Practice Problems on Phase Portraits

1) Discuss the nature of the general solution of linear system in a neighborhood of (0,0)

$$\boldsymbol{X}' = \begin{bmatrix} -1 & 5 \\ -1 & 1 \end{bmatrix} \boldsymbol{X}$$

2) Discuss the nature of the general solution of linear system in a neighborhood of (0,0)

$$\boldsymbol{X}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \boldsymbol{X}$$

3) Discuss the nature of the general solution of linear system in a neighborhood of (0,0), and plot the solution that satisfies $\boldsymbol{X}(0) = (1,1)$

$$\boldsymbol{X}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \boldsymbol{X}$$

For each of the following linear systems find general solution, classify the origin of the system . Also produce the phase portrait

4. $x' = 2x + 2y; y' = 5x - y$

5. $x' = x - 2y;\ \ y' = x + y$

6. X' = AX

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$$

1) $X' = \begin{bmatrix} -1 & 5 \\ -1 & 1 \end{bmatrix} X$

$$\begin{vmatrix} -1-\lambda & 5 \\ -1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \begin{array}{l} \lambda^2+4=0 \\ \lambda = \pm 2i \end{array}$$

(i) $\lambda_1 = \pm 2i$

$$\left[\begin{array}{cc|c} -1-2i & 5 & 0 \\ -1 & 1-2i & 0 \end{array}\right] \Rightarrow \begin{array}{l} (-1-2i)x + 5y = 0 \\ 5y = (1+2i)x \\ \text{let } x = 5 \\ y = (1+2i) \end{array}$$

$$\therefore K_1 = \begin{bmatrix} 5 \\ 1+2i \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$B_1 = Re(K_1) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad B_2 = Im(K_1) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$X_1 = \left[\begin{bmatrix} 5 \\ 1 \end{bmatrix} \cos 2t - \begin{bmatrix} 0 \\ 2 \end{bmatrix} \sin 2t\right] e^{0t} = \begin{bmatrix} 5\cos 2t \\ \cos 2t - 2\sin 2t \end{bmatrix}$$

$$X_2 = \left[\begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos 2t + \begin{bmatrix} 5 \\ 1 \end{bmatrix} \sin 2t\right] e^{0t} = \begin{bmatrix} 5\sin 2t \\ 2\cos 2t + \sin 2t \end{bmatrix}$$

$$\therefore X = c_1 \begin{bmatrix} 5\cos 2t \\ \cos 2t - 2\sin 2t \end{bmatrix} + c_2 \begin{bmatrix} 5\sin 2t \\ 2\cos 2t + \sin 2t \end{bmatrix}$$

Since, $x' = -x + 5y$

$y' = -x + y$

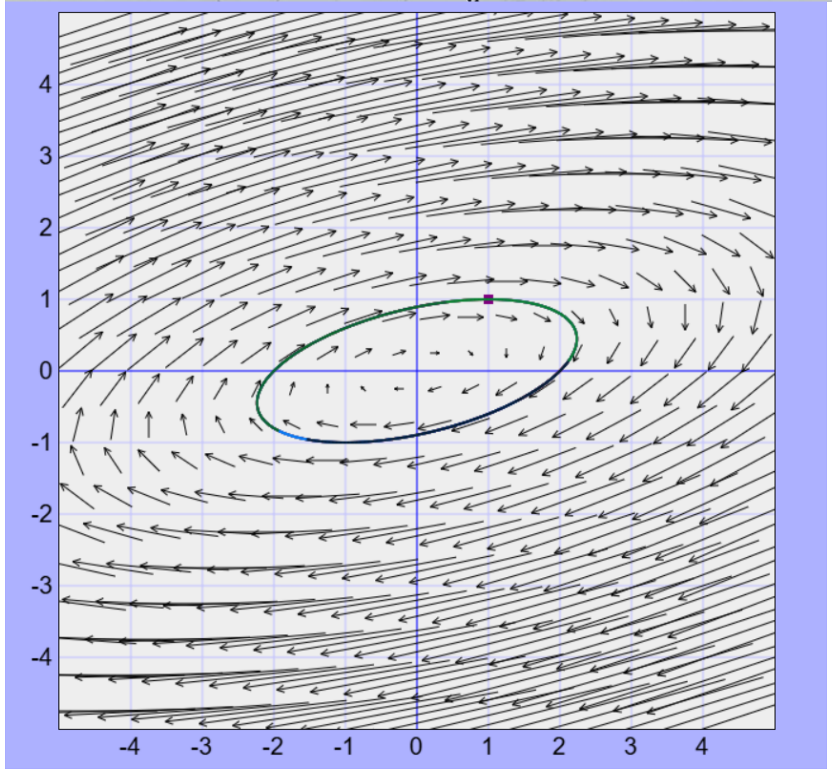
critical point is

$-x + 5y = 0, \quad -x + y = 0.$

$\therefore (0,0)$ is a critical point.

