(*vi*) Continue the process till the system is reduced to triangular form.

Step 3. Calculate x_i values by back substitution.

Step 4. Print results

Step 5. Stop.

3.5 Ill-Conditioned Equations

In some cases, when we make a very small change in the coefficients of the problem gives a very large change in the solution of the problem. Such type of problems are known as ill-conditioned problems. The system of equations which has such property is called ill-conditioned system. For ill-conditioned systems, the Gauss-elimination method is useful.

Well conditioned systems are those where a small change in one or more of the coefficients results in a similar change in solution.

Example 3.8 *Solve the following system:*

(book Q.No.-86)

Ill Conditioned Equations & Refinement of Solutions

Simultaneous Linear Equations :-

many real life problem in engineering give rise to a system of linear equations. For Example, such system occur in certain application of statistical analysis and in finding the numerical solution of partial equations and there are many problems involving the solution of system of algebric equations and so on. It is therefore natural to seek efficient methods for solving these equations numerically.

The general form of a system of m linear equations in n unknown $x_1, x_2, \ldots x_n$ can be represented in matrix form as under:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \text{--- (1)}$$

Using matrix notation, the above system can be written in compact form as.

$$[A](X) = (B) \quad \text{--- (2)}$$

The solution of the system of equation (2) given n unknown values $x_1, x_2, \ldots x_n$, which satisfies the system simultaneously. If $m > n$, we may not be able to find a solution, in principle which ~~st~~ satisfy all the equations. If $m < n$, the system usually will have an infinite number of solutions. However, we shall restrict to the case $m = n$. In, if $|A| \neq 0$, then the system will have a unique solution, while, if $|A| = 0$, then there exists no solution.

Solution of simultaneous Linear Equation –

The coefficient matrix of the system of linear equation is :

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \quad \text{——— (1)}$$

That is, the coefficient matrix is a matrix whose elements are the coefficients of the unknown in the system of linear equations. If the constants of the eq. (1) i.e. b's are appended to the coefficient matrix as a last column i.e. $(n+1)^{th}$

column. The resulting matrix is known as the augmented matrix as shown below.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_n \end{bmatrix} \quad \text{--- (2)}$$

In order to implement on computer, it is more convenient to express these b's as simply an additional column of the a_{ij}'s. In such case the matrix of eq. (2) might be expressed as.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & a_{1(n+1)} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & a_{2(n+1)} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & a_{3(n+1)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & a_{n(n+1)} \end{bmatrix} \quad \text{--- (3)}$$

This form will be used for solution of simultaneous L.E. ~~in discussions in the sections to follow.~~

Solution of Non-Homogeneous system of Linear Equations:

There are two types of methods to obtain solution of non-homogeneous system of linear equations. These are.

(1) Direct methods.
(2) Iterative methods.

(1) Direct methods:

The direct method also known as reduction methods. Under this category.

Matrix Inversion method

(a) Gauss elimination method
(b) Gauss - Jordan method.
(c) matrix Inversion method And
(d) Crout's method

(2) Iterative methods -

Under this category

(a) Jacobi's method
(b) Gauss - Seidel method.

Since a system of non-homogeneous linear eq. usually consists of linearly Independent equations, we will assume that a unique solution does exist. Remember that a unique solution for such as system of equations exists only if the coefficient matrix is non-singular, i.e. if the determinant of the coefficient matrix is non-zero.

Direct Methods –

The direct method give the solution of system of non-homogeneous linear equation in a fixed number of steps.

Gauss Elimination Method:

This is one of the most widely used methods. This method is a systematic process of eliminating unknowns from the linear equations. This method is divided into two parts.

(a) Triangularization

(b) Back substitution

The system of n equations in n unknown is reduced to an equivalent triangular system (an equivalent system is a system having identical solution) of equation of type.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$+ a_{nn}x_n = b_n$$

—— (1)

$$\left[a_{31} - \frac{a_{31}}{a_{11}} \cdot a_{11}\right] x_1 + \left[a_{32} - \frac{a_{31}}{a_{11}} \cdot a_{12}\right] x_2 + \cancel{[a_{3n}} - - - +$$
$$+ \left[a_{3n} - \frac{a_{31}}{a_{11}} \cdot a_{1n}\right] x_n = \left[b_3 - \frac{a_{31}}{a_{11}} \cdot b_1\right]$$

This new equivalent system of equation can be easily solved by using the back substitution procedure.

