q \frac{\partial}{\partial q_{\pm}} f\left(\frac{p}{q}\right) + (1-q) \frac{\partial}{\partial q_{\pm}} f\left(\frac{1-p}{1-q}\right) \\ &= f(x) - f(y) + yf'_{\pm}(y) - xf'_{\pm}(x) \\ &= \underbrace{f(x) - f(y) + f'_{\pm}(x)(y-x)}_{(a)} + \underbrace{y(f'_{\pm}(y) - f'_{\pm}(x))}_{(b)}. \end{aligned}$$

- In (a), for all $0 < t \leq 1$, we have by convexity of f ,

$$f(x + t(y-x)) \leq (1-t)f(x) + tf(y).$$

Dividing both sides by t ,

$$f(y) - f(x) \geq \frac{f(x + t(y-x)) - f(x)}{t}.$$

If we take the limit as $t \rightarrow 0$ from the right side of the x -axis, then the right directional derivatives in the direction of $y - x$ is:

$$f'_+(x)(y-x) = \lim_{t \rightarrow 0^+} \frac{f(x + t(y-x)) - f(x)}{t} \leq f(y) - f(x). \quad (36)$$

Similarly, for $-1 \leq t < 0$, we have,

$$\begin{aligned} f(x + t(x-y)) &= f(x + (-t)(y-x)) \leq (1+t)f(x) - tf(y), \\ (-t)(f(y) - f(x)) &\geq f(x + t(x-y)) - f(x), \\ f(y) - f(x) &\geq \frac{f(x) - f(x + t(x-y))}{t}. \end{aligned}$$

The left directional derivative in the direction of $y - x$ becomes:

$$f'_-(x)(y-x) = \lim_{t \rightarrow 0^-} \frac{f(x) - f(x + t(x-y))}{t} \leq f(y) - f(x). \quad (37)$$

Combining (36) and (37), we see that (a) $= f(x) - f(y) + f'_{\pm}(x)(y-x) \leq 0$.

- In (b), we know that $y = \frac{1-p}{1-q} > 0$. We also know that the optimal value p^* for problem (35) is greater than or equal to q . So, $x^* = \frac{p^*}{q} \geq 1$, $y^* = \frac{1-p^*}{1-q} \leq 1$, $x^* \geq y^*$. For the convex function f , the directional derivatives f'_{\pm} are nondecreasing. Therefore, for $x^* > y^*$, $f'_{\pm}(x^*) \geq f'_{\pm}(y^*)$. It then follows that (b) ≤ 0 .

Therefore, $\mathbb{D}'_{\pm}(p, q \pm \delta) \leq 0$, $\mathbb{D}(p, q)$ is non-increasing in q for fixed p , and $q \leq q' \Rightarrow \mathbb{D}(p, q) \geq \mathbb{D}(p, q')$. Thus, for all fixed $c \geq 0$, $\mathbb{D}(p, q) \leq c$ implies $\mathbb{D}(p, q') \leq c$. Noting that this is the constraint for the optimization problem (35), we see that any

feasible point p for $\mathbb{D}_f^{-1}(q|c)$ is also a feasible point for $\mathbb{D}_f^{-1}(q'|c)$. In other words, $q \leq q'$ implies $\mathbb{D}_f^{-1}(q|c) \leq \mathbb{D}_f^{-1}(q'|c)$, or f -inverse is monotonically non-decreasing in q for fixed c . ■

Theorem 1 (Single-step performance bound). *The best achievable reward is upper bounded by the task-relevant information potential (TRIP) of a sensor:*

$$\tau(\sigma_0; r_0) \geq R_{0 \rightarrow 1}^*(\sigma_0; r_0) = \sup_{\pi_0} \mathbb{E}_{s_0, o_0} [r_0(s_0, \pi_0(o_0))]. \quad (10)$$

Proof: For a given policy π_0 , define:

$$R_{0 \rightarrow 1} := \mathbb{E}_{p_0(s_0)} \mathbb{E}_{\sigma_0(o_0|s_0)} [r_0(s_0, \pi_0(o_0))],$$

and

$$\tilde{R}_{0 \rightarrow 1} := \mathbb{E}_{p_0(s_0)} \mathbb{E}_{q(o_0)} [r_0(s_0, \pi_0(o_0))].$$

The only difference between R and $\tilde{R}_{0 \rightarrow 1}$ is that the observations o_0 in $\tilde{R}_{0 \rightarrow 1}$ are drawn from a state-*independent* distribution q .

