Interpolation

In the mathematical subfield of numerical analysis, **interpolation** is a method of constructing new data points within the range of a discrete set of known data points.

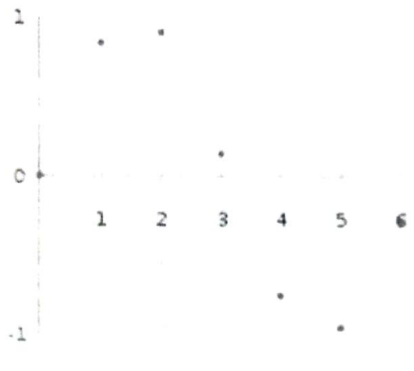
In engineering and science one often has a number of data points, as obtained by sampling or experimentation, and tries to construct a function which closely fits those data points. This is called curve fitting or regression analysis. Interpolation is a specific case of curve fitting, in which the function must go exactly through the data points.

A different problem which is closely related to interpolation is the approximation of a complicated function by a simple function. Suppose we know the function but it is too complex to evaluate efficiently. Then we could pick a few known data points from the complicated function, creating a lookup table, and try to interpolate those data points to construct a simpler function. Of course, when using the simple function to calculate new data points we usually do not receive the same result as when using the original function, but depending on the problem domain and the interpolation method used the gain in simplicity might offset the error.

It should be mentioned that there is another very different kind of interpolation in mathematics, namely the "interpolation of operators". The classical results about interpolation of operators are the Riesz–Thorin theorem and the Marcinkiewicz theorem. There are also many other subsequent results.

Example

For example, suppose we have a table like this, which gives some values of an unknown function *f*.



Plot of the data points as given in the table.

x	$f(x)$	
0	0	
1	0	.8415
2	0	.9093
3	0	.1411
4	−0	.7568
5	−0	.9589
6	−0	.2794

Interpolation provides a means of estimating the function at intermediate points, such as $x = 2.5$.

CALCULUS OF FINITE DIFFERENCES

4.1 Introduction : Interpolation : -

Let $y = f(x)$ be a single valued and continuous function defined over $[x_0, x_n]$. Then the value of $f(x)$ for $x_0, x_1, x_2, \ldots, x_n$ can easily be computed and tabulated. The central problem of numerical analysis is the converse one. Given the set of tabular values $(x_0, y_0), (x_1, y_1), \ldots (x_n, y_n)$ satisfying the relation $y = f(x)$, where the explicit nature of $f(x)$ is not known, it is required to find a simple functon say $\phi(x)$ such that $f(x)$ and $\phi(x)$ agree for the set of tabulated points. Such a process is called *interpolation*. If $\phi(x)$ is a polynomial, then the process is called *polynomial interpolation* and $\phi(x)$ is called the *interpolating polynomial*. ~~The Weierstrass (1885) theorem states that:~~ If $f(x)$ is continuous in $x_0 \le x \le x_n$, then for given any $\epsilon > 0$, there exists polynomial $P(x)$ such that:

$$|f(x) - P(x)| < \epsilon, \text{ for all } x \text{ in } (x_0, x_n).$$

Interpolation methods -

The following interpolation methods are used -

(i) Graphic method

(ii) Method of curve fitting

(iii) Application of the calculus of finite differences formula.

(i) Graphical method -

Let $y = f(x)$, then we can easily plot a graph between different values of x and corresponding values of y. From the graph so obtained, we can find out the value of y for given x, e.g. consider the data given below -

year (x) :	1891	1901	1911	1921	1931
Population (y) :	46	66	81	93	101

Equation (7) is almost useless in practical computations.

Particularly, we use it to determine errors in Newton's interpolating formulae.

III) Application of the calculus of finite difference

4.3 Finite Differences

Finite difference methods are applied to functions for which values are available at equidistant points. Assume that we have a table of values (x_i, y_i), $i = 0, 1, 2, \ldots, n$ of any function $y = f(x)$, the values of x being equally spaced, *i.e.*, $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, n$. Suppose that we are required to find the values of $f(x)$ for some intermediate values of x or to obtaine the derivative of $f(x)$, for some x in the range $x_0 \leq x \leq x_n$. The method for solution of these problems are based on the concept of the differences of a function.

4.3.1 Forward Differences

Let $y = f(x)$ be any function given by the values $y_0, y_1, y_2, \ldots y_n$, which correspond for the equidistant values $x_0, x_1, x_2, \ldots, x_n$ of the independent variable x, then $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots, y_n - y_{n-1}$ are called the first differences of the function y and these are denoted by Δy_0, Δy_1, Δy_2, ... etc.

$\therefore$ We have

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\cdots$$
$$\Delta y_{n-1} = y_n - y_{n-1}$$

first

The symbol Δ is called the *difference operator*. The second differences are

$$\Delta^2 y_0 = \Delta\ (\Delta y_0) = \Delta y_1 - \Delta y_0 = y_2 - 2y_1 + y_0$$
$$\Delta^2 y_1 = \Delta\ (\Delta y_1) = \Delta y_2 - \Delta y_1 = y_3 - 2y_2 + y_1$$

Δ^2 is called *second difference operator*. Continuing this we have,

$$\Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1}$$

The third forward differences are as under:

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0)$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$

and

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$$
$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0.$$

In general, $\boxed{\Delta^{m+1} y_n = \Delta^m y_{n+1} - \Delta^m y_n.}$

Table 4.1 Forward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_0	y_0					
		$\Delta y_0 = y_1 - y_0$				
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$			
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^2 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

4.3.2 Backward Differences

The differences $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \ldots, \nabla y_n$ respectively, are called the first backward differences where ∇ is the backward difference operator. Similarly, we define higher order backward differences. Thus, we have,

$$\nabla y_r = y_r - y_{r-1}, \quad \nabla^2 y_r = \nabla y_r - \nabla y_{r-1},$$

$$\nabla^3 y_r = \nabla^2 y_r - \nabla^2 y_{r-1} \text{ etc.}$$

In general $\nabla^{m+1} y_r = \nabla^m y_r - \nabla^m y_{r-1}$

The difference table is as follows:

Table 4.2 Backward Difference Table

x	y	∇	∇^2	∇^3	∇^4	∇^5
x_0	y_0					
		$y_1 - y_0 = \nabla y_1$				
x_1	y_1		$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$			
		$y_2 - y_1 = \nabla y_2$		$\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$		
x_2	y_2		$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$		$\nabla^3 y_4 - \nabla^3 y_3 = \nabla^4 y_4$	
		$y_3 - y_2 = \nabla y_3$		$\nabla^2 y_4 - \nabla^2 y_3 = \nabla^3 y_4$		$\nabla^4 y_5 - \nabla^4 y_4 = \nabla^5 y_5$
x_3	y_3		$\nabla y_4 - \nabla y_3 = \nabla^2 y_4$		$\nabla^3 y_5 - \nabla^3 y_4 = \nabla^4 y_5$	
		$y_4 - y_3 = \nabla y_4$		$\nabla^2 y_5 - \nabla^2 y_4 = \nabla^3 y_5$		
x_4	y_4		$\nabla y_5 - \nabla y_4 = \nabla^2 y_5$			
		$y_5 - y_4 = \nabla y_5$				
x_5	y_5					

4.3.3 Central Differences

The central difference operator δ is defined as

$$y_1 - y_0 = \delta y_{1/2}, \quad y_2 - y_1 = \delta y_{3/2}, \ldots, y_n - y_{n-1} = \delta y_{n-1/2},$$

Similarly, higher order central differences are defined as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1, \delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2, \dots \text{ etc.}$$

In general, $\delta^{m+1} y_{\frac{2n-1}{2}} = \delta^m y_n - \delta^m y_{n-1}$.

Table 4.3 Central Difference Table

x	y	δ	δ^2	δ^3	δ^4	δ^5
x_0	y_0					
		$\delta y_{1/2}$				
x_1	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
x_4	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
x_5	y_5					

From Table 4.3, we see that From the central differences Table on the same horizontal line have the same suffix. Also the differences of odd order are known only for half values of the suffix and those of even order for only integral values of the suffix:

INTERPOLATION WITH EQUAL INTERVALS

5.1 Introduction

In case of a certain measurement or observation, let $y_0, y_1, y_2, \ldots, y_n$ be the values of a dependent variable y corresponding to the values $x_0, x_1, x_2, \ldots, x_n$ of independent variable x. Suppose these values correspond to the best approximating function $y = f(x)$, not known. Then the process to compute the functional value y_k corresponding to the value x_k (an intermediate value of x in the domain *i.e.*, $x_0 < x_k < x_n$), is called *interpolation*. The process to compute the functional value of y corresponding to any value x' of independent variable x, outside the domain *i.e.*, $x' < x_0$ or $x' > x_n$ but near to x_0 or x_n not far away, is called *extrapolation*. If the values $x_0, x_1, x_2, \ldots, x_n$ are equally spaced *i.e.*, $x_1 - x_0 = x_2 - x_1 = \ldots$, then interpolation is said to be with equal intervals otherwise unequal. In this chapter, we shall discuss several methods of interpolation.

5.2 Newton Gregory Forward Interpolation Formula

Given the set of $(n + 1)$ values $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ of x and y, it is required to find $y_n(x)$, a polynomial of nth degree, so that y and $y_n(x)$ concide at tabulated points.

Let the values of x be equidistant so that,

$$x_i = x_0 + ih, \quad i = 0, 1, 2, 3, \ldots, n.$$

Since $y_n(x)$ is a polynomial of degree n, let us assume that

$$y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1}) \quad \ldots(1)$$

Using condition that y and $y_n(x)$ concide at the set of tabulated points, we obtain on putting, $x = x_0$; $y_n(x_0) = y_0$; $x_1 = x_0 + h$, $y_n(x_0 + h) = y_1$, ... in equation (1) From F.D.T.

$$a_0 = y_0, \quad a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$a_2 = \frac{\Delta^2 y_0}{h^2 \lfloor 2}, \quad a_3 = \frac{\Delta^3 y_0}{h^3 \lfloor 3}, \quad \ldots, \quad a_n = \frac{\Delta^n y_0}{h^n \lfloor n}.$$

Hence eqution (1) gives

$$y_n(x) = y_0 + \frac{x - x_0}{h} \Delta y_0 + \frac{(x - x_0)(x - x_1)}{h^2 \lfloor 2} \cdot \Delta^2 y_0$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2)}{h^3 \lfloor 3} \cdot \Delta^3 y_0 + \ldots + \frac{(x - x_0)(x - x_1) \ldots (x - x_{n-1})}{\lfloor n\, h^n} \cdot \Delta^n y_0 \quad \ldots(2)$$

On putting $x = x_0 + ph$, we have, from (2)

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2} \cdot \Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3} \Delta^3 y_0 + \ldots$$

$$\ldots + \frac{p(p-1)(p-2), \ldots (p-n+1)}{\lfloor n} \Delta^n y_0 \quad \ldots(3)$$

where $$p = \frac{x - x_0}{h}, \quad p \le 1.$$

Equation (3) is called *Newton Gregory forward interpolation formula.*

*

(i) Newton's Forward Difference Table :

