# 8

# NUMERICAL DIFFERENTIATION

## 8.1 Introduction

Numerical differentiation is the process of finding the value of the derivative of a function at some particular value of the independent variable when the values of the function corresponding to the given values of the independent variable are known. The problem of differentiation is solved by first approximating the function by a polynomial by means of an interpolation formula and then differentiating this formula as many times as desired. If the values of the arguments equally spaced, we represent the function by Newton Gregory formula. According to the requirement, we use the formula *i.e.* Newton's forward, Newton's backward, Stirling, Bessel's etc. If the entries are given for unequal intervals of the arguments, one can use, Newton's divided difference formula.

Now, we discuss different methods for finding the derivative of the interpolation the function using various technique.

## 8.2 Derivative using Newton's forward Interpolation Formula

Newton's forward interpolation formula is given by

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$y = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\cdot\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\cdot\Delta^3 y_0 + \dots \qquad \dots(1)$$

where $$p = \frac{x - x_0}{h} \qquad \dots(2)$$

Differentiating equation (1) w.r.t $p$, we get

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{\lfloor 2}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{\lfloor 3}\Delta^3 y_0 + \frac{4p^3 - 18p^2 + 22p - 6}{\lfloor 4}\Delta^4 y_0 + \dots \qquad \dots(2)$$

Now from [redacted] where condition -

$$p = \frac{x - x_0}{h}$$

differentiation w.r.t $x$

$$\Rightarrow \frac{dp}{dx} = \frac{1}{h}(1-0) = \frac{1}{h}$$

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}$$

$$= \frac{1}{h}\cdot\frac{dy}{dp}$$

Using this in equation (2), we get

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{\lfloor 2}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{\lfloor 3}\Delta^3 y_0 + \frac{4p^3 - 18p^2 + 22p - 6}{\lfloor 4}\Delta^4 y_0 + \dots\right] \qquad \dots(3)$$

Equation (3) gives the value of $\frac{dy}{dx}$ at any $x$ which is not tabulated.

When $x = x_0$, $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right] \quad \ldots(4)$$

Differentiating (4) w.r.t. $x$, we get

i.e.

$$\frac{d^2 y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\cdot\frac{dp}{dx} = \frac{1}{h}\cdot\frac{d}{dp}\left(\frac{dy}{dx}\right)$$

$$\frac{d^2 y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2 - 18p + 11}{12}\Delta^4 y_0 + \ldots\right] \quad \ldots(5)$$

On putting $x = x_0$ *i.e.*, $p = 0$, we get

$$\left(\frac{d^2 y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\right] \quad (6)$$

Eq. (4) & (6) are called first & second order derivatives. Continuing this process we can find higher order derivatives.

**Aliter**

We know that $E = e^{hD}$

$\Rightarrow \quad 1 + \Delta = e^{hD}$

or $\quad \log(1 + \Delta) = hD$

$$\Rightarrow \quad hD = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots$$

$$\Rightarrow \quad D = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\right]$$

$$D^2 = \frac{1}{h^2}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\right]^2$$

$$= \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \ldots\right]$$

Applying the above identities to $y_0$, we have,

$$Dy_0 = \left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

$$D^2 y_0 = \left(\frac{d^2 y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\right].$$

### 8.2.1 Algorithm for Numerical Differentiation using Newton Forward Interpolation Method

Step 1. Input the number of terms.

Step 2. Input the value of $x_i$ and $y_i$.

Step 3. Input the value of x for which dy/dx is calculated.

Step 4. Obtain h by using $x_i - x_{i-1}$.

Step 5. Construct the differences by using

$$\Delta^n y_{i-1} = \Delta^{n-1} y_i - \Delta^{n-1} y_{i-1}.$$

Step 6. Find $u = (x - x_0)/h$.

Step 7. Put all values in Newton -Gregory forward interpolation formula for differentiation.

Step 8. Print the results.

```
else
   sign = -1;
   ans=ans+(sign * diff[i][j]/(j+1));
  }
 ans=ans/(x[1]-x[0]);
 printf("\n\n\t The value of dy/dx when x=%f is (dy/dx)=%f",x0,ans);
 getch();
}
```

```
/* Output */
   Enter how many values you want: 6
   Enter the value of x0 and y0: 1  1
   Enter the value of x1 and y1: 2  8
   Enter the value of x2 and y2: 3  27
   Enter the value of x3 and y3: 4  64
   Enter the value of x4 and y4: 5  125
   Enter the value of x5 and y5: 6  216
   Now Enter the value of x for which u want value of (dy/dx):1
   The value of dy/dx when x=1.000000 is (dy/dx)=3.000000
```

## 8.3 Derivatives using Newton's Backward Interpolation Formula

We have the Newton's backward interpolation formula

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{\lfloor 2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{\lfloor 3}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{\lfloor 4}\nabla^4 y_n + \ldots \quad \ldots(1)$$

where $p = \frac{x - x_n}{h}$, ($h$ being the interval of differencing) ...(2)

Differentiating equation (1) w.r.t. $p$ we get

$$\frac{dy}{dp} = \nabla y_n + \frac{(2p+1)}{\lfloor 2}\nabla^2 y_n + \frac{(3p^2+6p+2)}{\lfloor 3}\nabla^3 y_n + \frac{(4p^3+18p^2+22p+6)}{\lfloor 4}\nabla^4 y_n + \ldots \quad \ldots(2)$$

From where Condition -

On differentiating (2) w.r.t. $x$, we get

$$\frac{dp}{dx} = \frac{1}{h}$$

where $p = \frac{x - x_0}{h} \Rightarrow \frac{dp}{dx} = \frac{1}{h}$

Now, $$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\cdot\frac{dy}{dp}$$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{(2p+1)}{\lfloor 2}\nabla^2 y_n + \frac{(3p^2+6p+2)}{\lfloor 3}\nabla^3 y_n + \frac{(4p^3+18p^2+22p+6)}{\lfloor 4}\nabla^4 y_n + \ldots\right] \quad \ldots(3)$$

Equation (3) gives us the value of $\frac{dy}{dx}$ at any $x$ which is not tabulated.