Now, assuming bounded rewards (Assumption 1), we have:

$$\begin{aligned} \mathbb{D}_{f, \mathcal{B}}(R_{0 \rightarrow 1} \| \tilde{R}_{0 \rightarrow 1}) &= \mathbb{D}_{f, \mathcal{B}} \left(\mathbb{E}_{p_0(s_0)} \mathbb{E}_{\sigma_0(o_0|s_0)} [r_0(s_0, \pi_0(o_0))] \middle\| \mathbb{E}_{p_0(s_0)} \mathbb{E}_{q(o_0)} [r_0(s_0, \pi_0(o_0))] \right) \\ &\leq \mathbb{E}_{p_0(s_0)} \mathbb{D}_{f, \mathcal{B}} \left(\mathbb{E}_{\sigma_0(o_0|s_0)} [r_0(s_0, \pi_0(o_0))] \middle\| \mathbb{E}_{q(o_0)} [r_0(s_0, \pi_0(o_0))] \right) \\ &\leq \mathbb{E}_{p_0(s_0)} \mathbb{D}_f \left(\sigma_0(o_0|s_0) \middle\| q(o_0) \right). \end{aligned} \quad (38)$$

The first inequality above follows from Jensen's inequality, while the second follows from the data processing inequality (see Corollary 2 by [6] for the specific version). For notational simplicity, we denote the right-hand side of equation (38) by $\mathbb{J}^q := \mathbb{E}_{p_0(s_0)} \mathbb{D}_f \left(\sigma_0(o_0|s_0) \middle\| q(o_0) \right)$. Then, inverting the bound using the f -inverse:

$$R_{0 \rightarrow 1} \leq \mathbb{D}_f^{-1}(\tilde{R}_{0 \rightarrow 1} | \mathbb{J}^q).$$

Taking supremum over policies on both sides gives:

$$\sup_{\pi_0} R_{0 \rightarrow 1} \leq \sup_{\pi_0} \mathbb{D}_f^{-1}(\tilde{R}_{0 \rightarrow 1} | \mathbb{J}^q).$$

On the RHS, $\tilde{R}_{0 \rightarrow 1}$ depends on π_0 but $\mathbb{J}^q(\sigma_0, p_0, q)$ is independent of policy. Thus, from the monotonicity of the f -inverse (Proposition 1) and by definition of the highest achievable expected reward, we have:

$$R_{0 \rightarrow 1}^*(\sigma_0; r_0) := \sup_{\pi_0} R_{0 \rightarrow 1} \leq \mathbb{D}_f^{-1} \left(\sup_{\pi_0} \tilde{R}_{0 \rightarrow 1} | \mathbb{J}^q \right). \quad (39)$$

Next, we show that $\sup_{\pi_0} \tilde{R}_{0 \rightarrow 1}$ is equal to the highest expected reward achieved by open-loop policies, and thus is independent of the distribution q . Using the Fubini-Tonelli theorem, we see:

$$\begin{aligned} \sup_{\pi_0} \tilde{R}_{0 \rightarrow 1} &= \sup_{\pi_0} \mathbb{E}_{p_0(s_0)} \mathbb{E}_{q(o_0)} [r_0(s_0, \pi_0(o_0))] \\ &= \sup_{\pi_0} \mathbb{E}_{q(o_0)} \mathbb{E}_{p_0(s_0)} [r_0(s_0, \pi_0(o_0))] \\ &\leq \mathbb{E}_{q(o_0)} \sup_{\pi_0} \mathbb{E}_{p_0(s_0)} [r_0(s_0, \pi_0(o_0))] \\ &= \mathbb{E}_{q(o_0)} \sup_{a_0} \mathbb{E}_{p_0(s_0)} [r_0(s_0, a_0)] \\ &= \sup_{a_0} \mathbb{E}_{p_0(s_0)} [r_0(s_0, a_0)]. \end{aligned}$$

