- Now use variation of parameters to solve the same question
- Get the fundamental matrix

$$\Omega = \begin{pmatrix} 1e^{-3t} & 1e^{t} & 0e^{t} \\ 3e^{-3t} & 0e^{t} & 1e^{t} \\ 1e^{-3t} & -1e^{t} & 1e^{t} \end{pmatrix} \qquad g = \begin{pmatrix} -3e^{-3t} \\ t \\ 0 \end{pmatrix}$$

$$\Omega^{-1} = \begin{pmatrix} -e^{3t} & 1e^{3t} & -e^{3t} \\ 2e^{-t} & -e^{-t} & 1e^{-t} \\ 3e^{-t} & -2e^{-t} & 3e^{-t} \end{pmatrix}$$

$$v = \int \Omega^{-1} g = \begin{pmatrix} -e^{-3t} & 1e^{3t} & -e^{3t} \\ 2e^{-t} & -e^{-t} & 1e^{-t} \\ 3e^{-t} & -2e^{-t} & 3e^{-t} \end{pmatrix} \begin{pmatrix} -3e^{-3t} \\ t \\ 0 \end{pmatrix}$$

$$\int \begin{pmatrix} 3 + te^{3t} \\ -6e^{-4t} - te^{-t} \\ -9e^{-4t} - 2te^{-t} \end{pmatrix} = \begin{pmatrix} 3t + t\,\frac{e^{3t}}{3} - \frac{e^{3t}}{9} \\ 6\,\frac{e^{-4t}}{4} + te^{-t} + e^{-t} \\ 9\,\frac{e^{-4t}}{4} + 2te^{-t} + 2e^{-t} \end{pmatrix}$$

$$\Omega v = \begin{pmatrix} 1e^{-3t} & 1e^{t} & 0e^{t} \\ 3e^{-3t} & 0e^{t} & 1e^{t} \\ 1e^{-3t} & -1e^{t} & 1e^{t} \end{pmatrix} \begin{pmatrix} 3t + t\dfrac{e^{3t}}{3} - \dfrac{e^{3t}}{9} \\ 6\dfrac{e^{-4t}}{4} + te^{-t} + e^{-t} \\ 9\dfrac{e^{-4t}}{4} + 2te^{-t} + 2e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} 3te^{-3t} + \frac{t}{3} - \frac{1}{9} + 6\dfrac{e^{-3t}}{4} + t + 1 \\ 9te^{-3t} + t - \frac{1}{3} + 9\dfrac{e^{-3t}}{4} + 2t + 2 \\ 3te^{-3t} + \frac{t}{3} - \frac{1}{9} - 6\dfrac{e^{-3t}}{4} - t - 1 + 9\dfrac{e^{-3t}}{4} + 2t + 2 \end{pmatrix}$$

- Now get the final solution using:

$$x = \Omega(t)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \Omega(t)v(t)$$

- Use the conditions given to obtain the values for C1, C2 and C3

Copy link to this solution

75% 90% 100% 110% 125%

System of ordinary differential equations

$$\begin{cases} x' = 4z - 4y + 5x - \frac{3}{e^{3t}} \\ y' = 12z - 11y + 12x + t \\ z' = 5z - 4y + 4x \end{cases}$$

General solution to the system of ODEs

$$\begin{cases} x = (C - C_1)\, e^t + \frac{3t}{e^{3t}} + \frac{C_2 + \frac{3}{4}}{e^{3t}} + \frac{4t}{3} + \frac{8}{9} \\ y = C\, e^t + \frac{9t}{e^{3t}} + \frac{3C_2}{e^{3t}} + 3t + \frac{5}{3} \\ z = C_1\, e^t + \frac{3t}{e^{3t}} + \frac{C_2}{e^{3t}} + \frac{4t}{3} + \frac{8}{9} \end{cases}$$

where $C,\ C_1,\ C_2$ are constants and

$x = x(t)\,,\ y = y(t)\,,\ z = z(t)$

Step-by-step solution

Solve

$$\begin{cases} x' = 4z - 4y + 5x - \frac{3}{e^{3t}} \\ y' = 12z - 11y + 12x + t \\ z' = 5z - 4y + 4x \end{cases}$$

Phase Portraits

- The solution representation in two dimensions.
- Origin x=0,y=0 is a special point since for the equation Ax=x' with x,y=0 at t=0 means there is no change in the position of the point
- Solution depends on eigen values and vectors
- Transformation from x to z converted a higher dimension solution to a single dimension one, i.e. the solutions were all decoupled when eigen vectors were independent
- A point on the eigen vector stays on the eigen vector forever

Consider:

$$x = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} e^{\lambda_1 t} c_1 + \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} e^{\lambda_2 t} c_2$$

Suppose for $t = 0$ i.e initial condition x_{01} *and* $x_{02}$are on eigen vector [v11 v12]. Then C2 =0. This means that a point on $\overline{v_1}$ always stays on $\overline{v_1}$.

