- Consider a system modelled as:

$$\dot{x} = Ax \qquad\qquad A = \begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix}$$

- First step is to get the eigen values

$$A - I\lambda = 0 \Rightarrow Det \begin{vmatrix} -2-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2+\lambda)(3-\lambda) + 1 = 0 \Rightarrow 7 + \lambda + \lambda^2 = 0$$

$$\Rightarrow \lambda = 3.1926;\ \lambda = -2.1926$$

- Obtain eigen values for $\lambda = 3.1926$

$$\Rightarrow \begin{bmatrix} -5.1926 & 1 \\ 1 & -0.1926 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 5.1926q_1 - q_2 = 0 \Rightarrow q_2 = 5.1926q_1$$

- Now consider $\lambda = -2.1926$

$$\Rightarrow \begin{bmatrix} 0 \cdot 1926 & 1 \\ 1 & 5 \cdot 1926 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};$$

$$\Rightarrow 0.1926 q_1 + q_2 = 0$$

- Now create B and compute its inverse:

$$\text{B} \equiv \begin{bmatrix} -5.1926 & 1 \\ 1 & 5.1926 \end{bmatrix}; B^{-1} = \begin{bmatrix} -0.1857 & 0.0358 \\ 0.0358 & 0.1857 \end{bmatrix}$$

$$x = Bz = \begin{bmatrix} -5.1926 & 1 \\ 1 & 5.1926 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$z = \mathrm{e}^{Dt} = \begin{bmatrix} \mathrm{e}^{-2\cdot 1926\mathrm{t}} C_1 \\ e^{3.1926t} C_2 \end{bmatrix}$$

$$x = B\mathrm{z} = \begin{bmatrix} v_{\lambda_1} \; v_{\lambda_2} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{-2\cdot 1926\mathrm{t}} C_1 \\ e^{3.1926t} C_2 \end{bmatrix}$$

$$x = C_1 \mathrm{e}^{\lambda_1 t} v_{\lambda_1} + C_2 \mathrm{e}^{\lambda_2 t} v_{\lambda_2}$$

- Use initial condition and obtain solution
- Eigen values can be imaginary and since matrix has real elements, the eigen values will be complex conjugates since characteristic equation has real coefficients.
- Let the eigen values be $\alpha \pm \mathrm{i}\beta$ and the corresponding eigen vectors be $u \pm \mathrm{i}v$

- Just like for real eigen values create the x vector using the same methodology:

$$x = C_1 e^{\lambda_1 t} v_{11} + C_2 e^{\lambda_2 t} v_{12}$$

$$x = C_1 e^{(\alpha + i\beta)t}[u + iv] + C_2 e^{(\alpha - i\beta)t}[u - iv]$$

C1ϕ1 C2ϕ2

- Consider the two terms separately

$$\phi_1 = e^{\alpha t}(u\cos\beta t - v\sin\beta t) + ie^{\alpha t}(u\sin\beta t + v\cos\beta t)$$
$$\phi_2 = e^{\alpha t}(u\cos\beta t - v\sin\beta t) + e^{\alpha t}(-u\sin\beta t - v\cos\beta t)i$$

- A linear combination of the solutions is also a solution for the set of equations. A typical element has been taken for ease in algebra

- Therefore:

$$\frac{\Phi_1 + \Phi_2}{2} = \widehat{\Phi}_1 \ = e^{\alpha t}[\cos\beta t U - \sin\beta t\, V]$$

$$\frac{\Phi_1 - \Phi_2}{2} = \widehat{\Phi}_2 = e^{\alpha t}[\sin\beta t U + \cos\beta t\, V]$$

- Now get the solution as:

$$\text{x=C1}\widehat{\Phi}_1 + \text{C2}\widehat{\Phi}_1$$

- Assume now that the eigen values are repeated then two solutions will not be obtained.

$$\text{x=C1}\widehat{\Phi}_1 + \text{C2}\widehat{\Phi}_2$$

$$\widehat{\Phi}_1 \ = e^{\lambda t}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$$

- As usual one can attempt the second solution by multiplying with independent variable but it does not work, so:

$$\widehat{\Phi}_2 = te^{\lambda t}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} + e^{\lambda t}\begin{bmatrix} \hat{v}_{11} \\ \hat{v}_{12} \end{bmatrix}$$