Since Eigen values are purely imaginary, all solutions are periodic with period $P = \pi$, and all solutions are ellipses with center at the origin.

The critical point $(0,0)$ is called center.



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2) $x' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} x$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1 \pm i$$

$\lambda_1 = 1 + i$

$$\left[\begin{array}{cc|c} 1-1-i & -1 & 0 \\ 1 & 1-1-i & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \Rightarrow -xi - y = 0$$

let $x = 1$

$\therefore y = -i$

$$K_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$B_1 = Re(K_1)$, $B_2 = Im(K_1)$

$$x_1 = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin t \right] e^t = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^t$$

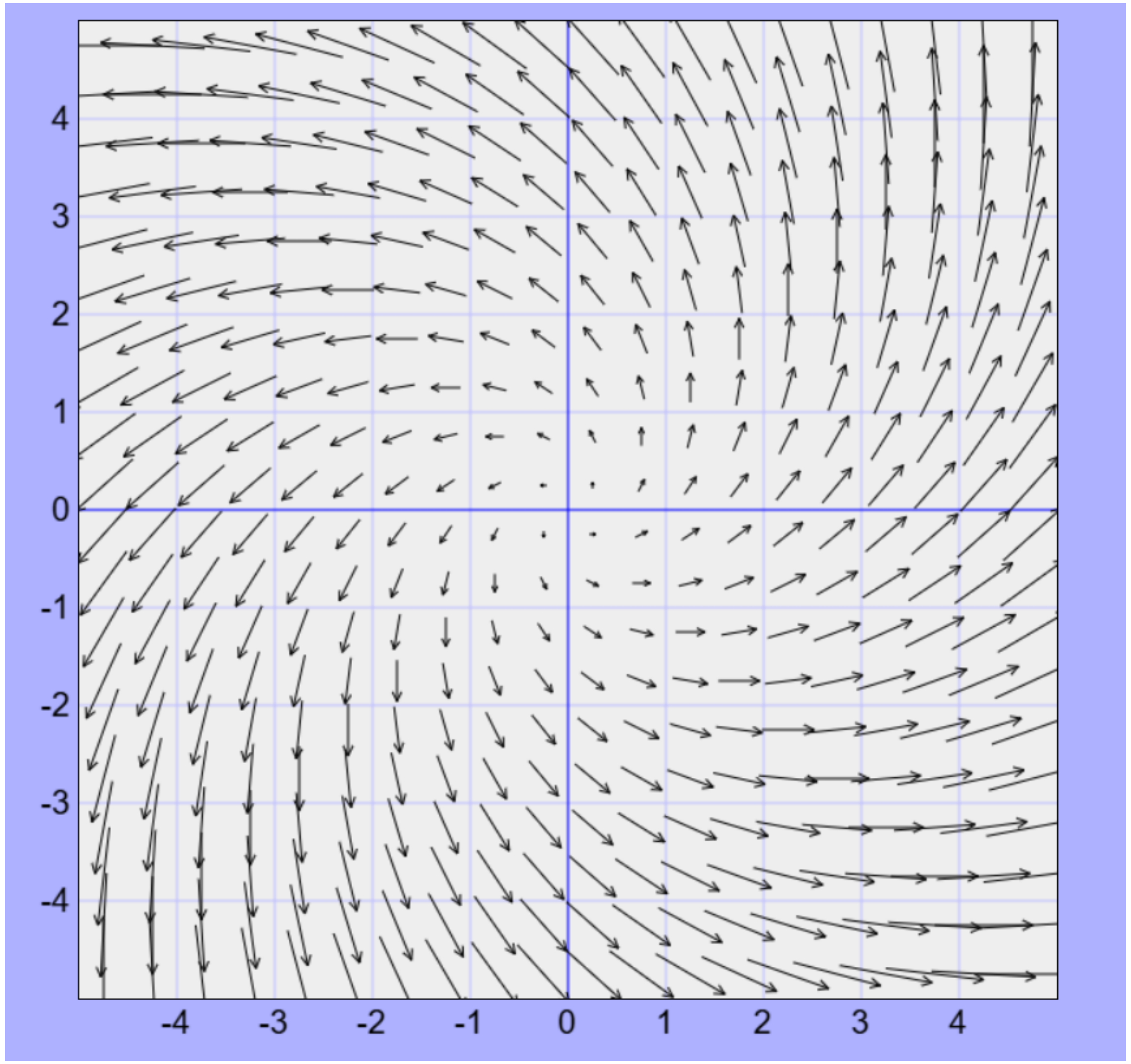
$$x_2 = \left[\begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t \right] e^t = \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} e^t$$