Similarly a point on $\overline{v_2}$ stays on $\overline{v_2}$.

Intermediate points will change position and create the phase portrait.

A point on the eigen vector at a given time must have been on the eigen vector at t=0 i.e. a point from outside cannot get onto the eigen vector

- Consider the spring mass damper system. Convert into a system of two equations:

$$\left(\frac{D^2}{\omega_n^2}+\frac{2\zeta D}{\omega_n}+1\right)x=0 \Rightarrow \begin{pmatrix} -2\zeta\omega_n & -\omega_n \\ 1 & 0 \end{pmatrix}\begin{bmatrix} v \\ x \end{bmatrix}=\begin{bmatrix} \dot{v} \\ \dot{x} \end{bmatrix}$$

- Let ω_n=1 and ζ=0. No damping.

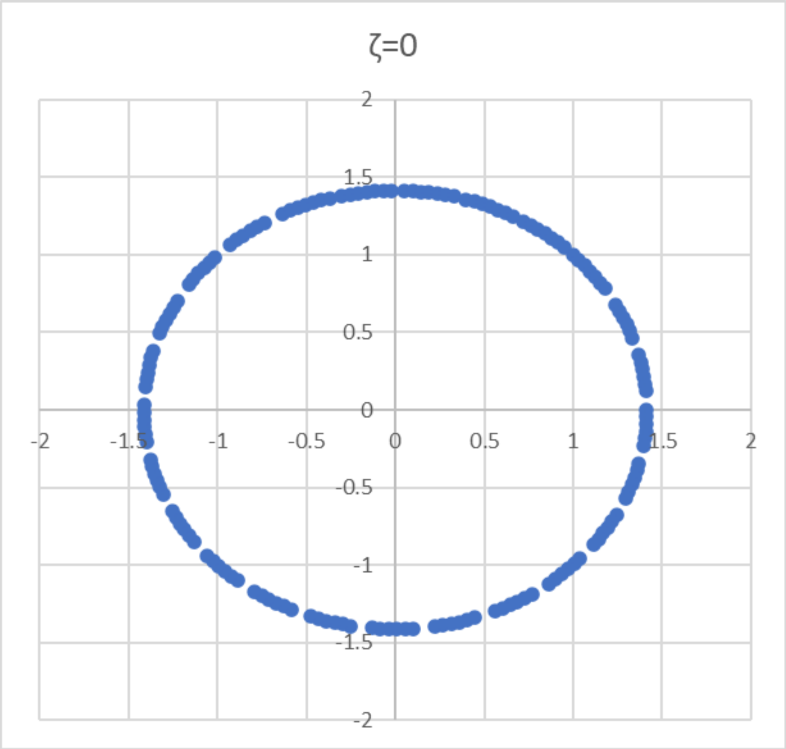
$$A=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \lambda = \underbrace{0}_{\alpha} + \mathrm{i}\underbrace{1}_{\beta} \qquad \begin{bmatrix} i \\ 1 \end{bmatrix} = \overset{U}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \mathrm{i}\overset{V}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

