- Sometimes approaches much simpler that those that have been discussed can be used.
- Consider the solution to the following equation using the homogenous and particular solution approach.

$$\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_o(t)y = b_o(t)$$

- The homogeneous part has two solutions and since it is linear the solution is expressed as:

$$y = C_1y_1 + C_2y_2$$

- y1 and y2 are linearly independent solutions for the diff. eqn.

- Consider a homogeneous equation with constant coefficients a_1 and a_2
- Two solutions are required so guess that the solution may be of the form $y=Ce^{rt}$ and substitute in the equation
- The following equation results:

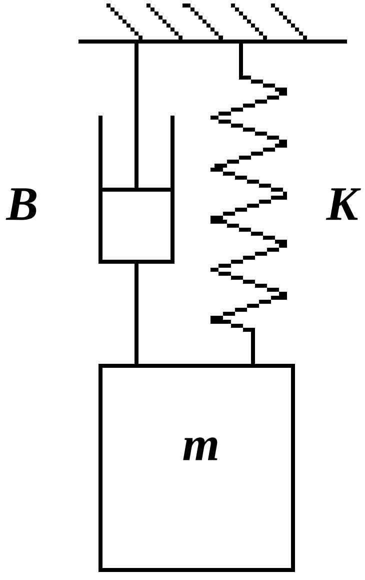
$$r^2 + a_1 r \; + a_o = 0$$

$$r = \frac{-a_1 \pm \sqrt{{a_1}^2 - 4a_o}}{2}$$

- Get solution corresponding to each root

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

- When the roots are real and distinct there is no problem
- A second order differential equation is a good model for a simple spring mass system



$$m\ddot{x} = F_i - Kx - B\frac{dx}{dt}$$

$$\left(mD^2 + BD + K\right)x = F_i$$

$$\left(\frac{m}{K}D^2 + \frac{B}{K}D + 1\right)x = \frac{F_i}{K}$$

$$\left(\frac{D^2}{\omega_n^2} + \frac{B\,D}{\sqrt{K}\;\omega_n\sqrt{m}}D + 1\right)x = \frac{F_i}{K}\,;\zeta = \frac{B}{2\sqrt{mK}}\,;\omega_n = \sqrt{\frac{K}{m}}$$

$$\left(\frac{D^2}{\omega_n^2} + \frac{2\zeta\,D}{\omega_n} + 1\right)x = \frac{F_i}{K}$$

- The spring mass system is modeled as a 2nd order system and using the same procedure as before, the solution is obtained as:

$$q_o = C_0 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_1 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} + Kq_i$$

- Since a physical system has been modeled it is not very comfortable to expect imaginary roots. Convert to physical variables.

- Let the complex roots be m+in and m-in. Since a linear combination of the roots is also a root consider first the sum of the roots

$$q_{oa} = q_{o1} + q_{o2} = e^{(m+in)t} + e^{(m-in)t}$$
$$= e^{mt}e^{int} + e^{mt}e^{-int}$$
$$= e^{mt}(e^{int} + e^{-int})$$

- Use the Euler identity $e^{int} = (cosnt + isinnt)$

$$q_{oa} = e^{mt}(Cosnt)$$

- Similarly q01-q02 is also a solution q0b which can be seen to be

$$\frac{q_{oa}}{i} = e^{mt}(Sinnt)$$

- Linear combination of solutions is also a solution even though the ‘i’ appears.
- Final solution is therefore

$$q_o = e^{mt}(C_1 Sinnt + C_2 Cosnt)$$

- Last case is for repeated roots. Here the two roots are the same, so only one solution becomes available.
- Need two solutions, so guess the solution to be of the form y= $e^{r_1 t}v(t)$

$$y' = v'e^{r_1 t} + vr_1 e^{r_1 t}$$
$$y'' = v''e^{r_1 t} + v'r_1 e^{r_1 t} + vr_1^2 e^{r_1 t} + v'r_1 e^{r_1 t}$$

- Substitute in original equation to get

$$v'' + (2r_1 + a_1)v' + (r_1^2 + a_1 r_1 + a_0)v = 0$$

- Note that $a_1^2 = 4a_0 \; and \; r_1 = -a_1/2$ which comes from the solution of the algebraic equation for the original equation
- Substituting in the above equation gives:

$$v'' = 0 \Rightarrow v = c + dt$$

- Choose c=0 with no loss of generality and therefore obtain y=$C_0 e^{rt}(1 + C_1 t)$

Example

- Solve:

$$y'' - 2y' - 8y = RHS; \quad y(0) = 1, y(0)' = 4$$

- Get solution to the homogeneous part. Get roots for characteristic equation

$$\lambda^2 - 2\lambda - 8 = 0 \qquad (\lambda - 4)(\lambda + 2) = 0$$

$$\lambda_1 = 4\,,\ \lambda_2 = -2 \qquad y_h = C_1 e^{-2x} + C_2 e^{4x}$$

Example

- Solve:

$$y'' - y' + y = RHS\,; \quad y(1) = 4, y(1)' = -2$$

- Get solution to the roots of char. equn:

$$\lambda^2 - \lambda + 1 = 0$$

$$\lambda_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i \quad , \quad \lambda_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$y_h = c_1 e^{\frac{x}{2}} cos\left(\frac{\sqrt{3}}{2}x\right) + \ c_2 e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

- Solve:

$$y'' - 6y' + 9y = RHS\,; \quad y(1) = 4, y(1)' = -2$$

- Get roots for char. Equn.

$$\lambda^2 - 6\lambda + 9 = 0 \qquad (\lambda - 3)(\lambda - 3) = 0$$

$$\lambda_1 = 3\,,\ \lambda_2 = 3 \qquad y_h = C_1 e^{3x} + C_2 x e^{3x}$$

- Need to determine if the solutions being obtained are really independent, since that is the basic requirement
- Compute the Wronskian(extend to nXn matrix)

$$w = \mathrm{Det}\begin{bmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{bmatrix}$$

- If $w \neq 0$ then functions are linearly independent. However w=0 does not make them dependent

- Linear independence means one function is not a linear sum of multiples of the others in a particular set

$$f_1 = C_2\, f_2 + C_3\, f_3 + C_4\, f_4 \ldots.$$

- In the present case for the unequal roots case we have only two functions

$$e^{r_1 t} = 1 + r_1 t + \frac{(r_1 t)^2}{2!} \ldots\ldots\ldots\ldots$$

$$e^{r_2 t} = 1 + r_2 t + \frac{(r_2 t)^2}{2!} \ldots\ldots\ldots\ldots$$

- It is quite obvious that they are linearly independent since you cannot obtain a C2 that will satisfy the above expression. So also for the equal root and imaginary roots cases.