Frobenius theorem

- If x_0 is a regular singular point at least one **power series solution** exists i.e. second solution is not necessarily a power series type one.
- A power series solution of the following form exists: (Usually x_0=0)

$$y = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

- r needs to be evaluated first and can be a fraction or negative number and if r1-r2 (r1>r2) is not an integer:

$$y_1 = \sum_0^\infty a_n x^{n+r_1} \text{ and } y_2 = \sum_0^\infty b_n x^{n+r_2}$$

- If r_1-r_2 is a positive integer two series solutions may not result and the following two are likely (A could be zero)

$$y_1 = \sum_0^\infty a_n x^{n+r_1},\ y_2 = Ay_1 \ln(\mathrm{x}) + \sum_0^\infty b_n x^{n+r_2}$$

- r_1=r_2 results in:

$$y_1 = \sum_0^\infty a_n x^{n+r_1},\ \ y_2 = y_1 \ln(\mathrm{x}) + \sum_1^\infty b_n x^{n+r_1}$$

- Algebra tends to be very tedious. Can use the Abel's theorem for obtaining the second solution from the first one.
- Note that a_0 and b_0 are non zero – if they turn out to be zero then the solution is not valid. Summation starts at 0 for all except r_1=r_2 case.

- Consider the following equation:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

- There is a regular singular point at x=0
- Use the Frobenius power series approach:

$$y = \sum a_n x^{n+r}; y' = \sum (n+r)a_n x^{n+r-1}$$

$$y'' = \sum (n+r)(n+r-1)a_n x^{n+r-2}$$

- Substitute in original equation

$$\sum (n+r)(n+r-1)a_n x^{n+r} + \sum (n+r)a_n x^{n+r}$$

$$+ \sum a_n x^{n+r+2} - \sum \nu^2 a_n x^{n+r} = 0$$

- Take x^r common to get

$$x^r\left\{\sum\left((n+r)^2-\nu^2\right)a_n x^n+\sum a_n x^{n+2}\right\}=0$$

$$\Rightarrow x^r\sum_{n=0}^{\infty}a_n x^n[(n+r)^2-\nu^2]+x^r\sum_{m=2}^{\infty}a_{m-2}x^m=0$$

- Rewrite summation of first term from 2 and combine:

$$x^r a_0(r^2-\nu^2)+a_1 x^{r+1}[(1+r)^2-\nu^2]+$$
$$\sum_{n=2}x^{n+r}[[(n+r)^2-\nu^2]a_n+a_{n-2}]=0$$

- Equate coefficients of powers of x to zero

- a0 cannot be zero therefore $r = \pm\nu$
- a1 now has to be zero since otherwise r becomes inconsistent
- Recurrence relation is

$$a_n = \frac{-1}{[(n+r)^2 - \nu^2]} a_{n-2}$$

- This implies $a_3 = a_5 = a_7 = \cdots = 0$
- Choose

$$a_0 = \frac{1}{2^\nu \nu!} \Rightarrow a_{2m} = (-1)^m \left(\frac{1}{2}\right)^{\nu+2m} \frac{1}{m!\,\nu m!}$$

- Substitute in the series to get

$$J_\nu = \sum (-1)^m \cdot \frac{1}{m!\, \nu m!} \cdot \left(\frac{x}{2}\right)^{2m+\nu}$$

- This is the Bessel function of order $\boldsymbol{\nu}$
- You can follow similar procedure to get the series for the r=- $\boldsymbol{\nu}$ root
- However, if the $\boldsymbol{\nu}$ happens to be an integer then you may have to use Abel's theorem or the variation of parameters method to get the second solution.
- …….

$$\int \frac{e^{-\int p(x)dt}}{y_1^2} = y_2$$

Example

- Solve about x=0:

$$xy'' + 2y' + y = 0$$

- Note that since x=0 is a regular singular point

$$y = \sum a_n x^{n+r}, y' = \sum (n+r) a_n x^{n+r-1}$$

$$y'' = \sum (n+r)(n+r-1) a_n x^{n+r-2}$$

- Substitute in the given equation:

$$\sum a_n(n+r)(n+r-1)x^{n+r-1} + 2\sum a_n(n+r)x^{n+r-1}$$

$$+\sum a_n x^{n+r} = 0$$

$$\Rightarrow \sum a_n x^{n+r-1}(n+r)[(n+r-1)+2] + \sum a_n x^{n+r} = 0$$

$$\Rightarrow \sum a_n x^{n+r-1}(n+r)(n+r+1) + \sum a_n x^{n+r} = 0$$

- Now adjust indices to group terms in a favourable manner i.e. n-1=m in the first term to get

$$\sum_{m=-1}^{\infty} a_{m+1}x^{m+r}(m+1+r)(m+2+r) + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

- Remove first term to get:

$$a_0 x^r x^{-1}(r)(r+1) + \sum_{m=0}^{\infty} a_{m+1} x^{m+r}(m+r+1)(m+r+2)$$

$$+\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

- Since a0=0 is not permitted

$$a_0 x^r x^{-1}(r)(r+1) + \sum_0^{\infty} x^{n+r}[a_n + a_{n+1}(m+r+1)(m+r+2)] = 0$$

$$r(r+1) = 0 \Rightarrow r = 0;\ r = -1$$

- Proceed with the recurrence relationship

Matrix Algebra

- Consider a system of linear equations:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$\vdots$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

