# Implementation of nCr for big values of n

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## 1 Introduction

$nCr$ is the coefficient of the $x^r$ term in the polynomial expansion of $(1+x)^n$ and is given by the formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

To calculate $nCr$ one can find the prime factors of $n!$, $r!$, and $(n-r)!$ and then cancel out the terms. So in essence all we have to do is find the powers of the prime factors of the above mentioned factorials and then subtract the powers of prime factors of $r!$ and $(n-r)!$ from that of $n!$

As an example

$$\binom{5}{3} = \frac{5!}{3!2!}$$

$$= \frac{5.4.3.2.1}{3.2.1 * 2.1}$$

$$= \frac{2^3.3^1.5^1}{2^2.3^1}$$

$$= 2^1.5^1$$

One can verify last equation at hackmath

## 2 Proof of Correctness

How do we know that the answer obtained is correct? One can always calculate it but it isn't practical. On the other hand giving a rigorous proof requires knowledge of Number Theory.

Perhaps the best way of going about this is by proving that the algorithm results in an integer, that is no power of any prime factor is negative.

In $nCr$ we have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Now a key observation here is that the sum of the numbers that we are taking the factorial of in the denominator is exactly equal to the number in the numberator of which we are taking the factorial. Meaning $n = r + (n-r)$. So we can if prove generally that

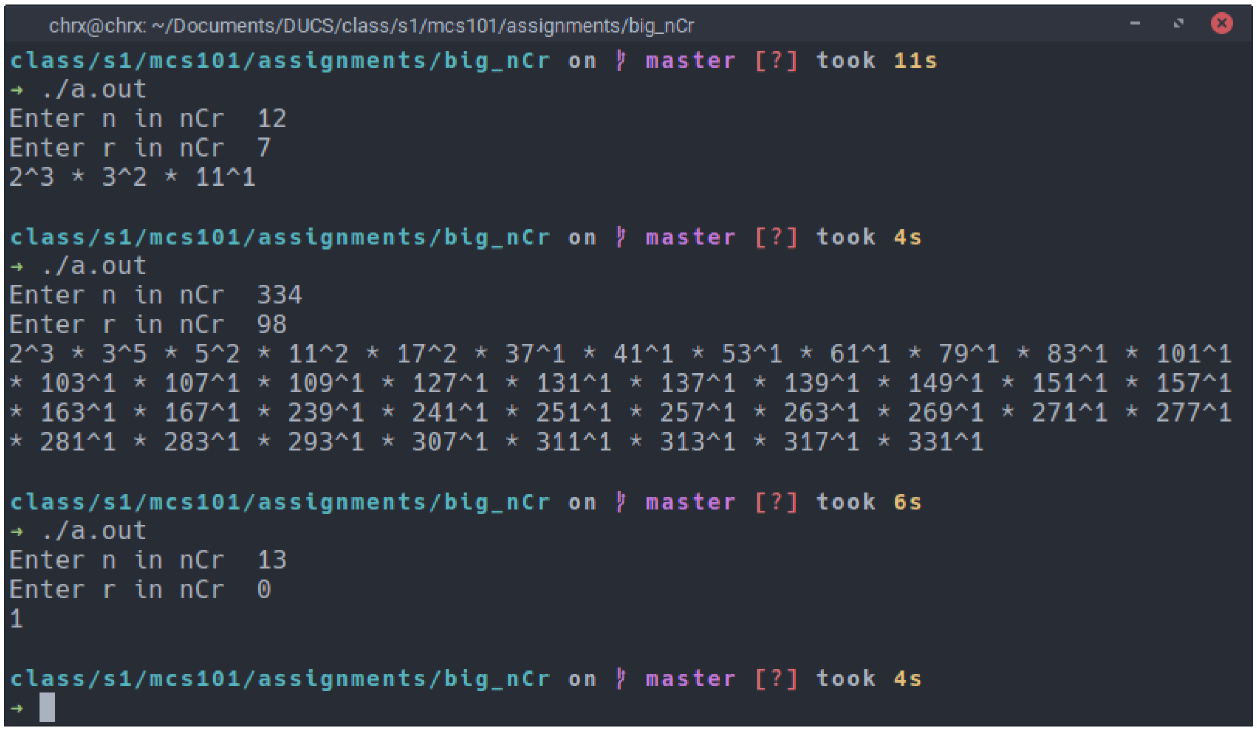
$$\frac{(n_1 + ... + n_k)!}{n_1!...n_k!}$$

is an integer(which can be proved by Lagrange's theorem) then we can conclude the nCr proof.