The details of the Gauss elimination method are given below.

step 1 :- Eliminate x_1 from 2^{nd} equation onwards. This is done as follows:

(i) Subtract from the second equation $\frac{a_{21}}{a_{11}}$ times the first equation. This result in

$$\left[a_{21} - \frac{a_{21}}{a_{11}} \cdot a_{11}\right] x_1 + \left[a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right] x_2 + - - - + \left[a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}\right] x_n = \left[b_2 - \frac{a_{21}}{a_{11}} b_1\right]$$

which on simplifying gives equation of type

$$a_{22} x_2 + a_{23} x_3 + - - - + a_{2n} x_n = b_2$$

(ii) Similarly, subtract from the third equation $\frac{a_{31}}{a_{11}}$ time the first equation. This will give equation of type

$$a_{32} x_2 + a_{33} x_3 + - - - + a_{3n} x_n = b_3$$

(iii) If we repeat this process till the n^{th} equation

$$\left[a_{32} - \frac{a_{32}}{a_{22}} \cdot a_{22}\right]x_2 + \left[a_{33} - \frac{a_{32}}{a_{22}} \cdot a_{23}\right]x_3 + \cdots + \left[a_{3n} - \frac{a_{32}}{a_{22}} \cdot a_{2n}\right]x_n = \left[b_3 - \frac{a_{32}}{a_{22}} \cdot b_2\right]$$

$$\left[a_{42} - \frac{a_{42}}{a_{22}} \cdot a_{22}\right]x_2 + \left[a_{43} - \frac{a_{42}}{a_{22}} \cdot a_{23}\right]x_3 + \cdots + \left[a_{4n} - \frac{a_{42}}{a_{22}} \cdot a_{2n}\right]x_n = \left[b_4 - \frac{a_{42}}{a_{22}} \cdot b_2\right]$$

is operated, we get the new system of equatn of type.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\vdots \qquad \vdots \qquad\quad \vdots &\qquad \vdots \qquad \text{——— ②} \\
a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

The solution of these equations is same as that of the original equations.

step 2: Eliminate x_2 from 3rd equation onwards from the system of eq. ② This is done as follows.

(i) Subtract from the third equation $\frac{a_{32}}{a_{22}}$ times the second equation.

$a_{33}x_3 + \cdots + a_{3n}x_n = b_3$

(ii) Subtract from the fourth equation $\frac{a_{42}}{a_{22}}$ times the second equation.

$+ \cdots + a_{4n} \cdot x_n = b_4.$

(iii) And so on.

We will thus get the new system of equation of type

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\vdots \qquad\qquad & \quad \vdots \\
a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{aligned} \qquad \text{—— (3)}
$$

This process will continue till the last equation contains only one unknown, namely x_n. The final form of equation will look like.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\vdots \qquad & \quad \vdots \\
a_{nn}x_n &= b_n
\end{aligned} \qquad \text{—— (4)}
$$

This process is called triangularization.

∴ We consider the augmented matrix of equation.

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & a_{1(n+1)} \\
0 & a_{22} & a_{23} & \cdots & a_{2n} & a_{2(n+1)} \\
0 & 0 & a_{33} & \cdots & a_{3n} & a_{3(n+1)} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{nn} & a_{n(n+1)}
\end{bmatrix}
$$

Exmp!- Solve the following system of linear equations.

$$2x_1 + 8x_2 + 2x_3 = 14$$

$$x_1 + 6x_2 - x_3 = 13$$

$$2x_1 - x_2 + 2x_3 = 5$$

Sol^n :- In order to eliminate x_1 from the second & third equation, first apply transformation

