If our instruments consistently underestimate or overestimate the velocity, we are dealing with an in accurate or biased device. On the other hand, if the measurements are randomly high and low, we are dealing with a question of precision.

Measurement errors can be quantified by summarizing the data with one or more well-chosen statistics that convey as much information as possible regarding specific characteristics of data. These descriptive statistics are most often selected to represent – (1) the location of the center of the distribution of the data and (2) the degree of the spread data. As such, they provide a measure of the bias and imprecision, respectively. Although you must be cognizant of blunders. Formulation errors and uncertain data, the numerical methods used for building models can be studied, for the most part independently of these errors.

3.6 SUMMARY

In this chapter, we have covered the following:

1. Taylor's Series

   $f(x) = f(a) + f'(a)\,(x-a) + f''(a)(x-a)^2/2! + f^{(3)}(a)(x-a)^3/3! + \ldots f^{(n)}(a)(x-a)^n/n!$

2. Error Propogation – Propogated Errors $(xwy - x^* wy^{*)}/xwy,$

   Total errors $(xwy - x^* wy^{*)}/xwy + (x^* wy^* - x^* w^* y^{*)}/x^* wy^*$

3. Stability and Condition

   The condition of a mathematical problem relates to its sensitivity to changes in its input values.

4. Total Numerical Error

5. Blunders, Formulation Errors and Data Uncertainty.

EXERCISE

1. Use the Taylor Series to estimate $f(x) = e^{-x}$ at $x_{i+1} = 1$ for $x_i = 0.2$. Employ the zero, first, second and third-order versions and compute the $|\varepsilon_i|$ for each case.

2. Use zero though third order Taylor series expansions to predict $f(3)$ for

   $f(x) = 25x^3 - 6x^2 + 7x - 88$

   Using a base point at $x = 1$. Compute the true percent relative error ε_1 for each approximation.

3. Use zero through fourth-order Taylor series expansion to predict $f(2.5)$ for $f(x) = \log x$ using a base point at $x = 1$. Compute the true percent relative error ε_1 for each approximation.

4. Evaluate and interpret the condition numbers for –

   (i) $f(x) = e^{-x}$ for $x = 10$

   (ii) $f(x) = \sqrt{x^2 + 1} - x$ for $x = 300$

   (iii) $f(x) = \dfrac{e^{-x} - 1}{x}$ for $x = 0.001$

   (iv) $f(x) = \dfrac{\sin x}{1 + \cos x}$ for $x = 1.0001\,\pi$

5. Given $f(x)$ sin x, construct the Taylor series approximations of orders 0 to 7 at $x = \pi/3$ and state their absolute errors.

6. Evaluate $f(1)$ using Taylors series for $f(x)$, where, $f(x) = x^3 - 3x^2 + 5x - 10$

7. Write down Taylor series expansion of:

   $f(x) = \cos x$ at $x = \pi/3$ in terms of $f(x)$ and its derivatives at $x = \pi/4$. Compute the approximations from the zero order to the fifth order and also state the absolute error in each case.

8. Estimate the error in evaluating the expression

   $x^3 - 2.5x^{-2} + 3.1\,x - 1.5$ at $x = 1.25$ using

   Taylor series to the fifth order

9. Estimate the error in evaluating $f(x, y) = x^2 + y^2$

   for $x = 3.00$ and $y = 4.00$

10. Find the absolute error in $w = xy + z$ if $x = 2.35$, $y = 6.74$ and $z = 3.45$.

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Example 16: Find the root of $f(x) = e^{-x} - x$ with initial value $x_0 = 0$ and $x_1 = 1$

Solution:

Given $f(x) = e^{-x} - x$

$x_0 = 0$ and $x_1 = 1$

$f(x_0) = 1$ and $f(x_1) = -0.63212$

$$x_2 = 1 - \frac{-0.63212(0-1)}{1-(-0.63212)}$$ [By using alternate formula of secant method]

$= 0.6127$

Now $x_1 = 1$ and $x_2 = 0.6127$

$f(x_1) = -0.63212$ and $f(x_2) = -0.07081$

Putting above values in formula we get,

$\therefore \; x_3 = 0.56384$

and $x_4 = 0.56717$.