$$x = \left\{ c_1 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} \right\} e^t$$

Since $\alpha > 0$, as t increases $e^t \to \infty$.

$\therefore$ An elliptical like solution is driven farther and farther from the origin, and the critical point is called as "unstable spiral point".

$$\left.\begin{array}{l} x - y = 0 \\ x + y = 0 \end{array}\right\} \Rightarrow (0,0) \text{ is critical point.}$$



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3) $X' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} X$

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm 1$$

(i) $\lambda_1 = 1$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - (3\times R_1)} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

let $y = \alpha$

$\therefore x = \alpha$.

$$K_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(ii) $\lambda_2 = -1$

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

let $y = \alpha$

$\therefore x = \alpha/3$

$$\therefore K_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$X(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$$

given $X(0) = (1,1)$

$\therefore$ $c_1 + c_2 = 1$

$c_1 + 3c_2 = 1$

$\therefore c_1 = 1, \; c_2 = 0$

$$\therefore X(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$$

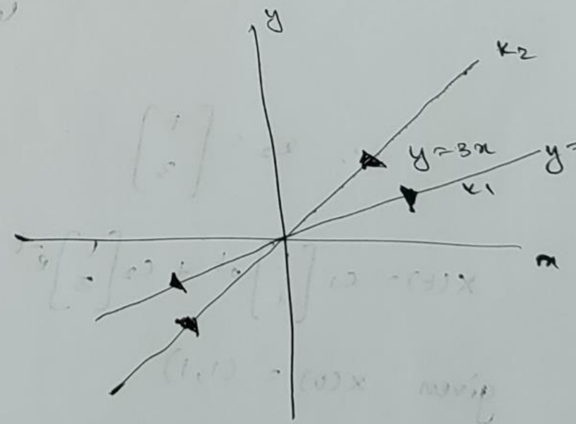
Since, $x' = 2x - y$

$y' = 3x - 2y$

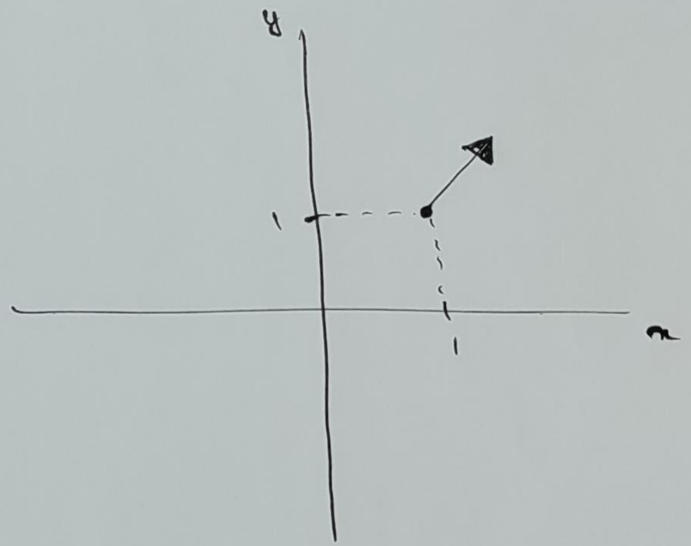
$2x - y = 0, \quad 3x - 2y = 0$

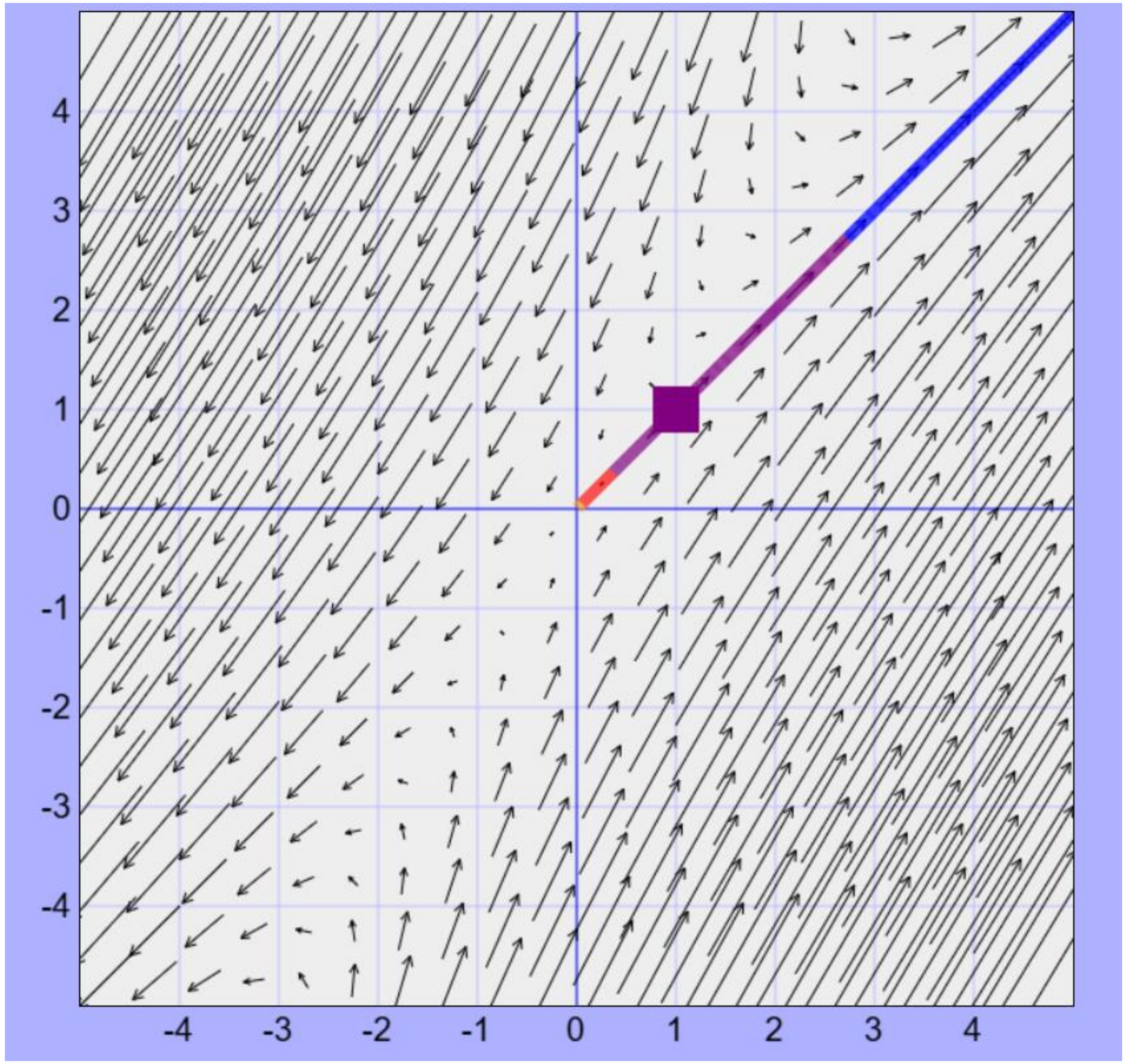
$\therefore (0,0)$ is critical point.

if both e^t, e^{-t} sustains than as t increases e^t becomes unbounded and e^{-t} reaches zero. Therefore particle in the direction of k_1 moves away from origin, particle in the direction of k_2 moves towards the origin.



initial condition is $x(0) = (1,1)$ on $y = x$ line, therefore moves away from origin.





