- Consider the differential equation with non constant coefficients

$$y'' + ay' + by = 0$$

- Assume the following solution

$$y = \sum_{0}^{\infty} a_n x^n = a_o + a_1 x + a_2 x^2 + \cdots$$

- Evaluate $y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$$y'' = 2a_2 + 6a_3 x + \cdots$$

- Now substitute in the main equation and equate the coefficients of powers of x on LHS and RHS

$$2a_2 + aa_1 + ba_o = 0; \quad ba_1 + 2a_2 + ba_3 = 0; \;\; \ldots$$

- Obtain all coefficients in terms of a0 and a1
- Evaluate a0 and a1 using the initial conditions
- Note that while the above algebra is with coefficients ‘a’ and ‘b’ as constants they can be functions of x but must be polynomials for the method to work
- Many functions will work. Use Taylor series

$$f(x+dx) = f(x) + f'dx + \frac{f''}{2!}(dx)^2 + \cdots$$

- Polynomial is a Taylor series exp$^n$ with x=0 and dx=x

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots; \;\; sinx = 0 + x + 0 - \frac{x^3}{3!} + \cdots$$

$$cosx = 1 + 0 - \frac{x^2}{2!} + 0 + \cdots etc.$$

- A power series can have convergence issues. Some may converge only in a particular range and some may diverge
- Consider the following Ordinary Diff. Equn.

$$y'' + p(x)y' + q(x)y = 0$$

- p(x) and q(x) are analytic functions i.e. they exist and their derivatives exist at a point $x=x_0$. The point $x_0$ is called a regular/ordinary point.
- If p(x) and/or q(x) are not analytic at x=x0 then it is called a singular point.
- However, if $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic then it is regular singular point else irregular singular point

$$y'' + xy' + \mathrm{y} = 0 \qquad x = 0 \; is \; a \;\; regular \; point$$
$$x^2y'' + xy' + \mathrm{y} = 0 \qquad x = 0 \; is \; a \;\; singular \; point$$
$$y'' + \frac{y'}{x} + \frac{y}{x^2} = 0 \qquad x = 0 \; is \; a \;\; regular \; sing.pt.$$
$$x^3y'' + xy' + \mathrm{y} = 0 \qquad x = 0 \; is \; an \;\; irrg. \; sing.pt.$$
$$y'' + \frac{y'}{x} + \frac{y}{x^3} = 0 \qquad x = 0 \; is \; an \;\; irrg.sing.pt.$$

- Theorem: If $x_0$ is an ordinary point, two linearly independent power series solutions exist centered at $x_0$ ie $y = \sum_0^\infty (x - x_0)^n$
- Solve the following about x=0

$$y'' - (1 + x)y = 0;$$

- Assume $y = \sum_0^\infty C_n x^n$

$$y' = \sum_0^\infty nC_n x^{n-1} \qquad y'' = \sum_0^\infty n(n-1)C_n x^{n-2}$$

- Substitute in the Differential equation to get:

$$\sum_0^\infty n(n-1)C_n x^{n-2} - \sum_0^\infty C_n x^n - \sum_0^\infty C_n x^{n+1} = 0$$

- Change variable m=n-2

$$\sum_{-2}^{\infty}(m+1)(m+2)C_{m+2}x^m - \sum_{0}^{\infty} C_n x^n - \sum_{1}^{\infty} C_{k-1}x^k = 0$$

- Now change limits noting that first two terms are zero

$$\sum_{0}^{\infty}(m+1)(m+2)C_{m+2}x^m - \sum_{0}^{\infty} C_m x^m - \sum_{1}^{\infty} C_{k-1}x^k = 0$$

- Now change limits on first two terms

$$\sum_{1}^{\infty}(m+1)(m+2)C_{m+2}x^m - \sum_{1}^{\infty} C_m x^m - \sum_{1}^{\infty} C_{m-1}x^m = 0$$
$$+C_2(2)(1)x^0 \qquad\qquad -C_0x^0$$