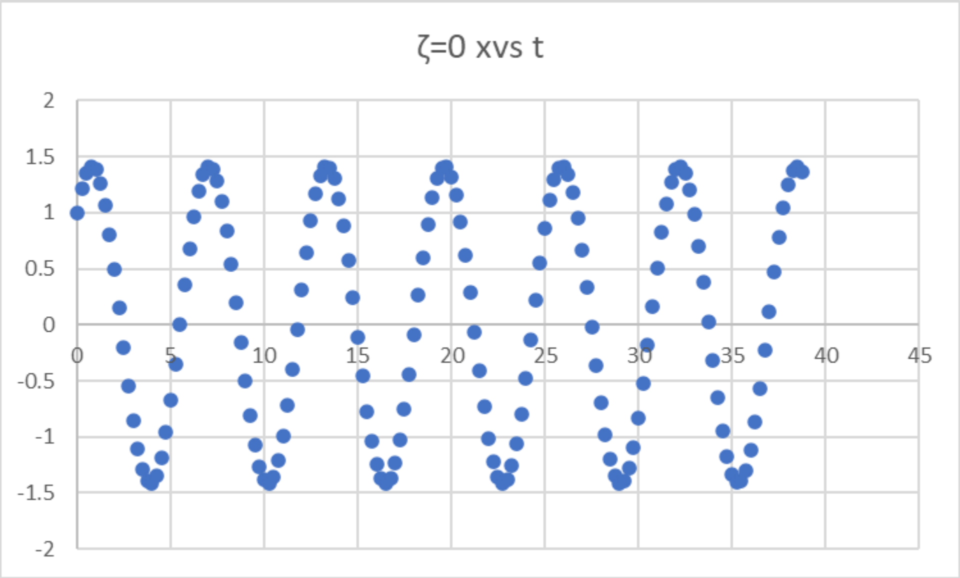
$$\lambda = 0 - \mathrm{i} \qquad \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \mathrm{i}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$X = C1e^{\alpha t}[U\cos\beta t - V\sin\beta t] + C2e^{\alpha t}[U\sin\beta t + V\cos\beta t]$$

$$\begin{bmatrix} v \\ x \end{bmatrix} = \begin{bmatrix} C1(-\sin t) + C2(\mathrm{Cos}\, t) \\ C1Cost + C2Sint \end{bmatrix}$$

- Purely imaginary eigen value results in a closed curve not necessarily circle





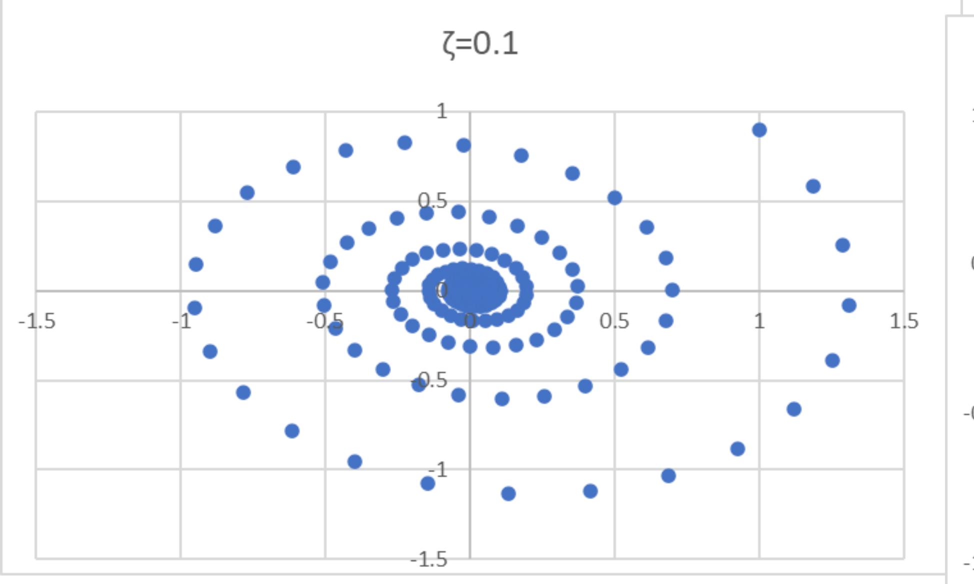
- Now suppose ω_n=1 and ζ=.1

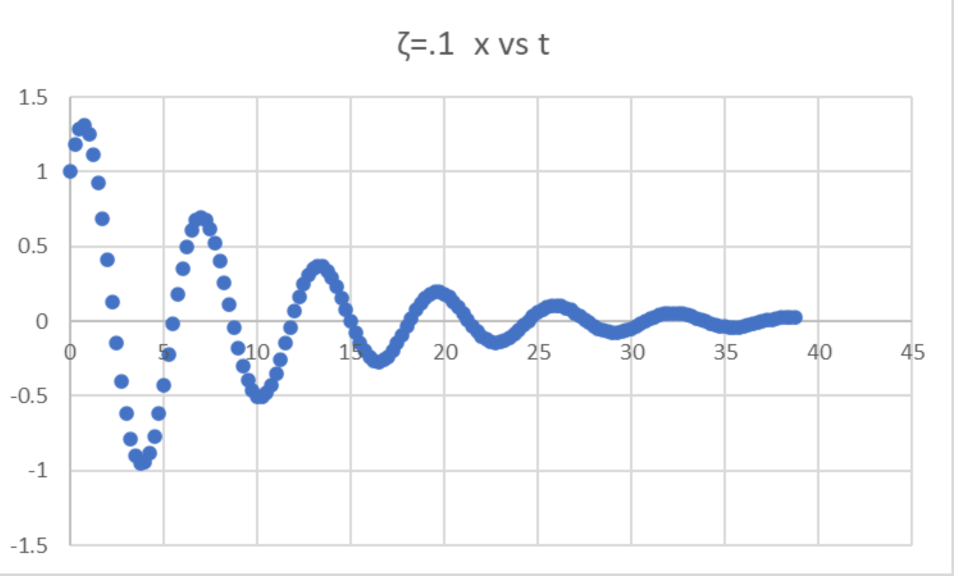
$$A=\begin{pmatrix} -.2 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{aligned} \lambda_1 &= -.1 + \mathrm{i}.995 \\ \lambda_2 &= -.1 - \mathrm{i}.995 \end{aligned} \quad \overset{U}{\begin{bmatrix} -.1 \\ 1 \end{bmatrix}} + \mathrm{i} \overset{V}{\begin{bmatrix} .995 \\ 0 \end{bmatrix}}$$