At $x = x_n$, on putting $p = 0$ in (4), we have

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right] \quad \ldots(4)$$

Again differentiating (3) w.r.t $x$, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx}$$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + (p+1)\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \ldots\right] \quad \ldots(5)$$

On putting $p = 0$ in (5), we have

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\cdot\nabla^4 y_n + \ldots\right] \quad \ldots(6)$$

Eq (4) & (6) are called first & Second order derivatives for N.B.D.

Continuing this process we can find higher order derivatives.

**Aliter** using $E = e^{hD} = \frac{1}{1-\nabla}$

We get $$D = -\frac{1}{h}\log(1-\nabla)$$

$\therefore$ $$D = \frac{1}{h}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \ldots\right]$$

$$D^2 = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \ldots\right]$$

$$D^3 = \frac{1}{h^3}\left[\nabla^3 + \frac{3}{2}\nabla^4 + \ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} \cdots\right]$$

By using the same procedure, we can find higher order derivatives.

## 8.5 Derivatives using Bessel's Formula

Let us consider the Bessel's formula:

$$y = \frac{y_0 + y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u\,(u-1)}{\lfloor 2} \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(u - \frac{1}{2}\right) u\,(u-1)}{\lfloor 3}\Delta^3 y_{-1} + \dots$$

where $$u = \frac{x - x_0}{h}.$$

Differentiating above equation w.r.t. $u$, we have,

$$\frac{dy}{du} = \Delta y_0 + \frac{(2u-1)}{\lfloor 2}\cdot\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{(3u^2 - 3u + \frac{1}{2})}{\lfloor 3}\Delta^3 y_{-1} + \dots$$

Also we have

$$\frac{du}{dx} = \frac{1}{h}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\cdot\frac{dy}{du}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{\lfloor 2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{\left(3u^2 - 3u + \frac{1}{2}\right)}{\lfloor 3}\Delta^3 y_{-1} + \dots\right]$$

By using the same procedure, we can find higher order derivatives.

*Illustrative Examples*

**Example 8.1** Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $x = 1$, using following table: by using Derivatives of newton's Forward Interpolation formula

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ : | 1 | 2 | 3 | 4 | 5 | 6 |
| $y$ : | 1 | 8 | 27 | 64 | 125 | 216 |

**Solution:** The difference table is

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| 1 | 1 | | | | |
| | | 7 | | | |
| 2 | 8 | | 12 | | |
| | | 19 | | 6 | |
| 3 | 27 | | 18 | | 0 |
| | | 37 | | 6 | |
| 4 | 64 | | 24 | | 0 |
| | | 61 | | 6 | |
| 5 | 125 | | 30 | | |
| | | 91 | | | |
| 6 | 216 | | | | |

We have $x_0 = 1,\ h = 1,\ x = 1$

$$\therefore \qquad P = \frac{x - x_0}{h} = \frac{1-1}{1} = 0.$$

$\therefore$ By Newton's forward formula: for first order derivatives N.B.I.F.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1} = \frac{1}{1}\left[7 - \frac{1}{2}\times 12 + \frac{1}{3}\times 6 - 0 + \ldots\right]$$

$$= [7 - 6 + 2] = 3.$$

= 108

and for second order derivatives

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \ldots\right] = \frac{1}{1^2}[12 - 6] = 6$$

= 36

$$\therefore \qquad \left(\frac{dy}{dx}\right)_{x=1} = 3 \text{ and } \left(\frac{d^2y}{dx^2}\right)_{x=1} = 6.$$

**Example 8.2** The population of a certain town is given below. Find the rate of growth of the population in 1931, 1941, 1961, and 1971, by using derivatives of newton's forward and backward interpolation formula ?

| Year ($x$) : | 1931 | 1941 | 1951 | 1961 | 1971 |
|---|---|---|---|---|---|
| *Population in thousands* ($y$) : | 40.62 | 60.80 | 79.95 | 103.56 | 132.65 |

**Solution:** The difference table is as follows:

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| 1931 | 40.62 | | | | |
| | | 20.18 | | | |
| 1941 | 60.80 | | −1.03 | | |
| | | 19.15 | | 5.49 | |
| 1951 | 79.95 | | 4.46 | | −4.47 |
| | | 23.61 | | 1.02 | |
| 1961 | 103.56 | | 5.48 | | |
| | | 29.09 | | | |
| 1971 | 132.65 | | | | |

Using Derivatives of N.F.I.F.: $f''(1931) = -0.106$

(*i*) To get $f'$ (1931) and $f'$ (1941) we use forward formula

$$x_0 = 1931, x_1 = 1941$$

$\therefore$ $$p = \frac{x - x_0}{h} = 0, \text{ when } x = 1931.$$ From eq. 4

$$\therefore \left(\frac{dy}{dx}\right)_{x=1931} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

$$= \frac{1}{10}\left[20.18 - \frac{1}{2}(-1.03) + \frac{1}{3}(5.49) - \frac{1}{4}(-4.47)\right]$$

$$= \frac{1}{10}[20.18 + 0.515 + 1.83 + 1.1175]$$

$$= 2.36425. \quad \ldots(i)$$

(*ii*) When $x = 1941$, $p = \dfrac{1941 - 1931}{10} = 1$, From eq. 3

Putting $p = 1$, in

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p - 1}{2}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{6}\Delta^3 y_0 + \frac{4p^3 - 18p^2 + 22p - 6}{24}\Delta^4 y_0 + \ldots\right]$$