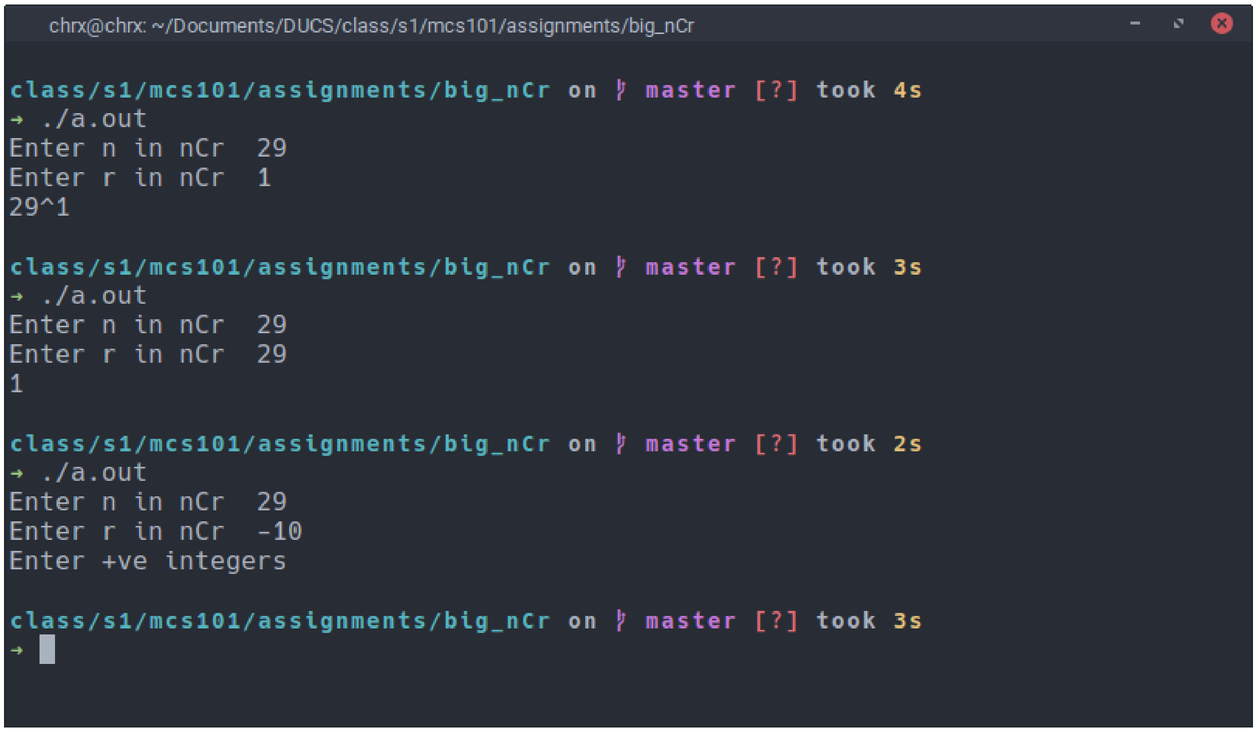
## 3 Code

```cpp
#include <iostream>
#include <cmath>
#include <vector>

bool isPrime(int x){
    if(x == 2 || x == 3) return true;
    for(int i = 2; i * i <= x; i++){
        if(x % i == 0) return false;
        else if((i + 1) > sqrt(x)) return true;
    }
}

//occurence of f in n!
int fact(int n, int f){
    int sum = 0;
    while(n >= f){
        sum += (int) n / f;
        n = (int) n / f;
    }
    return sum;
}

int main(){
    int n,r,nr;
    std::cout<<"Enter n in nCr\t";
    std::cin>>n;
    std::cout<<"Enter r in nCr\t";
    std::cin>>r;
    nr = n - r;

    if(n < 0 || r < 0){
        std::cout<<"Enter +ve integers\n";
        return 0;
    }
    if(r > n){
        std::cout<<"0\n";
        return 0;
    }
    if(r == 0 || n == r){
        std::cout<<"1\n";
        return 0;
    }

    //vectors for prime factors of n,r,and n-r
    std::vector<int> nVec;
    std::vector<int> rVec;
    std::vector<int> nrVec;

    std::vector<int> nFac(n);
    std::vector<int> rFac(n);
    std::vector<int> nrFac(n);

    for(int i = 2; i <= n; i++){
        //check if i is prime
        if(!isPrime(i)) continue;
        nVec.push_back(i);
    }

    for(int i = 2; i <= r; i++){
        //check if i is prime
        if(!isPrime(i)) continue;
        rVec.push_back(i);
    }

    for(int i = 2; i <= nr; i++){
        //check if i is prime
        if(!isPrime(i)) continue;
        nrVec.push_back(i);
    }

    for(int i = 0; i < nVec.size(); i++){
        nFac[nVec[i] - 2] = fact(n, nVec[i]);
    }

    for(int i = 0; i < rVec.size(); i++){
        rFac[rVec[i] - 2] = fact(r, rVec[i]);
    }

    for(int i = 0; i < nrVec.size(); i++){
        nrFac[nrVec[i] - 2] = fact(nr, nrVec[i]);
    }

    bool show = false;

    for(int i = 0; i < nFac.size(); i++){
        nFac[i] = nFac[i] - rFac[i] - nrFac[i];
        if(nFac[i] != 0){
            if(show) std::cout<<" * ";
            std::cout<<i + 2<<"^"<<nFac[i];
            show = true;
        }
    }
    std::cout<<"\n";
    return 0;
}
```