step:- 1:-

$$R_2 - \frac{a_{21}}{a_{11}} R_1$$

$$\Rightarrow \quad R_2 - \frac{1}{2} R_1$$

The coefficients of the second equation are computed as

$$a_{21} = a_{21} - \frac{1}{2} a_{11} = 1 - \frac{1}{2} \times 2 = 0$$

$$a_{22} = a_{22} - \frac{1}{2} a_{12} = 6 - \frac{1}{2} \times 8 = 2$$

$$a_{23} = a_{23} - \frac{1}{2} a_{13} = -1 - \frac{1}{2} \times 2 = -2$$

$$b_2 = b_2 - \frac{1}{2} b_1 = 13 - \frac{1}{2} \times 14 = 6$$

Now apply transformation

$$R_3 - \frac{a_{31}}{a_{11}} R_1$$

$$\Rightarrow R_3 - \frac{2}{2} R_1 \Rightarrow R_3 - R_1$$

The elements of the third equation are computed as

$a_{31} = a_{31} - a_{11} = 2 - 2 = 0$

$a_{32} = a_{32} - a_{12} = 1 - 8 = -9$

$a_{33} = a_{33} - a_{13} = 2 - 2 = 0$

$b_3 = b_3 - b_1 = 5 - 14 = -9$

Thus eliminating x_1 from second and third equation, we obtain the new system of linear equation as.

$$2x_1 + 8x_2 + 2x_3 = 14$$
$$2x_2 - 2x_3 = 6$$
$$-9x_2 + 0x_3 = -9$$

step 5.2: Finally in order to eliminate x_2 from the third equation, apply transformation.

$$R_3 - \frac{a_{32}}{a_{22}} R_2$$

$$\Rightarrow R_3 - \frac{-9}{2} R_2 \Rightarrow R_3 + \frac{9}{2} R_2$$

The eliminets of the third equation are computed as.

$a_{32} = a_{32} + \frac{9}{2} a_{22} = -9 + \frac{9}{2} \times 2 = 0$

$a_{33} = a_{33} + \frac{9}{2} a_{23} = 0 + \frac{9}{2} \times (-2) = -9$

$b_3 = b_3 + \frac{9}{2} b_2 = -9 + \frac{9}{2} \times 6 = 18$

Therefor the final system look as.

$$2x_1 + 8x_2 + 2x_3 = 14$$
$$2x_2 - 2x_3 = 6$$
$$-9x_3 = 18$$

Finally, through back substitution, beginning with the last equation, the following solution values are obtained:

$$x_3 = \frac{18}{-9} = -2$$

$$x_2 = \frac{(6 + 2x_3)}{2} = 1$$

$$x_1 = \frac{(14 - 2x_3 - 8x_2)}{2} = 5$$

Exmp:- Solve the system of equations:

$$3x_1 + x_2 - x_3 = 3$$
$$2x_1 - 8x_2 + x_3 = -5$$
$$x_1 - 2x_2 + 9x_3 = 8$$

Using Gauss elimination method.

Soln:- Ans) $x_1 = 1$, $x_2 = 1$, $x_3 = 1$

Exmp:- 3. Solve the system of equations.

$$28x_1 + 4x_2 - x_3 = 32,$$
$$x_1 + 3x_2 + 10x_3 = 24 \quad \text{And}$$
$$2x_1 + 17x_2 + 4x_3 = 35$$ by Gauss elimination method.

Solⁿ :- Ans :- $x_1 = \dfrac{-4x_2 + x_3 + 32}{28} = 0.993594$

$$x_2 = \frac{458 - 57x_3}{234} = 1.5069778$$

$$x_3 = \frac{56560}{30597} = 1.8485472.$$

Exmp:- 4! Using Gauss elimination method, solve the system

$$3.15x_1 - 1.96x_2 + 3.85x_3 = 12.95$$
$$2.13x_1 + 5.12x_2 - 2.89x_3 = -8.61$$
$$5.92x_1 + 3.05x_2 + 2.15x_3 = 6.88$$

Solⁿ :- Ans! $x_3 = \dfrac{0.6853}{0.6534} = 1.0488$, $x_2 = \dfrac{5.4933x_3 - 17.3667}{6.4453} =$

$$= -1.80057$$

$$x_1 = \frac{1.96x_2 - 3.85x_3 + 12.95}{3.15} = 1.7089$$

Exmp:- Solve the system, Using Gauss elimination method.

$$x_1 + x_2 + x_3 + x_4 = 2, \quad x_1 + x_2 + 3x_3 - 2x_4 = -6$$
$$2x_1 + 3x_2 - x_3 + 2x_4 = 7, \quad x_1 + 2x_2 + x_3 - x_4 = -2$$

Solⁿ :- Ans :- $x_1 = 1$, $x_2 = 0$, $x_3 = -1$, $x_4 = 2$.

{Note: After Completing step-I, we can not solve.}

Gauss Seidel method :

Even though the new values of unknowns are computed in each iteration, but the values of unknowns in the previous iterations are used in the subsequent iterations. That is although new values of x_1 is computed from the first equation in a current iteration, but it is not used to compute the new values of other unknowns in the current iteration.

