UP SECTION (INDIA)

3rd IEEE Conference on Information & Communication Technology (CICT-2019)

Indian Institute of Information Technology, Allahabad : Dec 6 - Dec 8, 2019

[99] Noise Analysis of Quantum Approximate Optimization Algorithm on Weighted MAX-CUT

[Track-9] Computer and Software Systems

Lakshya Priyadarshi, *Department of Computer Science & Engineering*, *IET Lucknow*
Utkarsh Azad, *Center for Computational Natural Sciences and Bioinformatics, IIIT-Hyderabad*

Presented by:
Lakshya Priyadarshi

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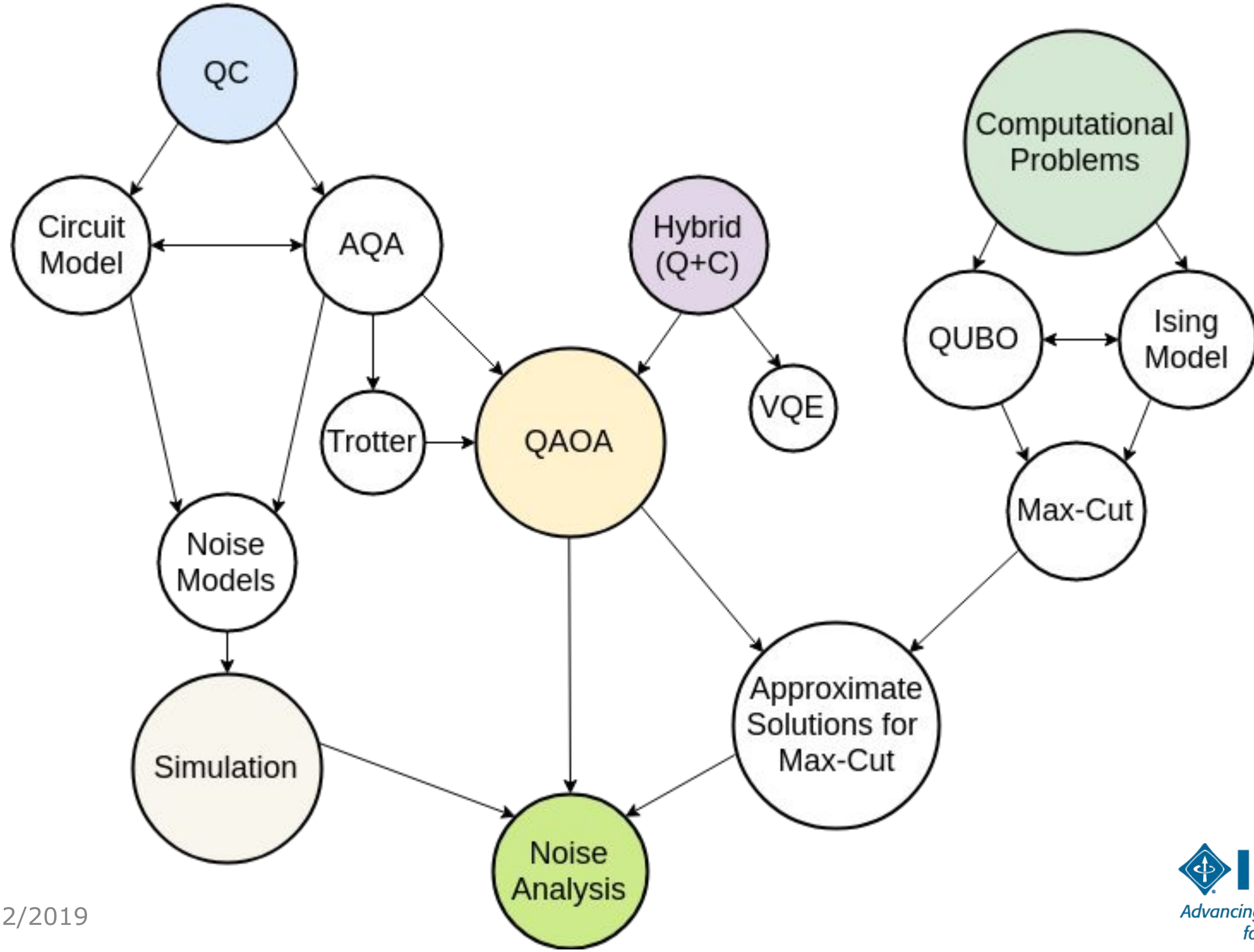
Introduction



Quantum Information and Computation

Quantum bit : ray in Hilbert space

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1, \quad \alpha, \beta \in \mathbb{C}$$

$$\{|0\rangle_1 \otimes |0\rangle_2,\ |0\rangle_1 \otimes |1\rangle_2,\ |1\rangle_1 \otimes |0\rangle_2,\ |1\rangle_1 \otimes |1\rangle_2\}$$

Unitary Evolution : deterministic, continuous, reversible

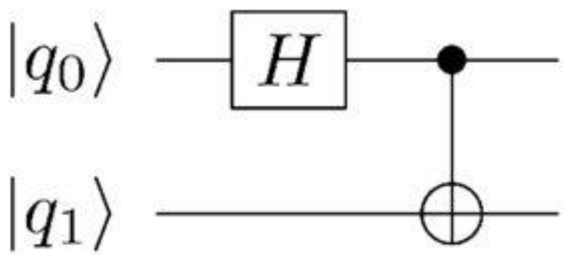
$$|\psi\rangle \rightarrow U|\psi\rangle$$

Measurements : Projection to an eigenstate of measurement operator *M*

$$\frac{M_m|\psi\rangle}{\sqrt{\left\langle\psi\left|M_m^\dagger M_m\right|\psi\right\rangle}}$$

Quantum Information and Computation

Quantum Computation in the Circuit Model



$$H \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$U_{CN} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$|00\rangle \mapsto \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle$$

$$|10\rangle \mapsto \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\Phi^-\rangle$$

$$|01\rangle \mapsto \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\Psi^+\rangle$$

$$|11\rangle \mapsto \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |\Psi^-\rangle$$

Quantum Information and Computation

Density Matrix Representation

$$\rho = \sum_{i}^{n} p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$$

Computation: Unitary Transformations

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \xrightarrow{U} \sum_i p_i U \left|\psi_i\right\rangle \left\langle\psi_i\right| U^\dagger = U \rho U^\dagger$$

Measurement: Projective Transformations

$$\rho_m = \frac{M_m \rho M_m^\dagger}{\mathrm{tr}(M_m^\dagger M_m \rho)}$$

Quadratic Unconstrained Binary Optimization and its equivalence to the Ising Model

QUBO Problems

$$\text{Maximize} : \sum_{i \in N} c_{ii} x_i + \sum_{(i,j) \in E} c_{ij} x_i x_j$$
$$\text{subject to } x_i = \{0, 1\} \text{ where } i \in N$$
$$\implies \quad Max \;\; x^t Q x : x \in \{0,1\}^n$$

