Linearization

- Sometimes it helps to linearize the equation about an equilibrium point y_p.

$$\dot{\bar{y}} = f(\bar{y}) \qquad \bar{y} \text{ is a vector}$$

Choose a point where $f\left(\overline{y_p}\right) = 0$

$$\dot{\bar{y}} = f(\bar{y}) = \overset{\mathbf{0}}{f\left(\overline{y_p}\right)} + f'_{y_p}(\Delta\bar{y}) + \frac{1}{2!} f''_{y_p}(\Delta\bar{y})^2 + \cdots \ldots\ldots\ldots\ldots\ldots.. \quad (\Delta\bar{y} \to 0)\ HO\ terms \to 0$$

$$\dot{\bar{y}} = f'_{y_p}\left(y - y_p\right) \qquad \dot{\overline{(y - y_p)}} = \dot{y}\ ; \text{ Since } y_p\text{=fixed}$$

$$\overline{\dot{(y - y_p)}} = \overline{f'_{y_p}}(y - y_p)$$

This is linear ODE

Note $\overline{f'_{y_p}} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}$

$$y_1(x_1\, x_2) = y_0 + \frac{\partial y_1}{\partial x_1} dx_1 + \frac{\partial y_1}{\partial x_2} dx_2 + HOT$$

$$y_2(x_1\, x_2) = y_0 + \frac{\partial y_2}{\partial x_1} dx_1 + \frac{\partial y_2}{\partial x_2} dx_2 + HOT$$

- Consider solution to the following nonlinear equation:

$$\ddot{y} = -y^3 + 4y \ (\equiv (-y^2 + 4)y)$$

$$\dot{y} = \mathrm{v} \qquad \dot{v} = -y^3 + 4y$$

$$\begin{bmatrix} \dot{y} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -y^3 + 4y \end{bmatrix} \qquad \frac{df}{dy} = \begin{bmatrix} 0 & 1 \\ -3y^2 + 4 & 0 \end{bmatrix}$$

- The equilibrium points for the above equation is y=2, v=0; y=-2,v=0 and y=0,v=0. Consider for the point y=2, v=0.

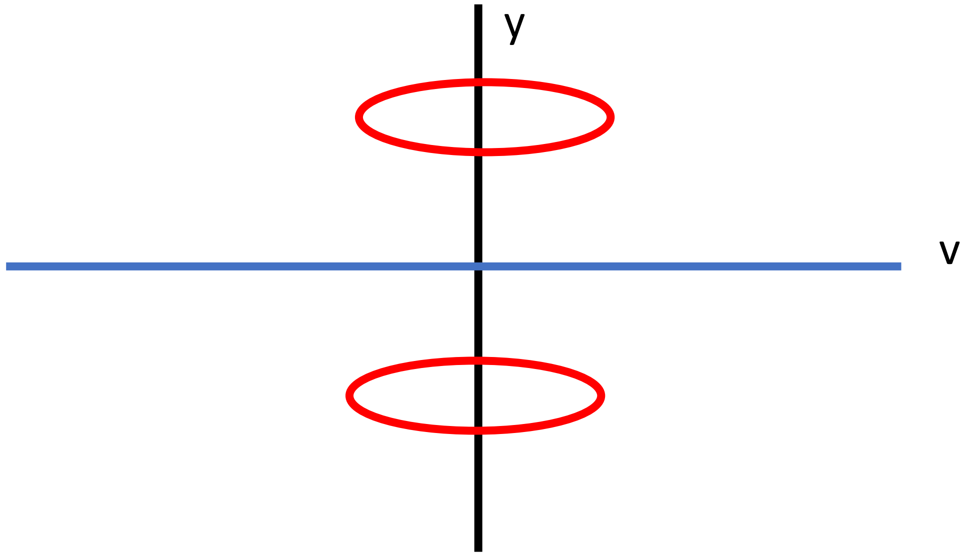
$$\left.\frac{df}{dx}\right| = \begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix} \qquad \begin{bmatrix} \dot{y} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}$$

$$\lambda = \pm\sqrt{8}\, i$$

- Purely imaginary roots give closed phase plot
- Consider now the y=0, v=0 point

$$\begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \rightarrow \lambda = \pm 2; Eigen\ vectors \equiv \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

- The eigen value shows one stable and one unstable value i.e. saddle



- Compute the potential energy of spring i.e.

$$\int F(= y^3 - 4y)\, dy$$