The sequence of steps contituting the Gauss-seidel method are as follows :

Initialisation step :

To begine the procedure, assign an initial value to each unknown, if ~~you~~ we can make a reasonable guess.

Step 1 :- (i) Find the value of x_1 from the first equation by substituting the initial values of other unknowns.

(ii) Find the value of x_2 from the second equation by substituting current value of x_1 and the initial values of other unknowns.

(iii) Find the value of x_3 from the third equation by substituting the current values of x_1 and x_2 and the initial values of other unknowns.

And so on, till the value of x_n is computed from the n^{th} equation using current values of $x_1, x_2, x_3, \ldots, x_{n-1}$.

step 2: (i) Find the value of x_1 from the first equation by substituting the values of other unknowns obtained in the first iteration.

(ii) Find the value of x_2 from the second equation by substituting current value of other unknowns

(iii) Find the value of x_3 from the third equation by substituting the current values other unknowns

⋮

And so on till the value of x_n is computed from the n^{th} equation using current values of $x_1, x_2, \ldots x_{n-1}$.

step 3: (i) Find the value of x_1 from the first equation by substituting the values of other unknowns obtained in the second iteration.

(ii) Find the value of x_2 from the second equation by substituting the current values of other unknowns.

(iii) Find the value of x_3 from the third equation by substituting the current values of other unknowns.

⋮

And so on till the value of x_n is computed from the n^{th} equation using current values of $x_1, x_2, \ldots, x_{n-1}$.

This iterative procedure is continued until the successive values of each unknown differs only within the permissible limits.

Exmp! - 1 :- Solve the following system of equations

$$10x_1 + x_2 + 2x_3 = 44$$
$$2x_1 + 10x_2 + x_3 = 51$$
$$x_1 + 2x_2 + 10x_3 = 61$$

accurate to ~~four~~ Three significant digits.

Soln :- We rewrite the given system of equation as.

$$x_1 = \frac{1}{10}(44 - x_2 - 2x_3)$$

$$x_2 = \frac{1}{10}(51 - 2x_1 - x_3)$$

$$x_3 = \frac{1}{10}(61 - x_1 - 2x_3)$$

We start with initial approximation as.

$$x_1 = x_2 = x_3 = 0$$

Step 1 :-

Substituting $x_2 = x_3 = 0$ in the first eq.

$$10x_1 = 44$$

$$x_1 = 4.4.$$

Substituting $x_1 = 4.4$ and $x_3 = 0$ in the second equation,

$$2 \times 4.4 + 10x_2 + 0 = 51$$

$$x_2 = 4.22$$

Substituting $x_1 = 4.4$ & $x_2 = 4.22$ in the third eq.

$$x_3 = 4.816.$$

Thus, we obtain $x_1 = 4.4$, $x_2 = 4.22$, $x_3 = 4.816$.

As new approximations at the end of the first iteration.

Step 2 :-

Now substituting $x_2 = 4.22$ & $x_3 = 4.816$ in the first eq.

$$x_1 = 4.0154$$

Next substituting $x_1 = 4.0154$ & $x_3 = 4.816$ in the second equation,

$$x_2 = 3.0148$$

Next substituting $x_1 = 4.0154$ & $x_2 = 3.0148$ in the third equal.

$$x_3 = 5.0955$$

Thus, we obtain $x_1 = 4.0154$, $x_2 = 3.0148$, $x_3 = 5.0955$.

step 3:-

Now substituting $x_2 = 3.0148$ & $x_3 = 5.0955$ in the first equation.

$$x_1 = 3.0794$$

Next substituting $x_1 = 3.0794$ & $x_3 = 5.0955$ in the second equation.

$$x_2 = 3.9746$$

Next substituting $x_1 = 3.0794$ & $x_2 = 3.9746$ in the third eq.

$$x_3 = 4.9979$$

Thus, we obtain $x_1 = 3.0794$, $x_2 = 3.9746$, $x_3 = 4.997$
As new approximations at the end of the third iterati

step 4:-

Now substituting $x_2 = 3.0794$ & $x_3 = 4.9971$ in the first equation.

$$x_1 = 3.0031$$

Next substituting $x_1 = 3.0031$ & $x_3 = 4.9979$ in the second equation

$$x_2 = 3.9997$$

Next substituting $x_1 = 3.0031$ & $x_2 = 3.9997$ in the third eq.