EXERCISE

(A) Obtain the root for each of the following equations using bisection Method.

1. $x^3 + x^2 - 1 = 0$
2. $x^3 - 4x + 9 = 0$ with $x_0 = 2$ and $x_1 = 3$
3. $x^3 + 2x^2 + 2.2x + 0.4$ with $x_0 = -1$ and $x_1 = 0$
4. $x^3 + x^2 - 100 = 0$ with $x_0 = 4$ and $x_1 = 5$
5. $x^3 + x - 1 = 0$ with $x_0 = 0$ and $x_1 = 1$
6. $x^3 - x^2 - 1 = 0$
7. $x^3 - 5x + 3 = 0$
8. $x^3 - x - 4 = 0$
9. $x^3 - 3x - 5 = 0$
10. $x^3 + x^2 + x + 7 = 0$

(B) Obtain the root for each of the following equations by Regula-Falsi Method (False Method)

11. $x^3 + x^2 + x - 100 = 0$ with $x_0 = 0$ and $x_1 = 1$
12. $x^3 - 8x + 40 = 0$ with $x_0 = -5$ and $x_1 = -4$
13. $x \tan x + 1 = 0$ with $x_0 = 2.5$ and $x_1 = 3$
14. $xe^x = 2$ with $x_0 = 0$ and $x_1 = 1$
15. $x^3 + x^2 + x + 7 = 0$
16. $x^3 - x - 4 = 0$
17. $x^3 - x^2 - 1 = 0$

18. $x^3 - x - 1 = 0$
19. $x^3 + x - 1 = 0$
20. $\cos x = 3x - 1$

(C) Obtain the root for each of the following equations by Newton Raphsón Method.

21. $x^4 - x - 10 = 0$ with root between 1 and 2
22. $x - e^{-x} = 0$
23. $xe^x - \cos x = 0$
24. $x^3 - x^2 + x + 100 = 0$
25. $\sin x = 1 - x$
26. $4(x - \sin x) = 1$
27. $x + \log x = 2$
28. $x - \cos x = 0$
29. $x^3 + 3x^2 - 3 = 0$
30. $x^4 + x^2 - 80 = 0$

(D) Obtain the root for each of the following equations by secant method.

31. $x^{2.2} - 69 = 0$ with $x_o = 5$ and $x_1 = 8$
32. $xe^x - 1 = 0$
33. $x^2 - 2x - 1 = 0$ with $x_o = 2.6$ and $x_1 = 2.5$
34. $x^3 - x + 3 = 0$ with $x_o = 1.7$ and $x_1 = 1.67$
35. $x^3 - x + 2 = 0$ with $x_o = -1.5$ and $x_1 = -1.52$
36. $\cos x + 2 \sin x + x^2$ with $x_o = -0.1$ and $x_1 = 0$
37. $x^3 - 20 = 0$ with $x_o = 4$ and $x_1 = 5.5$
38. $x^6 - x - 1 = 0$ with $x_o = 2$ and $x_1 = 1$

4.3. ITERATIVE METHOD AND CONVERGENCE CRITERIA

Let $\{x_n\}$ be a sequence of iterates of a required root α of the equation $f(x) = 0$ generated by a given method.

The error of the end of n^{th} iteration denoted by e_n is given by

$$e_n = |\alpha - x_n|$$

The sequence of iterates $\{x_n\}$ converges to α if and only if $e_n \to 0$ as $n \to \infty$ otherwise the sequence of iterates diverges.

4.3.1 Order of Convergence of Iterative Methods

If an iterative method converges, that is if $\{x_n\}$ converges to the desired root α_1 and two constants $p \geq 1$ and $C > 0$ exist such that

$$\lim_{h\to\infty} \left| \frac{e_{n+1}}{e_\eta^p} \right| = C \text{ (} C \text{ dose not depend on } n\text{)}$$