- Same conclusion can be obtained using the Laplace transform approach

$$\mathrm{g}(\mathrm{y}', \mathrm{y}) = \mathrm{f}(\mathrm{t}); \mathrm{y}(0) = 0$$

- Since the initial conditions are zero the Laplace transform of the equation will give:

$$\mathrm{Lg}\big((\mathrm{y}', y)\big) = \big(\mathrm{L}(\mathrm{y})\mathrm{h}(\mathrm{s})\big) = \mathrm{L}\big(\mathrm{f}(\mathrm{t})\big) = \mathrm{F}(\mathrm{s})$$

$$\mathrm{L}(\mathrm{y}) = \frac{F(s)}{\mathrm{h}(\mathrm{s})} \Rightarrow y = L^{-1}\left(\frac{F(s)}{\mathrm{h}(\mathrm{s})}\right)$$

- Now look at

$$g(y', C) = \delta(t); y(0) = 0$$

- Take Laplace transform on both sides to get:

$$L(y) = \frac{1}{h(s)} \Rightarrow \; y = L^{-1}\left(\frac{1}{h(s)}\right) \equiv y^*$$

- From previous slide

$$y = f(t) * L^{-1}\left(\frac{1}{h(s)}\right)$$

- Solution of the governing equation is the convolution of the RHS and the solution of the equation with the delta function on the RHS.

- Convolution of two functions is defined as

$$\mathrm{k}(t) * j(t) = \int_0^t k(\tau)j(t-\tau)d\tau = \int_0^t k(t-\tau)j(\tau)d\tau$$

- Above can be obtained by using a change of variables i.e. p=t- τ and a little algebra
- It is stated here without proof that

$$L^{-1}\left(\frac{F(s)}{\mathrm{h(s)}}\right) = L^{-1}(F(s))^* \, L^{-1}\left(\frac{1}{\mathrm{h(s)}}\right) = \mathrm{f(t)}^* \, L^{-1}\left(\frac{1}{\mathrm{h(s)}}\right)$$

- Laplace inverse of a product of functions is the convolution of the inverse of the individual functions

$$y^*(t) = L^{-1}\left(\frac{1}{\mathrm{h(s)}}\right) \Rightarrow \mathrm{y(t)} = \int_0^t f(\tau)y^*(t-\tau)d\tau$$

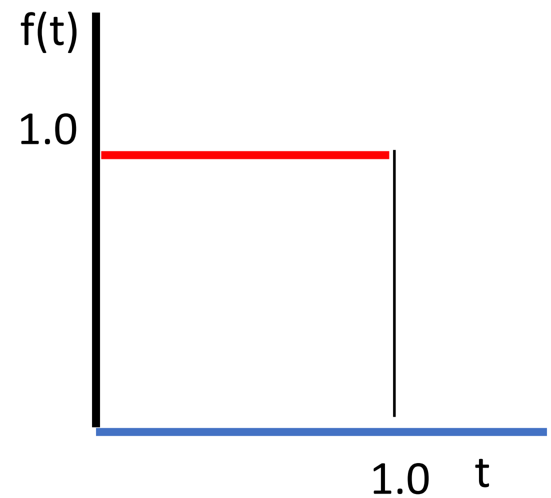
Consider the same example as earlier

$$y' = ay + f(t) \qquad \begin{aligned} f(t) &= 1 \qquad && 0 \leq t \leq 1 \\ &= 0 && t > 0 \end{aligned}$$

$$\frac{dy^*}{dt} = ay^* + \delta(t)$$

$$y^*(0) = 0 \quad y^* = \mathcal{L}^{-1} \frac{1}{(s-a)}$$

$$= e^{at}$$



$$\mathrm{y} = f(t) \star e^{a(t)} = \int_0^t f(\tau) e^{a(t-\tau)} d\tau$$

$$For\ (\ t\ <\ 1\ )$$

$$y = \int 1.\, y^{*}(t - \tau)\ d\tau$$

$$y = \int_0^t e^{a(t-\tau)}\ d\tau = e^{at}\left[\frac{e^{-a\tau}}{-a}\right]_0^t$$

$$y = \frac{e^{at}}{-a}\left[e^{-at} - 1\right] = \frac{e^{at}}{a}\left[1 - e^{-at}\right] = \frac{e^{at} - 1}{a}$$

$$For\ (\ t\ >\ 1\ )$$

$$y = \int_0^1 e^{a(t-\tau)}\ d\tau + \boxed{\int_1^t 0.\, e^{a(t-\tau)}\ d\tau}$$

In the present case is zero but in general will exist

$$y = e^{at}\left[\frac{e^{-a\tau}}{-a}\right]_0^1 = \frac{e^{at}}{-a}\left[e^{-a} - 1\right] = \frac{e^{a(t-1)}}{-a} + \frac{e^{at}}{a}$$

- Some special cases exist where the equation is neither linear nor separable but a solution still exists.
- Consider the differential equation

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

- A function ϕ=C can be considered to be the solution of this equation if one can make it satisfy:

$$\frac{\partial\varphi}{\partial x} = M(x,y); \frac{\partial\varphi}{\partial y} = N(x,y)$$

- The methodology works only if the diff. equ. Is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- Attempt to get integration factor for the equation

$$f(x,y)M(x,y)dx + f(x,y)N(x,y)dy = 0$$

- For exactness

$$\frac{\partial(f(x,y)M(x,y))}{\partial y} = \frac{\partial(f(x,y)N(x,y))}{\partial x}$$

- Suppose f is function of x only, then:

$$f(x)\frac{\partial M}{\partial y} = f(x)\frac{\partial N}{\partial x} + N\frac{df}{dx}$$

- Can get a solution if LHS is a function of x only:

$$\left(f(x)\frac{\partial M}{\partial y} - f(x)\frac{\partial N}{\partial x}\right)/N = \frac{df}{dx}$$

- Similarly if ‘f’ is a function of y only then

$$\left(-f(y)\frac{\partial M}{\partial y}+f(y)\frac{\partial N}{\partial x}\right)/M=\frac{df}{dy}$$

- Can get a solution only if LHS is a function of ‘y’ in which case it is an ordinary first order differential equation