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

Example 5.1 Evaluate $y = e^{2x}$ for $x = 0.05$ using the following table:

x	0.00	0.10	0.20	0.30	0.40
$y = e^{2x}$	1.000	1.2214	1.4918	1.8221	2.2255

Solution: The forward difference table is

x	$y = e^{2x}$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.00	1.000				
		0.2214			
0.10	1.2214		0.0490		
		0.2704		0.0109	
0.20	1.4918		0.0599		0.0023
		0.3303		0.0132	
0.30	1.8221		0.0731		
		0.4034			
0.40	2.2255				

We have $x_0 = 0.00, \quad x = 0.05, \quad h = 0.1$

$$\therefore \quad p = \frac{x - x_0}{h} = \frac{0.05 - 0.00}{0.1} = \frac{0.05}{0.1} = 0.5$$

Using Newton's forward, interpolation formula

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{\underline{|2}} \cdot \Delta^2 y_0 + \frac{p(p-1)(p-2)}{\underline{|3}} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{\underline{|4}} \Delta^4 y_0 + \ldots$$

We have, $y = 1.000 + 0.5 \times 0.2214 + \dfrac{0.5(0.5-1)}{2} \times (0.049) + \dfrac{0.5(0.5-1)(0.5-2)}{6} \times 0.0109$

$$+ \frac{0.5 \times (0.5-1)(0.5-2)(0.5-3)}{24} \times 0.0023$$

$$= 1.000 + 0.1107 - 0.006125 + 0.000681 - 0.000090$$

$$= 1.105166$$

$$\therefore \quad y \approx \mathbf{1.052.}$$

Example 5.2 Find the value of sin 30° 15′ 30″ from the following table:

Angle x^{o}	30°	31°	32°	33°	34°
$\sin x^{o} = y$	0.5000	0.5150	0.5299	0.5446	0.5592

Solution: The forward difference table is

x	$f(x) = \sin x^{o}$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
30°	0.5000				
		0.0150			
31°	0.5150		−0.0001		
		0.0149		−0.0001	
32°	0.5299		−0.0002		0.0002
		0.0147		0.0001	
33°	0.5446		−0.0001		
		0.0146			
34°	0.5592				

Given $x = 30^\circ\ 15'\ 30''$, $x_0 = 30^\circ$, $h = 1$

$$\therefore \quad p = \frac{x - x_0}{h} = \frac{30^\circ 15' 30'' - 30^\circ}{1} = 15'\ 30''$$

$$= 15'\ 30'' = \mathbf{0.2583^\circ}$$

$$f(x) = f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{\lfloor 2}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 f(x_0) + \frac{p(p-1)(p-2)(p-3)}{\lfloor 4}\Delta^4 f(x_0) + \ldots$$

i.e.,
$$f(30^\circ\ 15'\ 30'') = 0.5000 + 0.2583 \times 0.0150 + \frac{0.2583\,(0.2583-1)}{2}(-0.0001) + \frac{0.2583\,(0.2583-1)\,(0.2583-2)}{6} \times (-.0001) + \frac{0.2583\,(0.2583-1)\,(0.2583-2)(0.2583-3)\times(0.0002)}{24}$$

$$= 0.5000 + 0.0039 + 0.0000$$

$$= \mathbf{0.5039\ Ans.}$$

Example 5.3 Find the number of students whose weight is between 60 and 70 from the data given below:

Weights in lbs:	0–40	40–60	60–80	80–100	100–120
No. of students:	250	120	100	70	50

Solution: The difference table is follows:

x (Weight)	*y (No. of students)*	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
			–20		
Below 60	370			–10	
		100			
			–30		20
Below 80	470				
		70		10	
			–20		
Below 100	540				
		50			
Below 120	590				

Let us calculate the number of students whose weight is less than 70 lbs.

By using Newton forward difference formula

$$u = \frac{x - x_0}{h} = \frac{70 - 40}{20} = 1.5$$

$$y(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{\lfloor 2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{\lfloor 3}\Delta^3 y_0 + \ldots$$

$$y(70) = 250 + 1.5 \times 120 + \frac{(1.5)(0.5)}{2}(-20) + \frac{1.5(0.5)(-0.5)}{6}(-10) + \frac{(1.5)(0.5)(-0.5)(-1.5)}{24}(20)$$

$$= 250 + 180 - 7.5 + 0.625 + 0.46875$$

$$= 423.59$$

$$\approx 424$$

Hence, the number of students whose weight is between 60 and 70 is

$$= y(70) - y(60) = 424 - 370 = \mathbf{54}\ \textbf{Ans.}$$

Example 5.4 The following table gives the population of a town during the last six censuses. Estimate the increase in the population during the period from 1946 to 1948:

Year	1911	1921	1931	1941	1951	1961
Population (in thousands)	12	15	20	27	39	52

Solution: The difference table is as under:

Year x	Population y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1911	12					
		3				
1921	15		2			
		5		0		
1931	20		2		3	
		7		3		–10
1941	27		5		–7	
		12		–4		
1951	39		1			
		13				
1961	52					

Here,
$$u = \frac{x - x_0}{h} = \frac{1946 - 1911}{10} = 3.5$$

Applying Newton's forward difference formula we have

$$y_{(1946)} = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{\lfloor 2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{\lfloor 3}\Delta^3 y_0 + \ldots$$

$$= 12 + 3.5 \times 3 + \frac{3.5 \times 2.5}{\lfloor 2} \times 2 + 0 + \frac{3.5 \times 2.5 \times 1.5 \times 0.5}{\lfloor 4} \times 3$$

$$+ \frac{3.5 \times 2.5 \times 1.5 \times 0.5 \times (-0.5)}{\lfloor 5} \times -10$$

$$= 12 + 10.5 + 8.75 + 0.8203 + 0.2734 = 32.3437$$

For year 1948,
$$u = \frac{x - x_0}{h} = \frac{1948 - 1911}{10} = 3.7$$

Applying again Newton's forward difference formula, we get

$$y(1948) = 12 + 3.7 \times 3 + \frac{3.7 \times 2.7}{\lfloor 2} \times 2 + 0 + \frac{3.7 \times 2.7 \times 1.7 \times (0.7)}{\lfloor 4} \times 3$$

$$+ \frac{3.7 \times 2.7 \times 1.7 \times 0.7 \times (-0.3)}{\lfloor 5} \times (-10)$$

$$= 12 + 11.1 + 9.99 + 1.4860125 + 0.2972025$$

$$= 34.873215$$

Therefore, increase in the population during the period from 1946 to 1948 is

$$= 34.873215 - 32.3437 = 2.529515$$

$$\approx \mathbf{2.53\ thousands\ (Approx).}$$

Example 5.5 Using Newton's forward interpolation formula, find the area of a circle of diameter 82 from the given table of diameter and area of circle:

Diameter	80	85	90	95	100
Area	5026	5674	6362	7088	7854

Solution: The forward difference table is as follows:

Diameter (x)	Area (y)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
80	5026				
		648			
85	5674		40		
		688		−2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

$$u = \frac{x - x_0}{h} = \frac{82 - 80}{5} = \frac{2}{5} = 0.4$$

Applying Newton's forward difference formula, we have

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{\underline{|2}}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{\underline{|3}}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{\underline{|4}}\Delta^4 y_0 + \ldots$$

$$= 5026 + 0.4 \times 648 + \frac{(0.4)(0.4-1)}{2} \times 40 + \frac{0.4(0.4-1)(0.4-2)}{6} \times -2 + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24} \times 4$$

$$= 5026 + 259.2 - 4.8 - 0.128 - 0.416$$

$$= \mathbf{5279.856.}$$

5.3 Newton-Gregory Backward Difference Interpolation Formula

In this case, we write,

$$y_n(x) = A_0 + A_1(x - x_n) + A_2(x - x_n)(x - x_{n-1}) + \ldots + A_n(x - x_n)(x - x_{n-1}) \ldots (x - x_1) \quad \ldots(1)$$

where $A_0, A_1, A_2, \ldots, A_n$ are constants. From B.D.T, we obtain or

Putting $x = x_n$

$$y_n(x_n) = A_0 = y_n(x_n) \quad \text{(say)}$$

Again put $x = x_{n-1}$

$$y_n(x_{n-1}) = A_0 + A_1(x_{n-1} - x_n)$$

$$y_n(x_{n-1}) = y_n(x_n) + A_1(-h) \qquad \because x_n = x_{n-1} + h$$

$$y_n(x_n) - y_n(x_{n-1}) = A_1(+h) \Rightarrow A_1 = \frac{\nabla y_n}{h}$$

Again $$A_2 = \frac{y_n(x_n) - 2y_n(x_{n-1}) + y_n(x_{n-2})}{2h^2} = \frac{\nabla^2 y_n}{\lfloor 2\, h^2}$$

Similarly, $$A_3 = \frac{1}{\lfloor 3}\frac{1}{h^3}\cdot\nabla^3 y_n, \ldots, A_n = \frac{1}{\lfloor n . h^n}\nabla^n y_n$$ ⇒ In general we have

$$\therefore \quad y_n(x) = y_n(x_n) + \frac{1}{h}\nabla y_n(x_n) + \frac{\nabla^2 y_n}{\lfloor 2h^2}(x - x_n)(x - x_{n-1}) + \ldots$$

$$+ \frac{\nabla^n y_n}{\lfloor nh^n}(x - x_n)(x - x_{n-1}) \ldots (x - x_1) \quad \ldots(2)$$

Equation (2) is called *Newton-Gregory Backward difference interpolation formula.*

On putting $p = \dfrac{x - x_n}{h}$, we have $u = p$

$x = x_n + p.h$

$$\boxed{y_n(x) = y_n(x_n) + p\nabla y_n + \frac{p(p+1)}{\lfloor 2}\cdot\nabla^2 y_n + \ldots + \frac{p(p+1)\ldots(p+n-1)}{\lfloor n}\nabla^n y_n} \quad \ldots(3)$$

Note: This formula is particularly useful for interpolating the value of $f(x)$ near the end of the set of given values.

5.9.1 Algorithm for Newton-Gregory backward interpolation formula

(ii) Newton's Backward Difference Table :

x	y	~~Δ~~ ∇	∇~~Δ~~^{2}	∇^3	∇^4
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\nabla^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\nabla^4 y_0$
		Δy_2		$\nabla^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

```
The difference table is :-
     x          f(x)       ?^1        ?^2        ?^3
  20.0000     0.3420     0.0487    -0.0010    -0.0003
  23.0000     0.3907     0.0477    -0.0013
  26.0000     0.4384     0.0464
  29.0000     0.4848
Enter the value of 'x' : 28
The value of function at x = 28.000000 is 0.469496
```

Illustrative Examples

Example 5.6 The following data are taken from the steam table:

Temperature °C	140	150	160	170	180
Pressure kgf/cm^2	3.685	4.854	6.302	8.076	10.225

Find the pressure at temperature x = 175°C , by using N.B.I.F. ?