Ising Model

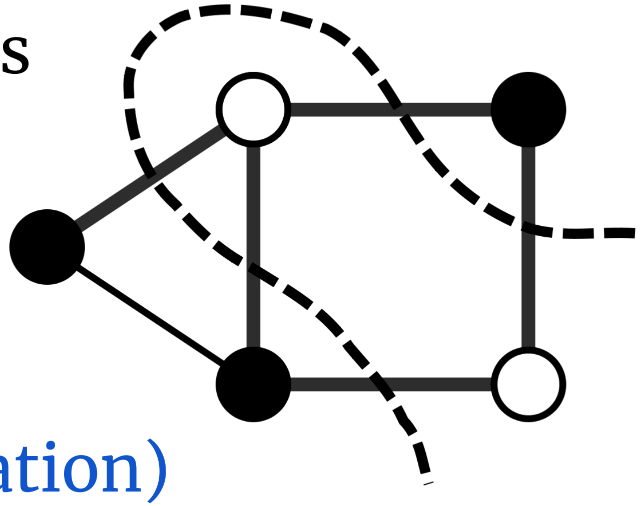
$$H(s_1, \ldots, s_N) = -\sum_{i<j} J_{ij} s_i s_j - \sum_{i=1}^{N} h_i s_i$$

Weighted Maximum-Cut Problem

Given: $G = (V, E)$, $w : E \mapsto R^+$ and $S \cup \overline{S} = V$

Objective: Find a cut which maximizes

$$\sum_{u \in S, v \in \overline{S}, (u,v) \in E} w_{uv}$$



Randomized Partitioning (Approximation)

$$A(I) = \frac{\sum_{(u,v) \in E} w_{uv}}{2} \geq \frac{OPT(I)}{2}$$

Weighted Maximum-Cut Problem

QUBO Formulation

$$\text{MAX} \sum_{(v_i,v_j)\in E} w_{ij} \frac{(1 - x_i x_j)}{2}$$

$$\text{subject to} \quad x_i \in \{-1, 1\} \ \forall v_i \in V$$

Relaxation to Semidefinite Programming

$$\text{MAX} \sum_{(u_i,u_j)\in E} w_{ij} \frac{(1 - u_i \cdot u_j)}{2}$$

$$\text{subject to} \quad u_i \cdot u_i = 1$$

This method gives cuts of expected value at least 0.87856 times the Maximum Cut.

[1] Michel X. Goemans, David P. Williamson "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming," *JACM Volume 42 Issue 6*, Nov. 1995 *Pages 1115-1145*

Quantum Approximate Optimization Algorithm

1. $$C(z) = \sum_{\alpha=1}^{m} C_\alpha(z)$$

2. $$U(C,\gamma) = e^{-i\gamma C} = \prod_{\alpha=1}^{m} e^{-i\gamma C_\alpha}$$

3. $$B = \sum_{j=1}^{n} \sigma_j^x \qquad U(B,\beta) = e^{-i\beta B} = \prod_{j=1}^{n} e^{-i\beta \sigma_j^x}$$

[1] Edward Farhi, Jeffrey Goldstone, Sam Gutmann "A Quantum Approximate Optimization Algorithm," *JACM Volume 42 Issue 6*, Nov. 1995 *Pages 1115-1145*

Quantum Approximate Optimization Algorithm

4. $$|s\rangle = \frac{1}{\sqrt{2^n}} \sum_z |z\rangle$$

5. $$|\boldsymbol{\gamma}, \boldsymbol{\beta}\rangle = U(B, \beta_p)\, U(C, \gamma_p) \cdots U(B, \beta_1)\, U(C, \gamma_1)\; |s\rangle$$

6. $$F_p(\boldsymbol{\gamma}, \boldsymbol{\beta}) = \langle \boldsymbol{\gamma}, \boldsymbol{\beta} |\, C \,| \boldsymbol{\gamma}, \boldsymbol{\beta} \rangle$$

7. $$M_p = \max_{\boldsymbol{\gamma}, \boldsymbol{\beta}} F_p\,(\boldsymbol{\gamma}, \boldsymbol{\beta})$$

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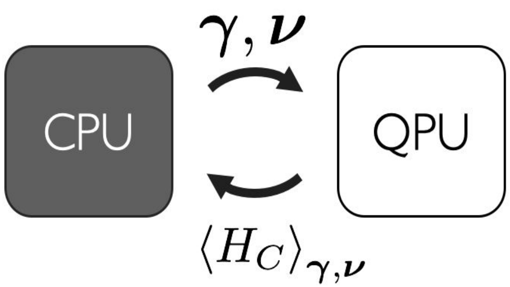
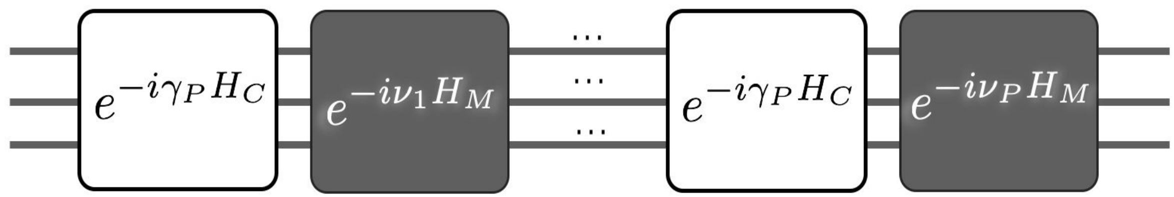
Quantum Approximate Optimization Algorithm



Mixer: $H_M = -\sum_j X_j$

Cost: $H_C = -\sum_{\langle jk \rangle} J_{jk} Z_j Z_k$

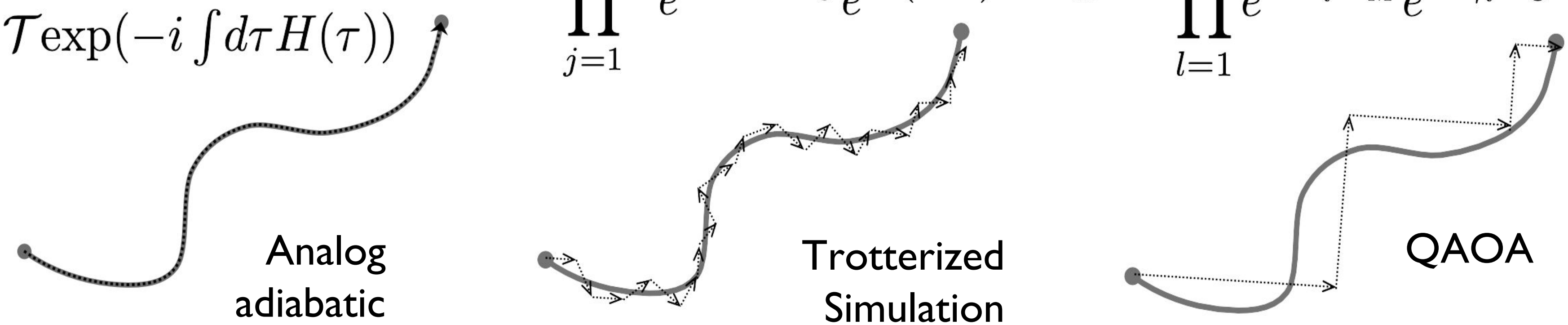


$$H(\tau) = (1-\tau)H_M + \tau H_C$$

$$\mathcal{T}\exp(-i\int d\tau H(\tau))$$

$$\prod_{j=1}^{\lceil 1/\Delta\tau \rceil} e^{-i\tau\Delta\tau H_C} e^{-i(1-\tau)\Delta\tau H_M}$$

$$\prod_{l=1}^{P} e^{-i\nu_l H_M} e^{-i\gamma_l H_C}$$



[1] Guillaume Verdon, Jacob Biamonte, Michael Broughton, “A quantum algorithm to train neural networks using low-depth circuits,” arXiv:1712.05304v2 [quant-ph]
[2] Guillaume Verdon, Jason Pye, Michael Broughton, “A Universal Training Algorithm for Quantum Deep Learning,” arXiv:1806.09729