## 4 Output



```
chrx@chrx: ~/Documents/DUCS/class/s1/mcs101/assignments/big_nCr
class/s1/mcs101/assignments/big_nCr on ᚠ master [?] took 11s
➜ ./a.out
Enter n in nCr  12
Enter r in nCr  7
2^3 * 3^2 * 11^1

class/s1/mcs101/assignments/big_nCr on ᚠ master [?] took 4s
➜ ./a.out
Enter n in nCr  334
Enter r in nCr  98
2^3 * 3^5 * 5^2 * 11^2 * 17^2 * 37^1 * 41^1 * 53^1 * 61^1 * 79^1 * 83^1 * 101^1
* 103^1 * 107^1 * 109^1 * 127^1 * 131^1 * 137^1 * 139^1 * 149^1 * 151^1 * 157^1
* 163^1 * 167^1 * 239^1 * 241^1 * 251^1 * 257^1 * 263^1 * 269^1 * 271^1 * 277^1
* 281^1 * 283^1 * 293^1 * 307^1 * 311^1 * 313^1 * 317^1 * 331^1

class/s1/mcs101/assignments/big_nCr on ᚠ master [?] took 6s
➜ ./a.out
Enter n in nCr  13
Enter r in nCr  0
1

class/s1/mcs101/assignments/big_nCr on ᚠ master [?] took 4s
➜ 
```



```
chrx@chrx: ~/Documents/DUCS/class/s1/mcs101/assignments/big_nCr

class/s1/mcs101/assignments/big_nCr on ᚠ master [?] took 4s
➜ ./a.out
Enter n in nCr  29
Enter r in nCr  1
29^1

class/s1/mcs101/assignments/big_nCr on ᚠ master [?] took 3s
➜ ./a.out
Enter n in nCr  29
Enter r in nCr  29
1

class/s1/mcs101/assignments/big_nCr on ᚠ master [?] took 2s
➜ ./a.out
Enter n in nCr  29
Enter r in nCr  -10
Enter +ve integers

class/s1/mcs101/assignments/big_nCr on ᚠ master [?] took 3s
➜ 
```

## 5 Runtime

RunTime of the algorithm in the worst cast is $O(n\sqrt{n})$ where n is the $n$ in $nCr$ or $\binom{n}{r}$. The run time can be validated by observing that for a number $n$ we find all the prime numbers upto $n$ which takes $n.O(\sqrt{n}) = O(n\sqrt{n})$ time. Now for numbers : $n$, $r$, and $n - r$ we calculate the powers of prime factors that constitute towards $n!$, $r!$, and $(n - r)!$ respectively. This is done in the function *fact* of the code and takes $O(logn)$ in the worst case. At last subtracting the powers of the prime factors take $\theta(n)$ time. Hence the entire algorithm runs in $O(n\sqrt{n})$ time.