Solution:

The difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.302		0.326		0.002
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

$$p = \frac{t - t_n}{h} = \frac{175 - 180}{10} = -0.5. \quad = \frac{x - x_n}{h}$$

By Newton's Backward difference interpolation formula

$$y_n(x) \; \cancel{P(t = 175°)} = y_n + p\nabla y_n + \frac{p(p+1)}{\lfloor 2} \nabla^2 y_n + \frac{p(p+1)(p+2)}{\lfloor 3} \nabla^3 y_n$$

$$+ \frac{p(p+1)(p+2)(p+3)}{\lfloor 4} \nabla^4 y_n + \ldots$$

$$= 10.225 + (-0.5)(2.149) + \frac{(-0.5)(0.5)}{2}(0.375)$$

$$+ \frac{(-0.5)(.5)(1.5)}{6}(.049) + \frac{(-0.5)(0.5)(1.5)(2.5)}{24}(0.002)$$

$$= 10.225 - 1.0745 - 0.046875 - 0.0030625 - 0.000078125$$

$$= 9.10048438$$

$$\approx \mathbf{9.100} \quad \textbf{Ans.}$$

Example 5.7 In an examination, the number of candidates who secured marks between certain limits were as follows:

Marks:	0–19	20–39	40–59	60–79	80–99
No. of Students:	41	62	65	50	17

Estimate the number of candidates getting marks less than 70. by using N.B.I.F.

Solution: Using given data, we construct the cummulative frequency table. By using commulative frequency, we construct the difference table.

Marks less than x	*Commulative frequency y*	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
19	41				
		62			
39	103		3		
		65		–18	
59	168		–15		0
		50		–18	
79	218		–33		
		17			
99	235				

$$p = \frac{x - x_n}{h} = \frac{70 - 99}{20} = \frac{-29}{20} = -1.45$$

Now, using Newton-Gregory backward difference formula

$$y_0 = y_n + p\nabla y_n + \frac{p(p+1)}{\lfloor 2} \nabla^2 y_n + \frac{p(p+1)(p+2)}{\lfloor 3} \nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{\lfloor 4} \nabla^4 y_n + \ldots$$

$$= 235 + (-1.45)(17) + \frac{(-1.45)(-1.45+1)}{\lfloor 2} (-33) + \frac{(-1.45)(-1.45+1)(-1.45+2)}{\lfloor 3} (-18) + 0$$

$$= 235 - 24.65 - 10.76625 - 1.076625$$

$$= 198.507126 \approx 199.$$

$\therefore$ The number of *candidates* getting marks less than 70 is **199.**

Example 5.8 The population of a town is as follows:

Year (x):	1941	1951	1961	1971	1981	1991
Population in lakhs (y):	20	24	29	36	46	51

Estimate the population increase during the period 1946 to 1976.

Solution: Let us find the population at x = 1946 and x = 1976.

$$u = \frac{x - x_0}{h} = \frac{1946 - 1941}{10} = \frac{1}{2}.$$

Then the difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1941	20					
		4				
1951	24		1			
		5		1		
1961	29		2		0	
		7		1		–9
1971	36		3		–9	
		10		–8		
1981	46		–5			
		5				
1991	51					

Using Newton forward difference formula

$$y(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{\underline{|2}}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{\underline{|3}}\Delta^3 y_0 + \ldots$$

$$y(1946) = 20 + \frac{1}{2}\times 4 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)(1)}{2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\cdot(1)$$

$$+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{24}(0) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{120}(-9)$$

$$= 20 + 2 - 0.125 + 0.0625 - 0.24609$$

$$= 21.69$$

Now, using Backward formula,

$$y(x) = y_n + v\,\Delta y_n + \frac{v(v+1)}{\underline{|2}}\nabla^2 y_n + \frac{v(v+1)(v+2)}{\underline{|3}}\nabla^3 y_n + \ldots$$

$$v = \frac{x - x_n}{h} = \frac{1976 - 1991}{10} = -\frac{3}{2}$$

$$\therefore \quad y(1976) = 51 - \frac{3}{2}(5) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}{2}(-5) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}{6}(-8)$$

$$+\frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{24}(-9) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{120}(-9)$$

$$= 51 - 7.5 - 1.875 - 0.5 - 0.2709375 - 0.10546875$$

$$= 40.8085938$$

Therefore, increase in population during the period

$$= 40.809 - 21.69 = 19.119 \textbf{ lakhs.}$$

Example 5.9 Find the cubic polynomial which takes the following values:

x:	0	1	2	3
$f(x)$:	1	2	1	10

Hence, evaluate $f(4)$. , by using N.B.I.F. ?

6

CENTRAL DIFFERENCE INTERPOLATION

6.1 Introduction

In the previous chapter, Newton's forward and backward interpolation formulae were discussed. These formulae are applicable for all values of interpolation but, infact are best suited for interpolation near the begning and end values of the tabulated data. In this chapter, we shall study some central difference interpolation formulae which are most suited for the values near the middle of the tabulated data.

Gauss Central Difference Formulae :

6.2 Gauss Forward Interpolation Formula

The Newton forward interpolation formula is

$$y = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{\lfloor 2}\cdot\Delta^2 y_0 + \frac{p\,(p-1)\,(p-2)}{\lfloor 3}\cdot\Delta^3 y_0 + \ldots \qquad \ldots(1)$$

where $p = \dfrac{x - x_0}{h}$ and $x > x_0$ is the origin. ; Central Diff. Table (Note.)

From the central difference table we have

$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1} \Rightarrow$ $$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$\Delta^4 y_{-1} = \Delta^3 y_0 - \Delta^3 y_{-1} \Rightarrow$ $$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$\Delta^5 y_{-1} = \Delta^4 y_0 - \Delta^4 y_{-1} \Rightarrow$ $$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$$

$$\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots$$

and

$\Delta^4 y_{-2} = \Delta^3 y_{-1} - \Delta^3 y_{-2} \Rightarrow$ $$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

$\Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2} \Rightarrow$ $$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$$

$$\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots$$

Substituting these values in (1), we get

$$y = y_0 + p\Delta y_0 + \frac{p\,(p-1)}{\lfloor 2}\,(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{\lfloor 3}\,(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$
$$+ \frac{p(p-1)(p-2)(p-3)}{\lfloor 4}\,(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots$$

The above formula may be written as

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\,\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{\lfloor 3}\,\Delta^3 y_{-1} + \frac{(p+1)p\,(p-1)\,(p-2)}{\lfloor 4}\,\Delta^4 y_{-1}$$
$$+ \ldots \qquad \ldots(2)$$

Equation (2) is called *Gauss's forward interpolation formula.*

$u = P$

```
Enter the value of X for which you want value of Y : - 32
The difference table is :-
   X          f(x)      ?^1       ?^2       ?^3
25.0000    0.2707    0.0320    0.0039    0.0010
30.0000    0.3027    0.0359    0.0049
35.0000    0.3386    0.0408
40.0000    0.3794
When X = 32.0000, Y = 0.31659201
Press Enter to Exit
```

6.3 Gauss Backward Interpolation Formula

Same as eq. ① ; central Diff. tab. (Notes) Pg. 19.

Substituting

$$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$$
$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$
$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$

and

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^2 y_{-2}$$
$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$$

in Newton forward interpolation formula, we have

$$y = y_0 + \frac{p}{\underline{|1}}(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{\underline{|2}}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \ldots$$

$$= y_0 + \frac{p}{\underline{|1}}\Delta y_{-l} + \frac{p(p+1)}{\underline{|2}}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{\underline{|3}}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{\underline{|4}}\Delta^4 y_{-2} + \ldots \quad \ldots(3)$$

Equation (3) is called *Gauss's Backward Interpolation formula.*

The Gauss's forward formula is used to interpolate the values of the function for the value of u such that $0 < u < 1$, and Gauss's backward formula is used to interpolate the value of function for a negative value of u which lies between -1 and 0 *i.e.*, $-1 < u < 0$.

6.3.1 Algorithm for Gauss backward Interpolation Method

Step 1. Enter the number of terms.

Step 2. Enter the value of X and Y variable.

Step 3. Evaluate backward differences.

Step 4. Print backward difference table.

Step 5. Enter the value of X for which the value of Y is calculated.

Step 6. Apply Gauss-backward interpolation formula to find the result.

Step 7. Print the result.

Step 8. Stop.

6.3.2 'C' Program to implement Gauss backward interpolation formula

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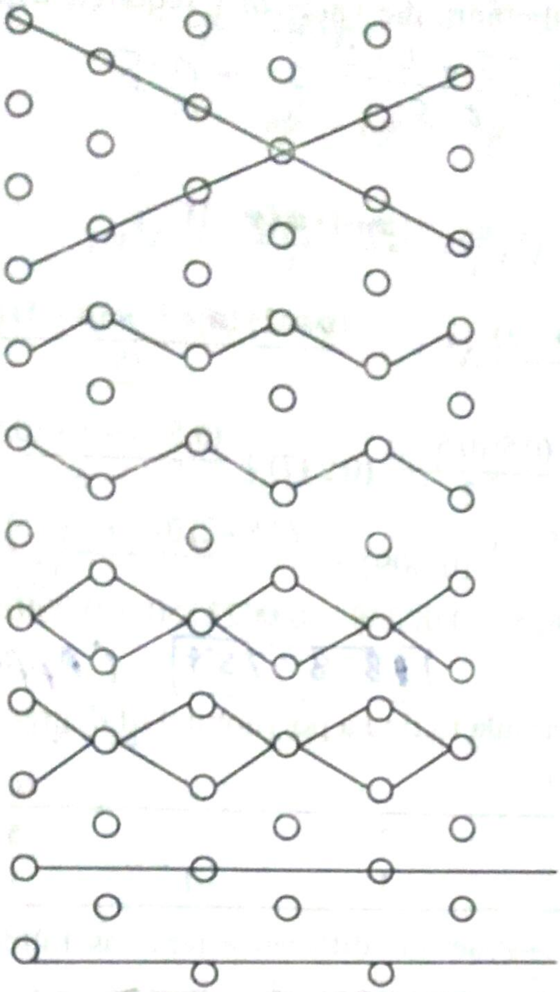
```
#include <stdio.h>
```



Illustrative Examples

• **Example 6.1** Use Gauss's forward formula to find the value of y when $x = 3.75$, from the following table:

x :	2.5	3.0	3.5	4.0	4.5	5.0
y :	24.145	22.043	20.225	18.644	17.262	16.047

Solution: The difference table is

p	x	y_u	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
−2	2.5	24.145					
			−2.102				
−1	3.0	22.043		0.284			
			−1.818		−0.047		
0	$\boxed{3.5}$	$\boxed{20.225}$		$\boxed{0.237}$		$\boxed{0.009}$	
			$\boxed{-1.581}$		$\boxed{-0.038}$		$\boxed{-0.003}$
1	4.0	18.644		0.199		0.006	
			−1.382		−0.032		
2	4.5	17.262		0.167			
			−1.215				
3	5.0	16.047					