Methodology : Simulation of QAOA

1. Initialize **four quantum bits** to represent 16 partitions of SQ2

2. ***Discretize*** the Hamiltonian Evolution into ***p*** steps :

$$\boxed{H(t) = (1 - t)H_0 + tH_1}$$

3. Select the ***Mixing Hamiltonian*** for Ground State Preparation :

$$\boxed{H_0 = -\sum_{i=0}^{3} \sigma_i^x}$$

4. Encode the Maximum-Cut Problem in ***Cost Hamiltonian*** :

$$\boxed{\begin{aligned} H_1 = \ & w_{01}(\sigma_1^z \otimes \sigma_0^z) + w_{12}(\sigma_2^z \otimes \sigma_1^z) \\ & + w_{23}(\sigma_3^z \otimes \sigma_3^z) + w_{03}(\sigma_3^z \otimes \sigma_0^z) \end{aligned}}$$

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Methodology : Simulation of QAOA

For $p = 2$ and the variational parameters $\{\beta, \gamma\}$, the total unitary U is approximated by two unitary operators as:

$$U = U(H_0, \beta_0) \cdot U(H_1, \gamma_0) \cdot U(H_0, \beta_1) \cdot U(H_1, \gamma_1)$$

We first prepare a state $|{+}{+}\rangle$ and then evolve as :

$$|\beta, \gamma\rangle = e^{-iH_0\beta_1} e^{-iH_1\gamma_1} e^{-iH_0\beta_0} e^{-iH_1\gamma_0} |{+}{+}\rangle$$

The optimal values of $\{\beta, \gamma\}$ is obtained by an iterative classical minimization of the objective function C defined as:

$$C = \langle\beta, \gamma| H |\beta, \gamma\rangle$$

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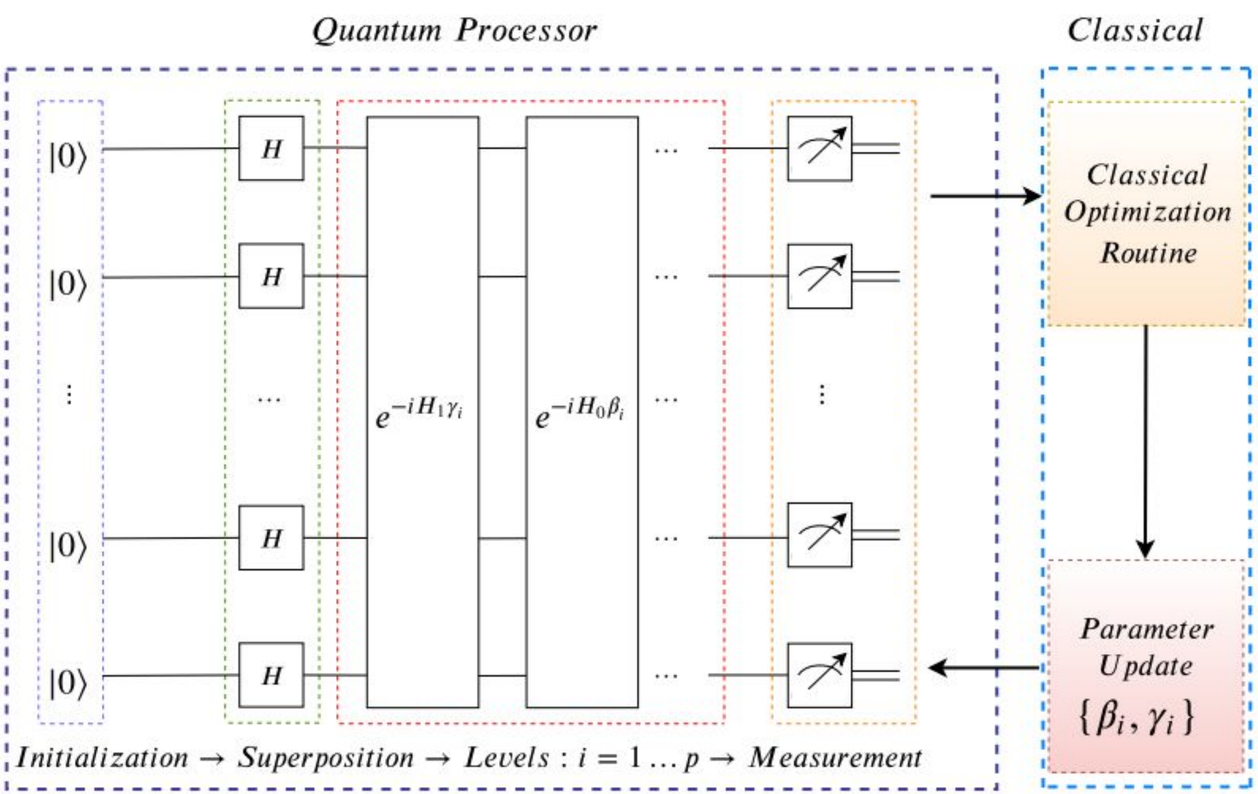
Variational Methods



Schematic representation of our simulation framework

Gates
2479

Depth
1576

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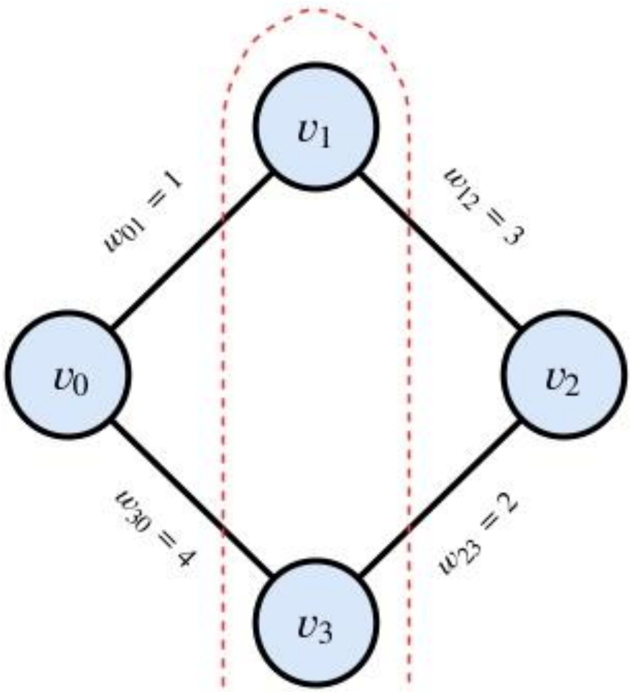
Weighted Maximum-Cut Problem on SQ2

