2^{nd} order equations

- Consider the following linear differential equation which is linear

$$\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_o(t)y = b_o(t)$$

- Requires two conditions for complete solution

which can be either initial or boundary conditions.

- Can use either Laplace transform or other methods for solution

Laplace Transform solution for second order Differential Equations

- Basics are all same as those discussed for first order

$$\ddot{x} + 5\dot{x} + 6x = \delta$$
$$x(0) = 0; \dot{x}(0) = 0$$

- Take Laplace transform

$$[s^2X - sx(0) - \dot{x}(0)] + 5[sX - x(0)\} + 6X = 1$$
$$[s^2 + 5s + 6]X = 1 \Rightarrow X = \frac{1}{(s+2)(s+3)}$$

- Use method of partial fractions to split the term:
- Take inverse Laplace transform(tables or online) to get

$$X = \frac{1}{s+2} - \frac{1}{s+3}; x = e^{-2t} - e^{-3t}$$

- Notice

$$\dot{x}(0) = [-2e^{-2t} - (-3)e^{-3t}]_{t=0} = 1$$

- Solution is valid for $t=0^+$. At zero there is a problem with the obtained solution but can be ignored since solution is needed only after t=0. The impulse function causes a discontinuity in the derivative term

Example

$$\ddot{x} + 5\dot{x} + 6x = sint; \;\; x(0) = 0; \dot{x}(0) = 0$$

$$\mathcal{L}(\ddot{x} + 5\dot{x} + 6x) = \mathcal{L}(sint)$$

$$X = \frac{1}{(s^2+1)(s+2)(s+3)}$$

- Need to split by using method of partial fractions

$$X = \frac{As+B}{s^2+1} + \frac{C}{s+2} + \frac{D}{s+3}$$

- Perform the algebra to get:(online tools available)

$$A = -\frac{1}{10}; B = \frac{1}{10}; C = \frac{1}{5}; D = -\frac{1}{10}$$

- Now use in the expression above to get:

$$x = \mathcal{L}^{-1}\left[\frac{\frac{1}{10} - \frac{1}{10}s}{s^2+1}\right] + \mathcal{L}^{-1}\left[\frac{\frac{1}{5}}{s+2} - \frac{\frac{1}{10}}{s+3}\right]$$

$$x = \frac{1}{10}sint - \frac{cost}{10} + \frac{1}{5}e^{-2t} - \frac{1}{10}e^{-3t}$$

- Can also use convolution integral approach while taking the inverse laplace transform:

$$x = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\frac{1}{s+2}\frac{1}{s+3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] * \mathcal{L}^{-1}\left[\frac{1}{s+2}\frac{1}{s+3}\right]$$

$$x = \int \sin(\tau)\left[e^{-2(t-\tau)} - e^{-3(t-\tau)}\right]d\tau$$

- If integration is ‘messy’ use online tools:

$$\int \sin(\tau)\left[e^{-2(t-\tau)} - e^{-3(t-\tau)}\right]d\tau$$

$$= -\frac{1}{5}\left[e^{2(\tau-t)}(\cos\tau - 2sin\tau)\right]_0^t$$

$$-\left(-\frac{1}{10}\right)\left[e^{3(\tau-t)}(\cos\tau - 3sin\tau)\right]_0^t$$

$$= -\frac{1}{5}[\cos t - 2sint - e^{-2t}] +$$

$$\frac{1}{10}[\cos t - 3sint - e^{-3t}]$$

$x = -\frac{cost}{10} + \frac{sint}{10} + \frac{e^{-2t}}{5} - \frac{e^{-3t}}{10}$; Same as earlier

- Assume a discontinuous function on the RHS. Represent using the Heaviside function. The following is a useful manipulation for this function.

$$\mathcal{L}[H(t-c)f(t-c)] = \int_0^\infty H(t-c)f(t-c)e^{-st}dt$$

$$= \int_c^\infty f(t-c)e^{-st}dt \quad Let \quad t-c=\tau$$

$$= \int_0^\infty f(\tau)e^{-s(\tau+c)}d\tau = e^{-sc}\int f(\tau)e^{-s\tau}d\tau$$

$$= e^{-sc}\,\mathcal{L}(f)$$

- If H(t-c)f(t) is there then use t-c=τ and get t in terms of τ and use the above expression

$$\mathcal{L}[H(t-c)(t^2)] = \mathcal{L}[H(\tau)(\tau^2 + 2c\tau + c^2)]$$
$$= \mathcal{L}\big[H(t-c)\big((t-c)^2 + 2c(t-c) + c^2\big)\big]$$

$$\mathcal{L}[H(t-2\pi)(sint)] = \mathcal{L}[H(\tau)\sin(\tau + 2\pi]$$
$$= \mathcal{L}[H(t-2\pi)\sin(t-2\pi)]$$

$$\sin(\tau + 2\pi) = sin\tau\, cos2\pi\ + cos\tau sin\, 2\pi = sin\tau$$

- The above may not work with complex functions and other strategies may be required.

Integro Differential equations

- Sometimes an integral is embedded in the differential equation
- We have already seen:

$$\mathcal{L}^{-1}[F(s)G(s)] = f(t) * g(t)$$
$$\Rightarrow \mathcal{L}\,(f(t) * g(t)) \;= [F(s)G(s)]$$

- Assume g(t)=1

$$\mathcal{L}\int_0^t f(\tau) * g(t-\tau)\;d\tau = F(s)G(s)$$

$$\mathcal{L}\int_0^t f(\tau)\;d\tau = F(s)/s$$

Example

$$\frac{dy}{dt} + 6y + 9\int y(\tau)d\tau = 1 \qquad\qquad y(0) = 0$$

- Take Laplace transform on both sides

$$sY - y(0) + 6Y + 9\frac{Y}{s} = \frac{1}{s}$$

$$Y\left(s + 6 + \frac{9}{s}\right) = \frac{1}{s} \Rightarrow Y = \frac{1}{s\left(s + 6 + \frac{9}{s}\right)}$$

$$y = e^{-3t}t$$

(Wolfram Alpha)