$$X = C1e^{\alpha t}[U\cos\beta t - V\sin\beta t] + C2e^{\alpha t}[U\sin\beta t + V\cos\beta t]$$

$$\begin{bmatrix} v \\ x \end{bmatrix} = \begin{bmatrix} C1e^{-.1t}(-.1cos.995t - .995\sin.995t) + C2e^{-.1t}(-.1sin.995t + .995cos.995t) \\ C1e^{-.1t}(1cos.995t \quad) + C2e^{-.1t}(1sin.995t) \end{bmatrix}$$

- Imaginary eigen value with negative real part. Inward spiral

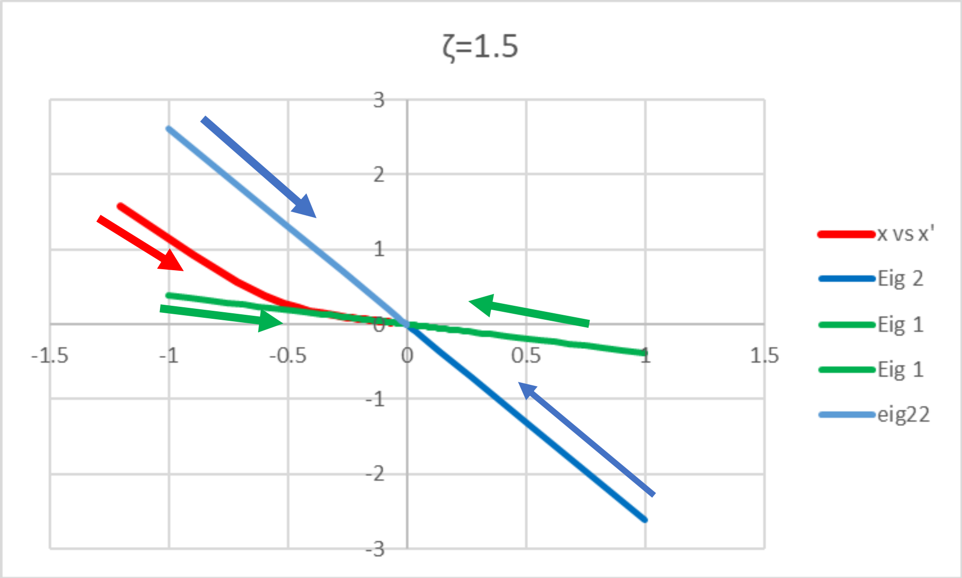


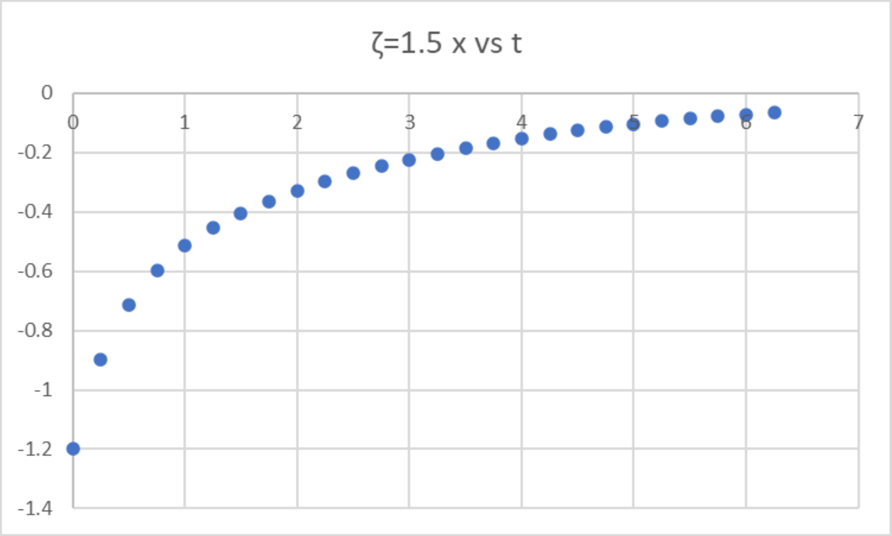


- Now suppose ω_n=1 and ζ=1.5. Get sink

A=$\begin{pmatrix} -3 & -1 \\ 1 & 0 \end{pmatrix}$ $\lambda = -2.61; -.38;$ $\begin{bmatrix} -2.61 \\ 1 \end{bmatrix}; \begin{bmatrix} -.38 \\ 1 \end{bmatrix}$

