Abel’s theorem

- Consider the second order differential equation

$$\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_o(t)y = 0$$

- If the solutions for the equation are y1 and y2 then the Wronskian can be expressed as:

$$w = Ce^{-\int a_1 dt}$$

- The Wronskian is therefore a solution to :

$$w' + a_1 w = 0$$

- The definition of Wronskian is used to get the derivative as:

$$w = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_2 y_1'$$

$$w' = (y_1' y_2' + y_1 y_2'') - (y_1' y_2' + y_2 y_1'')$$
$$= y_1 y_2'' - y_2 y_1''$$

- Now substitute in the differential equation for ‘w’ to get:

$$(y_1 y_2'' - y_2 y_1'') + a_1(y_1 y_2' - y_2 y_1')=0$$

- Now substitute for y1’’ and y2’’ to get:

$$y_1(y_2'' + a_1 y_2') - y_2(y_1'' + a_1 y_1') = 0$$

$$\Rightarrow\ y_1(-a_0 y_2) + y_2(-a_0 y_1) = 0$$

- The manipulations show that indeed the equation for Wronskian is verified to be true although it is not a proof

- Now solve for the Wronskian using the given equation:

$$w = ce^{-\int a_1 dt}$$

- The w thus is never zero except if c is zero, so it ensures independent solution for y2 if y1 is known.

$$\frac{w}{y_1} = \frac{y_1 y_2' - y_2 y_1'}{y_1} = y_2' - \frac{y_2 y_1'}{y_1}$$

- Therefore one can solve for y2 if y1 is known by

$$\frac{w}{y_1} = \frac{e^{-\int a_1 dt}}{y_1} = y_2' - \frac{y_2 y_1'}{y_1}$$

- C is taken unity since anyway later the unknown constants are calculated using boundary conditions

- Non homogeneous equations require a particular solution and inspection may not work always. Some methodologies are available.
- One is the method of undetermined coefficients
- Guess the particular solution to be the same form as the RHS(i.e. the non homogeneous part)

$$y_1' + \ a_0 y = g(t)$$

- Suppose g(t) is a polynomial $g_1 t^2$
- Assume the particular solution to be

$$PI = \hat{g}_1 t^2 + \hat{g}_2 t + \hat{g}_3$$

- Substitute in the governing equation

$$2\hat{g}_1 t + \hat{g}_2 + a_0(\hat{g}_1 t^2 + \hat{g}_2 t + \hat{g}_3) = g_1 t^2$$

- Equate coefficients of similar powers

$$a_0\hat{g}_1 = g_1 \Longrightarrow \hat{g}_1 = \frac{g_1}{a_0}$$

$$2\hat{g}_1 + a_0\hat{g}_2 = 0 \Longrightarrow \hat{g}_2 = \frac{-2\hat{g}_1}{a_0}$$

$$\hat{g}_2 + a_0\hat{g}_3 = 0 \Longrightarrow \hat{g}_3 = \frac{-\hat{g}_2}{a_0}$$

- Since g1 is known all coefficients are calculated
- Similarly assume RHS = $g_1 e^{g_2 t}$
- Assume particular solution is of the form $\hat{g}_1 e^{g_2 t}$
- Substitute in the governing equation and obtain $\hat{g}_1$

- Now suppose RHS is $g_1 \cos(g_2 t) \ + g_3 \sin(g_4 t)$
- Guess particular integral to be:

$$\hat{g}_1 \cos(g_2 t) + \hat{g}_{11} \sin(g_2 t) + \hat{g}_3 \sin(g_4 t) + \hat{g}_{31} \cos(g_4 t)$$

- Note that if RHS is either cos or sin function then PI has both sin and cos. Similarly if RHS is t^n then all powers of t less than n need to be included.
- Essentially guess the form of the solution to be same as that of the RHS but with unknown constants. Substitute in the governing equation and solve for unknowns

- Method will not work if the RHS is of the same form as the solution to the homogeneous equation
- Then multiply homogeneous solution be ‘t’ and attempt it as particular solution. If it does not work, then try multiplying by ‘t^2’ etc.

TABLE 2.1 *Functions to Try for $Y_p(x)$ in the Method of Undetermined Coefficients*

$f(x)$	$Y_p(x)$
$P(x)$	$Q(x)$
Ae^{cx}	Re^{cx}
$A\cos(\beta x)$	$C\cos(\beta x) + D\sin(\beta x)$
$A\sin(\beta x)$	$C\cos(\beta x) + D\sin(\beta x)$
$P(x)e^{cx}$	$Q(x)e^{cx}$
$P(x)\cos(\beta x)$	$Q(x)\cos(\beta x) + R(x)\sin(\beta x)$
$P(x)\sin(\beta x)$	$Q(x)\cos(\beta x) + R(x)\sin(\beta x)$
$P(x)e^{cx}\cos(\beta x)$	$Q(x)e^{cx}\cos(\beta x) + R(x)e^{cx}\sin(\beta x)$
$P(x)e^{cx}\sin(\beta x)$	$Q(x)e^{cx}\cos(\beta x) + R(x)e^{cx}\sin(\beta x)$