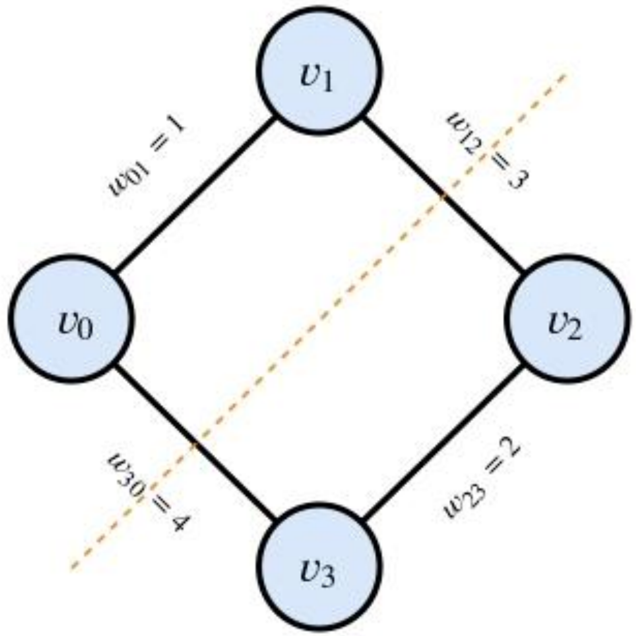
Partition "0101"



$\{\{v_0, v_2\}, \{v_1, v_3\}\}$

$W = 1 + 3 + 2 + 4 = 10$

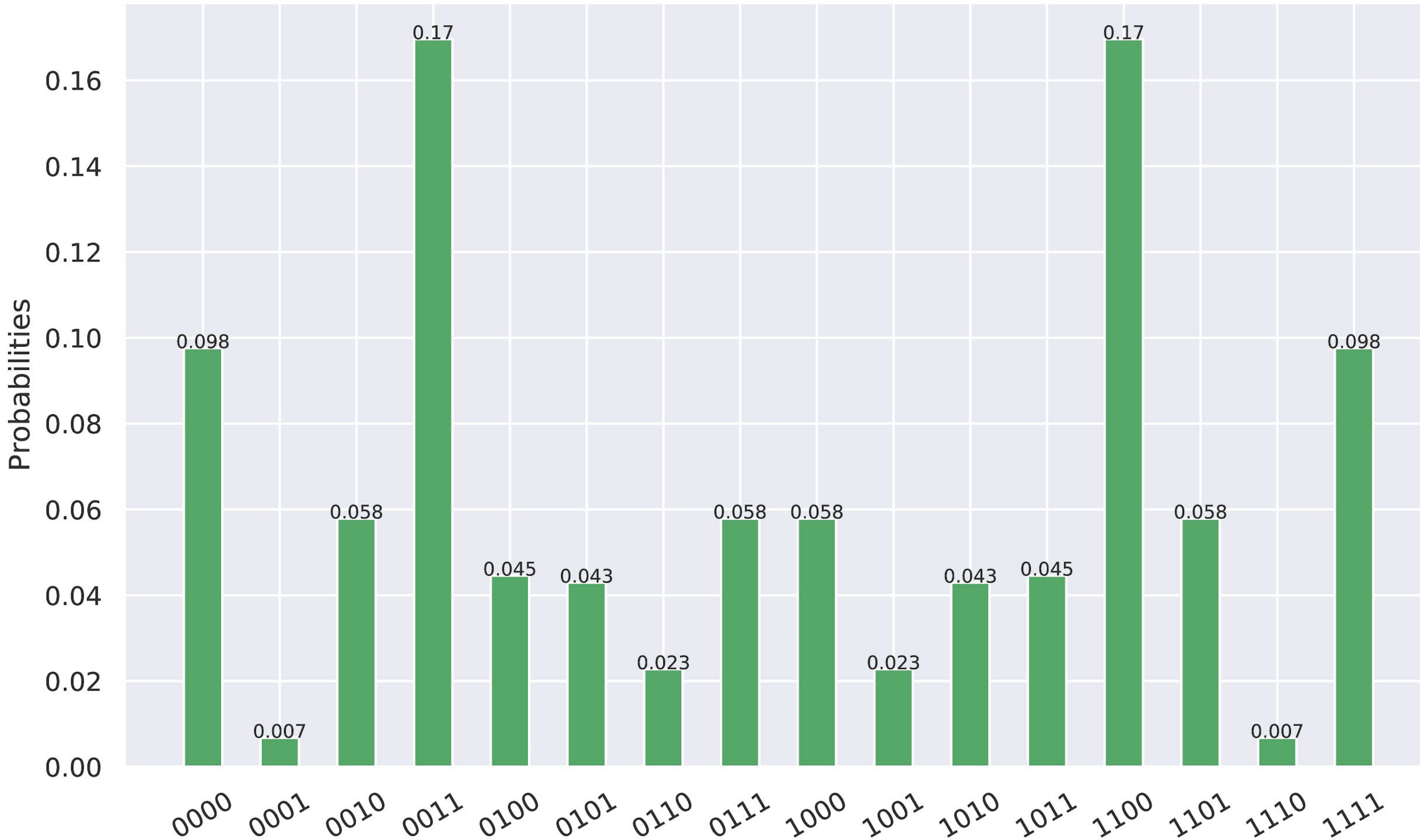
Partition "0011"