Since open-loop actions are special cases of policies, we also have:

$$\sup_{\pi_0} \tilde{R}_{0 \rightarrow 1} = \sup_{\pi_0} \mathbb{E}_{p_0(s_0)} \mathbb{E}_{q(o_0)} [r_0(s_0, \pi_0(o_0))] \geq \sup_{a_0} \mathbb{E}_{p_0(s_0)} [r_0(s_0, a_0)].$$

As a result, we see that:

$$\sup_{\pi_0} \tilde{R}_{0 \rightarrow 1} = \sup_{a_0} \mathbb{E}_{p_0(s_0)} [r_0(s_0, a_0)] = R_{0 \rightarrow 1}^\perp,$$

where $R_{0 \rightarrow 1}^\perp$ is as defined in (8). Now, taking the infimum over the state-independent distribution q on both sides of equation (39):

$$R_{0 \rightarrow 1}^*(\sigma_0; r_0) = \inf_q R_{0 \rightarrow 1}^*(\sigma_0; r_0) \leq \mathbb{D}_f^{-1}(R_{0 \rightarrow 1}^\perp | \inf_q \mathbb{J}^q). \quad (40)$$

From the definition of f -informativity [21], we note that:

$$\mathbb{I}_f(o_0; s_0) = \inf_q \mathbb{E}_{p_0(s_0)} \mathbb{D}_f \left(\sigma_0(o_0 | s_0) \parallel q(o_0) \right) = \inf_q \mathbb{J}^q.$$

Combining this with (40), we obtain the desired result:

$$R_{0 \rightarrow 1}^*(\sigma_0; r_0) \leq \mathbb{D}_f^{-1} \left(R_{0 \rightarrow 1}^\perp | \mathbb{I}_f(o_0; s_0) \right) =: \tau(\sigma_0; r_0).$$

■

Proposition 2 (Recursive bound). *For any $t = 0, \dots, T-1$, the following inequality holds for any open-loop sequence of actions $a_{0:t-1}$:*

$$\sup_{\pi_t^t, \dots, \pi_t^{T-1}} R_{t \rightarrow T} \leq \underbrace{(T-t) \cdot \mathbb{D}_f^{-1} \left(\frac{R_{t \rightarrow T}^{\perp*}}{T-t} \mid \mathbb{I}_f(o_t; s_t) \right)}_{=: \tau_t(\sigma_t; r_{t:T-1})}. \quad (16)$$

Proof: The proof follows a similar structure to that of Theorem 1.

First, note that $R_{t \rightarrow T}$ defined in (11) can be written as:

$$R_{t \rightarrow T} = \mathbb{E}_{\substack{s_t \\ a_{0:t-1}}} \mathbb{E}_{\substack{o_t | s_t \\ s_{t+1:T-1}, o_{t+1:T-1} \\ s_t, o_t}} \left[\sum_{k=t}^{T-1} r_k(s_k, \pi_t^k(o_{t:k})) \right].$$

Define:

$$\tilde{R}_{t \rightarrow T} := \mathbb{E}_{\substack{s_t \\ a_{0:t-1}}} \mathbb{E}_{\substack{q(o_t) \\ s_{t+1:T-1}, o_{t+1:T-1} \\ s_t, o_t}} \left[\sum_{k=t}^{T-1} r_k(s_k, \pi_t^k(o_{t:k})) \right].$$

The only difference between $\tilde{R}_{t \rightarrow T}$ and $R_{t \rightarrow T}$ is that the observations o_t in $\tilde{R}_{t \rightarrow T}$ are drawn from a state-independent distribution q .

For the sake of notational simplicity, we will assume that $R_{t \rightarrow T}$ and $\tilde{R}_{t \rightarrow T}$ have been normalized to be within $[0, 1]$ by scaling with $1/(T-t)$. The desired result (16) then follows from the bound we prove below by simply rescaling with $(T-t)$. Now,

