Practice 1 Solutions

1) $\boldsymbol{y' + 4y = e^{-4t}, y(0) = 2}$

Apply Laplace transform on the differential equation

$$sL(y) - y(0) + 4\,L(Y) = \frac{1}{s+4}$$

Solving for L {y}, we obtain

$$L\{y\} \; = \frac{2}{s+4} + \frac{1}{(s+4)^2}$$

Apply inverse Laplace transform

$$\boldsymbol{y(t) = (t+2)e^{-4t}}$$

2) $\boldsymbol{y' - y = 1 + te^t, y(0) = 0}$

Apply Laplace transform on the differential equation

$$sL(y) - y(0) - L(Y) = \frac{1}{s} + \frac{1}{(s-1)^2}$$

Solving for L {*y*}, we obtain

$$L\{y\} \quad = \frac{1}{s(s-1)} + \frac{1}{(s-1)^3} = \frac{-1}{s} + \frac{1}{s-1} + \frac{1}{(s-1)^3}$$

Apply inverse Laplace transform

$$\boldsymbol{y(t) = -1 + e^t + \frac{1}{2}t^2e^t}$$

3) $\boldsymbol{y' + y = f(t),\ y(0) = 0}$ **where** $\boldsymbol{f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 5, & t \ge 1 \end{cases}}$

The differential equation is non-homogeneous that has input function $f(t)$ unit step function starts from t >1.

Apply Laplace transform on the differential equation

$$\{sL(y) - y(0)\} + L(Y) = \frac{5}{s^2} e^{-s}$$

Solving for $L\ \{y\}$, we obtain

$$L\{y\} \quad = \frac{5e^{-s}}{s(s+1)} = 5e^{-s}\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

Apply inverse Laplace transform

$$\boldsymbol{y(t) = 5\mathcal{U}(t-1) - 5e^{-(t-1)}\,\mathcal{U}(t-1)}$$

4) $\boldsymbol{y' + y = f(t),\ y(0) = 0}$ **where** $\boldsymbol{f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & t \ge 1 \end{cases}}$

The differential equation is non-homogeneous that has input function $f(t)$ unit step function starts from $f(t) = -1, t > 1, f(t) = 1,0 < t < 1.$

Apply Laplace transform on the differential equation

$$\{sL(y) - y(0)\} + L(Y) = \frac{1}{s} - \frac{2}{s}e^{-s}$$

Solving for L {y}, we obtain

$$L\{y\} \;= \frac{1}{s(s+1)} - \frac{2e^{-s}}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1} - 2e^{-s}\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

Apply inverse Laplace transform

$$\boldsymbol{y(t) = 1 - e^{-t} - 2\left[1 - e^{-(t-1)}\right]\mathcal{U}(t-1)}$$

5) $y' + 2y = f(t),\ y(0) = 0$ where $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

The differential equation is non-homogeneous that has input function $f(t)$ unit step function starts from $f(t) = 0, t \ge 1, f(t) = t, 0 \le t < 1.$

Apply Laplace transform on the differential equation

$$\{sL(y) - y(0)\} + 2L(Y) = \frac{1}{s^2} - \frac{(s+1)}{s^2} e^{-s}$$

Solving for $L\ \{y\}$, we obtain

$$L\{y\} = -\frac{1}{4}\frac{1}{s} + \frac{1}{2}\frac{1}{s^2} + \frac{1}{4}\frac{1}{s+2} - e^{s}\left[\frac{1}{4}\frac{1}{s} + \frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s+2}\right]$$

Apply inverse Laplace transform

$$y(t) = -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t} - \left[\frac{1}{4} + \frac{1}{2}(t-1) - \frac{1}{4}e^{-2(t-1)}\right]$$

Q6) $xy' + y = e^x; y(1) = 2$

$$\frac{d}{dx}(xy) = e^x$$

Integrating on both side

$$xy = e^x + C$$

Put initial condition

$$C = 2 - e$$

$$xy = e^x + 2 - e$$

$$\boldsymbol{y = \frac{e^x + 2 - e}{x}}$$

Q7) $yx' - x = 2y^2; y(1) = 5$

$$\frac{yx' - x}{y^2} = 2$$

$$\frac{d}{dy}\left(\frac{x}{y}\right) = 2$$

Integrating on both side

$$\frac{x}{y} = 2y + C$$

Put initial condition

$$C = \frac{-49}{5}$$

$$\boldsymbol{x = 2y^2 - \frac{49}{5}y}$$

Q8) $y' + 2y = f(x); y(0) = 0; f(x) = \begin{cases} 1, & 0 \le x \le 3 \\ 0, & x > 3 \end{cases}$

$For\ 0 \le x \le 3$:

$$y' + 2y = 1$$
$$IF = e^{2x}$$
$$e^{2x}y' + 2ye^{2x} = e^{2x}$$
$$\frac{d}{dx}(e^{2x}y) = e^{2x}$$