$\{\{v_0, v_1\}, \{v_2, v_3\}\}$

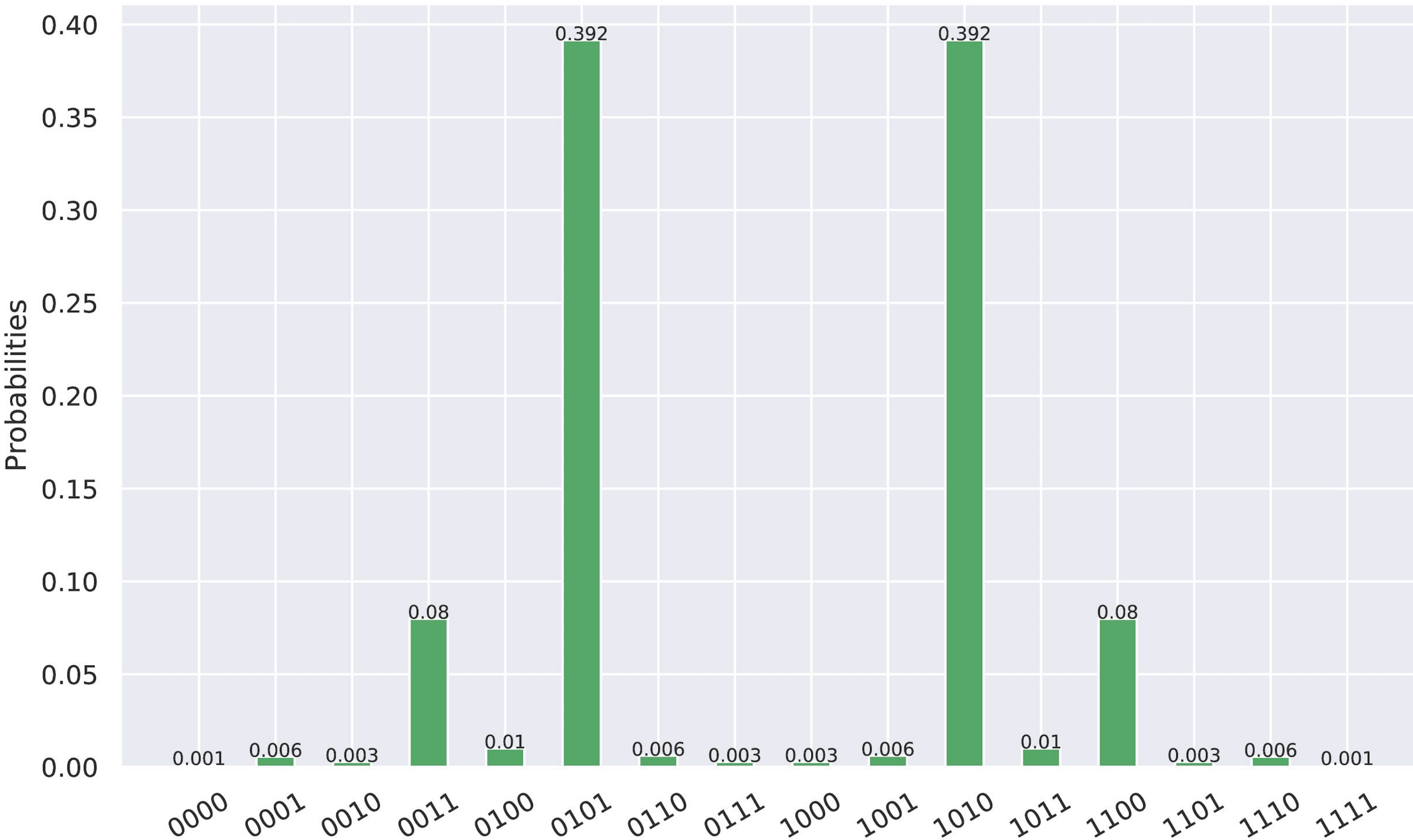
$W = 3 + 4 = 7$

PDF for Maximum-Cut on SQ2



$$\text{Initial } \{\beta, \gamma\} = [3.692, 5.317], [3.683, 2.479]$$

PDF for Maximum-Cut on SQ2



$$\text{Optimized } \{\beta, \gamma\} = [1.784, 5.308], [3.665, 2.732]$$

System Model: Quantum Simulator

Representation: Density Matrix in Pauli-Basis

$$\rho = \sum_{i_1, i_2, \ldots, i_n} a_{i_1 i_2 \ldots i_n} (\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_n})$$

Computation: Unitary Transformations

$$\begin{aligned}
\sigma_1 \rho \sigma_1 &= a_0 I + a_1 \sigma_1 - a_2 \sigma_2 - a_3 \sigma_3 \ , \\
\sigma_2 \rho \sigma_2 &= a_0 I - a_1 \sigma_1 + a_2 \sigma_2 - a_3 \sigma_3 \ , \\
\sigma_3 \rho \sigma_3 &= a_0 I - a_1 \sigma_1 - a_2 \sigma_2 + a_3 \sigma_3 \ ,
\end{aligned}$$

System Model: Quantum Simulator

Measurement : Projective Transformation

$$O = \sum_{i_1, i_2, \ldots, i_n} b_{i_1 i_2 \ldots i_n} (\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_n})$$

$$\langle O \rangle = Tr(O\rho) = 2^n \sum_{i_1, i_2, \ldots, i_n} a_{i_1 i_2 \ldots i_n} b_{i_1 i_2 \ldots i_n}$$

Apoorva D. Patel, et. al. "A Software Simulator for Noisy Quantum Circuits", arXiv:1908.05154 [quant-ph], 2019.

System Model: Simulation of Noise Models

Initialization error (Thermal State)

$$\rho_{\rm th} = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}^{\otimes n}, \quad \frac{p}{1-p} = \exp\left(\frac{E_1 - E_0}{kT}\right)$$

Logic Gate Imprecision (Rotation Error)

$$\cos\theta \to r\cos(\theta + \overline{\alpha})\ , \quad \sin\theta \to r\sin(\theta + \overline{\alpha})$$

Measurement Error (Depolarization)

$$\tfrac{1}{2}(1 \pm 2^n d_1 \hat{n} \cdot \vec{c})$$

System Model: Simulation of Noise Models

Memory Errors : Decoherence

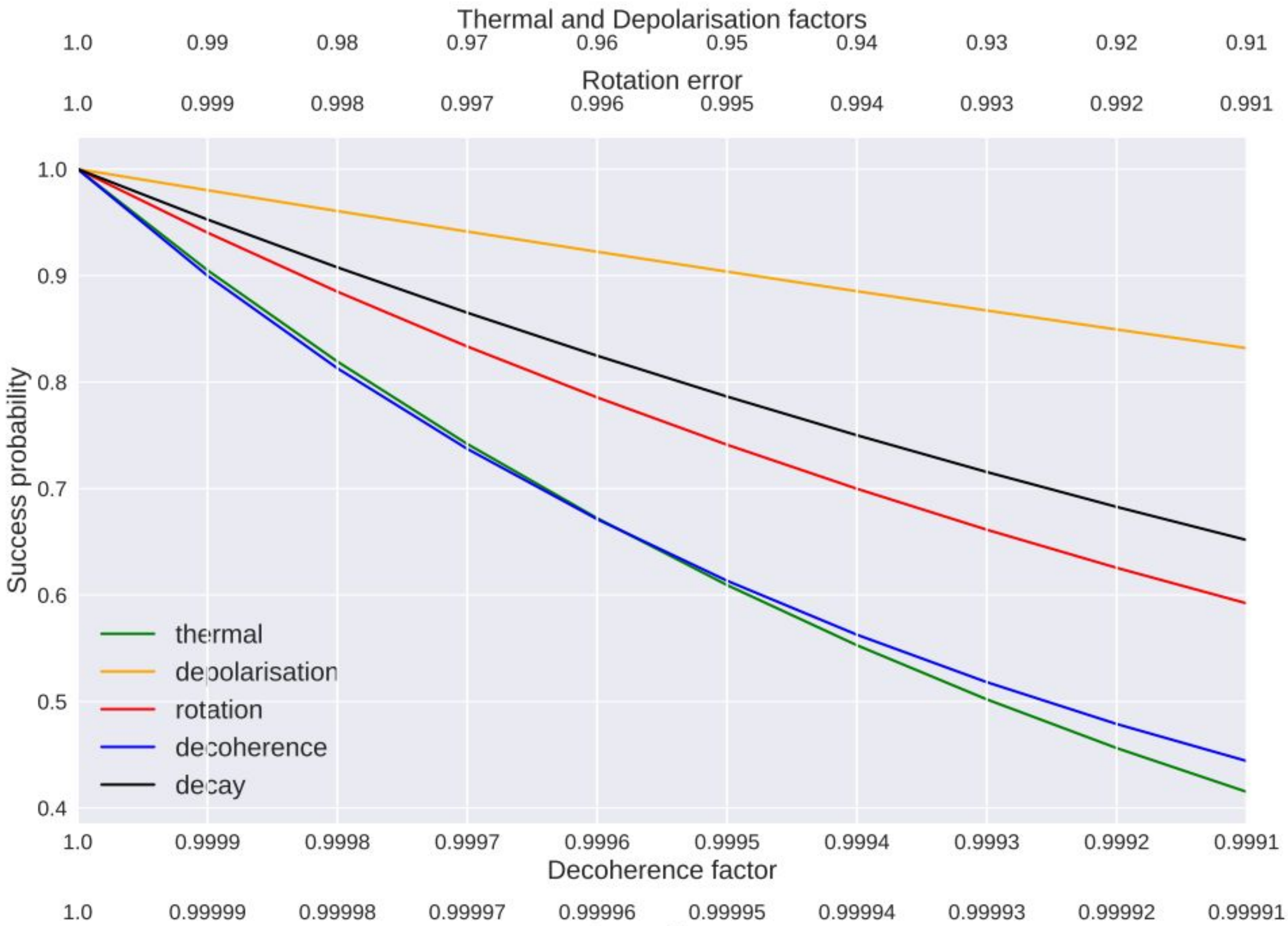
$$M_0 = \sqrt{\frac{1+f}{2}}\, I \;, \quad M_1 = \sqrt{\frac{1-f}{2}}\, \sigma_3$$