$$\begin{aligned} \mathbb{D}_{f, \mathcal{B}}(R_{t \rightarrow T} \| \tilde{R}_{t \rightarrow T}) &= \mathbb{D}_{f, \mathcal{B}} \left(\mathbb{E}_{\substack{s_t \\ a_{0:t-1}}} \mathbb{E}_{\substack{o_t \\ s_t}} \mathbb{E}_{\substack{s_{t+1:T-1}, o_{t+1:T-1} \\ s_t, o_t}} \left[\sum_{k=t}^{T-1} r_k(s_k, \pi_t^k(o_{t:k})) \right] \parallel \mathbb{E}_{\substack{s_t \\ a_{0:t-1}}} \mathbb{E}_{\substack{q(o_t) \\ s_{t+1:T-1}, o_{t+1:T-1} \\ s_t, o_t}} \left[\sum_{k=t}^{T-1} r_k(s_k, \pi_t^k(o_{t:k})) \right] \right) \\ &\leq \mathbb{E}_{\substack{s_t \\ a_{0:t-1}}} \mathbb{D}_{f, \mathcal{B}} \left(\mathbb{E}_{\substack{o_t \\ s_t}} \mathbb{E}_{\substack{s_{t+1:T-1}, o_{t+1:T-1} \\ s_t, o_t}} \left[\sum_{k=t}^{T-1} r_k(s_k, \pi_t^k(o_{t:k})) \right] \parallel \mathbb{E}_{\substack{q(o_t) \\ s_{t+1:T-1}, o_{t+1:T-1} \\ s_t, o_t}} \left[\sum_{k=t}^{T-1} r_k(s_k, \pi_t^k(o_{t:k})) \right] \right) \\ &\leq \mathbb{E}_{\substack{s_t \\ a_{0:t-1}}} \mathbb{D}_f \left(\sigma_t(o_t | s_t) \parallel q(o_t) \right). \end{aligned} \quad (41)$$

The first inequality above follows from Jensen's inequality, while the second follows from the data processing inequality (see Corollary 2 by [6] for the specific version). We denote the RHS of (41) by $\mathbb{J}^q := \mathbb{E}_{\substack{s_t \\ a_{0:t-1}}} \mathbb{D}_f \left(\sigma_t(o_t | s_t) \parallel q(o_t) \right)$ and invert the bound:

$$R_{t \rightarrow T} \leq \mathbb{D}_f^{-1} \left(\tilde{R}_{t \rightarrow T} \mid \mathbb{J}^q \right).$$

Taking supremum over policies on both sides gives:

$$\sup_{\pi_t^t, \dots, \pi_t^{T-1}} R_{t \rightarrow T} \leq \sup_{\pi_t^t, \dots, \pi_t^{T-1}} \mathbb{D}_f^{-1} \left(\tilde{R}_{t \rightarrow T} \mid \mathbb{J}^q \right).$$

Notice that the LHS is precisely the quantity we are interested in upper bounding in Proposition 2. From the monotonicity of the f -inverse (Proposition 1), we have:

$$\sup_{\pi_t^t, \dots, \pi_t^{T-1}} R_{t \rightarrow T} \leq \mathbb{D}_f^{-1} \left(\tilde{R}_{t \rightarrow T}^* \mid \mathbb{J}^q \right), \quad (42)$$

where

$$\tilde{R}_{t \rightarrow T}^* := \sup_{\pi_t^t, \dots, \pi_t^{T-1}} \tilde{R}_{t \rightarrow T} = \sup_{\pi_t^t, \dots, \pi_t^{T-1}} \mathbb{E}_{a_{0:t-1}} \mathbb{E}_{s_t \mid q(o_t)} \mathbb{E}_{s_{t+1:T-1}, o_{t+1:T-1} \mid s_t, o_t} \left[\sum_{k=t}^{T-1} r_k(s_k, \pi_t^k(o_{t:k})) \right].$$

We then show that $\tilde{R}_{t \rightarrow T}^*$ is the highest expected total reward achieved by open-loop policies, and thus independent of q . Using the Fubini-Tonelli theorem:

