$$\mathcal{L}[\delta(t-\tau)] = \int_0^\infty e^{-st}\delta(t-\tau)dt =$$

$$= \int_0^\infty e^{-st} Lim_{\varepsilon\to 0} \frac{H(t-(\tau-\varepsilon))-H(t-(\tau+\varepsilon))}{2\varepsilon} dt$$

$$= Lim_{\varepsilon\to 0} \frac{e^{-(s(\tau-\varepsilon))} - e^{-(s(\tau+\varepsilon))}}{2\varepsilon s} = e^{-s\tau}$$

$$Use\ e^{\varepsilon s} = 1 + \varepsilon s + \frac{(\varepsilon s)^2}{2!} \dots or\ \mathrm{L'Hospital's\ rule}$$

$$\mathcal{L}[\delta(t)] = \int_0^\infty e^{-st}\delta(t)dt = e^0 = 1$$

- Note that

$$\int_0^\infty kf(t)\delta(t-\tau)dt = kf(\tau)$$

- Consider an impulse forcing function, i.e. delta function for the following first order differential equation:

$$y' - ay = \delta(t); y(0) = 0$$

- Take Laplace transform on both sides

$$\mathrm{L}[f'] = -f(0) + sL(f)$$

$$-y(0) + sL(y) - \mathrm{aL(y)} = \mathrm{L}\big(\delta(t)\big) = 1$$

$$L(y)(s-a) = 1; L(y) = \frac{1}{s-a}$$

- Use tables or online resources to get the solution as y= e^{at}
- Note that if the forcing function(i.e. RHS) is made kδ(t) then the solution is simply y= ke^{at}
- k δ(t) can also be thought of as sum of δ(t) k times.
- Now suppose the equation is

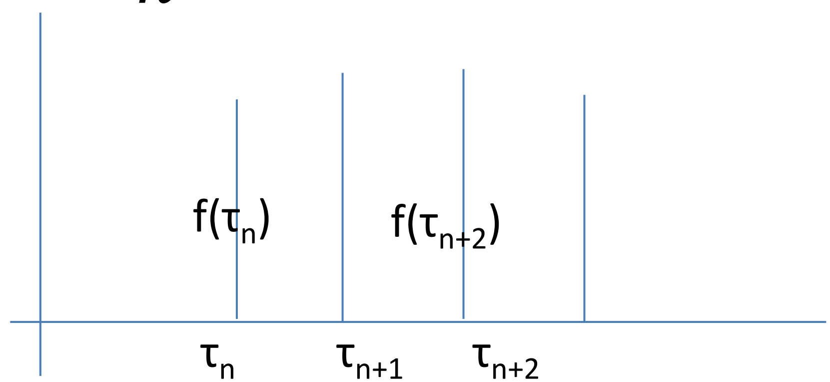
$$y' - ay = \delta(t - \tau); y(0) = 0$$

- Proceed in a similar fashion to get the solution as y= $e^{a(t-\tau)}H(t - \tau)$

- The solution for the delta function at $t=\tau$ was obtained by simply replacing t with $t-\tau$ in the solution.
- This is not surprising since the initial condition is homogeneous.
- Now instead of delta function consider a specified input as the forcing term

$$y' - ay = f(t); y(0) = 0$$

- f(t) is an arbitrary function as shown below
- RHS can be written as the following summation:
- $$f(t) = \sum_{n=-\infty}^{n=\infty} f(\tau_n)\,\hat{\delta}\,(t - \tau_n)\Delta\tau$$



- $\hat{\delta}\,(t - \tau_n)$ is the unit impulse function and as Δτ approaches zero the impulse function becomes the delta function

$$y' - ay = f(t) = \sum_{n=-\infty}^{n=\infty} f(\tau_n)\,\hat{\delta}\,(t - \tau_n)\Delta\tau$$

- Assume the function starts at τ1 and write only first two terms explicitly

$$y' - ay = f(\tau_1)\hat{\delta}\,(t - \tau_1)\Delta\tau + f(\tau_2)\,\hat{\delta}\,(t - \tau_2)\Delta\tau \ldots$$

- Since the governing equation is linear it can be easily verified that the solution can be assumed to be a sum as: y=y1+y2+.....
- Substitute in the governing equation and also the boundary condition

- The governing equation and the boundary conditions therefore become:

$$y_1' - ay_1 + y_2' - ay_2 \dots \dots =$$
$$f(\tau_1)\hat{\delta}\,(t - \tau_1)\Delta\tau + f(\tau_2)\hat{\delta}\,(t - \tau_2)\Delta\tau \dots$$
$$y_1 + y_2 \dots . = 0$$

- Now generalize the above into ***n***(tending to infinity) terms and then split the above into ***n*** differential equations with appropriate boundary conditions.
- Notice that homogeneous boundary condition is important to be able to do this.

$$y_1' - ay_1 = f(\tau_1)\hat{\delta}\,(t - \tau_1)\Delta\tau$$
$$y_1 = 0$$
$$y_2' - ay_2 = f(\tau_2)\hat{\delta}\,(t - \tau_2)\Delta\tau$$
$$y_2 = 0$$

$$y_n' - ay_n = f(\tau_n)\hat{\delta}\,(t - \tau_n)\Delta\tau$$
$$y_n = 0$$

- n equations can be written where n is very large number in which case $\hat{\delta} \equiv \delta$ and the earlier equations for the solution can be used.

- The solution for this equation has already been established as $y_1 = f(\tau_1)\Delta\tau\, e^{a(t-\tau_1)}$ Note that $e^{a(t-\tau_1)}$ is the solution for the simple delta function
- Generalize by not restricting to the equation taken as an example but a general differential equation. The solution for the simple delta function is assumed to be y_1^* which is a general function of (t-τ) i.e. $y_1^*(t-\tau)$

- The solution is therefore

$$y = \sum_1^n f(\tau_n)\Delta\tau\, y^*(t - \tau_n)$$

- As n tends to a large number the summation can be replaced with an integral

$$y = \int_0^t f(\tau)\, y^*(t - \tau)\, d\tau$$

- The solution is therefore integral of the RHS and the solution for the delta function