Memory Errors : Decay

$$M_0 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{g} \end{pmatrix},\; M_1 = \sqrt{p}\begin{pmatrix} 0 & \sqrt{1-g} \\ 0 & 0 \end{pmatrix},$$

$$M_2 = \sqrt{1-p}\begin{pmatrix} \sqrt{g} & 0 \\ 0 & 1 \end{pmatrix},\; M_3 = \sqrt{1-p}\begin{pmatrix} 0 & 0 \\ \sqrt{1-g} & 0 \end{pmatrix}$$

Experiment: Simulation of Noise Models



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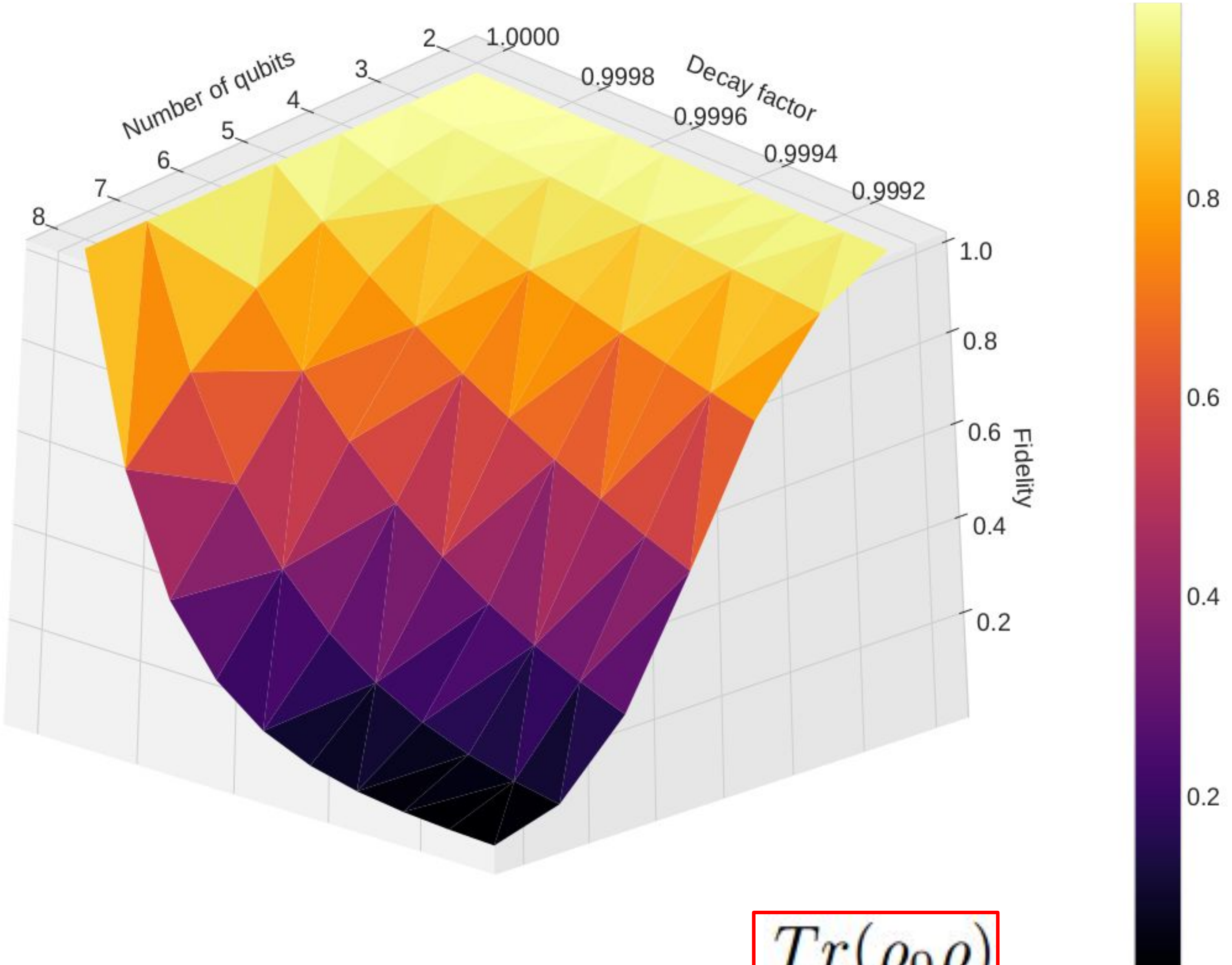
Experiment: Noise modeling of QFT circuit