- Second vector is unknown still
- Substitute in original equation $\dot{\bar{X}} = AX$

$$\dot{\bar{X}} = \boxed{Ate^{\lambda t}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}} + Ae^{\lambda t}\begin{bmatrix} \hat{v}_{11} \\ \hat{v}_{12} \end{bmatrix}$$

$$= \boxed{\lambda te^{\lambda t}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}} + e^{\lambda t}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} + \lambda e^{\lambda t}\begin{bmatrix} \hat{v}_{11} \\ \hat{v}_{12} \end{bmatrix}$$

- Terms in blue box are equation since ‘v’ is an eigen vector for matrix A. This is why simply multiplying by t did not work.

- Therefore

$$[A - \lambda I]\hat{v} = \bar{v}$$

- Solve for $\hat{v}$
- Example: $\dot{x} = Ax \quad A = \begin{bmatrix} 2 & 0 \\ 5 & 2 \end{bmatrix} \Rightarrow Eig\ values = 2,2$
- Compute the eigen vectors:

$$\begin{bmatrix} 2 & 0 \\ 5 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$5x_1 = 0 \Rightarrow x_1 = 0, \qquad x_2 = 1(say)$$

- Only one eigen vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- Solution is:

$$te^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + e^{2t}\begin{bmatrix} \hat{v}_{11} \\ \hat{v}_{12} \end{bmatrix}$$

$$\dot{x} = 2te^{2t}\begin{bmatrix}0\\1\end{bmatrix} + e^{2t}\begin{bmatrix}0\\1\end{bmatrix} + 2e^{2t}\begin{bmatrix}\hat{v}_{11}\\\hat{v}_{12}\end{bmatrix}$$

$$Ax = \begin{bmatrix}2 & 0\\5 & 2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}te^{2t} + \begin{bmatrix}2 & 0\\5 & 2\end{bmatrix}\begin{bmatrix}\hat{v}_{11}\\\hat{v}_{12}\end{bmatrix}e^{2t}$$

- Blue terms cancel off due to $Ax = \lambda x$. Now solve for unknown vector

$$\begin{bmatrix}0\\1\end{bmatrix} + 2\begin{bmatrix}\hat{v}_{11}\\\hat{v}_{12}\end{bmatrix} = \begin{bmatrix}2 & 0\\5 & 2\end{bmatrix}\begin{bmatrix}\hat{v}_{11}\\\hat{v}_{12}\end{bmatrix}$$

$$\hat{v}_{11} = 1/5; \quad \hat{v}_{12} = \alpha$$

$$x = C_1\left[e^{2t}\begin{bmatrix}0\\1\end{bmatrix}\right] + C_2\left[te^{2t}\begin{bmatrix}0\\1\end{bmatrix} + e^{2t}\begin{bmatrix}1/5\\\hat{v}_{12}\end{bmatrix}\right]$$

$$x = C_1 \left[e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] + C_2 \left[t e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 1/5 \\ \hat{v}_{12} (= \alpha) \end{bmatrix} \right]$$

$$x = \left[e^{2t} \begin{bmatrix} C_2/5 \\ C_1 + C_2 t + C_2 \alpha \end{bmatrix} \right]$$

- Evaluate C_2 and $C_1+C_2\alpha$ using the two BC/IC
- Sometimes there is another repetition of the root. Then we need two additional solutions to get the three independent solutions for the system.
- Similar approach is followed. Consider an example where a root is repeated three times

$$\Phi_1 = e^{\lambda t}\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} \qquad \Phi_2 = te^{\lambda t}\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} + e^{\lambda t}\begin{bmatrix} \hat{v}_{11} \\ \hat{v}_{12} \\ \hat{v}_{13} \end{bmatrix}$$

$$\Phi_3 = \frac{t^2 e^{\lambda t}}{2!}\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} + te^{\lambda t}\begin{bmatrix} \hat{v}_{11} \\ \hat{v}_{12} \\ \hat{v}_{13} \end{bmatrix} + e^{\lambda t}\begin{bmatrix} \check{v}_{11} \\ \check{v}_{12} \\ \check{v}_{13} \end{bmatrix}$$

- Finally the general solution is X=C1ϕ1+C2ϕ2+C3ϕ3 and use the boundary conditions to evaluate C1,C3 and C2.