- Use terms within the summation sign to get

$$C_{m+2} = \frac{C_m + C_{m-1}}{(m+2)(m+1)}$$

- Use first two terms to get

$$C_0 = 2C_2 \Rightarrow C_2 = \frac{C_0}{2}$$

- Now use the ‘recurrence’ relation to get

$$C_3 = \frac{C_1 + C_0}{3(2)}\ ; C_4 = \frac{C_2 + C_1}{4(3)} ; C_5 = \frac{C_3 + C_2}{5(4)} \ldots\ldots$$

- Note that C2 is already obtained in terms of C0. therefore C4, C6 etc. are known in terms of C0
- Similarly C3, C5 etc. are known in terms of C1
- The solution is therefore obtained in terms of C0 and C1 which can be evaluated using the Boundary/Initial conditions.

- If the complete algebra is done then the following is the solution:

$$y = c_0\left(1 + \frac{x^2}{2} + \frac{x^3}{6} +..\right) + c_1\left(x + \frac{x^3}{6} + \frac{x^4}{12} +..\right)$$

- Rewrite the solution as

$$y = c_0\, y_1(x) + c_1\, y_2(x)$$

- Consider the Legendre equation( *l* is a constant):

$$(1-x^2)y'' - 2xy' + (l)(l+1)\mathrm{y} = 0$$

- Attempt power series solution

$$y = \sum_0^\infty C_n x^n \quad y' = \sum_0^\infty nC_n x^{n-1}$$
$$y'' = \sum_0^\infty n(n-1)C_n x^{n-2}$$

- Substitute in the original equation to get:

$$\sum_0^\infty n(n-1)C_n x^{n-2} - \sum_0^\infty n(n-1)C_n x^n - \sum_0^\infty 2nC_n x^n + \sum_0^\infty (l)(l+1)C_n x^n$$

- Again substitute n=m+2 in first term

$$\sum_0^\infty (m+2)(m+1)C_{m+2}x^m - \sum_0^\infty m(m-1)a_m x^m - \sum_0^\infty 2mC_m x^m + \sum_0^\infty (l)(l+1)C_m x^m$$

- Simplify to get:

$$C_{m+2} = \frac{m(m+1) - l(l+1)}{(m+2)(m+1)} C_m \; m = 0,1,2 \ldots$$

- Use the above relationship to get $C_2$, $C_3$ etc. in terms of $C_0$ and $C_1$ which will remain unknowns
- When $l$ is an integer one gets Legendre polynomials
- $l$ =m results in a zero for $C_{l+2}$ and all subsequent even coefficients.
- e.g. $l$ =2

$$c_2 = -\frac{6}{2} c_0, c_4 = \frac{2(2+1) - 2(2+1)}{D^m} c_2 = 0,$$

$$c_6 = c_8 = c_{10} = \cdots = 0$$

- Similarly (for $l$ =2 contd.) get

$$c_3 = \frac{2-6}{6} c = \frac{-4}{6} c_1; c = \frac{3(4)-2(3)}{5(4)} c_3; .....$$