$$X = \begin{bmatrix} v \\ x \end{bmatrix} = C1e^{-2.61t}\begin{bmatrix} -2.61 \\ 1 \end{bmatrix} + C2e^{-.38t}\begin{bmatrix} -.38 \\ 1 \end{bmatrix}$$





Consider example in previous class

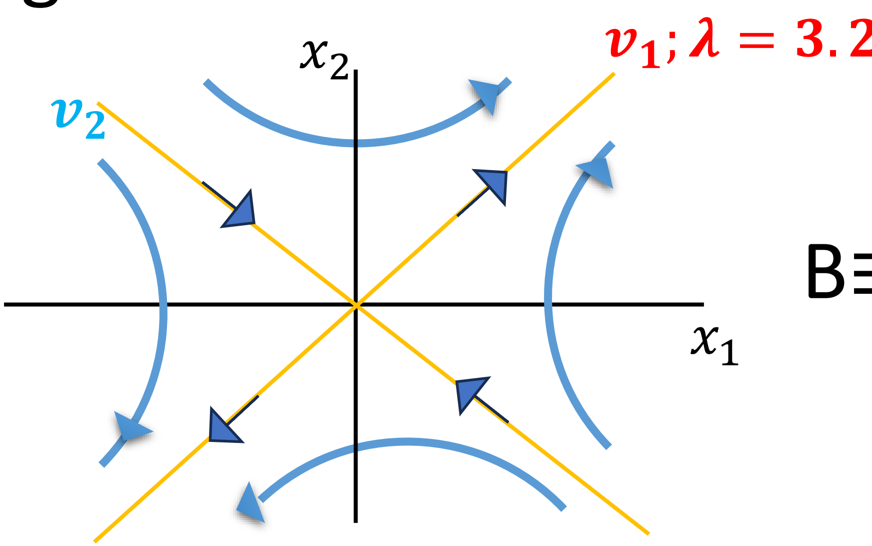
$$x = c_1 e^{\lambda_1 t}[v_1] + c_2 e^{\lambda_2 t}[v_1]$$

At $t = 0$ If a point t is on v_1

Then $c_2 = 0$ $\lambda_1 = +3.2$; $\lambda_2 = -2.2$

One eigen value positive and the other negative - saddle

$$A = \begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix}$$



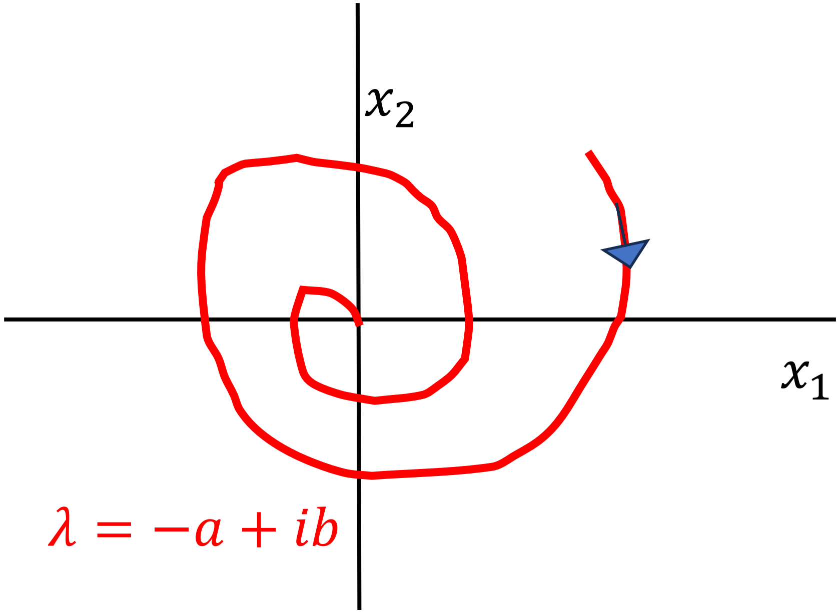
$$\mathrm{B} \equiv \begin{bmatrix} -5.1926 & 1 \\ 1 & 5.1926 \end{bmatrix}$$

