- Assume a non homogeneity exists

$$\dot{x} = Ax + g$$

- Proceed exactly in the same way as earlier

$$x = Bz \qquad \dot{z} = B^{-1}ABz + B^{-1}g$$

- Now these are a set of first order equations with non homogeneous part which can be solved as earlier with Integrating factor or otherwise.
- Use of variation of parameters: $\dot{x} = Ax + g$
- Let x_h be solution of homogeneous part and let x_p be the particular solution

$$(\dot{x}_p) = Ax_p + g \qquad (\dot{x}_h) = Ax_h$$

- As earlier in variation of parameters approach hypothesize that the particular solution is obtainable from homogeneous solution

$$x_p = \Omega(t)v(t)$$

- Substitute in the governing equation to get:

$$(\dot{\Omega v}) = A\Omega v + g \Rightarrow \quad (\dot{\Omega})v + \Omega\dot{v} = A\Omega v + g$$

$$(\dot{\Omega})v + \Omega\dot{v} = A\Omega v + g \ \Rightarrow \Omega\dot{v} = g$$

- Above coloured terms cancel since Ω is fundamental solution of homogeneous equation

$$\dot{v} = \Omega^{-1}g \Rightarrow v = \int \Omega^{-1}gdt$$

Section 10.3

18.

$$x_1' = 5x_1 - 4x_2 + 4x_3 - 3e^{-3t}$$
$$x_2' = 12x_1 - 11x_2 + 12x_3 + t$$
$$x_3' = 4x_1 - 4x_2 + 5x_3$$
$$x_1(0) = 1, x_2(0) = -1, x_3(0) = 2$$

Solution:

Let $\boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\boldsymbol{X}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$

Let's rewrite the given system in terms of matrices:

$$\boldsymbol{X}' = A\boldsymbol{X} + G$$

with **A** being a real, constant, 3×3 matrix

$$\boldsymbol{X}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 5x_1 - 4x_2 + 4x_3 \\ 12x_1 - 11x_2 + 12x_3 \\ 4x_1 - 4x_2 + 5x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \underbrace{\begin{pmatrix} -3e^{-3t} \\ t \\ 0 \end{pmatrix}}_{G}$$

Let's calculate eigenvalues of **A**.

$$|\,\lambda I_2 - A\,| = \begin{vmatrix} \lambda - 5 & 4 & -4 \\ -12 & \lambda + 11 & -12 \\ -4 & 4 & \lambda - 5 \end{vmatrix}$$

$$|\,\lambda I_2 - A\,| \;=\; (\lambda - 5)\begin{vmatrix} \lambda + 11 & -12 \\ 4 & \lambda - 5 \end{vmatrix} - 4\begin{vmatrix} -12 & -12 \\ -4 & \lambda - 5 \end{vmatrix} - 4\begin{vmatrix} -12 & \lambda + 11 \\ -4 & 4 \end{vmatrix}$$

$$(\lambda + 3)(\lambda - 1)(\lambda - 1) = 0$$

$$\lambda_1 = -3 \quad , \qquad \lambda_2 = 1 \quad , \qquad \lambda_3 = 1$$

$$eigenvector\ corresponding\ to\ \lambda_1 = -3 \qquad E_1 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$$eigenvector\ corresponding\ to\ \lambda_2 = 1\ and\ \lambda_3 = 1\ \ E_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} and\ E_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

3×3 matrix **A** has 3 linearly independent eigenvectors and therefore it's **diagonalizable** matrix

$$P^{-1} A\, P = D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 2 & -1 & 1 \\ 3 & -2 & 3 \end{pmatrix}$$

$$put\ X = PZ \quad in \qquad \boldsymbol{X}' = A\boldsymbol{X} + G$$

$$Z' = DZ + P^{-1}G$$

$$\begin{pmatrix} Z_1' \\ Z_2' \\ Z_3' \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} + \begin{pmatrix} -1 & 1 & -1 \\ 2 & -1 & 1 \\ 3 & -2 & 3 \end{pmatrix} \begin{pmatrix} -3e^{-3t} \\ t \\ 0 \end{pmatrix}$$

$$Z_1' = -3Z_1 + 3e^{-3t} + t$$

$$Z_2' = Z_2 - 6e^{-3t} - t$$

$$Z_3' = Z_3 - 9e^{-3t} - 2t$$

$$Z_1' + 3Z_1 = 3e^{-3t} + t$$

After Multiply e^{3t} on both side

$$(e^{3t}Z_1)' = 3 + te^{3t}$$

$$e^{3t}Z_1 = \int (3 + te^{3t})\ dt$$

$$Z_2' - Z_2 = -6e^{-3t} - t$$

After Multiply e^{-t} on both side

$$(e^{-t}Z_2)' = -6e^{-3t} - t$$

$$e^{-t}Z_2 = \int -6e^{-4t} - t\, e^{-t} dt$$

$$Z_3' - Z_3 = -9e^{-3t} - 2t$$

After Multiply e^{-t} on both side

$$(e^{-t}Z_3)' = -9e^{-3t} - 2t$$

$$e^{-t}Z_3 = \int -9e^{-4t} - 2te^{-t}\ dt$$

$$Z_1 = \frac{t}{3} - \frac{1}{9} + 3te^{-3t} + c_1 e^{-3t}$$

$$Z_2 = (\mathrm{t} + 1) + \frac{3}{2} e^{-3t} + c_2 e^{t}$$

$$Z_3 = 2(\mathrm{t} + 1) + \frac{9}{4} e^{-3t} + c_3 e^{t}$$

$$Z(t) = \begin{pmatrix} \frac{t}{3} - \frac{1}{9} + 3te^{-3t} + c_1 e^{-3t} \\ (\mathrm{t} + 1) + \frac{3}{2} e^{-3t} + c_2 e^{t} \\ 2(\mathrm{t} + 1) + \frac{9}{4} e^{-3t} + c_3 e^{t} \end{pmatrix}$$

Finally, we can calculate **X**=**PZ**.

$$x_1(0) = 1 \qquad x_2(0) = -1 \qquad x_3(0) = 2$$

$$c_1 + c_2 + \frac{8}{9} + \frac{3}{2} = 1 \qquad 3c_1 + c_3 + \frac{5}{3} + \frac{9}{4} = -1 \qquad c_1 - c_2 + c_3 + \frac{8}{9} + \frac{3}{4} = 2$$

$$c_1 + c_2 = -\frac{25}{18} \qquad 3c_1 + c_3 = -\frac{59}{12} \qquad c_1 - c_2 + c_3 = \frac{13}{36}$$

Solving equation (i) , (ii) and (iii) we get

$$c_1 = -\frac{35}{9} \quad \text{and} \quad c_2 = \frac{5}{2} \quad \text{and} \quad c_3 = \frac{27}{4}$$

Put the value of c_1 and c_2 in X(t)

$$X(t) = \begin{pmatrix} \frac{5}{2}e^t + \frac{4}{3}t + \frac{8}{9} + (3t - \frac{43}{18})e^{-3t} \\ \frac{27}{4}e^t + 3t + \frac{5}{3} + (9t - \frac{113}{12})e^{-3t} \\ \frac{17}{4}e^t + \frac{4}{3}t + \frac{8}{9} + (3t - \frac{113}{36})e^{-3t} \end{pmatrix}$$