$\Rightarrow y = 18.644 + 0.7905 + 0.0746 - 0.002375 - 0.000234 + 0.0000351$

$\Rightarrow y = 19.506$ (Approx)

Taking 3.5 as the origin and 0.5 as the unit, the value of y required will be value for

$$p = \frac{3.75 - 3.5}{0.5} = 0.5 \quad , \quad p = \frac{3.75 - 4.0}{0.5} = -0.5$$

Using Gauss's forward formula, *i.e.*,

$$y_4 = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{\lfloor 3}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{\lfloor 4}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{\lfloor 5}\Delta^5 y_{-2} + \ldots$$

$$y_{0.5} = 20.225 + 0.5(-1.581) + \frac{(0.5)(0.5-1)}{2}(0.237) + \frac{(0.5+1)(0.5)(0.5-1)}{6}(-0.038)$$

$$+ \frac{(0.5+1)(0.5)(0.5-1)(0.5-2)}{24}(0.009) + \frac{(0.5+2)(0.5+1)(0.5)(0.5-1)(0.5-2)}{120}(-0.003)$$

$$= 20.225 - 0.7905 - 0.029625 + 0.00238 + 0.0023750 + 0.0002106$$

$$= 19.40 \text{ approx.} \qquad \boxed{18.80759} \text{ (Approx)}$$

Example 6.2 Use Gauss forward formula to find a polynomial of degree four or less which takes the following values of the formula $f(x)$.

x:	1	2	3	4	5
$f(x)$:	1	−1	1	−1	1

Solution: ~~Let u = x − 3. Then we construct~~ the difference table as follows:

p	x	$y = f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
−2	1	1				
			−2			
−1	2	−1		4		
			2		−8	
0	3	$\boxed{1}$		$\boxed{-4}$		$\boxed{16}$
			$\boxed{-2}$		$\boxed{8}$	
1	4	−1		4		
			2			
2	5	1				

The Gauss forward formula is

$$y(x) = y_0 + p\,.\,\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{\lfloor 3}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{\lfloor 4}\Delta^4 y_{-2} + \ldots$$

$$= 1 + p(-2) + \frac{p(p-1)}{2}(-4) + \frac{(p+1)p(p-1)}{6}(8) + \frac{(p+1)p(p-1)(p-2)}{24}(16)$$

u = P

$$= 1 - 2p - 2p(p-1) + \frac{4p(p^2-1)}{3} + \frac{2p(p^2-1)(p-2)}{3}$$

$$= \frac{1}{3}\left[3 - \cancel{6p} - 6p^2 + \cancel{6p} + 4p^3 - 4p + 2p(p^3 - 2p^2 - p + 2)\right]$$

$$= \frac{1}{3}\left[3 - \cancel{4p} - 6p^2 + \cancel{4p^3} + 2p^4 - \cancel{4p^3} - 2p^2 + 4p\right]$$

$$= \frac{1}{3}\left[2p^4 - 8p^2 + 3\right].$$

Put $\quad p = x - 3$

We have $\quad y = f(x) = \frac{1}{3}\left[2(x-3)^4 - 8(x-3)^2 + 3\right].$

Example 6.3 Given that

$\sqrt{12500} = 111.803399$, $\sqrt{12510} = 111.848111$, $\sqrt{12520} = 111.892806$

$\sqrt{12530} = 111.937483$.

Show by Gauss's backward formula that $\sqrt{12516} = 111.874930$.

Solution: Taking 12520 as the origin and $h = 10$ as the unit, the value of y required is for

$$u = \frac{12516 - 12520}{10} = -0.4.$$

The difference table is

u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
−2	111.803399			
		0.044712		
−1	111.848111		−0.000017	
		$\boxed{0.044695}$		$\boxed{-0.000001}$
0	$\boxed{111.892806}$		$\boxed{-0.000018}$	
		0.044677		
1	111.937483			

Now, by Gauss's backward formula, we have

$$y_u = y_0 + u\,\Delta y_{-1} + \frac{(u+1)\,u}{\underline{|2}}\,\Delta^2 y_{-1} + \frac{(u+1)\,u\,(u-1)}{\underline{|3}}\,\Delta^3 y_{-2} + \ldots$$

$$y_{-0.4} = 111.892806 + (-0.4)(0.044695) + \frac{(0.6)(-0.4)}{2}(-0.000018) + \frac{(0.6)(-0.4)(-1.4)}{6}(-0.000001)$$

$$= 111.892806 - 0.01788 + \text{negligible quantities}$$

$$= 111.874930$$

i.e., $\quad \sqrt{12516} = 111.874930.$

• **Example 6.4** By using Gauss's backward formula find the sales by a concern for the year 1936, given,

Year	1901	1911	1921	1931	1941	1951
Sales (in thousands)	12	15	20	27	39	52

Solution: On taking 1931 as the origin and $h = 10$ years as the unit, then sale of the concern is to be found for

$$u = \frac{1936-1931}{10} = 0.5$$

The difference table is as under:

x	$p = \frac{x-1931}{10}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
1901	–3	12					
			3				
1911	–2	15		2			
			5		0		
1921	–1	20		2		3	
			7		3		–10
1931	0	27		5		–7	
			12		–4		
1941	1	39		1			
			13				
1951	2	52					

(31.5)

Again Gauss's backward formula is

$$y_u = y_0 + {}^uC_1\ \Delta y_{-1} + {}^{u+1}C_2\ \Delta^2 y_{-1} + {}^{u+1}C_3\ \Delta^3 y_{-2} + \ldots$$

i.e.,

$$y_u = y_0 + p\Delta y_{-1} + \frac{(p+1)\,u}{\lfloor 2} \Delta^2 y_{-1} + \frac{(p+1)\,p\,(p-1)}{\lfloor 3} \Delta^3 y_{-2} + \frac{(p+2)(p+1)\,p\,(p-1)}{\lfloor 4} \Delta^4 y_{-2} + \ldots$$

$$= 27 + 0.5 \times 7 + \frac{(1.5)(.5)}{2} \times 5 + \frac{(1.5)(0.5)(-0.5)}{6} \times 3 + \frac{(2.5)(1.5)(0.5)(-0.5)}{24} \times (-7) + \frac{(2.5)(1.5)(0.5)(-0.5)}{120} (-10)$$

$$= 27 + 3.5 + 1.875 - 0.1875 + 0.2734 - 0.11718$$

$$= 32.6484 - 0.30468 = 32.3437 \text{ thousands.}$$

using Gauss's Backward Answer is = 36.28125

Example 6.5 Compute the value of $\frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$, when $x = 0.6538$ by using

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(*a*) Gauss's forward formula.

(*b*) Gauss's backward formula.

(*c*) Stirling's formula, given that

x	0.62	0.63	0.64	0.65	0.66	0.67	0.68
y	0.6194114	0.6270463	0.634857	0.6420292	0.6493765	0.6566275	0.6637820

Solution: The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^4 y$	$\Delta^5 y$
0.62	0.6194114						
		0.007649					
0.63	0.6270463		–0.0000955				
		0.0075394		–0.0000004			
0.64	0.6345857		–0.0000959		0.0000001		
		0.0074435		–0.0000003		0.0000001	
$\boxed{0.65}$	$\boxed{0.6420292}$		$\boxed{-0.0000962}$		$\boxed{0.0000002}$		$\boxed{-0.0000004}$
		$\boxed{0.0073473}$		$\boxed{-0.0000001}$		$\boxed{-0.0000003}$	
0.66	0.6493765		–0.0000963		–0.0000001		
		0.0072510		–0.0000002			
0.67	0.6566275		–0.0000965				
		0.0071545					
0.68	0.6637820						

We have $h = 0.01, \quad x = 0.6538, \quad x_0 = 0.65$

$$\therefore \quad p = \frac{0.6538 - 0.65}{0.01} = 0.38$$

(*a*) Using Gauss's forward interpolation formula

$$y = y_0 + u\,\Delta y_0 + \frac{u\,(u-1)}{\lfloor 2}\,\Delta^2 y_{-1} + \frac{(u+1)\,u\,(u-1)}{\lfloor 3}\,\Delta^3 y_{-1} + \ldots$$

$$y_{0.38} = 0.6420292 + 0.3 \times 0.0073473 + \frac{(0.38)\,(0.38-1)}{2}\,(-0.0000962) + \ldots$$

$$= 0.6448325.$$

(*b*) Using Gauss's backward interpolation formula

$$y = y_0 + u\,\Delta y_{-1} + \frac{u\,(u+1)}{\lfloor 2}\,\Delta^2 y_{-1} + \frac{(u+1)\,u\,(u-1)}{\lfloor 3}\,\Delta^3 y_{-2} + \ldots$$

$$y_{0.38} = 0.6420292 + 0.00282853 - 0.0000252$$

$$= 0.6448325.$$

(*c*) Using Stirling's formula

The arithmetic mean of Gauss's forward and Gauss's backward formula is

$$y_{0.38} = \frac{0.6448325 + 0.6448325}{2}$$

$$y = 0.6448325.$$

Example 6.6 Given $y_{20} = 24, y_{24} = 32, y_{28} = 35, y_{32} = 40$, find y_{25} by Bessel's formula.

Solution: Take origin at $x = 24$ and 4 as the unit, we are to find y_u for $u = \dfrac{25-24}{4} = \dfrac{1}{4}$ or 0.25.