$$x_3 = 4.8001$$

Thus, we obtain $x_1 = 3.0031$, $x_2 = 3.9997$, $x_3 = 4.8001$

As new approximations at the end of the fourth iterat^n^

step 5 :-

Now substituting $x_2 = 3.9997$ & $x_3 = 4.8001$. in the first equation.

$$x_1 = 3.0040$$

Next substituting $x_1 = 3.0400$ & $x_3 = 4.8001$ in the second eq.

$$x_2 = \cancel{4.0120}\ 3.9947$$

Next substituting $x_1 = 3.0400$ & $x_2 = 4.0120$ in the third eq.

$$x_3 = \cancel{4.8360}\ 4.8036$$

Thus, we obtain, $x_1 = 3.00400$, $x_2 = \cancel{4.0120}\ 3.9947$, $x_3 = 4.8060$

By comparing the 4th & 5th approximation, we find that there is ~~As New approximations at the end of the fifth iteration~~ no variation in first three significante digits

Therefor we take the solution obtained as -

$x_1 = 3.00$, $x_2 = 3.99$, $x_3 = 4.80$ //

Step :- 6:

Now substituting $x_2 = 4.0120$ & $x_3 = 4.8360$ in the first eq.

$$x_1 = 3.0316$$

Next substituting $x_1 = 3.0316$ & $x_3 = 4.8360$ in the second equation,

$$x_2 = 4.0101$$

Next subs ti' $x_1 = 3.0316$ & $x_2 = 4.0101$ T.E.

$$x_3 = 4.9948$$

Thus $x_1 = 3.0316$, $x_2 = 4.0101$, $x_3 = 4.9948$

As new approximation at the end of the 6th itn.

Step 7:-

Now s. $x_2 = 4.0101$ & $x_3 = 4.9948$ F.E.

$$x_1 = 3.0000$$

Next s. $x_1 = 3.0000$ & $x_3 = 4.9948$ S.E.

$$x_2 = 4.0001$$

Next s $x_1 = 3.0000$ & $x_2 = 4.0001$ T.E.

$$x_3 = 5.0000$$

Thus $x_1 = 3.0000$, $x_2 = 4.0001$, $x_3 = 5.0000$

Step 8:- Now S. $x_2 = 4.0001$ & $x_3 = 5.0000$ F.E.

$x_1 = 3.0000$

Next S $x_1 = 3.0000$ & $x_3 = 5.0000$ S.E.

$x_2 = 4.0000$

M. S. $x_1 = 3.0000$ & $x_2 = 4.0000$ T.E.

$x_3 = 5.0000$

Thus $x_1 = 3.0000$, $x_2 = 4.0000$, $x_3 = 5.0000$

As new approximation at the end of the eight iteration.

By comparing the approximation of the 7^{th} & 8^{th} iterations we find that there is no variation in first four significant digits, therefore we take the solution obtained at the end of the eighth iteration as the desired solution

Hence, the solution correct to four significant digits is:

$x_1 = 3.0000$

$x_2 = 4.0000$

$x_3 = 5.0000$.

Prob :- 2 :- Solve the equation by Gauss - Seidel Iterative method.

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

Soln :- We write the given eq. in the form

$$x_1 = \frac{1}{10}(3 + 2x_2 + x_3 + x_4) \quad \text{--- (1)}$$

$$x_2 = \frac{1}{10}(15 + 2x_1 + x_3 + x_4) \quad \text{--- (2)}$$

$$x_3 = \frac{1}{10}(27 + x_1 + x_2 + 2x_4) \quad \text{--- (3)}$$

& $$x_4 = \frac{1}{10}(-9 + x_1 + x_2 + 2x_3) \quad \text{--- (4)}$$

First Iteration -

Substituting $x_2 = x_3 = x_4 = 0$ in the R.H.S. of eq. (1) we get.

$$x_1 = \frac{3}{10} = 0.3$$

Substituting $x_1 = 0.3$, $x_3 = x_4 = 0$ in R.H.S. of eq. (2) we get -

$$x_2 = \frac{1}{10}(15 + 2 \times 0.3) = 1.56$$

Substituting $x_1 = 0.3$, $x_2 = 1.56$, $x_4 = 0$ in R.H.S. of eq. (3) we get -

$$x_3 = \frac{1}{10}(27 + 0.3 + 1.56 + 0) = 2.886$$

Substituting $x_1 = 0.3$, $x_2 = 1.56$ & $x_3 = 2.886$ in R.H.S. of eq. (4), we get.