- Next method is ‘variation of parameters’. This method was used earlier to tackle the multiple roots case. The particular solution now has two unknowns.
- Hypothesize that the solution is made up of the homogenous solutions combined in the form below where u and v are unknowns

$$y_p = uy_1 + vy_2$$

- Start the process of substituting in the main equation

$$y_p' = u'y_1 + uy_1' + v'y_2 + vy_2'$$

- Arbitrarily state: $u'y_1 + v'y_2 = 0$ …..(1)
- Then $y_p' = uy_1' + vy_2'$
- Now substitute in governing equation

$$y_p'' + a_1\, y_p' + a_o y = g(t)$$

$$uy_1'' + u'y_1' + v'y_2' + vy_2'' + a_1(uy_1' + vy_2') + a_0(uy_1 + vy_2) = g(t)$$

- Group terms to get

$$u\left(\cancelto{0}{y_1''+a_1\ y_1' + a_0 y_1}\right) + v\left(\cancelto{0}{y_2'' + a_1\ y_p' + a_0 y_2}\right) + u'y_1' + v'y_2' = g(t)$$

$$\Rightarrow u'\ y_1' + v'y_2' = g(t) \ldots\ldots\ldots\ldots..(2)$$

- Equations (1) and (2) are solved to get:

$$u' = \frac{-y_2 g}{y_1 y_2' - y_2 y_1'}\ ,\ v' = \frac{y_1 g}{y_1 y_2' - y_2 y_1'}$$

- The above can be integrated to get u and v. Since y1 and y2 are independent and the denominator is the Wronskian it is never zero.

- The variation of parameters is in general more powerful than the method of undetermined coefficients since it will give a solution all the time.
- Only must be able to perform the integration which may not be easy

- Next let us relax the constant coefficient approximation – solutions can be obtained sometimes for special cases.
- Consider the Euler Cauchy equation:

$$x^2y'' + xy' - y = 0$$

- Try x^r as the solution. Substitute in diff. eq. to get:

$$r(r-1) + r - 1 = 0 \quad \text{solve for } r$$

$$y = c_1x^{r_1} + c_2x^{r_2}$$

- Suppose the roots are complex: $a \pm ib$. Use the same algebra as earlier to get

$$y = x^a(c_1\cos(b\ln x) + c_2\sin(b\ln x))$$

- To get the above note that

$$x^r = (e^{\ln x})^r = e^{(a+ib)\ln x}$$

- When there are repeated roots use:

$$y = c_1 x^r + c_2 (\ln x) x^r$$

- Often the coefficients may not be in this form which is sometimes called ‘equidimensional’.
- A more general solution methodology based on the Taylor series is used when the substitution used in the above does not help.

Example

$$y'' - 2y' - 8y = 10e^{-x} + 8e^{2x}\,; \quad y(0) = 1, y(0)' = 4$$

Solution:

We first need to find two linearly independent solutions of the associated homogeneous equation

$$y'' - 2y' - 8y = 0$$

The characteristic equation is

$$\lambda^2 - 2\lambda - 8 = 0$$

$$(\lambda - 4)(\lambda + 2) = 0$$

$$\lambda_1 = 4\,,\ \lambda_2 = -2$$

Hence, the solution of the homogeneous equation is

$$y_h = C_1 e^{-2x} + C_2 e^{4x}$$

For particular solution y_p of the nonhomogeneous equation.

$$f(x) = 10e^{-x} + 8e^{2x}$$

The particular solution has the form

$$y_p = Ae^{-x} + Be^{2x}$$

We plug this solution into our initial nonhomogeneous equation to get

$$(Ae^{-x} + Be^{2x})'' - 2(Ae^{-x} + Be^{2x})' - 8(Ae^{-x} + Be^{2x}) = 10e^{-x} + 8e^{2x}$$

$$Ae^{-x} + 4Be^{2x} + 2Ae^{-x} - 4Be^{2x} - 8Ae^{-x} - 8\mathrm{B}e^{-x} = 10e^{-x} + 8e^{2x}$$

$$(-5A)e^{-x} - 8Be^{2x} = 10e^{-x} + 8e^{2x}$$

$$A = -2$$

$$B = -1$$

$$y_p = -2e^{-x} - e^{2x}$$

The general solution is $y = y_h + y_p$

$$y = C_1e^{-2x} + \ C_2e^{4x} - 2e^{-x} - e^{2x}$$

Now use the initial conditions to solve for the constants.