The Bessel's formula is

$$y_u = \frac{y_0 + y_1}{2} + (u - 1/2)\,\Delta y_0 + \frac{u\,(u-1)}{\lfloor 2}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(u-1/2)\,u\,(u-1)}{\lfloor 3}\,\Delta^3 y_{-1} + \ldots$$

The difference table is

x	$u = \dfrac{x-24}{4}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
20	–1	24			
			8		
				–5	
24	0	32			
			3		7
				2	
28	1	35			
			5		
32	2	40			

On putting the values in Bessel's formula, we get

$$y_{1/4} = \frac{32+35}{2} + \left(\frac{1}{4}-\frac{1}{2}\right) \times 3 + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2} \cdot \left(\frac{-5+2}{2}\right) + \frac{\left(\frac{1}{4}-\frac{1}{2}\right)\frac{1}{4}\left(\frac{1}{4}-1\right)}{6} \times 7$$

$$= 33.5 - 0.75 + 0.140625 + 0.0546875$$

$$= 32.9453125.$$

Example 6.7 From the following table, find the value of log 337.5 by Gauss, Stirling, Bessel and Everett formulae:

x:	310	320	330	340	350	360
$\log_{10} x$:	2.4913617	2.5051500	2.5185139	2.5314789	2.5440680	2.5563025

Solution: On taking 330 as the origin and 10 as the unit, the value of $\log_{10}$ 337.5 required will be for $u = \dfrac{337.5-330}{10} = 0.75.$

The difference table is as under:

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
310	–2	2.4913617					
			0.0137883				
320	–1	2.5051500		–0.0004244			
			0.0133639		0.0000255		
330	0	2.5185139		–0.0003989		-25×10^{-7}	
			0.0129650		0.0000230		8×10^{-7}
340	1	2.5314789		–0.0003759		-17×10^{-7}	
			0.0125891		0.0000213		
350	2	2.5440680		–0.0003546			
			0.0122345				
360	3	2.5563025					

(*i*) Gauss's forward formula is

$$y_u = y_0 + {}^uC_1\, \Delta y_0 + {}^uC_2\, \Delta^2 y_{-1} + {}^{u+1}C_3 \cdot \Delta^3 y_{-1} + {}^{u+1}C_4\, \Delta^4 y_{-2} + {}^{u+2}C_5\, \Delta^5 y_{-2} + \cdots$$

$$y_{.75} = 2.5185139 + \frac{3}{4}(0.0129650) + \frac{\frac{3}{4}\left(-\frac{1}{4}\right)}{2}(-0.0003989)$$

$$+\frac{\frac{7}{4}\times\frac{3}{4}\left(-\frac{1}{4}\right)}{6}(0.0000230)+\frac{\frac{7}{4}\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{24}(-0.0000025)$$

$$+\frac{\frac{11}{4}\times\frac{7}{4}\left(-\frac{3}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{120}(0.0000008)$$

$$=2.52827374.$$

(*ii*) Gauss's backward formula is

$$y_u = y_0 + {}^uC_1\,\Delta y_{-1} + {}^{u+1}C_2\,\Delta^2 y_{-1} + {}^{u+1}C_3\,\Delta^3 y_{-2} + {}^{u+1}C_4\,\Delta^4 y_{-2} + \ldots$$

$$=2.5185139+\frac{3}{4}(0.0133639)+\frac{\frac{7}{4}\times\frac{3}{4}}{2}(-0.0003989)+\frac{\frac{7}{4}\times\frac{3}{4}\left(-\frac{1}{4}\right)}{6}(0.0000255)$$

$$+\frac{\frac{11}{4}\times\frac{7}{4}\times\frac{3}{4}\times\left(-\frac{1}{4}\right)}{24}(-0.0000025)$$

$$=2.52827375.$$

(*iii*) Stirling's formula is

$$y_u = y_0 + \frac{u\,(\Delta y_0 + \Delta y_{-1}]}{2} + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u\,(u^2-1)}{\underline{|3}}\,\frac{1}{2}\,[\Delta^3 y_{-1} + \Delta^3 y_{-2}]$$

$$+\frac{u^2\,(u^2-1)}{\underline{|4}}\cdot\Delta^4 y_{-2} + \ldots$$

$$=2.5185139+\frac{3}{4}\times\frac{1}{2}\,[0.0129650+0.0133639]+\frac{\left(\frac{3}{4}\right)^2(-0.0003989)}{2}$$

$$+\frac{\frac{3}{4}\left(\frac{9}{16}-1\right)}{6}\cdot\frac{1}{2}\,(0.0000230+0.000255]+\frac{\frac{9}{16}\left(\frac{9}{16}-1\right)}{24}(-0.0000025)$$

$$=2.52827374.$$

(*iv*) Bessel's formula is

$$y_u = \frac{1}{2}\,[y_0 + y_1] + \left(u-\frac{1}{2}\right)\Delta y_0 + \frac{u\,(u-1)}{\underline{|2}}\cdot\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$

$$+\frac{(u-1/2)\,u\,(u-1)}{\underline{|3}}\,\Delta^3 y_{-1} + \frac{(u+1)\,u\,(u-1)\,(u-2)}{\underline{|4}}\cdot\frac{1}{2}[\Delta^4 y_{-1} + \Delta^4 y_{-2}]$$

$$+\frac{(u-1/2)\,(u+1)\,u\,(u-1)\,(u-2)}{\underline{|5}}\,\Delta^5 y_{-2} + \ldots$$

$$=\frac{1}{2}\,[2.5185139+2.5314789]+\frac{1}{4}\,(0.0129650)$$

7

INTERPOLATION WITH UNEQUAL INTERVALS

7.1 Introduction

~~In previous chapters~~, on interpolation, we considered the interval of differencing to be a constant h. In case, the values of the arguments are given at unequal intervals, then the various differences will be effected with the change in values of the arguments and consequently definition of various differences for equal intrvals will not hold good. The differences defined by taking into consideration the change in the values of the arguments are called *divided differences.*

7.2 Divided Differences

Let the function $y = f(x)$ assumes the values $f(x_0), f(x_1), \ldots, f(x_n)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$ respectively where the intervals $x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}$ need not be equal.

The first divided difference of $f(x)$ for the arguments x_0, x_1, is defined as $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$.

~~It is denoted~~ by $f(x_0, x_1)$ or $[x_0, x_1]$ or $\underset{x_1}{\Delta} f(x_0)$ *i.e.*. we may Formatted as -

$$f(x_0, x_1) = [x_0, x_1] = \underset{x_1}{\Delta} f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \text{by definition} \quad \ldots(1)$$

In the ~~same notation~~ other differences, we have

$$f(x_1, x_2) = \underset{x_2}{\Delta} f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

and in general, $f(x_{n-1}, x_n) = \underset{x_n}{\Delta} f(x_{n-1}) = \dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$, $n = 1, 2, \ldots, n$. — 1

The second divided difference of $f(x)$ for three arguments x_0, x_1, x_2 is defined as

$$f(x_0, x_1, x_2) = \underset{x_1, x_2}{\Delta^2} f(x_0) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} \quad \ldots(2)$$

(x_{n-2}, x_{n-1}, x_n) In general — 2

This shows that to find a second divided difference, we require three continuous arguments.

In the same way, we define the third order divided difference of $f(x)$ for the four arguments x_0, x_1, x_2, x_3 as

$$\underset{x_1, x_2, x_3}{\Delta^3} f(x_0) = f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0} \quad \ldots(3)$$

In general — 3 $(x_{n-3}, x_{n-2}, x_{n-1}, x_n)$

Equations (1), (2), and (3) refer to divided differences of order one, two and three respectively.

The divided difference table is as follows (Table 7.1).

D. D. **Table 7.1**

Argument (x)	*Entry* $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
x_0	$f(x_0)$				
		$f(x_0, x_1)$			
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$		
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$	
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$		$f(x_0, x_1, x_2, x_3, x_4)$
		$f(x_2, x_3)$		$f(x_1, x_2, x_3, x_4)$	
x_3	$f(x_3)$		$f(x_2, x_3, x_4)$		
		$f(x_3, x_4)$			
x_4	$f(x_4)$				

7.3 Properties of Divided Difference

Property 1 Divided differences are symmetric functions of their arguments *i.e.*, the value of any divided difference is independent of the order of the argument.

~~**Proof:**~~ statement –

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$f(x_1, x_1) = f(x_1, x_0) = \underset{x_1}{\Delta} f(x_0)$$

Also, $$f(x_0, x_1) = \frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1} = \sum \frac{f(x_0)}{x_0 - x_1}$$

Again $$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$= \frac{1}{x_2 - x_0}\left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}\right]$$

$$= \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)}$$

$$= \sum \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)}$$

which is a symmetric function of the arguments x_0, x_1 and x_2. Thus,

$$f(x_0, x_1, x_2) = f(x_2, x_1, x_0) = f(x_1, x_2, x_0) \text{ etc.}$$

By mathematical induction, we can prove that

$$f(x_0, x_1, x_2, \ldots, x_n) = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} + \frac{f(x_{n-1})}{(x_{n-1} - x_0)(x_{n-1} - x_1)\ldots(x_{n-1} - x_n)}$$

$$+ \ldots + \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} = f$$

(any one of the perputations of the arguments $x_0, x_1, \ldots x_n$)

Thus, we observe that the value of a divided difference depends only on the values of the arguments involved and not on the order in which they involve.

Property 2 The nth divided differences of a polynomial of the nth degree are constant

Proof ~~Consider a function~~ statement $f(x) = x^n$.

The first divided difference of $f(x)$ for the arguments x_r, x_{r+1} are

$$f(x_r, x_{r+1}) = \frac{f(x_{r+1}) - f(x_r)}{x_{r+1} - x_r} = \frac{x_{r+1}^n - x_r^n}{x_{r+1} - x_r}$$

$$= x_{r+1}^{n-1} + x_r \,.\, x_{r+1}^{n-2} + \ldots + x_r^{n-2}\, x_{r+1} + x_r^{n-1}$$

This is a symmetric function of the $(n-1)$th degree in x_r, x_{r+1}.

The second divided difference of $f(x)$ for the arguments x_r, x_{r+1}, x_{r+2} is given by

$$f(x_r, x_{r+1}, x_{r+2}) = \frac{f(x_r, x_{r+1}) - \cancel{f(x_{r+1}, x_{r+2})}}{x_r - x_{r+2}}$$

$$= \frac{f(x_{r+1}, x_{r+2}) - f(x_r, x_{r+1})}{x_{r+2} - x_r}$$

$$= [(x_{r+2}^{n-1} + x_{r+1} \,.\, x_{r+2}^{n-2} + \ldots + x_{r+1}^{n-2}\, x_{r+2} + x_{r+1}^{n-1})$$

$$- (x_{r+1}^{n-1} + x_r \,.\, x_{r+1}^{n-2} + \ldots + x_r^{n-2}\, x_{r+1} + x_r^{n-1})]/(x_{r+2} - x_r)$$

$$= \frac{x_{r+2}^{n-1} - x_r^{n-1}}{x_{r+2} - x_r} + x_{r+1} \frac{(x_{r+2}^{n-2} - x_r^{n-2})}{x_{r+2} - x_r} + \ldots + \frac{x_{r+1}^{n-2}\,(x_{r+2} - x_r)}{x_{r+2} - x_r}$$

$$= (x_{r+2}^{n-2} + \ldots + x_r^{n-2}) + x_{r+1}\,(x_{r+2}^{n-3} + \ldots + x_r^{n-3}) + \ldots + x_{r+1}^{n-2}.$$

This is a symmetric function of $(n-2)^{\text{th}}$ degree in x_r, x_{r+1} and x_{r+2}.

By mathematical induction, it can be shown that $f(x_r, x_{r+1}, \ldots, x_{r+m})$ is a symmetric function of degree $n - m$. In particular, the nth divided difference of $f(x) = x^n$ is constant and therefore independent of the values of $x_r, x_{r+1}, x_{r+2}, \ldots, x_{r+n}$.