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Page No.
Date

→ *

$x_4 = \frac{1}{10}(-9 + 0.3 + 1.56 + 2.886) = -0.1368.$

Second Iteration, we have

$x_1 = \frac{1}{10}(3 + 2 \times 1.56 + 2.886 - 0.1368) = 0.887$

$x_2 = \frac{1}{10}(15 + 2 \times 0.887 + 2.886 - 0.1368) = 1.952$

$x_3 = \frac{1}{10}(27 + 1.952 + 0.887 + 2 \times (-0.1368))$

$x_3 = 2.957$

$x_4 = \frac{1}{10}(-9 + 0.887 + 1.952 + 2 \times 2.957) = -0.025$

Third Iteration, we have

$x_1 = 0.984$

$x_2 = 1.99$

$x_3 = 2.99$

$x_4 = -0.0046$

Fourth Iteration, we have -

$x_1 = 0.997$

$x_2 = 1.998$

$x_3 = 2.999$

$x_4 = -0.0007$

Fifth Approximation, we have -

$x_1 = 0.999$

$x_2 = 1.999$

$x_3 = 2.999$

$x_4 = -0.00044$

Hence Fourth Iteration = Fifth Iteration =

$x_1 = 1$, $x_2 = 2$, $x_3 = 3$ & $x_4 = 0$

5. $4x + y + 3z = 11;\ \ 3x + 4y + 2z = 11;\ \ 2x + 3y + z = 7.$
6. $2x + 6y - z = -12;\ \ 5x - y + z = 11;\ \ 4x - y + 3z = 10.$
7. $x + 2y - 12z + 8w = 27;\ \ 5x + 4y + 7z - 2w = 4;$
   $6x - 12y - 8z + 3w = 49;\ \ 3x - 7y - 9z - 5w = -11.$
8. $x + y + z - w = 2;\ \ 7x + y + 3z + w = 12;$
   $8x - y + z - 3w = 5;\ \ 10x + 5y + 3z + 2w = 20.$
9. $2x + 4y + 8z = 41;\ \ 4x + 6y + 10z = 56,\ 6x + 8y + 10z = 64.$
10. $2x + 2y - z + w = 4;\ \ 4x + 3y - z + 2w = 6;$
    $8x + 5y - 3z + 4w = 12;\ \ 3x + 3y - 2z + 2w = 6.$

ANSWERS

1. −2, 3, 6.	2. 1, 1, 1.	3. 1, 1, 1.
4. 1, 1, 1.	5. 1, 1, 2.	6. 1.64, −2.49, 0.32.
7. 3, −2, 1, 5.	8. 1, 1, 1, 1.	9. 1.5, 2.5, 3.5.
10. 1, 1, −1, −1.		

3.4 GAUSS ELIMINATION METHOD WITH PIVOTING

In Gauss elimination method, the element a_{ij} when $i = j$ is known as *pivot element*. Each row is normalised by dividing the coefficients of that row by its pivot element *i.e.*, $\frac{a_{kj}}{a_{kk}}$, $j = 1, 2, 3, \ldots, n$.

The problem arises when any one of the pivot elements becomes zero. In such a case, we rewrite the equations in a different order so that the pivots are non-zero. The reordering of the equations are to be made in such a way, that rounding off error's is minimized. It is suggested that the row with zero pivot element should be interchanged with the row having the largest (absolute value) coefficient in that position. In fact, the reordering of equations is done to improve the accuracy even if the pivot element is non-zero.

The reordering is done in a way such that:

(*i*) The first row, first column coefficient must be largest. If not so, interchange the row with the row having largest coefficient.

(*ii*) When, first unknown (variable) is eliminated from second and next equations, locate the largest coefficient of second variable (2^{nd} column). Interchange that row having largest coefficient for second variable to second place.

(*iii*) Continue the procedure till $(n - 1)$ variables are eliminated.

The process in which largest pivot element is fixed by the above procedure is called ***partial pivoting***. If columns as well as rows are searched for the largest element and then switched, the procedure is called *complete pivoting*. (Full pivoting) Exmp : Notes Pg No.- 2

3.4.1 Algorithm for Gauss Elimination method with Pivoting

Step 1. Input n, a_{ij} and b_i values.