$$\boxed{Tr(\rho_0 \rho)}$$

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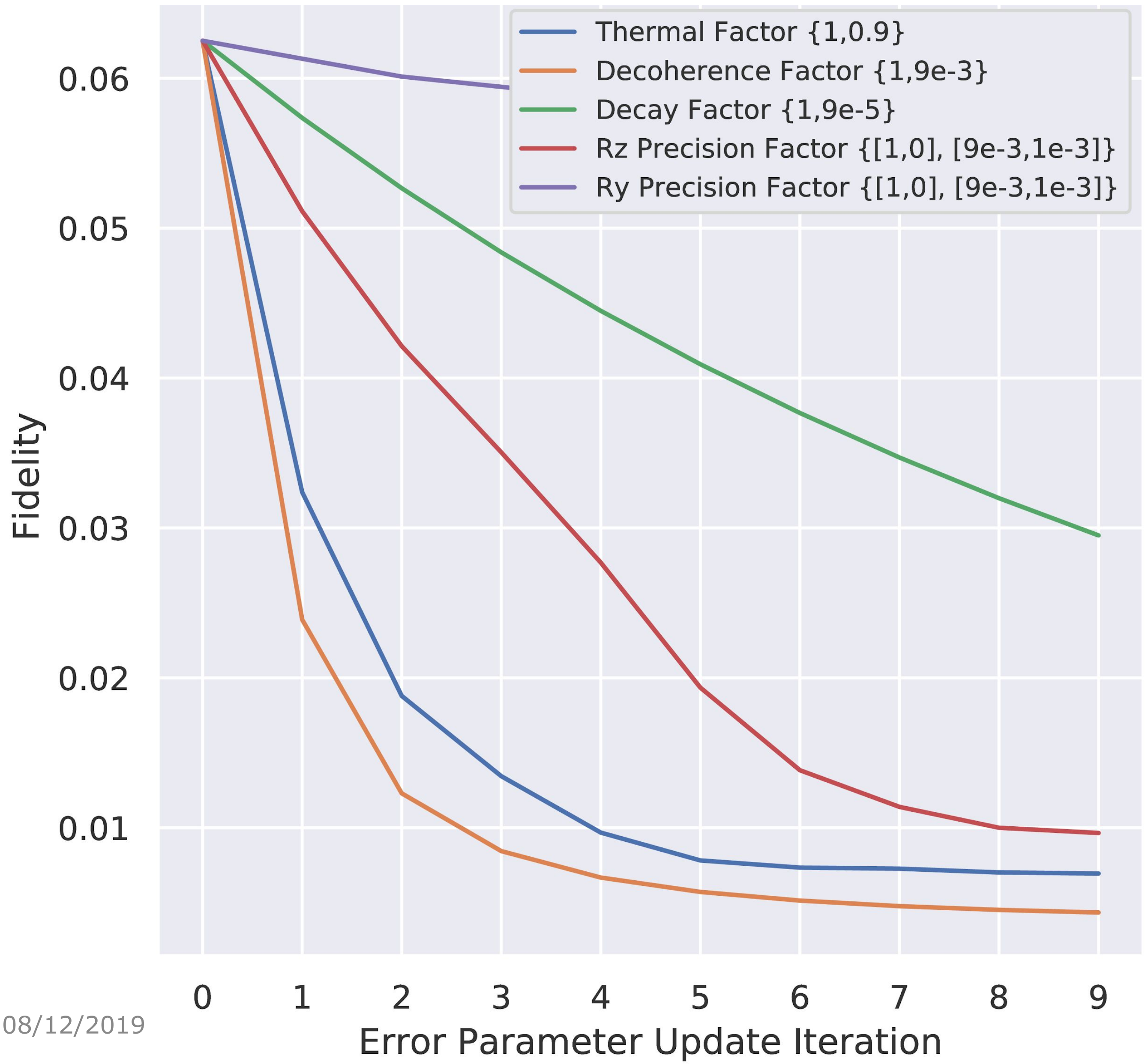
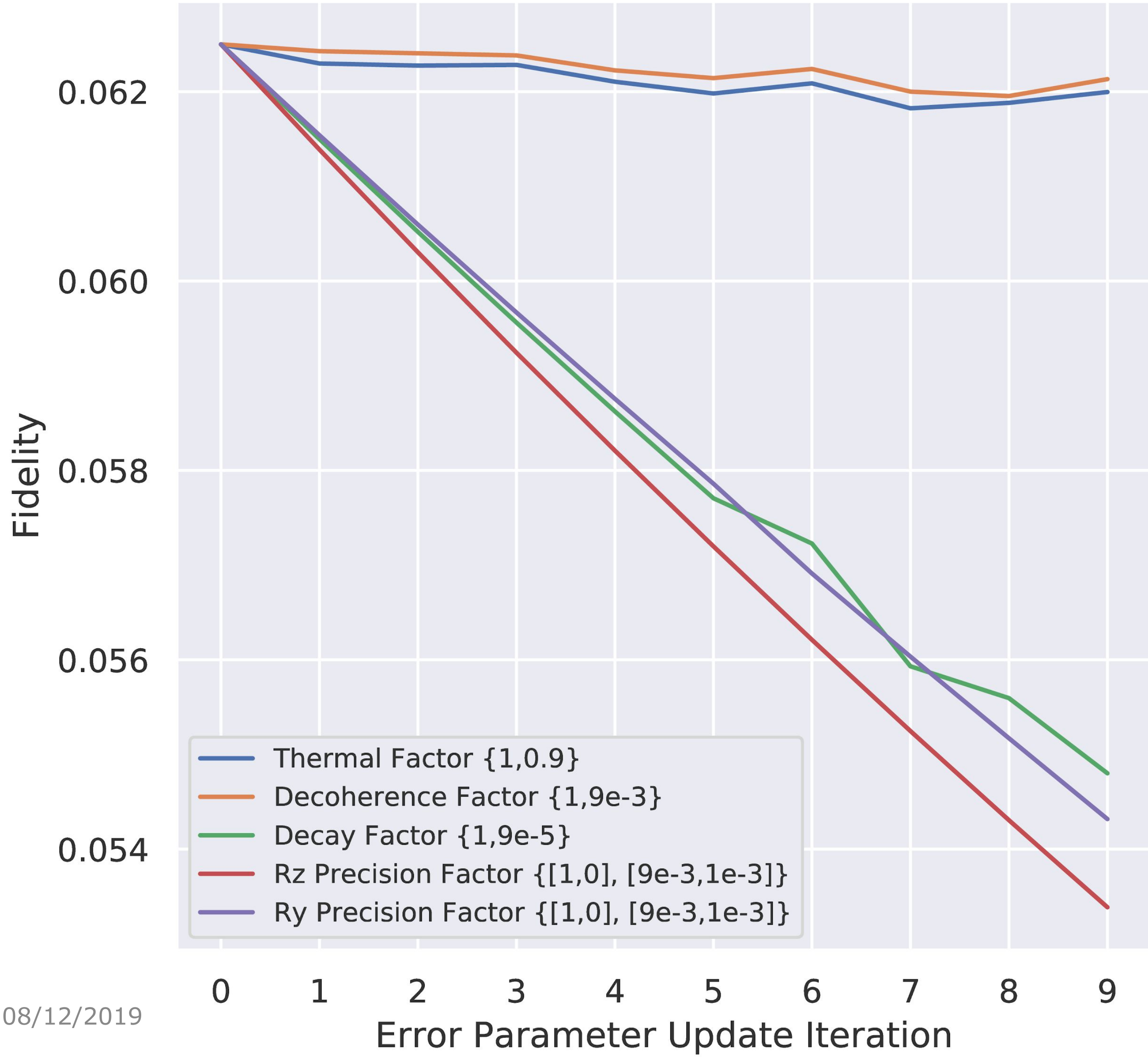
Results: Simulation of QAOA



Results: Simulation of VQE



Conclusion

Output state fidelity, the cost function, and its gradient obtained from QAOA decrease exponentially with respect to the number of gates and noise strength of decoherence, amplitude decay and thermalization factors.

Dependence of success probability of QAOA depends on noise :
Theoretically, $p \to \infty$ is equivalent to adiabatic quantum optimization.

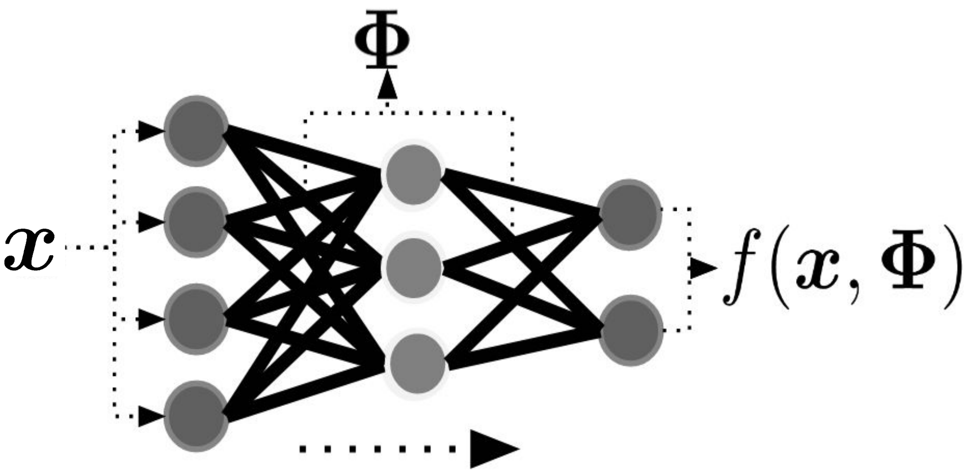
Design exploration of NISQ processors dedicated to executing hybrid quantum-classical algorithms

