- Often an easier methodology which gives the solution is used for first order equations
- Multiply the equation with an integrating factor which is defined as $e^{\int a_0 dt}$ and then perform algebra:

$$\frac{dq_o}{dt} + \frac{1}{\tau} q_o = + \frac{K}{\tau} q_i$$

$$e^{\int \frac{1}{\tau} dt} \frac{dq_o}{dt} + e^{\int \frac{1}{\tau} dt} \quad \frac{1}{\tau} q_o = + e^{\int \frac{1}{\tau} dt} \frac{K}{\tau} q_i$$

$$\frac{d}{dt}\left(q_o e^{\int \frac{1}{\tau} dt}\right) = + e^{\int \frac{1}{\tau} dt} \frac{K}{\tau} q_i$$

- Consider the earlier example where Kq_i is a constant and $1/\tau$ is a constant

$$e^{\int \frac{1}{\tau} dt} = e^{t/\tau}$$

$$e^{t/\tau} \, \frac{dq_o}{dt} + e^{t/\tau} \frac{1}{\tau} q_o = +e^{t/\tau} \frac{K}{\tau} q_i$$

$$\mathrm{q}_0 = Ce^{-\frac{t}{\tau}} + \mathrm{K}q_i$$

- Final solution is the same as the one obtained from inspection

- Consider x as the independent variable

$$y' + y = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

- $e^{\int dx}$ is an integrating factor.

$$y'e^{x} + ye^{x} = \frac{1}{2}\left(e^{2x} - 1\right)$$

$$\left(ye^{x}\right)' = \frac{1}{2}\left(e^{2x} - 1\right)$$

- This equation can be integrated

$$ye^{x} = \frac{1}{4}e^{2x} - \frac{1}{2}x + \mathrm{c}$$

$$y = \frac{1}{4}e^{x} - \frac{1}{2}xe^{-x} + \mathrm{c}e^{-x}$$

- Integrating factor method works for first order non homogeneous differential equations.
- Constant of integration is not included in evaluation of $e^{\int \frac{1}{\tau} dt}$ since integrating factor is multiplied on both sides of equation and so will cancel off. Also note that integrating factor method gives both homogeneous and particular solutions – integration constant is with the homogeneous solution.

$$y' - ay = f(t); \quad f(t) = 1 \quad 0 \le t \le 1;$$
$$= 0 \quad t > 1$$

$$y(0) = 0$$

$$IF = e^{\int -a dt} = e^{-at}$$

$$(y' - ay = 1)\, e^{-at} \equiv \left((ye^{-at})' = e^{-at}\right)$$
$$\Rightarrow ye^{-at} = \left(\frac{e^{-at}}{-a}\right) + C$$
$$\Rightarrow y = \left(\frac{1}{-a}\right) + Ce^{at}$$

Particular integral

Homogeneous solution

$$y = 0 \; at \; t = 0 \Rightarrow C = \frac{1}{a}$$

$$y = \frac{-1}{a} + \frac{1}{a} e^{at} \; solution \; till \; t = 1$$

- The solution at t=1 is boundary condition for the next step

$$y' - ay = 0; y = \frac{-1}{a} + \frac{1}{a} e^{a} \; at \quad t = 0 \;\; for \; t \geq 1;$$

$$y = \hat{C} e^{at} \; ; \hat{C} = \frac{1}{a} - \frac{1}{a} e^{-a}$$

- A discontinuity in the RHS needs special treatment. Here the continuity of the function at the jump point was used

Laplace Transform methods

- Laplace transform methods are often used to get solution to differential equations
- The transform is defined as

$$\mathrm{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt$$

- For this course we will assume that the transform exists

Example

- Let a be any real number, and $f(t) = e^{at}$.

$$F(s) = \int_0^\infty e^{-st} f(t) dt = \int_0^\infty e^{-st} e^{at} dt$$

$$= \int_0^\infty e^{(a-s)t} dt = \lim_{k\to\infty} \int_0^k e^{(a-s)t} dt$$

$$= \lim_{k\to\infty} \left[\frac{1}{a-s} e^{(a-s)t} \right] = -\frac{1}{a-s} = \frac{1}{s-a}$$

Provided that s>a.

Laplace Transform table

- L[F(t)] = F(s)

S. No.	F(t)	F(s)
1	1	$\frac{1}{s}$
2	t^n	$\frac{n!}{s^{n+1}}$
3	e^{at}	$\frac{1}{s-a}$
4	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
5	$e^{at} - e^{bt}$	$\frac{a-b}{(s-a)(s-b)}$
6	$sin(at)$	$\frac{a}{s^2+a^2}$
7	$cos(at)$	$\frac{s}{s^2+a^2}$

Laplace Transform

S. No.	F(t)	F(s)
1	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
2	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
3	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
4	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
5	$\sinh(at)$	$\frac{a}{s^2-a^2}$
6	$\cosh(at)$	$\frac{s}{s^2-a^2}$
7	$\delta(t-a)$	e^{-as}