We get, $$\left(\frac{dy}{dx}\right)_{p=1} = \frac{1}{10}\left[20.18 + \frac{1}{2}(-1.03) - \frac{1}{6}(5.49) + \frac{1}{12}(-4.47)\right]$$

$$= \frac{1}{10}[20.18 - 0.515 - 0.915 - 0.3725]$$

$$= 1.83775. \quad \ldots(ii)$$

**Note:** If we neglect the data against 1931 and take 1941 as $x_0$, we have

$$\Delta y_0 = 19.15, \ \Delta^2 y_0 = 4.46, \ \Delta^3 y_0 = 1.02$$

Now, using;

X

$$\left(\frac{dy}{dx}\right)_{x=1941} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \ldots\right]$$

$$= \frac{1}{10}\left[19.15 - \frac{1}{2}(4.46) + \frac{1}{3}(1.02)\right]$$

$$= 1.7260. \quad \ldots(iii)$$

Evidently the values given by (*ii*) and (*iii*) are not the same. In getting the answer by (*ii*) we have assumed a polynomial of degree four whereas in getting the answer by (*iii*), we have assumed the interpolating polynomial of degree three only. In fact, both polynomials considered above are different. Hence we see the difference in answers.

(*iii*) To get $f'$ (1971) we use the Newton, Backward formula, *i.e.* , From eq 4

Using Derivatives of N.B.I.F.:

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right]$$

$\therefore$ $p = \dfrac{x - x_n}{h}$

$p = \dfrac{1971 - 1971}{10}$

$p = 0$

$$= \frac{1}{10}\left[29.09 + \frac{1}{2}(5.48) + \frac{1}{3}(1.02) + \frac{1}{4}(-4.47)\right]$$

$$\left(\frac{dy}{dx}\right)_{1971} = \frac{1}{10}[31.0525] = 3.10525.$$

(iv) To find $f'$ (1961), we use , From eq. 3

$$p = \frac{x - x_n}{h} = \frac{1961 - 1971}{10} = -1$$

$$\left(\frac{dy}{dx}\right)_{x=1961} = \left(\frac{dy}{dx}\right)_{v=-1} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2}\nabla^2 y_n + \frac{3p^2 + 6p + 2}{6}\nabla^3 y_n + \ldots\right]$$

$$= \frac{1}{10}\left[29.09 - \frac{1}{2}(5.48) - \frac{1}{6}(1.02) - \frac{1}{12}(-4.47)\right]$$

$$= \frac{1}{10}[29.09 - 2.74 - 0.17 + 0.3725]$$

$$= 2.65525.$$

**Example 8.3** *A rod is rotating in a plane about one of its ends. If the following table gives the angle radians through which the rod has turned for different values of time t seconds, find it's angular velocity and acceleration when t = 0.7 sec.*

| $t$ (seconds) : | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $\theta$ (radians) : | 0.0 | 0.12 | 0.48 | 1.10 | 2.0 | 3.20 |

**Solution:** The difference table is as follows:

| $t$ | $\theta$ | $\nabla\theta$ | $\nabla^2\theta$ | $\nabla^3\theta$ | $\nabla^4\theta$ |
|---|---|---|---|---|---|
| 0.0 | 0.0 | | | | |
| | | 0.12 | | | |
| 0.2 | 0.12 | | 0.24 | | |
| | | 0.36 | | 0.02 | |
| 0.4 | 0.48 | | 0.26 | | 0 |
| | | 0.62 | | 0.02 | |
| 0.6 | 1.10 | | 0.28 | | 0 |
| | | 0.90 | | 0.02 | |
| 0.8 | 2.0 | | 0.30 | | |
| | | 1.20 | | | |
| 1.0 | 3.20 | | | | |

Here $x_n = t_n = 1.0$, $h = 0.2$, $x = t = 0.7$

$$u = \frac{x - x_n}{h} = \frac{0.7 - 1.0}{0.2} = -1.5$$

From the Newton's backward interpolation formula, we have

$$\left(\frac{d\theta}{dt}\right)_{t=0.7} = \frac{1}{h}\left[\nabla\theta_0 + \frac{2u+1}{2}\nabla^2\theta_0 + \frac{3u^2 + 6u + 2}{6}\nabla^2\theta_n\right]$$

$$= \frac{1}{0.2}\left[1.20 - 0.30 + \frac{3(-1.5)^2 - 6(-1.5) + 2}{6}(0.02)\right]$$

$$= 5\,[1.20 - 0.30 - 0.0008]$$

and $$= 4.496 \text{ radian/sec}$$

$$\left(\frac{d^2\theta}{dt^2}\right)_{t=0.7} = \frac{1}{h^2}\,[\nabla^2\theta_0 + (u+1)\nabla^3\theta_0]$$

$$= \frac{1}{(0.2)^2}\,[0.30 - 0.5 \times 0.02]$$

$$= 25 \times 0.29 = 7.25 \text{ radian/sec}^2.$$

∴ Angular velocity = 4.496 radian/sec.

Angular acceleration = 7.25 radian/sec$^2$.