$$\begin{aligned} & \sup_{\pi_t^t, \dots, \pi_t^{T-1}} \mathbb{E}_{a_{0:t-1}} \mathbb{E}_{s_t \mid q(o_t)} \mathbb{E}_{s_{t+1:T-1}, o_{t+1:T-1} \mid s_t, o_t} \left[\sum_{k=t}^{T-1} r_k(s_k, \pi_t^k(o_{t:k})) \right] \\ &= \sup_{\pi_t^t, \dots, \pi_t^{T-1}} \mathbb{E}_{q(o_t)} \mathbb{E}_{a_{0:t-1}} \mathbb{E}_{s_{t+1:T-1}, o_{t+1:T-1} \mid s_t, o_t} \left[\sum_{k=t}^{T-1} r_k(s_k, \pi_t^k(o_{t:k})) \right] \\ &= \sup_{\pi_{t+1}^t, \dots, \pi_t^{T-1}} \left[\sup_{\pi_t^t} \mathbb{E}_{q(o_t)} \mathbb{E}_{a_{0:t-1}} \mathbb{E}_{s_{t+1:T-1}, o_{t+1:T-1} \mid s_t, o_t} \left[\sum_{k=t}^{T-1} r_k(s_k, \pi_t^k(o_{t:k})) \right] \right] \\ &\leq \sup_{\pi_{t+1}^t, \dots, \pi_t^{T-1}} \left[\mathbb{E}_{q(o_t)} \sup_{\pi_t^t} \mathbb{E}_{a_{0:t-1}} \mathbb{E}_{s_{t+1:T-1}, o_{t+1:T-1} \mid s_t, o_t} \left[\sum_{k=t}^{T-1} r_k(s_k, \pi_t^k(o_{t:k})) \right] \right]. \end{aligned} \quad (43)$$

Notice that:

$$\begin{aligned} & \mathbb{E}_{q(o_t)} \sup_{\pi_t^t} \mathbb{E}_{a_{0:t-1}} \mathbb{E}_{s_{t+1:T-1}, o_{t+1:T-1} \mid s_t, o_t} \left[\sum_{k=t}^{T-1} r_k(s_k, \pi_t^k(o_{t:k})) \right] \\ &= \mathbb{E}_{q(o_t)} \sup_{\pi_t^t} \mathbb{E}_{a_{0:t-1}} \left[r_t(s_t, \pi_t^t(o_t)) + \mathbb{E}_{s_{t+1}, o_{t+1} \mid s_t, \pi_t^t(o_t)} \left[r_{t+1}(s_{t+1}, \pi_t^{t+1}(o_{t+1})) + \dots \right] \dots \right] \end{aligned} \quad (44)$$

$$= \mathbb{E}_{q(o_t)} \underbrace{\sup_{a_t} \mathbb{E}_{a_{0:t-1}} \left[r_t(s_t, a_t) + \mathbb{E}_{s_{t+1}, o_{t+1} \mid s_t, a_t} \left[r_{t+1}(s_{t+1}, \pi_{t+1}^{t+1}(o_{t+1})) + \dots \right] \dots \right]}_{\text{Does not depend on } o_t} \quad (45)$$

$$= \sup_{a_t} \mathbb{E}_{a_{0:t-1}} \left[r_t(s_t, a_t) + \mathbb{E}_{s_{t+1}, o_{t+1} \mid s_t, a_t} \left[r_{t+1}(s_{t+1}, \pi_{t+1}^{t+1}(o_{t+1})) + \dots \right] \dots \right].$$

Here, (45) follows (44) since q is a fixed distribution that does not depend on the state.

We thus see that (43) equals:

$$\begin{aligned} & \sup_{\pi_{t+1}^t, \dots, \pi_t^{T-1}} \underbrace{\left[\sup_{a_t} \mathbb{E}_{a_{0:t-1}} \left[r_t(s_t, a_t) + \mathbb{E}_{s_{t+1}, o_{t+1} \mid s_t, a_t} \left[r_{t+1}(s_{t+1}, \pi_{t+1}^{t+1}(o_{t+1})) + \dots \right] \dots \right] \right]}_{\text{Does not depend on } o_t} \\ &= \sup_{\pi_{t+1}^{t+1}, \dots, \pi_t^{T-1}} \left[\sup_{a_t} \mathbb{E}_{a_{0:t-1}} \left[r_t(s_t, a_t) + \mathbb{E}_{s_{t+1}, o_{t+1} \mid s_t, a_t} \left[r_{t+1}(s_{t+1}, \pi_{t+1}^{t+1}(o_{t+1})) + \dots \right] \dots \right] \right] \\ &= \sup_{\pi_{t+1}^{t+1}, \dots, \pi_t^{T-1}} \left[\sup_{a_t} \mathbb{E}_{a_{0:t-1}} \left[r_t(s_t, a_t) \right] + \mathbb{E}_{s_{t+1:T-1}, o_{t+1:T-1} \mid a_{0:t}} \left[\sum_{k=t+1}^{T-1} r_k(s_k, \pi_{t+1}^k(o_{t+1:k})) \right] \right] \\ &= \sup_{\pi_{t+1}^{t+1}, \dots, \pi_t^{T-1}} R_{t \rightarrow T}^\perp \\ &= R_{t \rightarrow T}^{\perp*}. \end{aligned}$$