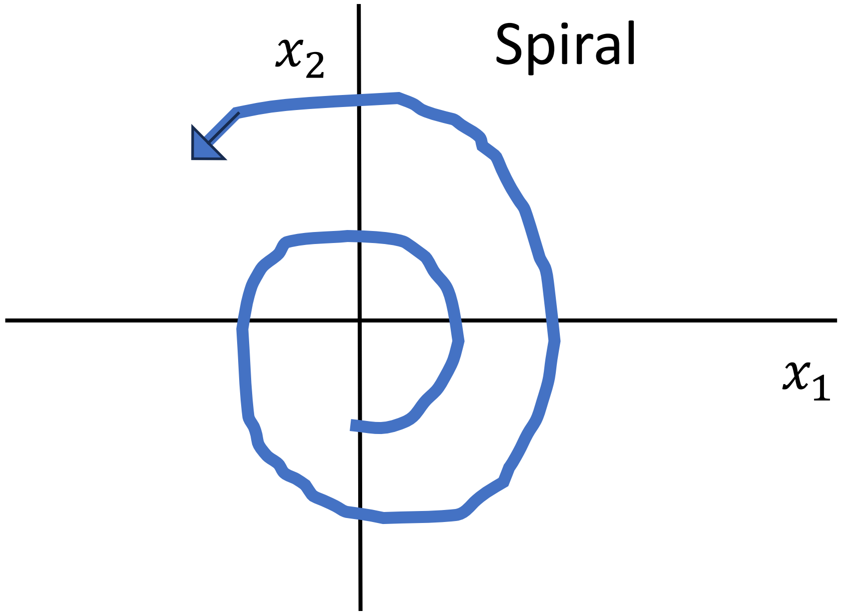
- Consider another example

$$A = \begin{bmatrix} -1 & -2 \\ 4 & 3 \end{bmatrix} \qquad \lambda_1 = 1 + i\,;\ \lambda_2 = 1 - i$$

- Get direction by a small calculation at a convenient point

$$X' = \begin{bmatrix} -1 & -2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -12 \end{bmatrix}$$

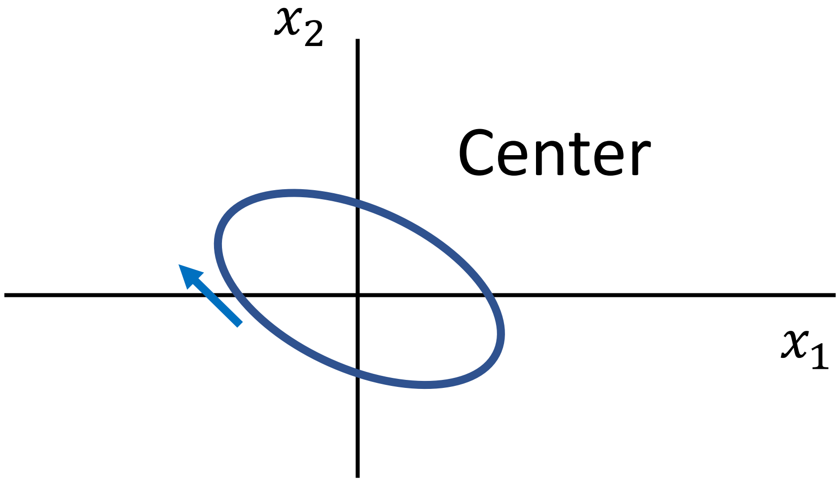




- Another example

$$A = \begin{bmatrix} 3 & 18 \\ -1 & -3 \end{bmatrix} \qquad \lambda = \pm 3i$$

$$X' = \begin{bmatrix} 3 & 18 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \end{bmatrix}$$



- We consider only phase plots where distinct eigen vectors exist