- Represent in matrix form as follows:

$$\begin{bmatrix} a_{11} & a_{12} \cdots \cdots & a_{1\mathrm{n}} \\ & \vdots & \vdots \\ & \vdots & \vdots \\ a_{n1} & a_{\mathrm{n}2} \cdots \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

Elementary Row Operations

- Interchange two rows
- Multiply row by non zero scalar
- Addition of scalar multiple of one row to another
- Row operations do not change the properties of the matrix
- Use row operations to put matrix in the Row Echelon Form
- If a row has all zeros then it should be last row
- The first non zero element of a row should be to the right of the one above it, i.e. a staircase pattern is observed
- Rank of matrix is the number of rows that do not have all zeros

- A matrix is singular if the corresponding determinant is zero
- A singular matrix has linearly dependent rows, ie. Rank is less than the number of rows
- Determinant physically represents the ‘volume’ of the vectors contained in it – zero volume indicates one vector is not independent

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix} \qquad \begin{matrix} R_1 * 3 + R_2 \\ R_1 * (-5) + R_3 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.1 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.1 \\ 2.5 \end{bmatrix}$$ Interchange R_2 and R_3

$$\Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.1 \end{bmatrix} \qquad R_2 * 0.04 + R_3$$

$$\Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.2 \end{bmatrix}$$ Row Echelon form

- Matrix inversion

$$\left[\begin{array}{ccc:ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 1 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

- Solution to the above matrix using Gauss elimination would yield three vectors which when put together represent the inverse of the matrix
- To get inverse therefore make an enhanced matrix with the identity matrix

$$\begin{bmatrix} 2 & 3 & -1 & 1 & 0 & 0 \\ 4 & 4 & -3 & 0 & 1 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 4 & 4 & -3 & 0 & 1 & 0 \\ 2 & 3 & -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{array}\right]$$

R_1* (-1/2) + R_2
R_1 *(-1/2) + R_3

$$\left[\begin{array}{ccc|ccc} 4 & 4 & -3 & 0 & 1 & 0 \\ 0 & 1 & 1/2 & 1 & -1/2 & 0 \\ 0 & -5 & -5/2 & 0 & -1/2 & 1 \end{array}\right]$$

Row interchange

$$\left[\begin{array}{ccc|ccc} 4 & 4 & -3 & 0 & 1 & 0 \\ 0 & -5 & -5/2 & 0 & -1/2 & 1 \\ 0 & 1 & 1/2 & 1 & -1/2 & 0 \end{array}\right]$$

R_2 *(1/5) + R_3

$$\left[\begin{array}{ccc|ccc} 4 & 4 & -3 & 0 & 1 & 0 \\ 0 & -5 & -5/2 & 0 & -1/2 & 1 \\ 0 & 0 & 1 & 1 & -3/5 & 1/5 \end{array}\right]$$

Get X,Y,Z

- Inverse is the combination of the three solutions

$$X = \begin{bmatrix} 1/4 \\ 1/2 \\ 1 \end{bmatrix} \quad Y = \begin{bmatrix} 0 \\ -1/5 \\ -3/5 \end{bmatrix} \qquad \mathrm{Z} = \begin{bmatrix} 1/4 \\ -1/10 \\ 1/5 \end{bmatrix}$$

- Eigen values and vectors characterize a matrix
- $AX = \lambda X$; X= eigen vectors λ = eigen values
- If non trivial solution is required for x then matrix should be singular.

$$|A - \lambda I| = 0$$

- This results in the equation that gives the eigen value
- Each value has a corresponding eigen vector.
- Repeated eigen values may have different eigen vectors

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad A - \lambda I = \begin{bmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{bmatrix}$$

Det(A- λ I)=(1- λ)[(1- λ)(-1- λ)]=0; λ=1; -1,1

- Three eigen values are obtained and two are identical. Assume λ=1; (A-I λ)X=0

$$0x_1 - 1x_2 + 0x_3 = 0 \Rightarrow x_2 = 0$$
$$0x_1 + 0x_2 + x_3 = 0 \Rightarrow x_3 = 0$$
$$0x_1 - 0x_2 - 2x_3 = 0 \Rightarrow x_1 = \alpha$$

$$V1 = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$$

- In this case even though λ=1 is a multiple eigen value only one eigen vector –also possible to get two eigen vectors

- Now assume $\lambda = -1 \Rightarrow (A - I\lambda)$x=0
- $$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$2x_1 - 1x_2 + 0x_3 = 0 \Rightarrow x_1 = x_2/2$$
$$0x_1 + 2x_2 + x_3 = 0 \Rightarrow x_2 = -x_3/2$$
$$0x_1 + 0x_2 + 0x_3 = 0 \Rightarrow x_3 = \alpha$$

v2= $\begin{bmatrix} -\alpha/4 \\ -\alpha/2 \\ \alpha \end{bmatrix}$