$$y = c_0(1 - 3x^2) + c\left(x - \frac{2}{3}x^3 \; ....\right)$$

$$y = c_0 \, y_1(x) + c_1 \, y_2(x)$$

- Similarly for $l$ =3

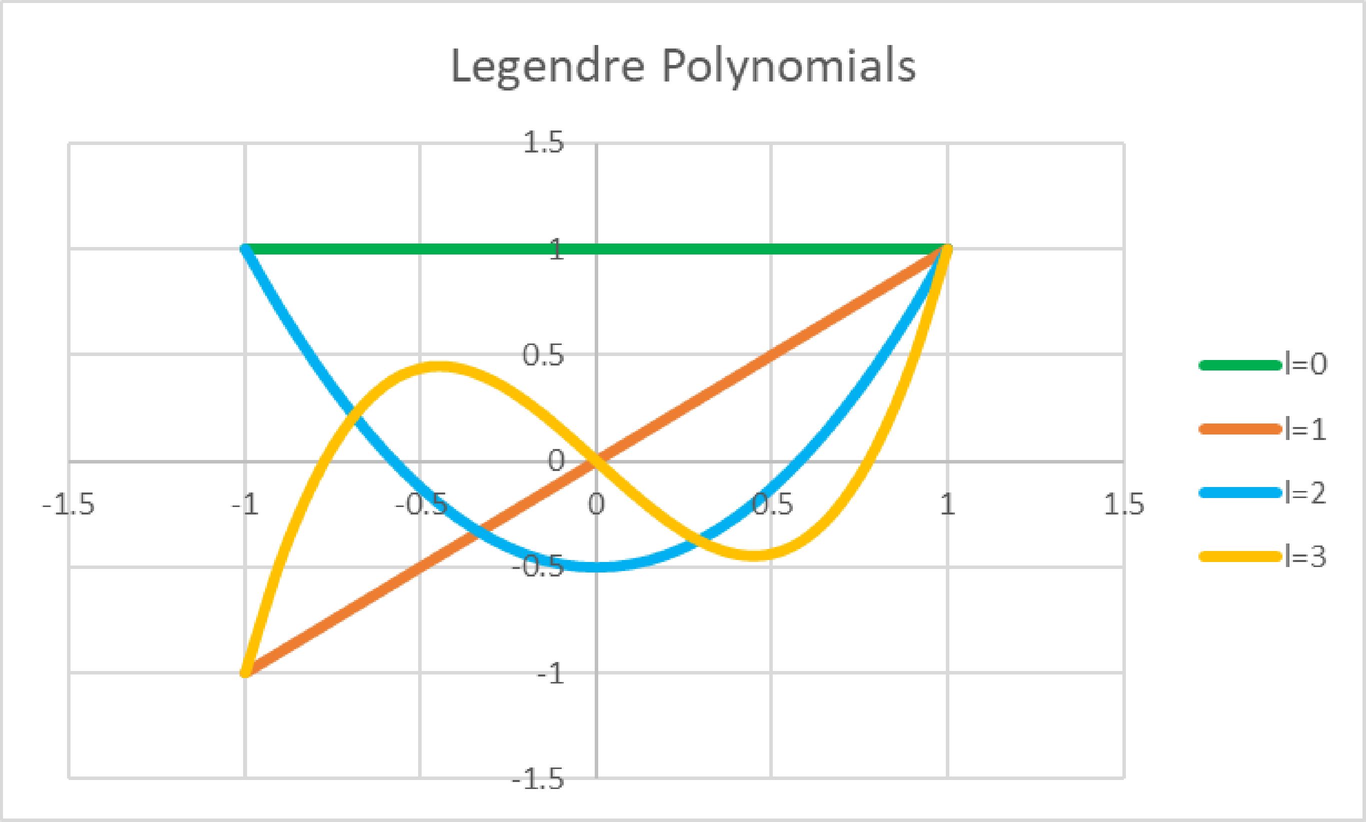
$$a_3 = \frac{2-6}{6} a_1 = \frac{-4}{3} a_1$$

$$a_5 = \frac{3(4)-3(4)}{5(4)} a_3 = 0 \, ; a_7 = a_9 = \ldots = 0$$

- Now all even coefficients will survive and

$$y = c_0(\dots) + c_1\left(x - \frac{5}{3}x^3\right)$$

- Can get for other l values
- For even $l$ values the even coefficients will survive uptil a particular value $m(m+1) - l(l+1)$ whereas similarly odd $l$ values will have odd coefficients survive till a particular value
- The polynomials with finite number of terms are called the Legendre polynomials i.e. coefficients of $c_0$, $c_1$. Typically keep the value=1 at the x=1 location.



Legendre Polynomials
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
l=0
l=1
l=2
l=3