$f'(4.0)$ & $f''(4.0)$

**Example 8.4** Find $f'$ (1.5) and $f''$ (1.5) from the following table: by using N.F.

| $x$ : | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
|---|---|---|---|---|---|---|
| $f(x)$ : | 3.375 | 7.000 | 13.625 | 24.000 | 38.875 | 59.000 |

**Solution:** The difference table is:

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ |
|---|---|---|---|---|---|
| 1.5 | 3.375 | | | | |
| | | 3.625 | | | |
| 2.0 | 7.000 | | 3.000 | | |
| | | 6.625 | | 0.75 | |
| 2.5 | 13.625 | | 3.75 | | 0 |
| | | 10.375 | | 0.75 | |
| 3.0 | 24.000 | | 4.5 | | 0 |
| | | 14.875 | | 0.75 | |
| 3.5 | 38.875 | | 5.25 | | |
| | | 20.125 | | | |
| 4.0 | 59.000 | | | | |

Here, we have $x_0 = 1.5$, $h = 0.5$

$$f'(1.5) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

$$= \frac{1}{0.5}\left[3.625 - \frac{1}{2}(3.000) + \frac{1}{3}(0.75)\right]$$

Newton's Backward

$$= 2\,[3.625 - 1.5 + 0.25]$$

$$f'(1.5) = 4.75$$

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and $$f''(1.5) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0\right]$$

$$= \frac{1}{(0.5)^2}\cdot[3.000 - 0.75]$$

$= 4\,(2.25)$

$f''(1.5) = 9.$ 24

**Example 8.5** Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ of $y = x^{1/3}$ at $x = 50$ from the following table:

| $x$ : | 50 | 51 | 52 | 53 | 54 | 55 | 56 |
|---|---|---|---|---|---|---|---|
| $y = x^{1/3}$ : | 3.6840 | 3.7084 | 3.7325 | 3.7563 | 3.7798 | 3.8030 | 3.8259 |

**Solution:** The difference table is:

| $x$ | $y = x^{1/3}$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| 50 | 3.6840 | | | |
| | | 0.0244 | | |
| 51 | 3.7084 | | − 0.0003 | |
| | | 0.0241 | | 0 |
| 52 | 3.7325 | | − 0.0003 | |
| | | 0.0238 | | 0 |
| 53 | 3.7563 | | − 0.0003 | |
| | | 0.0235 | | 0 |
| 54 | 3.7798 | | − 0.0003 | |
| | | 0.0232 | | 0 |
| 55 | 3.8030 | | − 0.0003 | |
| | | 0.0229 | | |
| 56 | 3.8259 | | | |

Here, $x_0 = 50$, $h = 1$. Then $u = \dfrac{x - x_0}{h} = \dfrac{50 - 50}{1} = 0$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \ldots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=50} = \frac{1}{1}\left[0.0244 - \frac{1}{2}(-0.0003) + \frac{1}{3}(0)\right]$$

$$= [0.0244 + 0.00015] = 0.02455$$

and
$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\,[\Delta^2 y_0 - \Delta^3 y_0 + \ldots]$$

$$= \frac{1}{(1)^2}\,(-0.0003) = -0.0003.$$

**Example 8.6** A slider in a machine moves along a fixed straight rod. Its distance $x$ cm. along the rod is given below for various values of the time $t$ seconds. Find the velocity of the slider and it's acceleration when $t = 0.3$ sec.

| $t$ : | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 |
|---|---|---|---|---|---|---|---|
| $x$ : | 30.13 | 31.62 | 32.87 | 33.64 | 33.95 | 33.81 | 33.24 |

**Solution:** As the derivatives are required near the middle of the table, we use Stirling formulae:

$$\left(\frac{dx}{dt}\right)_{t_0} = \frac{1}{h}\left[\left(\frac{\Delta x_0 + \Delta x_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 x_{-2} + \Delta^5 x_{-3}}{2}\right) + \ldots\right]$$

# 9

# NUMERICAL INTEGRATION

## 9.1 Introduction

It is the process of finding or evaluating a definite integral $I = \int_a^b f(x)\,dx$ from a set of numerical values of the integrand function $f(x)$. If it is applied to the integration of function of a single variable, the process is known as *quadrature*. (N.I.)

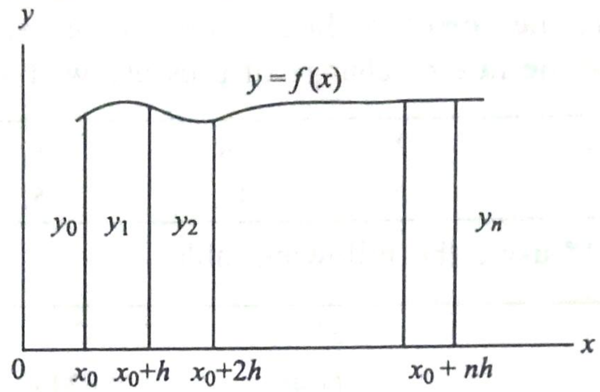


Fig. 9.1

General Formula for Numerical Integration OR

## 9.2 A General Quadrature Formula for Equidistant Ordinates: (Newton-Cotes quadrature formula)

Let $I = \int_a^b y\,dx$, where $y = f(x)$. Let $f(x)$ be given for certain equally distance values of arguments, say $x_0, x_0 + h, x_0 + 2h, \ldots, x_n$. Let the range $(b - a)$ be divided into $n$ equal parts each of which is of width $h$. *i.e.*,

$$b - a = nh.$$

Let $\quad x_0 = a, \quad x_1 = x_0 + h = a + h, \quad x_2 = x_1 + h = a + 2h$

$$x_n = a + nh = b$$

Here, we have assumed that $(n + 1)$ ordinates $y_0, y_1, y_2, \ldots, y_n$ are at equal intervals.