7.4 Newton's General Divided Difference Interpolation Formula

Let the function $f(x)$ be given for the $(n+1)$ values $x_0, x_1, x_2, \ldots, x_n$ as $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ where $x_0, x_1, x_2, \ldots, x_n$ are not necessarily equally spaced. From the definition of divided difference

$$[x, x_0] = f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow \quad f(x) = f(x_0) + (x - x_0)\,[x, x_0] \qquad \ldots(1)$$

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

$$\therefore \quad [x, x_0] = (x - x_1)\,[x, x_0, x_1] + [x_0, x_1].$$

and $[x, x_0, x_1, x_2] = \dfrac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$

$\Rightarrow [x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$ (3)

Substituting these in (1), we get

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x, x_0, x_1] \qquad \ldots(2)$$

On proceeding, we have

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2]$$
$$+ (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3]$$
$$+ \ldots + (x - x_0)\,(x - x_1) \ldots (x - x_n) \,.\, [x, x_0, x_1, \ldots, x_n] \qquad \ldots(4)$$

If $f(x)$ is a polynomial of degree n, then the $(n+1)$th divided difference of $f(x)$ will be zero.

$\therefore \; f(x, x_0, x_1, \ldots, x_n) = 0.$

$\therefore$ Equation (4) can be written as

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1) \cdot [x_0, x_1, x_2] + \ldots + (x - x_0)\,(x - x_1) \ldots (x - x_{n-1}) \,.\, [x_0, x_1, \ldots, x_n] \qquad \ldots(5)$$

$$f(x) = f(x_0) + (x - x_0) \cdot \Delta f(x_0, x_1) + (x - x_0)(x - x_1) \cdot \Delta^2 f(x_0, x_1, x_2) + \ldots + (x - x_0)(x - x_1) \ldots (x - x_{n-1}) \cdot \Delta^n f(x_0, x_1, x_2, \ldots, x_n)$$

Interpolation

Equation (~~4~~ 5) is called *Newton's General divided difference formula.*

7.4.1 Algorithm to implement Newton divided difference formula

Step 1. Input the number of terms n.

Step 2. Input the value of x and y.

Step 3. Form the divided difference table.

Step 4. Apply Newton divided difference formula i.e.

$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + \ldots + (x - x_0)\,(x - x_1) \ldots (x - x_{n-1}) \cdot [x_0, x_1, \ldots, x_n]$

Step 5. Print the result

Step 6. Stop.

7.4.2 'C' program to implement Newton divided difference formula

```
#include <stdio.h>
#include <conio.h>
void main()
{
    int i,j,n;
    float xy[10][11],h,p,px=1,x,y;
    clrscr();
    /*Data read opration*/
    printf("\nEnter the number of data : ");
    scanf("%d",&n);
    printf("\nEnter the data : \n");
    for(i=0;i<n;i++)
    {
        printf("x(%d) and y(%d) : ",i+1,i+1);
        scanf("%f%f",&xy[i][0],&xy[i][1]);
    }
    for(j=2;j<n+1;j++)                    /* Forming difference table */
    for(i=0;i<n-j+1;i++)
    xy[i][j]=(xy[i+1][j-1]-xy[i][j-1])/(xy[i+(j-1)][0]-xy[i][0]);
    printf("\nThe difference table is :-");     /* Printing table */
    printf("\nx          f(x)      ");
    for(i=0;i<n-1; i++)
    printf("^%d       ",i+1);
    for(i=0;i<n;i++)
    {
        printf("\n");
        for(j=0;j<n+1-i;j++)
    printf("%.4f   ",xy[i][j]);
    }
                                  /* Take the valuen of x for f(x) */
    printf("\nEnter the value of 'x' : ");
    scanf("%f",&x);
    y=xy[0][1];
                                  /* Calculate the value of f(x) */
```

```
    for(i=1;i<n;i++)
    {
        px *= (x-xy[i-1][0]);
        y += xy[0][i+1]*px;
    }
                                    /* Printing the result */
    printf("\nThe value of function at x = %f is %f",x,y);
    getch();
    }
/* Output */
Enter the number of data : 4
Enter the data :
x(1) and y(1) : 0    1
x(2) and y(2) : 1    4
x(3) and y(3) : 3    88
x(4) and y(4) : 6    1309
The difference table is :-
      x         f(x)       ?^1        ?^2        ?^3
   0.0000    1.0000     3.0000    13.0000    10.0000
   1.0000    4.0000    42.0000    73.0000
   3.0000   88.0000   407.0000
   6.0000 1309.0000
Enter the value of 'x' : 5
The value of function at x = 5.000000 is 676.000000
```

Example 7.1 Using the ~~table given below~~ N.G.D.D.I. Formula find the value of $f(8)$ for following data set. (UPTU MCA 2008-09)

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

Solution: The divided difference table is as under:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\frac{100-48}{5-4}=52$			
5	100		$\frac{97-52}{7-4}=15$		
		$\frac{294-100}{7-5}=97$		$\frac{21-15}{10-4}=1$	
7	294		$\frac{202-97}{10-5}=21$		0
		$\frac{900-294}{10-7}=202$		$\frac{27-21}{11-5}=1$	
10	900		$\frac{310-202}{11-7}=27$		0
		$\frac{1210-900}{11-10}=310$		$\frac{33-27}{13-7}=1$	
11.	1210		$\frac{409-310}{13-10}=33$		
		$\frac{2028-1210}{13-11}=409$			
13	2028				

By Newton's divided difference Interpolation formula,

$$f(x) = f(x_0) + (x - x_0)\, \Delta f(x) + (x - x_0)(x - x_1)\, \Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)\, \Delta^3 f(x) + \ldots$$

$$= 48 + (x - 4)\, 52 + (x - 4)(x - 5)\, 15 + (x - 4)(x - 5)(x - 7)\, 1$$

$$f(8) = 48 + (8 - 4)\, 52 + (8 - 4)(8 - 5)\, 15 + (8 - 4)(8 - 5)(8 - 7)\, 1$$

$$= 48 + 4 \times 52 + 4 \times 3 \times 15 + 4 \times 3 \times 1$$

$$= \mathbf{448} \quad \textbf{Ans.}$$

Example 7.2 From the table given as below, obtain $f(x)$ as a cubic polynomial in powers of $(x - 5)$ by using Newton's ~~method~~ G. D. D. I. Formula -

x:	0	2	3	4	7	9
$f(x)$:	4	26	58	112	466	922

Solution: The divided difference table is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	4				
		11			
2	26		7		
		32		1	
3	58		11		0
		54		1	
4	112		16		0
		118		1	
7	466		22		
		228			
9	922				

Using Newton's divided difference interpolation formula, we have,

$$f(x) = f(0) + (x - 0)\, \Delta f(x_0) + (x - 0)(x - 2) \,.\, \Delta^2 f(x_0) + (x - 0)(x - 2)(x - 3) \,.\, \Delta^3 f(x_0) + \ldots$$

$$= 4 + x \,.\, 11 + x(x - 2) \,.\, 7 + x(x - 2)(x - 3) \,.\, 1$$

$$= 4 + 11x + 7x^2 - 14x + x^3 - 5x^2 + 6x$$

$$f(x) = x^3 + 2x^2 + 3x + 4$$

$\because$ $(x - 5)$ is the factor of the polynomial so,

x

5	1	2	3	4
		5	35	190
5	1	7	38	194
		5	60	
5	1	12	98	
		5		
5	1	17		
5	1			

$\therefore$ The polynomial in terms of $(x - 5)$ is

$f(x-5) = (x-5)^3 + 2(x-5)^2 + 3(x-5) + 4$

$$f(x) = (x - 5)^3 + 17(x - 5)^2 + 98(x - 5) + 194$$

Example 7.3 Determine $f(x)$ as a polynomial in x and Find the value of $f(3.5)$ for the following data: by using N.G.D.D.I. Formula -

x:	-4	-1	0	2	5
$f(x)$:	1245	33	5	9	1335

Solution: The divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-4	1245				
		-404			
			94		
-1	33				
				-14	
		-28			
					3
0	5				
			10		
		2			
				13	
2	9				
			88		
		442			
5	1335				

Applying Newton's divided difference formula

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0)\,\Delta f(x_0) + (x - x_0)(x - x_1)\,\Delta^2 f(x_0) + \ldots \\ &= 1245 + (x + 4)(-404) + (x + 4)(x + 1)\,94 \\ &\quad + (x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)(x - 0)(x - 2)(3) \\ &= 3x^4 - 5x^3 + 6x^2 - 14x + 5. \end{aligned}$$

Example 7.4 Prove that $\underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = -\frac{1}{abcd}$.

Solution: We construct the following divided difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
a	$\frac{1}{a}$			
		$\dfrac{\frac{1}{b} - \frac{1}{a}}{b - a} = -\frac{1}{ab}$		
			$\dfrac{-\frac{1}{bc} - \frac{1}{ab}}{c - a} = \frac{1}{abc}$	
b	$\frac{1}{b}$			
				$\dfrac{\frac{1}{bcd} - \frac{1}{abc}}{d - a} = -\frac{1}{abcd}$
		$\dfrac{\frac{1}{c} - \frac{1}{b}}{c - b} = -\frac{1}{bc}$		
			$\dfrac{-\frac{1}{cd} - \frac{1}{bc}}{d - b} = \frac{1}{bcd}$	
c	$\frac{1}{c}$			
		$\dfrac{\frac{1}{d} - \frac{1}{c}}{d - c} = -\frac{1}{cd}$		
d	$\frac{1}{d}$			

Hence, from table, one can see that $\underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = -\frac{1}{abcd}$.

Example 7.5 Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 66$ 2.8202, find by divided difference formula $\log_{10} 656$.

Solution: Shift the origin to 654, then we can consider the data given for $x = 0, 4, 5$ and and are required to find $f(x)$ at $x = 2$.

The divided difference table is as under:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	2.8156			
		$\frac{2.8182-2.8156}{4-0} = 0.00065$		
4	2.8182		$\frac{0.00070-0.00065}{5-0} = 0.00001$	
		$\frac{2.8189-2.8182}{5-4} = 0.00070$		$\frac{-0.000017-0.00001}{7-0} = -0.0000$
5	2.8189		$\frac{0.00065-0.00070}{7-4} = -0.000017$	
		$\frac{2.8202-2.8189}{7-5} = 0.00065$		
7	2.8202			

$$\log_{10} 656 = 2.8156 + (2-0)(0.00065) + (2-0)(2-4)(0.00001) + (2-0)(2-4)(2-5)(-0.0000$$

$$= 2.8156 + 0.0013 - 0.00004 - 0.0000048$$

$$= 2.8168 \textbf{ Approx.}$$

7.5 Error in Newton's Divided Difference Formula

The error occurred in Newton's divided difference interpolation formula can be obtained by usi formula

$$E_n(x) = (x-x_0)(x-x_1)\ldots(x-x_n)\frac{f^{n+1}(\xi)}{\lfloor n+1}$$

where ξ is some point of the least interval containing all the points x_i $(i = 0, 1, 2, \ldots, n)$ an point x.

