- To get transpose of a matrix interchange rows and columns
- A symmetric matrix is one with $A = A^T$
- A diagonal matrix is one with all off diagonal elements = 0
- A square matrix with linearly independent eigen vectors can be diagonalized – eigen values can be real or complex or repeated.
- Eigen vectors associated with distinct eigen values are linearly independent.
- Eigen values of real symmetric matrix are real. Eigen vectors are orthogonal for distinct eigen values.

- Diagonalization: $D = P^{-1}AP$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \end{bmatrix}, P = \begin{bmatrix} \vdots & \vdots & & \\ v_1 & v_2 & \cdots & \cdots \\ \vdots & \vdots & & \\ \vdots & \vdots & & \end{bmatrix}$$

- A diagonal matrix with eigen values as the elements is obtained if the above operation is performed where P is the matrix with columns as the corresponding eigen vectors. <u>A matrix is diagonalizable only if eigen vectors are independent.</u>

Linear System of Equations

- Higher order differential equations are solved as a system of equations. Consider:

$$\ddot{y} + a_1\dot{y} + a_0 y = 0$$

- Let $\dot{y} = p \;\Rightarrow \ddot{y} = \dot{p} = -a_1 p - a_0 y$
- Now express the original equation as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p(=\dot{y}) \\ y \end{bmatrix} = \begin{bmatrix} -a_1 & -a_0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p \\ y \end{bmatrix}$$

- Consider the following third order equation

$$y''' + a_2 y'' + a_1 y' + a_0 y = 0$$

- Again express the equation in terms the first derivatives only, i.e.

$$y' = p;\ y'' = q;\ y''' = -a_2 q - a_1 p - a_0 y = \dot{q}$$

- The equation is thus expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y \\ p(\equiv y') \\ q(\equiv y'') \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}\begin{bmatrix} y \\ p \\ q \end{bmatrix}$$

- Sometimes there could be more than one variable and there could be coupling between equations:

$$\ddot{y} + a_1\dot{y} + a_0 y = 0$$
$$\ddot{z} + b_1\dot{z} + b_0(z - y) = 0$$

- Again choose variables such that only the first derivative exists

$$\dot{y} = p; \ddot{y} = \dot{p} = -a_0 y - a_1\dot{y} = -a_1 p - a_0 y$$
$$\dot{z} = q; \ddot{z} = \dot{q} = -b_1\dot{z} - b_0(z - y)$$

- Note the coupling terms in the following formulation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p(\equiv \dot{y}) \\ y \\ q(\equiv \dot{z}) \\ z \end{bmatrix} = \begin{bmatrix} a_1 & -a_0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & b_0 & -b_1 & -b_0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p(\dot{y}) \\ y \\ q(\dot{z}) \\ z \end{bmatrix}$$

- Now consider the following:

$$\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$$

- Solution of first order equations has already been seen earlier:

$$y = \mathrm{e}^{c_1 t} y_0; \; z = \mathrm{e}^{c_2 t} z_0$$

- The set of equations as above are uncoupled and the solution is easy to get
- Desire to convert a set of equations from the normal form to one that has a diagonal matrix so that solution is easily obtained.

- Consider the following equation set:

$$\dot{x} = Ax \Rightarrow x = x(0)e^{At}$$
$$e^{At} = I + At + \frac{(At)^2}{2!} + \cdots \ldots$$

- Simply substitute and verify that indeed it works just as if A were a constant, except that it is a matrix
- Computing the powers of a matrix is not a very easy task. However, if the matrix is a diagonal one then it is very comfortable
- So try to diagonalize a matrix

- If B is chosen such that the columns are the **distinct** eigen vectors of A then $B^{-1}AB$ is a diagonal matrix with eigen values as the elements of the matrix

$$\mathrm{D} = B^{-1}AB$$

- Choose the following transformation and substitute in the original equation to get:

$$x = Bz\,;\, \dot{x} = B\dot{z} \Rightarrow B\dot{z} = ABz$$

- Multiply throughout by B^{-1}:

$$\dot{z} = B^{-1}ABz$$

- Solution is easily obtained i.e. $\dot{z} = Dz$

$$z = e^{Dt}z_0$$

- Using the transformations

$$x = B\mathrm{e}^{Dt}B^{-1}x_0$$

- Aside: The methodology is also useful for determining the power of a matrix provided it can be diagonalized

$$A = BDB^{-1}; A^2 = BDB^{-1}BDB^{-1} = BDDB^{-1}$$

- Extend the argument to get:

$$A^n = BDDDD \ldots . B^{-1} = BD^nB^{-1}$$

- If the eigen vectors are distinct it can be diagonalized.

- An alternate way to look at the solution is:

$$x = Bz \Rightarrow x = \begin{bmatrix} \overline{v_1} & \overline{v_2} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} C_1 \\ e^{\lambda_2 t} C_2 \end{bmatrix}$$

$$x = \begin{bmatrix} e^{\lambda_1 t} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} & e^{\lambda_2 t} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

- This also works even if the matrix is not diagonalizable $\dot{x} = Ax$
- Assume a solution exists of the form $x = \boldsymbol{v} e^{\lambda t}$
- Substitute in the original equation:

$$\dot{x} = \lambda e^{\lambda t} \boldsymbol{v} = A \boldsymbol{v} e^{\lambda t} \Rightarrow \lambda \boldsymbol{v} = A \boldsymbol{v}$$

- Implies $\boldsymbol{v}$ is an eigen vector, i.e. same solution as above. Diagonalization not absolutely necessary.

- One can check if the solutions are linearly independent

$$\Omega = \begin{bmatrix} \begin{bmatrix} e^{\lambda_1 t} v_{11} \\ e^{\lambda_1 t} v_{12} \end{bmatrix} & \begin{bmatrix} e^{\lambda_2 t} v_{21} \\ e^{\lambda_2 t} v_{22} \end{bmatrix} \end{bmatrix}$$

- Above matrix is referred to as the fundamental matrix. Determinant of the above matrix should be non zero for linear independence. Choose any convenient ‘t’ for computation.
- Concept is similar to independence of functions, i.e. multiply each by a constant and equate to zero and solve for constants and if constants are non zero then linearly independent
- Proof not here.