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4. For each of the following linear systems find general solution, classify the origin of the system . Also produce the phase portrait

$$x' = 2x + 2y; y' = 5x - y$$

$$\begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}$$

Finding the eigen values.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 2 \\ 5 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - \lambda - 12 = 0$$

$$(\lambda+3)(\lambda-4) = 0 \qquad \lambda = -3, \lambda = 4$$

Now finding eigen vectors for $\lambda = -3$.

$$(A+3I)v = 0.$$

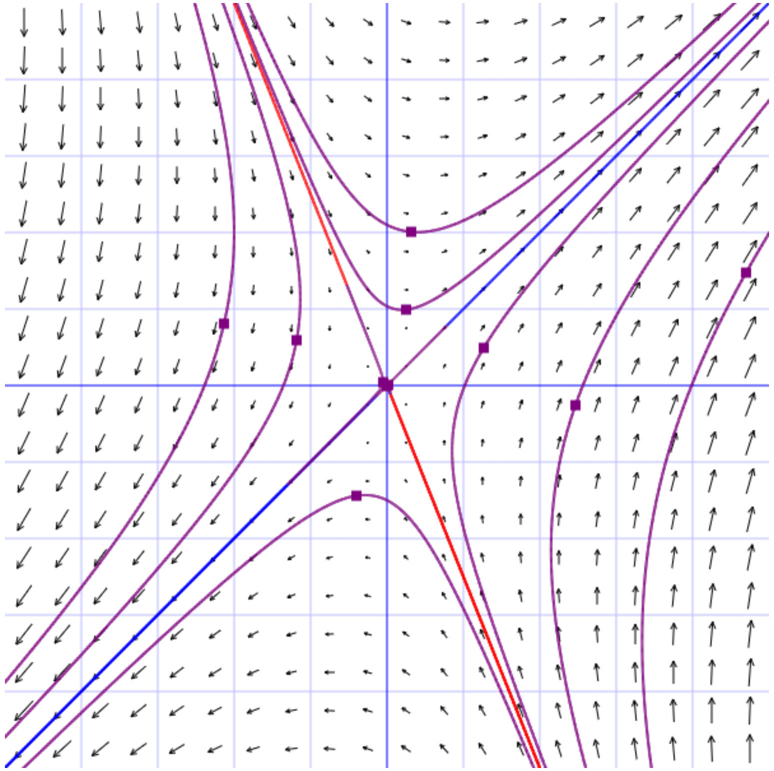
$$\begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 & 2/5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} -2/5 \\ 1 \end{bmatrix}$$

For $\lambda = 4$ $(A-4I)v = 0$

$$\begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X(t) = c_1\begin{pmatrix} -2/5 \\ 1 \end{pmatrix} e^{-3t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

Saddle pt. at origin



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5. $x' = x - 2y; \;\; y' = x + y$

$x' = x - 2y$

$y' = x + y$

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

Finding the eigen values. $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = 0.$$

$\Rightarrow \lambda^2 - 2\lambda + 3$ $\quad \lambda_1 = 1 - \sqrt{2}i$

$\lambda_2 = 1 + \sqrt{2}i$

Eigen vectors: $(A - \lambda_1 I)v = 0.$

$$\begin{bmatrix} \sqrt{2}i & -2 \\ 1 & \sqrt{2}i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \sqrt{2}i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} -\sqrt{2}i \\ 1 \end{bmatrix}$$

For λ_2 $(A-\lambda_2 I)v=0$

$$\begin{bmatrix} -\sqrt{2}i & -2 \\ 1 & -\sqrt{2}i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad v=\begin{bmatrix} \sqrt{2}i \\ 1 \end{bmatrix}$$

let $\lambda=\alpha+i\beta$ be an eigen vector with $\alpha\neq 0$ & eigen vectors $U+iV$

General solution is.

$$X(t)=c_1 e^{\alpha t}\left[U\cos\beta t - V\sin(\beta t)\right]+c_2 e^{\alpha t}\left[U\sin\beta t + V\sin\beta t\right]$$

If $\alpha<0$ $x(t)$ decreases to zero as $t\longrightarrow\infty$

$\rightarrow$ Spiral inward (spiral sink)

If $\alpha>0$ $x(t)$ spiral outwards.

Origin $\longrightarrow$ Spiral source.