we know that $p = \frac{x - x_0}{h}$

Then $\quad I = \int_a^b y\,dx = \int_{0 \; \cancel{x_0}}^{n \; \cancel{x_0+nh}} y_{\cancel{x_0+ph}}\, h\,dp \qquad (\because x = x_0 + ph \Rightarrow dx = h\,dp)$

D. w. r. to p

Approximating y by N. F. D. F. we obtain

$$\therefore \quad I = h \int_0^n \Big[ y_0 + p\Delta y_0 + \frac{p(p-1)}{\underline{|2}}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\underline{|3}}\Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{\underline{|4}}\Delta^4 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{\underline{|5}}\Delta^5 y_0$$

$$+ \ldots + \frac{p(p-1)\ldots(p-n+1)}{\underline{|n}}\Delta^n y_0 \Big]\,dp$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{\lfloor 2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{\lfloor 3}\right.$$

$$+ \ldots + \left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2\right)\frac{\Delta^4 y_0}{\lfloor 4} + \cancel{\left(\frac{n^5}{5} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\Delta^5 y_0}{\lfloor 5}} +$$

$$+ - - - .$$

$$\left. \ldots + \text{ upto } (n+1) \text{ terms}\right] \quad \ldots(1)$$

The equation (1) is known as *General quadrature formula*. One can obtain a number of quadrature formulae from this by putting $n = 1, 2, \ldots$ .

## 9.3 Trapezoidal Rule

When $n = 1$, the interval of integration will be from $x_0$ to $x_0 + h$, and there are only two functional values $y_0$ and $y_1$ in this interval. With only two values, there can be no differences higher than the first. Thus on putting $n = 1$ in (1) and neglecting the terms containing second and higher degree, we obtain

G.F.N.I

$$\int_{x_0}^{x_0+h} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0 + \frac{y_1 - y_0}{2}\right]$$

$$= \frac{h}{2}[y_0 + y_1].$$

Similarly, $$\int_{x_0+h}^{x_0+2h} y\,dx = \frac{h}{2}[y_1 + y_2]$$

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$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx = \frac{h}{2}[y_{n-1} + y_n]$$

Adding these intervals, we get

$$\int_{x_0}^{x_0+nh} y\,dx = h\left[\frac{1}{2}(y_0 + y_n) + (y_1 + y_2 + y_3 + .. + y_{n-1})\right]$$

$$= \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1})] \quad \ldots(2)$$

= Distance between two consecutive ordinates × [mean of the first and the last ordinates + sum of all the intermediate ordinates]

The equation (2) is called *Trapezoidal rule.*

**Note:** The area of each strip (trapezium) is calculated separately. The area under the curve and the ordinates at $x_0$ and $x_0 + nh$ is approximately equal to the sum of the areas of the $n$ trapeziums.

The polynomial $y(x)$ is a linear function of $x$ which represent a straight line.

## TRAPEZOIDAL Rule -

The General Formulae of Numerical Integration.

$$I = \int_{x_0}^{x_0+n\cdot h} f(x)dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\cdot\frac{\Delta^3 y_0}{6} + \left(\frac{n^5}{5} - \frac{3}{2}n^4 + \frac{4}{3}n^3 - 3n^2\right)\frac{\Delta^4 y_0}{24} + \cdots\right]$$

Substituting $n = 1$ in the general eq. & neglecting all differences greater than the first we get.

$$I_1 = \int_{x_0}^{x_0+h\cdot h} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] \qquad \because \Delta y_0 = y_1 - y_0$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= h\left[y_0 + \frac{1}{2}y_1 - \frac{1}{2}y_0\right]$$

$$I_1 = \int_{x_0}^{x_0+h} f(x)dx = \frac{h}{2}\left[y_0 + y_1\right]$$

for the first subinterval $[x_0, x_0+h]$.

Similarly, we get.

$$I_2 = \int_{x_0+h}^{x_0+2\cdot h} f(x)dx = \frac{h}{2}\left[y_1 + y_2\right].$$

$$I_3 = \int_{x_0+2\cdot h}^{x_0+3\cdot h} f(x)dx = \frac{h}{2}\left[y_2 + y_3\right]$$

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$$I_n = \int_{x_0+(n-1)h}^{x_0+(n)\cdot h} f(x)dx = \frac{h}{2}\left[y_{n-1} + y_n\right]$$

Adding $I_1, I_2, I_3, \ldots, I_n$, we get.

$$I_1 + I_2 + I_3 + \cdots + I_n = \int_{x_0}^{x_0+h} f(x)\,dx + \int_{x_0+h}^{x_0+2\cdot h} f(x)\,dx + \int_{x_0+2\cdot h}^{x_0+3\cdot h} f(x)\,dx + \cdots + \int_{x_0+(n-1)h}^{x_0+(n)\cdot h} f(x)\,dx$$

$$\Rightarrow I_n = \int_{x_0}^{x_0+n\cdot h} f(x)\,dx = \frac{h}{2}[y_0+y_1] + \frac{h}{2}[y_1+y_2] + \frac{h}{2}[y_2+y_3] + \cdots + \frac{h}{2}[y_{n-1}+y_n]$$

$$\Rightarrow I_n = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right] \quad \text{--- (1)}$$

The eq. (1) is called Trapezoidal rule for Numerical integration.

Note :- Trapezoidal rule can be applied to any number of subintervals odd or even.