We have thus proved that $\tilde{R}_{t \rightarrow T}^* \leq R_{t \rightarrow T}^{\perp*}$ (indeed, since open-loop policies are special cases of feedback policies, we also have $\tilde{R}_{t \rightarrow T}^* \geq R_{t \rightarrow T}^{\perp*}$ and thus $\tilde{R}_{t \rightarrow T}^* = R_{t \rightarrow T}^{\perp*}$). The RHS of (42) then becomes:

$$\sup_{\pi_t^t, \dots, \pi_t^{T-1}} R_{t \rightarrow T} \leq \mathbb{D}_f^{-1}\left(R_{t \rightarrow T}^{\perp*} \mid \mathbb{J}^q\right).$$

Now, taking the infimum over the state-independent distribution q on both sides and using monotonicity of f -inverse (proposition 1):

$$\sup_{\pi_t^t, \dots, \pi_t^{T-1}} R_{t \rightarrow T} = \inf_q \sup_{\pi_t^t, \dots, \pi_t^{T-1}} R_{t \rightarrow T} \leq \mathbb{D}_f^{-1}\left(R_{t \rightarrow T}^{\perp*} \mid \inf_q \mathbb{J}^q\right). \quad (46)$$

From the definition of f -informativity, [21], we note that:

$$\mathbb{I}_f(o_t; s_t) := \inf_q \mathbb{E}_{\substack{s_t \\ a_{0:t-1}}} \mathbb{D}_f\left(\sigma_t(o_t | s_t) \| q(o_t)\right) = \inf_q \mathbb{J}^q.$$

Combining with (46), we obtain the desired result:

$$\sup_{\pi_t^t, \dots, \pi_t^{T-1}} R_{t \rightarrow T} \leq \mathbb{D}_f^{-1}\left(R_{t \rightarrow T}^{\perp*} \mid \mathbb{I}_f(o_t; s_t)\right).$$

■

Theorem 2 (Multi-step performance bound). *Algorithm 1 returns an upper bound on the best achievable reward $R_{0 \rightarrow T}^*$.*

Proof: Using (backwards) induction, we prove that for all $t = T-1, \dots, 0$,

$$\sup_{\pi_t^t, \dots, \pi_t^{T-1}} R_{t \rightarrow T} \leq \bar{R}_t(a_{0:t-1}), \quad \forall a_{0:t-1}. \quad (47)$$

Thus, in particular,

$$R_{0 \rightarrow T}^* = \sup_{\pi_0^0, \dots, \pi_0^{T-1}} R_{0 \rightarrow T} \leq \bar{R}_{0 \rightarrow T}.$$

We prove (47) by backwards induction starting from $t = T-1$. In particular, Proposition 2 leads to the inductive step. We first prove the base step of induction using $t = T-1$. Using (16), we see:

$$\sup_{\pi_{T-1}^{T-1}} R_{T-1 \rightarrow T} \leq \mathbb{D}_f^{-1}\left(R_{T-1 \rightarrow T}^{\perp*} \mid \mathbb{I}_f(o_{T-1}; s_{T-1})\right). \quad (48)$$

By definition (see (15)),

$$\begin{aligned} R_{T-1 \rightarrow T}^{\perp*} &= \sup_{a_{T-1}} \mathbb{E}_{s_{T-1} | a_{0:T-2}} \left[r_{T-1}(s_{T-1}, a_{T-1}) \right] + \underbrace{R_{t \rightarrow T}}_{=0} \\ &= \sup_{a_{T-1}} \mathbb{E}_{s_{T-1} | a_{0:T-2}} \left[r_{T-1}(s_{T-1}, a_{T-1}) \right] \\ &= \bar{R}_{T-1 \rightarrow T}^{\perp*}. \end{aligned}$$