$\lambda_1 = 1 - \sqrt{2}i$

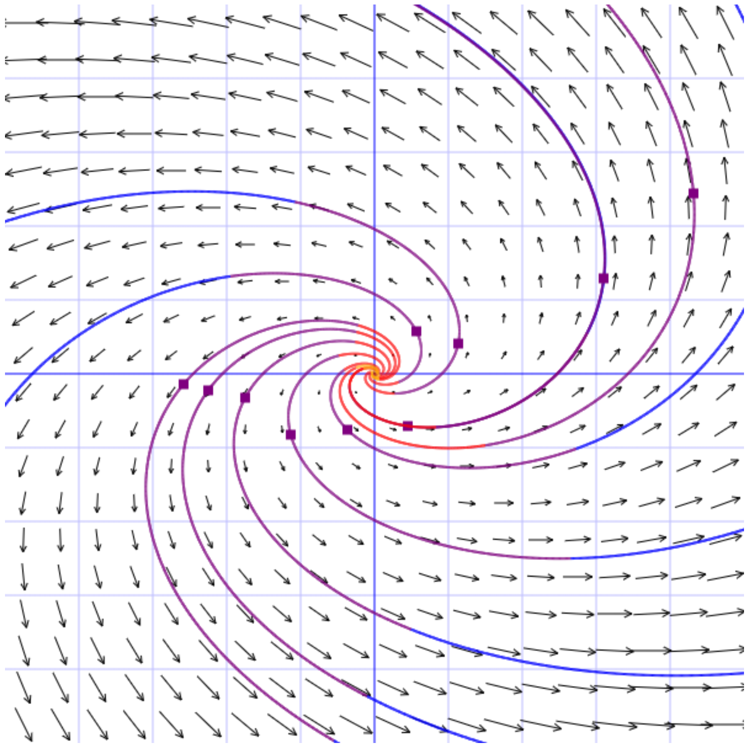
$\lambda_2 = 1 + \sqrt{2}i$

$u = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad V = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}$

$1-$

$$x(t) = c_1 e^t \left[u \cos\sqrt{2}t - V\sin(\sqrt{2}t) \right] + c_2 e^t \left[u \sin\sqrt{2}t + V\cos(\sqrt{2}t) \right]$$

Spiral out from origin



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6. Find the general solution,classify the origin of the system **X’** = **AX** for the given coefficient matrix . Also produce the phase portrait

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$$

Eigen values $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$= -1 + \lambda - \lambda + \lambda^2 + 5 = 0$$

$$\Rightarrow \lambda^2 + 4 = 0$$

$$\lambda = \pm 2i$$

$\lambda = \alpha \pm i\beta$ Eigen vectors $(u + iV)$

$$x(t) = c_1 e^{\alpha t}\left[u \cos\beta t - v \sin\beta t\right] + c_2 e^{\alpha t}\left[u \sin(\beta t) + V \sin(\beta t)\right]$$

$\alpha > 0$ Spirals outward from origin.

$\alpha < 0$ length of $x(t)$ reduces to zero.

$\alpha = 0$ Closed curve with origin.

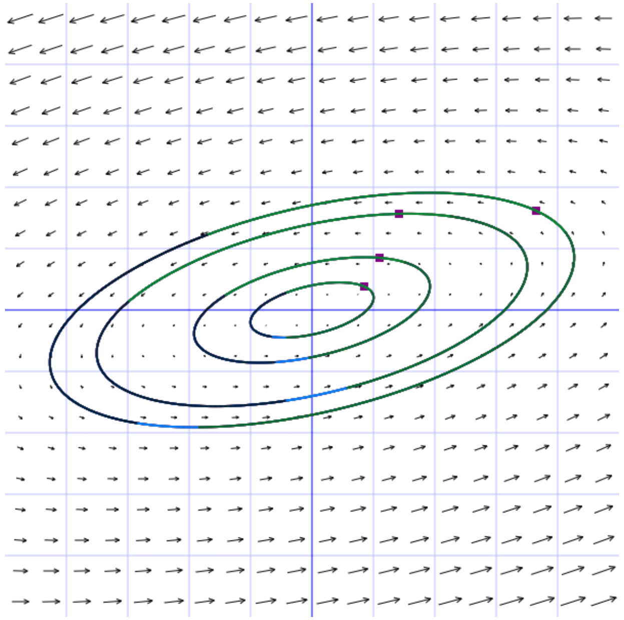
The origin is a center

Eigen vectors : $\begin{pmatrix} 1+2i \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1-2i \\ 1 \end{pmatrix}$

$\lambda=2i$ $\lambda=-2i$

The general solution is

$$X = \begin{bmatrix} (c_1-2c_2)\sin 2t + (2c_1+c_2)\cos 2t \\ c_1\sin(2t) + c_2\cos(2t) \end{bmatrix}$$



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