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## SIMPSON'S ONE-THIRD Rule

The General Formulae

$$I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{6} + \left(\frac{n^5}{5} - \frac{3}{2}n^4 + \frac{11}{3}n^3 - 3n^2\right)\frac{\Delta^4 y_0}{24} + \cdots\right]$$

Substituting $n=2$ in the General formulae given by Numerical Integration in eq. And neglecting the third & other higher order differences.

we get.

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$I_1 = \int_{x_0}^{x_0+2h} f(x)\,dx = h\left[2y_0 + 2\Delta y_0 + \left(\frac{8}{3} - 2\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$\therefore I_1 = \int_{x_0}^{x_0+2h} f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Similarly:

$$I_2 = \int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

$$I_3 = \int_{x_0+4h}^{x_0+6h} f(x)\,dx = \frac{h}{3}\left[y_4 + 4y_5 + y_6\right]$$

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$$I_n = \int_{x_0+(n-2)h}^{x_0+(n)h} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

Adding $I_1, I_2, I_3, \ldots, I_n$, we get,

$$I_1 + I_2 + \cdots + I_n = \int_{x_0}^{x_0+2h} f(x)\,dx + \int_{x_0+2h}^{x_0+4h} f(x)\,dx + \int_{x_0+4h}^{x_0+6h} f(x)\,dx + \cdots + \int_{x_0+(n-2)h}^{x_0+(n)h} f(x)\,dx$$

$$I_n = \int_{x_0}^{x_0+(n)h} f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] + \frac{h}{3}\left[y_2 + 4y_3 + y_4\right] + \frac{h}{3}\left[y_4 + 4y_5 + y_6\right] + \cdots + \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

$$I_n = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4\times(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2\times(y_2 + y_4 + y_6 + \cdots + y_{n-2})\right]$$

where $y_0$ & $y_n$ are the first & last ordinates.

The above rule is known as Simpson's one-third rule.

Note :- Simpson's one-third rule can be applied only when the given interval $[a,b]$ is subdivided into even number of subintervals each of width $h$ & within any two consecutive subintervals the interpolating polynomial $\phi(x)$ is of degree 2.

— x —

## SIMPSON'S THREE - EIGHTS Rule -

The General Formula -

$$I = \int_{x_0}^{x_0+n\cdot h} f(x)\,dx = h\left[ n y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{6} + \left(\frac{n^5}{5} - \frac{3}{2}n^4 + \frac{11}{3}n^3 - 3n^2\right)\frac{\Delta^4 y_0}{24} + \cdots \right]$$

substituting $n = 3$ in the general formula & neglecting all the differences above $\Delta^3$, we get.

$$I_1 = \int_{x_0}^{x_0+3\cdot h} f(x)\,dx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \left(9 - \frac{9}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{81}{4} - 27 + 9\right)\frac{\Delta^3 y_0}{6}\right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 + y_0)\right]$$

$$I_1 = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

similarly

$$I_2 = \int_{x_0+3h}^{x_0+6\cdot h} f(x)\,dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$$

$$I_3 = \int_{x_0+6\cdot h}^{x_0+9h} f(x)\,dx = \frac{3h}{8}\left[y_6 + 3y_7 + 3y_8 + y_9\right]$$

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$$I_n = \int_{x_0+(n-3)\cdot h}^{x_0+(n)\cdot h} f(x)dx = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

Adding $I_1, I_2, I_3, \ldots, I_n$

$$I_1 + I_2 + I_3 + \cdots + I_n = \int_{x_0}^{x_0+3h} f(x)dx + \int_{x_0+3h}^{x_0+6h} f(x)dx + \int_{x_0+6h}^{x_0+9h} f(x)dx + \cdots + \int_{x_0+(n-3)h}^{x_0+nh} f(x)dx$$

$$\Rightarrow I_n = \int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right] + \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$$
$$+ \frac{3h}{8}\left[y_6 + 3y_7 + 3y_8 + y_9\right] + \cdots + \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

$$\Rightarrow I_n = \int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\Big[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) +$$
$$+ 2(y_3 + y_6 + \cdots + y_{n-3})\Big]$$

The above rule is known as Simpson's Three-Eight rule

Note :- Simpson's Three-Eight rule can be applied when the range $[a, b]$ is divided into a number of subintervals which must be a multiple of 3.

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## BOOLE'S Rule

The General Formulae

$$I = \int_{x_0}^{x_0+n\cdot h} f(x)dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\cdot\frac{\Delta^3 y_0}{6} + \left(\frac{n^5}{5} - \frac{3}{2}n^4 + \frac{11}{3}n^3 - 3n^2\right)\cdot\frac{\Delta^4 y_0}{24} + \cdots\right]$$

Substituting $n = 4$ in the numerical Integration And Neglecting all the differences above $\Delta^4$, we get.

$$I_1 = \int_{x_0}^{x_0+4h} f(x)dx = h\left[4y_0 + \frac{16}{2}\Delta y_0 + \left(\frac{64}{3} - \frac{16}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{256}{4} - 64 + 16\right)\right.$$
$$\left. \cdot \frac{\Delta^3 y_0}{6} + \left(\frac{1024}{5} - \frac{3}{2}\times 256 + \frac{4}{3}\times 64 - 3\times 16\right)\cdot\frac{\Delta^4 y_0}{24}\right]$$

$$I_1 = h\left[4y_0 + 8\Delta y_0 + \underbrace{\left(\frac{64}{3} - 8\right)}_{\left(\frac{40}{3}\right)}\frac{\Delta^2 y_0}{2} + (16)\frac{\Delta^3 y_0}{6} + \left(\frac{1024}{5} - 384 + \frac{704}{3} - 48\right)\frac{\Delta^4 y_0}{24}\right]$$

$$\left(\frac{112}{15}\right) = \frac{28}{90}$$

$$I_1 = \int_{x_0}^{x_0+4\cdot h} f(x)dx = \frac{2\cdot h}{45}\left[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4\right]$$

similarly -

$$I_2 = \int_{x_0+4\cdot h}^{x_0+8\cdot h} f(x)dx = \frac{2\cdot h}{45}\left[7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8\right]$$