EXERCISE 7.1

1. Find the divided differences of $f(x) = x^3 + x + 2$ for the arguments 1, 3, 6, 11.
2. By means of Newton's general divided difference Interpolation formula, find the values of $f(2)$, $f(8)$ and $f(15)$ fro the following table:

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

3. Using the following table, find $f(x)$ as a polynomial in powers of $(x-6)$:

x :	−1	0	2	3	7	10
$f(x)$:	−11	1	1	1	141	561

Also find $f'(6)$, $f''(6)$ and $f'''(6)$.

4. Find the pressure of steam at 142°C using Newton's general formula:

Temperature °C :	140	150	160	170	180
Pressure kgf/cm^2 :	3.685	4.854	6.302	8.076	10.225

5. Find y ($x = 5.60275$) from the table:

x :	5.600	5.602	5.605	5.607	5.608
y :	0.77556588	0.77682686	0.77871250	0.77996571	0.78059114

6. Find $\log_{10} 323.5$ given:

x :	321.0	322.8	324.2	325.0
$\log_{10} x$:	2.50651	2.50893	2.51081	2.51188

7. From the following table find $f(5)$:

x :	0	1	3	6
$f(x)$:	1	4	88	1309

8. Find $\Delta^4 f(x)$ if $f(x) = x(x+1)(x+2)(x+3)$.

9. Find the function $y(x)$ in powers of $(x-1)$ given

$$y(0) = 8, \quad y(1) = 11, \quad y(4) = 68, \quad y(5) = 123.$$

10. If $f(x) = u(x)\,v(x)$, show that

$$f(x_0, x_1) = u(x_0)\,v(x_0, x_1) + u(x_0, x_1)\,v(x_1).$$

ANSWERS

1. $\Delta^3 f(x) = 1$.
2. $f(2) = 4, f(8) = 448\ f(15) = 3150$.
3. $73 + 54(x-6) + 13(x-6)^2 + (x-6)^3 \quad f'(6) = 54, \quad f''(6) = 26, \quad f'''(6) = 6$.

7.6 Lagrange's Formula for Unequal Intervals

When the values of independent variable are not equally spaced and the differences of dependent variable are not small, ultimately, we will use lagrange interpolation formula.

Let $y = f(x)$ be a function such that $f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$, corresponding to $x = x_0, x_1, x_2, \ldots, x_n$ *i.e.*,

$$y_i = f(x_i), \quad i = 0, 1, 2, \ldots, n.$$

Now, there are $(n+1)$ paired values (x_i, y_i), $i = 0, 1, 2, \ldots, n$ and hence $f(x)$ can be represented by a polynomial function of degree n in x.

We ~~One~~ can assume $f(x)$ as follows:

$$\begin{aligned} f(x) = A_0 (x - x_1)(x - x_2) \ldots (x - x_n) &+ A_1 (x - x_0)(x - x_2) \ldots (x - x_n) \\ &+ A_2 (x - x_0)(x - x_1)(x - x_3) \ldots (x - x_n) + \ldots \\ &+ A_i (x - x_0)(x - x_1) \ldots (x - x_{i-1})(x - x_{i+1}) + \ldots \\ &+ A_n (x - x_0)(x - x_1) \ldots (x - x_{n-1}) \quad \ldots(1) \end{aligned}$$

This is true for all values of x.

Substituting in (*1*), $x = x_0$, $y = y_0$, we get

$$y_0 = A_0 (x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)$$

$$\therefore \qquad A_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)}$$

Similarly, setting $x = x_1$, $y = y_1$, we have $\Rightarrow y_1 = A_1(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)$

$$A_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)\dots(x_1 - x_n)}$$

similarly, setting $x = x_2, y = y_2$, we have $\Rightarrow y_2 = A_2(x_2 - x_0)(x_2 - x_1)\cdots(x_2 - x_n)$

and again
$$A_2 = \frac{y_2}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)\dots(x_2 - x_n)}$$

In General,

....

and similarly setting $x = x_n$, $y = y_n$, we have, $\Rightarrow y_n = A_n(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})$

$$A_n = \frac{y_n}{(x_n - x_0)(x_n - x_1)\dots(x_n - x_{n-1})}$$

Substituting these values in equation (1), we have

$$y = f(x) = \frac{(x - x_1)(x - x_2)\dots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)}\, y_0 + \frac{(x - x_0)(x - x_2)\dots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_n)}\, y_1$$
$$+ \frac{(x - x_0)(x - x_1)(x - x_3)\dots(x - x_n)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)\dots(x_2 - x_n)}\, y_2 + \dots$$
$$+ \frac{(x - x_0)(x - x_1)\dots(x - x_{i-1})(x - x_{i+1})\dots(x - x_n)}{(x_i - x_0)(x_i - x_1)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)}\, y_i$$
$$+ \dots + \frac{(x - x_0)(x - x_1)\dots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\dots(x_n - x_{n-1})} \cdot y_n \qquad \dots(2)$$

Equation (2) is called *Lagrange's interpolation formula* for unequal intervals.

Corrollary. Dividing both sides of equation (2) by $(x - x_0)(x - x_1) \dots (x - x_n)$, we get

$$\frac{f(x)}{(x - x_0)(x - x_1)\dots(x - x_n)} = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)}\,\frac{1}{x - x_0} +$$
$$\frac{y_1}{(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_n)}\,\frac{1}{(x - x_1)} + \dots$$
$$+ \frac{y_n}{(x_n - x_0)(x_n - x_1)\dots(x_n - x_{n-1})} \cdot \frac{1}{(x - x_n)}$$

7.7 Different Forms of Lagrange's Interpolation Formula

The Lagrangian interpolation formula can also be written as

$$f(x) = \sum_{i=0}^{n} \frac{\Pi_n(x)}{(x - x_i)\,\Pi'_n(x_i)}\, y_i,$$

where $\Pi_n(x) = (x - x_0)(x - x_1) \dots (x - x_n)$ and $\Pi'_n(x) = \dfrac{d}{dx}[\Pi_n(x)]$

Solution: $\Pi_n(x) = (x - x_0)(x - x_1) \dots (x - x_n)$

Differentiating this and substituting $x = x_i$, we get

$$\Pi'_n(x_i) = (x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x - x_n)$$

```
    printf("\n Value of function at x=%f is : %f",xx,yy);
    getch();
  }
/*Output*/
 Enter the number of data : 6
 Enter the data :
      x(1) and y(1): 4  48
      x(2) and y(2): 5  100
      x(3) and y(3): 7  294
      x(4) and y(4): 10 900
      x(5) and y(5): 11 1210
      x(6) and y(6): 13 2028
 Enter the value of 'x' : 8
Value of function at x=8.000000 is : 448.000000
```

Example 7.6 Given the values:

	x_0	x_1	x_2	x_3	x_4
x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202
	y_0	y_1	y_2	y_3	y_4

evaluate $f(9)$ using Lagrange's formula.

Solution: The Lagrange's interpolation formula is

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \cdot f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} f(x_2)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} f(x_3)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} f(x_4)$$

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$f(9) = -16.666 + 209.066 + 1290.666 - 788.666 + 115.6$

$$= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = \mathbf{810} \quad \textbf{Ans.}$$

Example 7.7 Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$. Find $\log_{10} 656$., by using Lagrange's formula.

y_2 y_3

Solution: Here, $f(x) = \log_{10} 656$, $x_0 = 654$, $x_1 = 658$, $x_2 = 659$, $x_3 = 661$

Thus, by Lagrange's interpolation formula

$$\log_{10} 656 = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \times (2.8156)$$

$$+ \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} \times 2.8182$$

$$+ \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} \times 2.8189$$

$$+ \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} \times 2.8202$$

$$= \frac{3}{14}(2.8156) + \frac{5}{2}(2.8182) - 2(2.8189) + \frac{2}{7}(2.8202)$$

$$= 0.6033 + 7.045 - 5.6378 + 0.8057$$

$$= 2.8170.$$

Example 7.8 Find the form of the function given by, using Lagrange's formula.

x:	3	2	1	–1
$f(x)$:	3	12	15	–21

Solution: In this case $x_0 = 3$, $x_1 = 2$, $x_2 = 1$, $x_3 = -1$.

∴ By Lagrange's formula, we have

$$f(x) = \frac{(x-2)(x-1)(x+1)}{(3-2)(3-1)(3+1)} \times 3 + \frac{(x-3)(x-1)(x+1)}{(2-3)(2-1)(2+1)} \times 12$$

$$+ \frac{(x-3)(x-2)(x+1)}{(1-3)(1-2)(1+1)} \times 15 + \frac{(x-3)(x-2)(x-1)}{(-1-3)(-1-2)(-1-1)} \times (-21)$$

On solving this, we get $f(x) = x^3 - 9x^2 + 17x + 6$.

Example 7.9 Using Lagrange's formula, prove

$$Y_1 = Y_3 - 0.3(Y_5 - Y_{-3}) + 0.2(Y_{-3} - Y_{-5}) \text{ nearly.}$$

Solution: Y_{-5}, Y_{-3}, Y_3, Y_5 occur in the answers. So, we can have the table

x:	–5	–3	3	5
Y:	Y_{-5}	Y_{-3}	Y_3	Y_5

∴ By Lagrange's formula

$$Y(x) = \frac{(x+3)(x-3)(x-5)}{(-5+3)(-5-3)(-5-5)} \cdot Y_{-5} + \frac{(x+5)(x-3)(x-5)}{(-3+5)(-3-3)(-3-5)} \cdot Y_{-3}$$

$$+ \frac{(x+5)(x+3)(x-5)}{(3+5)(3+3)(3-5)} \cdot Y_3 + \frac{(x+5)(x+3)(x-3)}{(5+5)(5+3)(5-3)} Y_5$$

$$y_1 = \frac{(4)(-2)(-4)}{(-2)(-8)(-10)} \cdot y_{-5} + \frac{(6)(-2)(-4)}{(2)(-6)(-8)} \cdot y_{-3} + \frac{(6)(4)(-4)}{(8)(6)(-2)} y_3$$

$$+ \frac{(6)(4)(-2)}{(10)(8)(2)} \cdot y_5$$

$$= -0.2y_{-5} + 0.5y_{-3} + y_3 - 0.3y_5$$

$$y_1 = y_3 - 0.3\,(y_5 - y_{-3}) + 0.2\,(y_{-3} - y_{-5}).$$

7.8 Inverse interpolation: In interpolation, we have discussed various methods of fitting a missing value of $y = f(x)$ for the corresponding value x in the given interval. Here, we discuss the problem of inverse interpolation in which one can find the value of x corresponding to the value of y lying between two tabulated values of y.

Definition: Inverse interpolation is the process of finding the value of the argument corresponding to a given value of the function when the latter is intermediate between two tabulated values.

There are following methods for inverse interpolation:

(*i*) Use of Lagrange's formula.

(*ii*) Method of successive approximations or iterations.

(*iii*) Method of reversion of series.

