Asymptotic Notations

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1 Small-O(o)

$f(n) = o(g(n))$ means for all $c > 0$ there exists some $n_0 > 0$ such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0$.

Example 1.1 $f(n) = n^2$ is $O(n^2)$ and $\theta(n^2)$. Is $f(n) = o(n^2)$?

Sol: No, $f(n)$ is not $o(n^2)$.

Proving by contradiction method.

Claim: Suppose if possible, $f(n) = o(n^2)$ for all, $c > 0$ there exists some $n_0 > 0$ such that $0 \leq f(n) < cn^2\ \forall\ n \geq n_0$.

Proof: $f(n) < cn^2$ which implies $n^2 < cn^2$whenever $c > 1$.
But we said that this should be true $\forall\ c > 0$. So, let us check for some $0 < c < 1$.

Let $c = \frac{1}{2}$

We get,

=> $$n^2 < \frac{1}{2}n^2$$

but this is not true for any n. Thus, we can say that $f(n) \neq o(n^2)$.

Example 1.2 $f(n) = n$. $f(n) = O(n^2)$ and $f(n) \neq \theta(n^2)$. Is $f(n) = o(n^2)$?

Sol: By definition, we know that

$$f(n) = o(n^2),$$

if $\exists$ some $n_0 > 0$, $\forall\ c > 0$

such that,

$$0 \leq f(n) < cn^2,\ \forall\ n \geq n_0.$$

Let $c > 0$, be any arbitrary constant. Then, for $f(n) = o(n^2)$ we must have

=> $$f(n) < cn^2$$

=> $$n < cn^2$$

=> $$1 < cn$$

=> $$\frac{1}{c} < n$$

For every c,

=> $$nc = \frac{1}{c}$$

Thus we can say,

=> $$n < cn^2\ \forall\ n > \frac{1}{c}$$

Hence,

$$f(n) = o(n^2).$$

Remark: If a function $f(n)$ is $O(g(n))$ but not $\theta(g(n))$, then $f(n)$ is $o(g(n))$.

Function 1 $f(n)$	**Function 2** $g(n)$	**Big O** O	**Big theta** θ	**Small O** o
n	$n+100$	T	T	F
2^n	3^n	T	F	T
$n \log n^2$	$n \log n^4$	T	T	F
$2^{\log n}$	n	T	T	F
$2^{\log \sqrt{n}}$	$2^{\log n}$	T	F	T

Theorem 2.1: If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then $f(x) = O(g(x))$ and also $f(x) = o(g(x))$.

Theorem 2.2: If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = c$, for some constant c, then $f(x) = O(g(x))$ and also $f(x) = \theta(g(x))$.

Example 2.1: $f(n) = \log n$, $g(n) = n$. Is $f(n) = O(g(n))$ and $f(n) = o(g(n))$?

Sol: By using limits, finding

$$\lim_{n\to\infty} \frac{f(n)}{g(n)},$$

i.e.

=> $$\lim_{n\to\infty} \frac{\log n}{n}$$

=> $$\lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

(using L'Hôpital's rule)

So, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, Hence f(n)=O(g(n)) and also f(n)=o(g(n)).

Example 2.2: $f(n) = \log^5 n$, $g(n) = \sqrt{n}$. Is $f(n) = o(g(n))$?

Sol: By using limits, finding

$$\lim_{n\to\infty} \frac{f(n)}{g(n)},$$

i.e.

=> $$\lim_{n\to\infty} \frac{\log^5 n}{\sqrt{n}}$$

=> $$\lim_{n\to\infty} \frac{5.\log^4 n.\frac{1}{n}}{\frac{1}{2\sqrt{n}}}$$

(using L'Hôpital's rule)

=> $$\lim_{n\to\infty} \frac{5.\log^4 n}{\frac{1}{2}.\sqrt{n}}$$

=> $$\lim_{n\to\infty} \frac{5.4.\log^3 n.\frac{1}{n}}{\frac{1}{2}.\frac{1}{2\sqrt{n}}}$$

(using L'Hôpital's rule)

=> $$\lim_{n\to\infty} \frac{5.4.\log^3 n}{(\frac{1}{2})^2.\sqrt{n}}$$

...

...

=> $$\lim_{n\to\infty} \frac{5!}{(\frac{1}{2})^5 . \frac{1}{\sqrt{n}}}$$

So, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, Hence, $f(n) = o(g(n))$.

Home Work

3.1 Prove that, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ implies, $f(n) = o(g(n))$.

3.2 Prove that, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$, c is a constant, implies $f(n) = \theta(g(n))$.

3.3 Assume that $\log n = o(n)$ and prove that, $\log^M n = o(n^\epsilon)$, however large M is and however small the value of ϵ is, $\epsilon > 0$.