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$$I_n = \int_{x_0+(n-4)\cdot h}^{x_0+(n)h} f(x)dx = \frac{2\cdot h}{45}\left[7(y_{n-4}) + 32y_{n-3} + 12y_{n-2} + 32y_{n-1} + 7y_n\right]$$

Adding $I_1, I_2, \ldots, I_n$. we get -

$$I_1 + I_2 + \cdots + I_n = \int_{x_0}^{x_0+4\cdot h} f(x)dx + \int_{x_0+4\cdot h}^{x_0+8\cdot h} f(x)dx + \cdots + \int_{x_0+(n-4)\cdot h}^{x_0+n\cdot h} f(x)dx$$

$$\Rightarrow I_n = \int_{x_0}^{x_0+n\cdot h} f(x)dx = \frac{2\cdot h}{45}\left[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4\right]$$
$$+ \frac{2\cdot h}{45}\left[7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8\right]$$
$$+ \cdots + \frac{2h}{45}\left[7y_{n-4} + 32y_{n-3} + 12y_{n-2} + 32y_{n-1} + 7y_n\right]$$

$$\boxed{\Delta^4 y_0 = (y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)}$$

$\frac{112}{15}$ 432

$3072 + 3520/15 = 6592/15$

$$\Rightarrow I_n = \int_{x_0}^{x_0+n\cdot h} f(x)dx = \frac{2\cdot h}{45}\Big[7(y_0+y_n) + 32\,(y_1+y_3+y_5+y_7+\cdots+$$

$$+\,y_{n-3}+y_{n-1}) + 12\,(y_2+y_6+\cdots+y_{n-2})\Big]$$

This is known as Boole's Rule for Integration –

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## WEDDLE'S Rule

The General Formulae -

$$I = \int_{x_0}^{x_0+n\cdot h} f(x)dx = h\Big[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{n^4}{4} - n^3 + n^2\right)$$

$$\cdot\frac{\Delta^3 y_0}{6} + \left(\frac{n^5}{5} - \frac{3}{2}n^4 + \frac{11}{3}n^3 - 3n^2\right)\frac{\Delta^4 y_0}{24} + \cdots\Big]$$

substituting $n = 6$ in the General Formulae given by eq. And Neglecting all differences above $\Delta^6$, we get -

$$I_1 = \int_{x_0}^{x_0+6\cdot h} f(x)dx = h\Big[6y_0 + \frac{36}{2}\Delta y_0 + \left(\frac{216}{3} - \frac{36}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{1296}{4} - 216 + 36\right)$$

$$\cdot\frac{\Delta^3 y_0}{6} + \left(\frac{7776}{5} - \frac{3}{2}\times 1296 + \frac{11}{3}\times 216 - 3\times 36\right)\frac{\Delta^4 y_0}{24} + \cdots\Big]$$

$$I_1 = h\Big[6y_0 + 18\Delta y_0 + (72-18)\frac{\Delta^2 y_0}{2} + (324 - 216 + 36)\frac{\Delta^3 y_0}{6} +$$

$$+ (1555 - 1944 + 792 - 108)\frac{\Delta^4 y_0}{24} + \cdots\Big]$$

$$I_1 = h\Big[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0$$

$$+ \frac{41}{140}\Delta^6 y_0\Big]$$

Since $\frac{3}{10} - \frac{41}{140} = \frac{1}{140}$

we take the coefficient of $\Delta^6 y_0$ as $\frac{3}{10}$ except $\frac{41}{140}$, so that the error committed is $\frac{1}{140}$ And we write

$$I_1 = \int_{x_0}^{x_0+6h} f(x)\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right].$$

Similarly -

$$I_2 = \int_{x_0+6h}^{x_0+12\cdot h} f(x)\,dx = \frac{3h}{10}\left[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}\right]$$

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$$I_n = \int_{x_0+(n-6)\cdot h}^{x_0+n\cdot h} f(x)\,dx = \frac{3h}{10}\left[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n\right]$$

Adding $I_1, I_2, \ldots, I_n$ $\therefore$ $I_1 + I_2 + \cdots + I_n =$

$$I_n = \int_{x_0}^{x_0+6\cdot h} f(x)\,dx + \int_{x_0+6\cdot h}^{x_0+12\cdot h} f(x)\,dx + \cdots + \int_{x_0+(n-6)h}^{x_0+n\cdot h} f(x)\,dx$$

$$I_n = \int_{x_0}^{x_0+n\cdot h} f(x)\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right] +$$

$$+ \frac{3h}{10}\left[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}\right] +$$

$$+ \cdots + \frac{3h}{10}\left[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n\right]$$

$$I_n = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 \right.$$
$$+ y_{10} + 6y_{11} + 2y_{12} + \cdots + 2y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} +$$
$$\left. + y_{n-2} + 5y_{n-1} + y_n\right]$$

$$\Rightarrow I_n = \frac{3h}{10}\left[(y_0 + y_n) + (y_2 + y_4 + y_8 + y_{10} + y_{14} + y_{16} + \cdots + y_{n-4} + y_{n-2})\right.$$
$$+ 5(y_1 + y_5 + y_7 + y_{11} + \cdots + y_{n-5} + y_{n-1}) + 6(y_3 + y_9 + y_{15} + \cdots$$
$$\left. \cdots + y_{n-3}) + 2(y_6 + y_{12} + \cdots + y_{n-6})\right]$$

This is known as Weddle's Rule for Integration.