Combining this with (48) and the monotonicity of the f -inverse (Proposition 1), we see:

$$\begin{aligned} \sup_{\pi_{T-1}^{T-1}} R_{T-1 \rightarrow T} &\leq \mathbb{D}_f^{-1}\left(\bar{R}_{T-1 \rightarrow T}^{\perp*} \mid \mathbb{I}_f(o_{T-1}; s_{T-1})\right) \\ &= \bar{R}_{T-1 \rightarrow T}(a_{0:T-2}). \end{aligned}$$

Next, we prove the induction step. Suppose it is the case that for $t \in \{0, \dots, T-2\}$, we have

$$\sup_{\pi_{t+1}^{t+1}, \dots, \pi_{t+1}^{T-1}} R_{t+1 \rightarrow T} \leq \bar{R}_{t+1 \rightarrow T}(a_{0:t}). \quad (49)$$

We then need to show that

$$\sup_{\pi_t^t, \dots, \pi_t^{T-1}} R_{t \rightarrow T} \leq \bar{R}_{t \rightarrow T}(a_{0:t-1}). \quad (50)$$

To prove this, we first observe that

$$\begin{aligned} R_{t \rightarrow T}^{\perp \star} &:= \sup_{\pi_{t+1}^{t+1}, \dots, \pi_{t+1}^{T-1}} \sup_{a_t} \left[\mathbb{E}_{s_t | a_{0:t-1}} \left[r_t(s_t, a_t) \right] + R_{t+1 \rightarrow T} \right] \\ &= \sup_{a_t} \left[\mathbb{E}_{s_t | a_{0:t-1}} \left[r_t(s_t, a_t) \right] + \sup_{\pi_{t+1}^{t+1}, \dots, \pi_{t+1}^{T-1}} R_{t+1 \rightarrow T} \right]. \end{aligned}$$

Combining this with the induction hypothesis (49), we see

$$\begin{aligned} R_{t \rightarrow T}^{\perp \star} &\leq \sup_{a_t} \left[\mathbb{E}_{s_t | a_{0:t-1}} \left[r_t(s_t, a_t) \right] + \bar{R}_{t+1 \rightarrow T}(a_{0:t}) \right] \\ &=: \bar{R}_{t \rightarrow T}^{\perp \star}. \end{aligned}$$

Finally, combining this with (16) and the monotonicity of the f -inverse (Prop. 1), we obtain the desired result (50):

$$\begin{aligned} \sup_{\pi_t^t, \dots, \pi_t^{T-1}} R_{t \rightarrow T} &\leq (T-t) \cdot \mathbb{D}_f^{-1} \left(\frac{\bar{R}_{t \rightarrow T}^{\perp \star}}{T-t} \mid \mathbb{I}_f(o_t; s_t) \right) \\ &= \bar{R}_{t \rightarrow T}(a_{0:t-1}). \end{aligned}$$

■

APPENDIX B CHERNOFF-HOEFFDING BOUND

In our numerical examples (Section VII), we utilize a slightly tighter version of Hoeffding's inequality than the one presented in Theorem 3. In particular, we use the following Chernoff-Hoeffding inequality (see Theorem 5.1 by [38]).

Theorem 4 (Chernoff-Hoeffding inequality [38]). *Let z be a random variable bounded within $[0, 1]$, and let z_1, \dots, z_n denote i.i.d. samples. Then, with probability at least $1 - \delta$ (over the sampling of z_1, \dots, z_n), the following bound holds with probability at least $1 - \delta$:*

$$\mathbb{D} \left(\frac{1}{n} \sum_{i=1}^n z_i \parallel \mathbb{E}[z] \right) \leq \frac{\log(2/\delta)}{n}. \quad (51)$$

We can obtain an upper bound on $\mathbb{E}[z]$ using (51) as follows:

$$\mathbb{E}[z] \leq \sup \left\{ p \in [0, 1] \mid \mathbb{D}_{f, \mathcal{B}} \left(\frac{1}{n} \sum_{i=1}^n z_i \parallel p \right) \leq \frac{\log(2/\delta)}{n} \right\}. \quad (52)$$

The optimization problem in the RHS of (52) is analogous to the f -inverse defined in Section III-B, and can be thought of as a “right” f -inverse (instead of a “left” f -inverse). In the case of KL divergence, (similar to the f -inverse in Section III-B), we can solve the optimization problem in (52) using geometric programming.