Determination of effects of noise on specific experimental realisations

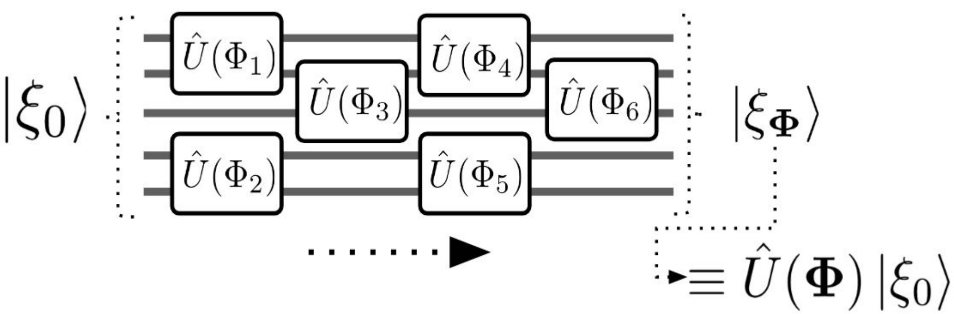
Development of error correction codes

Optimized Circuit Designs for NISQ hardwares

Appendix I : Structural Similarity of Variational Circuits and Classical Feed-Forward Neural Networks



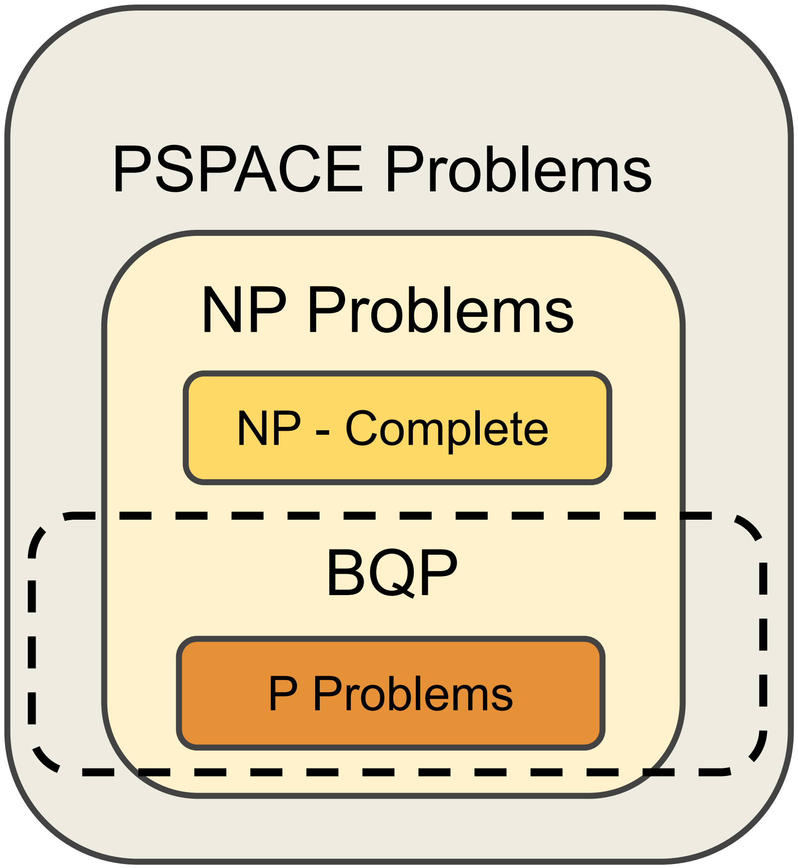
- Input $\boldsymbol{x}$
- Parameters $\boldsymbol{\Phi}$
- Feedforward operation f
- Loss function $L(f(\boldsymbol{x}, \boldsymbol{\Phi}), y)$
- Goal: find $\underset{\boldsymbol{\Phi}}{\operatorname{argmin}}\ L(f(\boldsymbol{x}), \boldsymbol{\Phi}), y)$

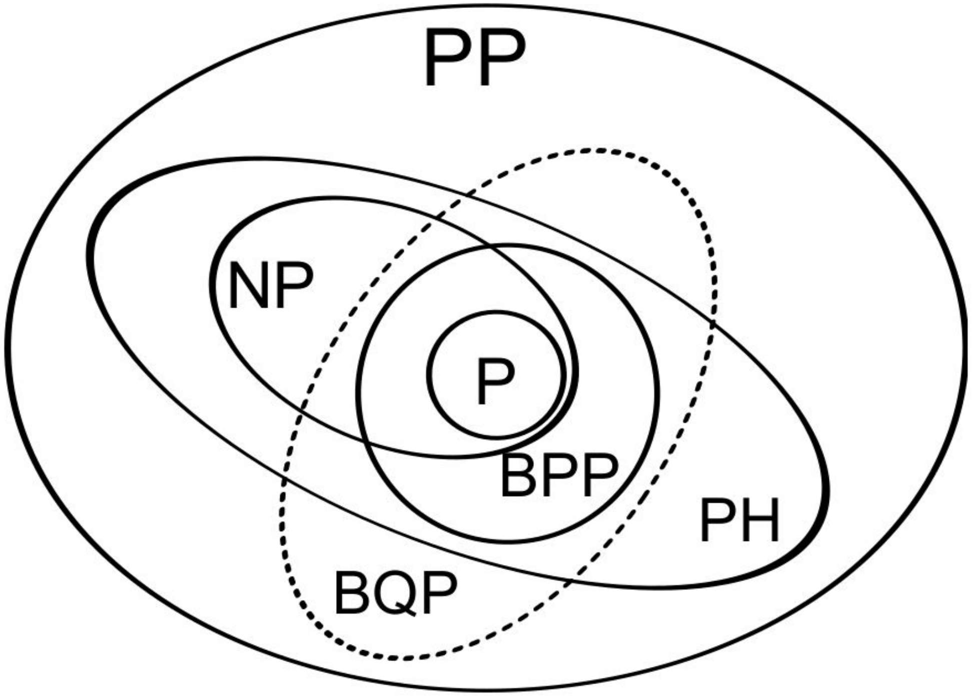


- Input $|\xi_0\rangle$
- Parameters $\boldsymbol{\Phi}$
- Feedforward operation $\hat{U}(\boldsymbol{\Phi})$
- Loss function $\hat{L}$
- Goal: find $\underset{\boldsymbol{\Phi}}{\operatorname{argmin}}\ \langle \xi_{\boldsymbol{\Phi}}|\, \hat{L}\, |\xi_{\boldsymbol{\Phi}}\rangle$

[1] Guillaume Verdon, Jason Pye, Michael Broughton, "A Universal Training Algorithm for Quantum Deep Learning," arXiv:1806.09729

Appendix II : Complexity Classes: P, NP, BQP, QMA, PSPACE





Source: Y. Nakata and M. Murao, "Diagonal quantum circuits: Their computational power and applications", *The European Physical Journal Plus*, vol. 129, no. 7, 2014.

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Thank you!