Step 2. Start from the first equation,

(*i*) Check for the pivot element.

(*ii*) If it is the largest among the elements below it, obtain the derived system.

(*iii*) Otherwise, identity the largest element and make it the pivot element.

(*iv*) Interchange the original pivot equation with the one containing the largest element, so that the later becomes new pivot equation.

(*v*) Obtain the derived system.

Using Gaussian elimination method, the solution is found to be $x_1 = 1, x_2 = 1$ which is a meaningful & perfect Res

In Full pivoting which is also known as complete pivoting, we interchange rows as well as columns, such that the largest element in the matrix of the system becomes the pivot elements. In this process the position of the unknown variable also get changed. Full pivoting is more complicated than the partial pivoting.

Exmp :- Solve the system of equation

$$x + y + z = 7 \qquad x = x_1$$
$$3x + 3y + 4z = 24 \qquad y = x_2$$
$$2x + y + 3z = 16 \qquad z = x_3$$

by Gaussian elimination method with partial pivoting.

Sol^n :-

Step 1 :- In order to eliminate x_1 from eq. ② & Third eq. First apply transformation.

$$R_2 - \frac{a_{21}}{a_{11}} R_1 \Rightarrow R_2 - \frac{3}{1} R_1$$

The coefficients of the second equation are computed a

$$a_{21} = a_{21} - \frac{3}{1} a_{11} = 3 - 3 = 0$$
$$a_{22} = a_{22} - \frac{3}{1} a_{12} = 3 - 3 = 0$$

$$a_{23} = a_{23} - \frac{3}{1} a_{13} = 4 - 3 = 1$$

$$b_2 = b_2 - \frac{3}{1} b_1 = 24 - 21 = 3$$

Now apply transformation

$$R_3 - \frac{a_{31}}{a_{11}} R_1 \Rightarrow R_3 - \frac{2}{1} R_1$$

The elements of the third equation are computed as

$$a_{31} = a_{31} - \frac{2}{1} a_{11} = 2 - 2 = 0$$

$$a_{32} = a_{32} - \frac{2}{1} a_{12} = 1 - 2 = -1$$

$$a_{33} = a_{33} - \frac{2}{1} a_{13} = 3 - 2 = 1$$

$$b_3 = b_3 - \frac{2}{1} b_1 = 16 - 14 = 2$$

Thus elimination x_1 from ~~eq~~ II & III equation, we obtain the new system of linear equation as

$$x_1 + x_2 + x_3 = 7$$

$$+ 0x_2 + x_3 = 3$$

$$- x_2 + x_3 = 2$$

~~step 2~~ : Finally, through back substitution, beginning with the last equation, the following solution values are obtained

$$x_3 = 3, \quad -x_2 + 3 = 2 \Rightarrow x_2 = 1$$

$$x_1 + 1 + 3 = 7 \Rightarrow x_1 = 3$$

Thus x_1 & x_3 constitute the solution to the given system of equations.

Examp:-2: Solve by Gaussian elimination method with partial pivoting, the following system of equations:

Sq. Not Correct

$$0x_1 + 4x_2 + 2x_3 + 8x_4 = 24$$
$$4x_1 + 10x_2 + 5x_3 + 4x_4 = 32$$
$$4x_1 + 5x_2 + 6.5x_3 + 2x_4 = 26$$
$$9x_1 + 4x_2 + 4x_3 + 0x_4 = 21$$

Soln:- $x_1 = 1.0$, $x_2 = 1.0$, $x_3 = 2.0$, $x_4 = 2.0$.

Thus, Eq. x_1 & x_2 And x_3 & x_4 constitute the solution to the given system of equations.

Ex:- 3:- Using Gauss elimination with partial pivoting solve the system of eq.

$x_1 + x_2 - 2x_3 = 3$, $4x_1 - 2x_2 + x_3 = 5$

$3x_1 - x_2 + 3x_3 = 8$.

Soln:- $R_2 \to R_2 - \frac{1}{4} R_1$ & $R_3 \to R_3 - \frac{3}{4} R_1$

& $R_3 \to R_3 - \frac{1}{3} R_2$

we get the solution is -

$x_1 = \frac{22}{9}$, $x_2 = 3$, $x_3 = \frac{11}{9}$.