Note :

1. Weddle's rule requires atleast seven consecutive equispaced ordinates with in the given interval $(a, b)$.
2. It is more accurate than the Trapezoidal & Simpson's rules.
3. If $f(x)$ is a polynomial of degree 5 or lower, Weddle's rule gives an exact result.

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**Prob 1:** - Given the following table -

| $x$ : | 4 | 4.2 | 4.4 | 4.6 | 4.8 | 5.0 | 5.2 |
|---|---|---|---|---|---|---|---|
| $y = f(x)$ : | 1.3863 | 1.4351 | 1.4816 | 1.5261 | 1.5686 | 1.6094 | 1.6487 |

Evaluate (i) Trapezoidal rule, (ii) Simpson's 1/3 rule (iii) Weddle's rule. in which start point $x_0 = 4$ & end point $x_n = 5.2$.

Soln :- Here the number of equal interval is not given so we take $n = 6$.

$$\therefore \quad h = \frac{b-a}{n} = \frac{5.2-4}{6} = 0.2$$

(i) Trapezoidal Rule :- By Trapezoidal's rule, we have -

$$I_n = \int_{x_0}^{x_0+n\cdot h} f(x)dx = \frac{h}{2}\left[ (y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right].$$

$$\Rightarrow \int_4^{5.2} f(x)dx = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{0.2}{2}\left[(1.3863+1.6487) + 2(1.4351+1.4816+1.5261 + 1.5686+1.6094)\right]$$

$$I_n = 1.82766$$

(ii) Simpson's 1/3 Rule :-

$$I_n = \int_{x_0}^{x_0+n \cdot h} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4\times(y_2+y_4+\cdots+y_{n-2})+ 2\times(y_1+y_3+\cdots+y_{n-1})\right]$$

$$I_n = \int_4^{5.2} f(x)\,dx = \frac{h}{3}\left[(y_0+y_6)+2(y_2+y_4)+4(y_1+y_3+y_5)\right]$$

$$= \frac{0.2}{3}\left[(1.3863+1.6487)+2(1.4816+1.5686)+ 4(1.4351+1.5261+1.6094)\right]$$

$$\Rightarrow I_n = \int_4^{5.2} f(x)\,dx = 1.8278.$$

(iii) Weddle's Rule -

$$I_n = \int_{x_0}^{x_0+n \cdot h} f(x)\,dx = \frac{3h}{10}\left[(y_0+y_n)+(y_2+y_4+y_8+y_{10}+\cdots+y_{n-4}+y_{n-2}) +5(y_1+y_5+y_7+\cdots+y_{n-5}+y_{n-1})+6(y_3+y_9+y_{15}+\cdots + y_{n-3})+2(y_6+y_{12}+\cdots+y_{n-6})\right]$$

$$I_n = \int_4^{5.2} f(x)\,dx = \frac{3h}{10}\left[(y_0+y_6)+5(y_1+y_5)+(y_2+y_4) + 6(y_3)\right]$$

$$= \cancel{\frac{3\times0.2}{10}\left(1.3863+5\times1.4\right.}$$

$$= \frac{3\times0.2}{10}\left[(1.3863+1.6487)+5(1.4351+1.6094)+ (1.4816+1.5686)+6\times1.5261\right]$$

$$\Rightarrow I_n = \int_4^{5.2} f(x)\,dx = 1.82786$$

—x—

Prob :- Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using (i) Trapezoidal rule (ii) Simpson's 1/3 rule (iii) Simpson's 3/8 rule (iv) Weddle's rule. (v) Boole's Rule, where the interval of integration is subdivided into six equal parts?

Soln :- We have $y = f(x) = \frac{1}{1+x^2}$ , $a=0$, $b=6$

Here the number of equal interval is not given, so we take $n=6$

$\therefore \quad h = \frac{b-a}{n} = \frac{6-0}{6} = 1$

Now we obtain the following table -

| $x$ : | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $f(x)$ : | 1 | 0.5 | 0.2 | 0.1 | 0.0588 | 0.0385 | 0.027 |

(i) Trapezoidal Rule - By Trapezoidal rule, we have

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.027)+2(0.5+0.2+0.1+0.0588+0.0385)\right]$$

$$\int_0^6 \frac{1}{1+x^2}dx = 1.4108$$

(ii) Simpson's 1/3 rule - By Simpson's 1/3 rule, we have

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(1+0.027)+4(0.5+0.1+0.0385)+2(0.2+0.0588)\right]$$

$$= \frac{1}{3}\left[1.027+2.554+0.5176\right] =$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.3662$$

(iii) Simpson's 3/8 Rule - By Simpson's 3/8 rule, we have

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$= \frac{3}{8}\left[(1+0.027)+3(0.5+0.2+0.0588+0.0385)+2\times0.1\right]$$

$$= \frac{3}{8}\left[1.027+2.3919+0.2\right]$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.3571.$$

(iv) Weddle's Rule - By weddle's rule, we have -

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\right]$$

$$= \frac{3}{10}\left[1+5\times0.5+0.2+6\times0.1+0.0588+5\times0.0385+0.027\right]$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.3735$$

— x —