Integrating on both side

$$e^{2x}y = \frac{e^{2x}}{2} + C$$

Put initial condition

$$C = \frac{-1}{2}$$
$$\boldsymbol{y = \frac{1}{2}(1 - e^{-2x})}$$

$For\ x > 3$:

$$y' + 2y = 0$$

$$\frac{y'}{y} = -2$$

$$\ln y = -2x + D$$

$$y = De^{-2x}$$

For continuity at x=3: $De^{-6} = \frac{1}{2}(1 - e^{-6})$

$$D = \frac{1}{2}(e^{6} - 1)$$

$$\boldsymbol{y = \frac{1}{2}\left(e^{6} - 1\right)e^{-2x}}$$

$$\boldsymbol{y = \begin{cases} \frac{1}{2}(1 - e^{-2x}), & 0 \le x \le 3 \\ \frac{1}{2}\left(e^{6} - 1\right)e^{-2x}, & x > 3 \end{cases}}$$

Q9) $y' + y = f(x); y(0) = 1; f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ -1, & x > 1 \end{cases}$

$For\ 0 \le x \le 1$:

$$y' + y = 1$$

$$IF = e^x$$

$$e^x(y' + y) = e^x$$

$$\frac{d}{dx}(e^x y) = e^x$$

Integrating on both side and put Initial condition $y(0) = 1$

$(e^x y)$= $(e^x) + C; Put\ initial\ condition$

$$C = 0$$

$$\therefore \boldsymbol{y = 1}$$

$For\ x > 1$:

$$y' + y = -1$$

Follow same procedure as above only difference is -1

$$\frac{d}{dx}(e^x y) = -e^x$$

Integrating on both side

$$(e^x y) = -e^x + D$$

Continuity as x=1

Put y=1 and x=1 to get continuity

$$D = 2e$$

$$\therefore \boldsymbol{y = 2e^{1-x} - 1}$$

$$\boldsymbol{y} = \begin{cases} \boldsymbol{1}, & \boldsymbol{0 \le x \le 1} \\ \boldsymbol{2e^{1-x} - 1}, & \boldsymbol{x > 1} \end{cases}$$

Q10. $y' - 3y = -12y^2; y(0) = 2$

Substitute $y = \frac{1}{u} \Rightarrow y' = -\frac{u'}{u^2} = \frac{3}{u} - \frac{12}{u^2}$

Multiplication by u^2 gives the linear ODE : $u' = -3u + 12$

Solution: $u = ce^{-3x} + 4$

General solution of the ODE = $y = \frac{1}{u} = \frac{1}{ce^{-3x} + 4}$

Using initial condition: $y(0) = \frac{1}{c+4} = 2 \Rightarrow c = -3.5$

Which gives:

$$y = \frac{1}{-3.5e^{-3x} + 4}$$

Q11. $y' + \pi y = 2bcos(\pi x); y(0) = 0$

- Linear ODE
- Solution to homogeneous part $y = ce^{-\pi x}$
- For the non-homogeneous equation: Let $y = Acos(x) + Bsin(x)$

$\Rightarrow y' = -Asin(x) + Bcos(x)$

Now, equate the result to the right side; that is,

$y' + \pi y = (B + A)\pi\cos(\pi x) + (-A + B)\pi\sin(\pi x) = 2bcos\pi x$

$\Rightarrow A = B = \frac{b}{\pi}$. The general solution is

$$y = ce^{-\pi x} + \frac{b}{\pi}(\cos(\pi x) + \sin(\pi x))$$

Using initial condition to find c:

$$y(0) = c + \frac{b}{\pi} = 0, \Rightarrow c = -\frac{b}{\pi}$$

Q12. $y' + 3y = 5e^{2x} - 6; \quad y(0) = 2$

$e^{\int 3dx} = e^{3x}$ is an integrating factor.

Multiplying the differential equation by e^{3x} to obtain

$$y'e^{3x} + 3ye^{3x} = 5e^{5x} - 6e^{3x}$$
$$(ye^{3x})' = 5e^{5x} - 6e^{3x}$$

Integrating to obtain the general equation

$$ye^{3x} = e^{5x} - 2e^{3x} + c$$

The general solution is

$$y = e^{2x} - 2 + ce^{-3x}$$

Putting the initial condition $y(0) = 2$

$$y(0) = 1 - 2 + c = 2$$

So, $c = 3$

The final solution is $y = e^{2x} - 2 + 3e^{-3x}$

Q13. $y' + \frac{2}{x+1} y = 3; \qquad y(0) = 5$

IF=$e^{\int \frac{2}{x+1} dx} = e^{2\ln|x+1|} = (x+1)^2$

Multiplying the differential equation by $(x+1)^2$ to obtain

$$(x+1)^2 y' + 2(x+1)y = 3(x+1)^2$$
$$((x+1)^2 y)' = 3(x+1)^2$$

Integrating to obtain the general equation

$$(x+1)^2 y = (x+1)^3 + c$$

Then

$$y = (x+1) + \frac{c}{(x+1)^2}$$

Now$y(0) = 5$ $\qquad 5 = 1 + c$

So, $c = 4$

The final solution is

$$y = (x+1) + \frac{4}{(x+1)^2}$$