Here we discuss the Lagrange's formula for inverse interpolation. In Lagrange's interpolation y is expressed as a function of x as given below:

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} \cdot y_0$$

$$+ \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} \cdot y_1$$

$$+ \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} \cdot y_n$$

By interchanging x and y, we can express x as a function of y as follows:

$$x = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_n)}\, x_0 +$$

$$\frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)}\, x_1 + \ldots$$

$$+ \frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots(y_n-y_{n-1})}\, x_n$$

The above formula may be used for inverse interpolation.

Example 7.10 Find the age corresponding to the annuity value 13.6 from the given table:

Age (x) :	30	35	40	45	50
Annuity value (y) :	15.9	14.9	14.1	13.3	12.5

Solution: By using Lagrange's interpolation formula, we have

Inverse

26·934

$$x = \frac{(13.6-14.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(15.9-14.9)(15.9-14.1)(15.9-13.3)(15.9-12.5)} \times 30$$

$$+\frac{(13.6-15.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(14.9-15.9)(14.9-14.1)(14.9-13.3)(14.9-12.5)} \times 35$$

$$+\frac{(13.6-15.9)(13.6-14.9)(13.6-13.3)(13.6-12.5)}{(14.1-15.9)(14.1-14.9)(14.1-13.3)(14.1-12.5)} \times 40$$

$$+\frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-12.5)}{(13.3-15.9)(13.3-14.9)(13.3-14.1)(13.3-12.5)} \times 45$$

$$+\frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-13.3)}{(12.5-15.9)(12.5-14.9)(12.5-14.1)(12.5-13.3)} \times 50$$

$$= \frac{(-1.3)(-0.5)(0.3)(1.1)}{(1)(1.8)(2.6)(3.4)} \times 30 + \frac{(-2.3)(-0.5)(0.3)(1.1)}{(-1)(0.8)(1.6)(2.4)} \times 35$$

$$+\frac{(-2.3)(-1.3)(0.3)(1.1)}{(-1.8)(-0.8)(0.8)(1.6)} \times 40 + \frac{(-2.3)(-1.3)(-0.5)(1.1)}{(-2.6)(-1.6)(-0.8)(0.8)} \times 45$$

$$+\frac{(-2.3)(-1.3)(-0.5)(0.3)}{(-3.4)(-2.4)(-1.6)(-0.8)} \times 50$$

$$= 0.404412 - 4.323730 + 21.412760 + 27.795410 - 2.147001$$

$$x = 43.141851 \approx \mathbf{43}.$$

Example 7.11 Find the value of x when $y = 0.3$ by applying Lagrange's inversion formula.

x:	0.4	0.6	0.8
y:	0.3683	0.3332	0.2897

Solution: By using Lagrange's inverse interpolation formula, we get

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)}\, x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)}\, x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)}\, x_2$$

i.e.,

$$x = \frac{(0.3-0.3332)(0.3-0.2897)}{(0.3683-0.3332)(0.3683-.2897)} \times (0.4)$$

$$+\frac{(0.3-0.3683)(0.3-0.2897)(0.6)}{(0.3332-0.3683)(0.3332-0.2897)}$$

$$+\frac{(0.3-0.3683)(0.3-0.3332)}{(0.2897-0.3683)(0.2897-0.3332)} \times (0.8)$$

$$= \mathbf{0.757358.}$$

Example 7.12 Find the value of θ given $f(\theta) = 0.3887$ where $f(\theta) = \int_0^\theta \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$, using table:

θ :	21°	23°	25°
$f(\theta)$:	0.3706	0.4068	0.4433

Solution: We take $f(\theta)$ as independent and θ as dependent. By Lagrange's inverse interpolation formula we get

$$\theta = \frac{(0.3887-0.4068)(0.3887-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)} \times (21)$$

$$+ \frac{(0.3887-0.3706)(0.3887-0.4433)}{(0.4068-0.3706)(0.4068-0.4433)}(23) + \frac{(0.3887-0.3706)(0.3887-0.4068)}{(0.4433-0.3706)(0.4433-0.4068)} \times 25$$

$$= 7.885832 + 17.202739 - 3.086525$$

$$= 22.0020^\circ.$$

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EXERCISE 7.2

1. Find the form of the function assuming it to be a polynomial in x, from the following: $u_0 = -18$, $u_1 = 0$, $u_3 = 0$, $u_5 = -248$, $u_6 = 0$ and $u_9 = 13104$.
2. Using Lagrange's formula of interpolation find $Y(9.5)$ given

x :	7	8	9	10
y :	3	1	1	9

3. Given $u_1 = 22$, $u_2 = 30$, $u_4 = 82$, $u_7 = 106$, $u_8 = 206$, find u_6.
4. Using Lagrange's formula find $f(6)$ given & $f(15)$

(x) :	2	5	7	10	12
$f(x)$:	18	180	448	1210	2028

5. Find $Y(6)$ given $Y(1) = 4$, $Y(2) = 5$, $Y(7) = 5$, $Y(8) = 4$. Also find x for which $y(x)$ is maximum or minimum.
6. If $Y_0 = 1$, $Y_3 = 19$, $Y_4 = 49$ and $Y_6 = 181$ find Y_5.
7. If log (300) = 2.4771, log (304) = 2.4829, log (305) = 2.4843, log (307) = 2.4871, find log (301).
8. Given $f(30) = -30$, $f(34) = -13$, $f(38) = 3$ and $f(42) = 18$ find x so that $f(x) = 0$
9. Find the value of tan 33° by using Lagrange's formula of interpolation given

x :	30°	32°	35°	38°
tan x :	0.5774	0.6249	0.7002	0.7813

10. Prove that the Lagrange's formula can be put in the form

$$P_n(x) = \sum_{r=0}^{n} \frac{\phi(x) f(x_r)}{(x-x_r)\phi'(x_r)},$$

where $$\phi(x) = \prod_{r=0}^{n} (x - x_r).$$

11. The following table gives the sales of a concern for the five years. Estimate the sales for the year (a) 1986, (b) 1992:

Year	1985	1987	1989	1991	1993
Sales	40	43	48	52	57

12. Compute sin 39° form the table:

x^o	0	10	20	30	40
sin x^o	0	1.1736	0.3420	0.5000	0.6428

13. Use Lagrange's interpolation formula and find $f(0.35)$:

x :	0.0	0.1	0.2	0.3	0.4
$f(x)$:	1.0000	1.1052	1.2214	1.3499	1.4918

14. Use Lagrange's formula and compute $f(0.25)$

x :	0.20	0.22	0.24	0.26	0.28	0.30
$f(x)$:	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

Inverse Interpolation

15. Given $f(0) = 16.35, f(5) = 14.88, f(10) = 13.59, f(15) = 12.46$ and $f(x) = 14.00$ find x.

16. Given:

x :	4.80	4.81	4.82	4.83	4.84
$f(x) = \sin hx$	60.7511	61.3671	61.9785	62.6015	63.2307

in $\sin hx = 62$. Find x.

17. From the data given below, find the value of x when $y = 13.5$:

x :	93.0	96.2	100.0	104.2	108.7
y :	11.38	12.80	14.70	17.07	19.91

18. Find the value of θ, given $f(\theta) = 0.3887$ where $f(\theta) = \int_0^\theta \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}}$, using the table:

θ:	21°	23°	25°
$f(\theta)$:	0.3706	0.4068	0.4433

19. Find the age corresponding to the annuity value 13.6 given the table:

Age (x) :	30	35	40	45	50
Annuity value (y) :	15.9	14.9	14.1	13.3	12.5

$\boxed{20.}$ Find x corresponding to $y = 100$ given

x :	3	5	7	9	11
y :	6	24	58	108	174

ANSWERS

1. $(x - 1)(x - 3)(x - 6)(x^2 + x + 1)$. 2. 3.625. 3. 83.515. 4. 294.
5. 5.6, $x = 4.5$. 6. 101. 7. 2.4786. 8. 37.23. 9. 0.6494
11. (a) 41.02 (b) 54.46. 12. 0.6293. 13. 1.4191. 14. 1.6751. 15. 8.1440.
16. 4.820347. 17. 97.6557503. 18. 20.0020°. 19. $x = 43$. 20. $x = 8.656$.

3.12.1 Method of Successive Approximations

We start with Newton's forward difference formula [see equation (3.10), § 3.6] written as

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \dots \qquad \cancel{(3.117)}\ 1$$

From this we obtain

$$p = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p(p-1)}{2}\Delta^2 y_0 - \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 - \dots\right] \qquad \cancel{(3.118)}\ 2$$

Neglecting the second and higher differences, we obtain the first approximation to p and this we write as follows

$$p_1 = \frac{1}{\Delta y_0}(y_p - y_0) \qquad 3 \quad (3.\cancel{119})$$

Next, we obtain the second approximation to p by including the term containing the second differences. Thus,

$$p_2 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p_1(p_1 - 1)}{2}\Delta^2 y_0\right] \qquad 4 \quad (3.\cancel{120})$$

where we have used the value of p_1 for p in the coefficient of $\Delta^2 y_0$. Similarly, we obtain

$$p_3 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p_2(p_2 - 1)}{2}\Delta^2 y_0 - \frac{p_2(p_2 - 1)(p_2 - 2)}{6}\Delta^3 y_0\right] \qquad 5 \quad (3.\cancel{121})$$

and so on. This process should be continued till two successive approximations to p agree with each other to the required accuracy. The method is illustrated by means of the following example.

Example 3.26 Tabulate $y = x^3$ for $x = 2, 3, 4$ and 5, and calculate the cube root of 10 correct to *three* decimal places.

x	y	Δ	Δ^2	Δ^3
2	8			
		19		
3	27		18	
		37		6
4	64		24	
		61		
5	125			

Here $y_p = 10$, $y_0 = 8$, $\Delta y_0 = 19$, $\Delta^2 y_0 = 18$ and $\Delta^3 y_0 = 6$. The successive approximations to p are therefore

$$p_1 = \frac{1}{19}(2) = 0.1$$

$$p_2 = \frac{1}{19}\left[2 - \frac{0.1(0.1 - 1)}{2}(18)\right] = 0.15$$

$$p_3 = \frac{1}{19}\left[2 - \frac{0.15(0.15 - 1)}{2}(18) - \frac{0.15(0.15 - 1)(0.15 - 2)}{6}(6)\right]$$
$$= 0.1532$$

$$p_4 = \frac{1}{19}\left[2 - \frac{0.1532(0.1532-1)}{2}(18) - \frac{0.1532(0.1532-1)(0.1532-2)}{6}(6)\right]$$

$$= 0.1541$$

$$p_5 = \frac{1}{19}\left[2 - \frac{0.1541(0.1541-1)}{2}(18) - \frac{0.1541(0.1541-1)(0.1541-2)}{6}(6)\right]$$

$$= 0.1542.$$

We therefore take $u = 0.154$ correct to three decimal places. Hence the value of x (which corresponds to $y = 10$), i.e., the cube root of 10 is given by $x_0 + uh = \underline{2.154}$.

This example demonstrates the relationship between the inverse interpolation and the solution of algebraic equations.