$$y(0) = 1, \quad y(0)' = 4$$

$$y(0) = 1 \ \Rightarrow \ C_1 + \ C_2 - 2 - 1 = 1 \ \Rightarrow \ C_1 + \ C_2 = 4 \quad \ldots\ldots(i)$$

$$y' = -2C_1e^{-2x} + 4C_2e^{4x} + 2e^{-x} - 2e^{2x}$$

$$y(0)' = 4 \quad \Rightarrow \quad -2C_1 + 4C_2 = 4 \quad \ldots\ldots(ii)$$

After solving equation (i) and (ii) we get

$$C_1 = 2 \;\; ; \; C_2 = 2$$

Therefore, the solution is

$$y = 2e^{-2x} + 2e^{4x} - 2e^{-x} - e^{2x}$$

Section 2.3

4. $y'' - 2y' - 3y = 2sin^2(x)$

Solution:

We first need to find two linearly independent solutions of the associated homogeneous equation

$$y'' - 2y' - 3y = 0$$

The characteristic equation is

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda^2 - 3\lambda + \lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\lambda_1 = 3 \quad , \qquad \lambda_2 = -1$$

Hence, the solution of the homogeneous equation is

$$y_H = c_1 e^{3x} + c_2 e^{-x}$$

We can take $y_1 = e^{3x}$ and $y_2 = e^{-x}$ as two solutions

To use the variation of parameters approach, first compute the Wronskian $W(x) \neq 0$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = y_1 y_2{}' - y_2 y_1{}'$$

$$W(x) = \begin{vmatrix} e^{3x} & e^{-x} \\ 3e^{3x} & -e^{-x} \end{vmatrix} = -e^{3x}e^{-x} - 3e^{3x}e^{-x} = -4e^{2x} \neq 0$$

Thus, y_1 and y_2 are linearly independent

The general solution is given by

$$y = c_1 e^{3x} + c_2 e^{-x} + Y_P$$

Where, $\qquad Y_P = u_1 y_1 + u_2 y_2$

$$u_1(x) = -\int \frac{y_2(x) f(x)}{W(x)} dx \qquad u_2(x) = \int \frac{y_1(x) f(x)}{W(x)} dx$$

$$u_1(x) = -\int \frac{e^{-x}.\, 2sin^2(x)}{-4e^{2x}} dx$$

$$= \int \frac{e^{-3x} sin^2(x)}{2} dx$$

$$u_1(x) = \frac{1}{2}\left(-\frac{e^{-3x} sin^2(x)}{3} - \frac{1}{39} e^{-3x}(3\sin(2x) + 2\cos(2x)\right) + C$$

Now compute the other integral

$$u_2(x) = -\int \frac{e^x sin^2(x)}{2} dx$$

$$u_2(x) = \frac{1}{2}(e^x sin^2(x) + \frac{1}{5}e^x(2\cos(2x) - \sin(2x))) + D$$

The general solution is

$$y = c_1 e^{3x} + c_2 e^{-x} + Y_P$$

$$= c_1 e^{3x} + c_2 e^{-x} + (\frac{1}{2}(-\frac{e^{-3x} sin^2(x)}{3} - \frac{1}{39}e^{-3x}(3\sin(2x) + 2\cos(2x)) + C)\, e^{3x} + (\frac{1}{2}(e^x sin^2(x) + \frac{1}{5}e^x(2\cos(2x) - \sin(2x))) + D)e^{-x}$$

$$y = (c_1 + C)e^{3x} + (c_2 + D)e^{-x} - \frac{2sin^2(x)}{3} + \frac{4}{65}\sin(2x) - \frac{44}{195}\cos(2x)$$

$$y = C_1 e^{3x} + C_2 e^{-x} - \frac{2}{3}\left(\frac{1 - \cos(2x)}{2}\right) + \frac{4}{65}\sin(2x) - \frac{44}{195}\cos(2x)$$

$$y = C_1 e^{3x} + C_2 e^{-x} - \frac{1}{3} + \frac{4}{65}\sin(2x) + \frac{7}{65}\cos(2x)$$

$\boldsymbol{Section\ 2.5}$

14. $x^2y'' + 25xy' + 144y = 0\,; \quad y(1) = -4, y(1)' = 0$

Solution: Homo. Equn. $x^2y'' + 25xy' + 144y = 0$

Use $y = Cx^r \quad \Rightarrow r(r-1) + 25r + 144 = 0$

The characteristic equation is

$$r^2 + 24r + 144 = 0$$

$$(r+12)(r+12) = 0$$

$$r_1\,,\ r_2 = -12$$

λ has multiplicity 2, hence, the solution of the homogeneous equn is $y(x) = C_1 x^{-12} + \ C_2 x^{-12} lnx$

We now use the initial conditions to solve for constants.

$$y(1) = -4, \qquad y(1)' = 0$$

$$y(1) = -4 \quad \Rightarrow \qquad C_1 = -4 \quad \ldots \ldots (i)$$

$$y(x)' = -12 C_1 x^{-13} - 12\, C_2 x^{-13} lnx + C_2 x^{-13}$$

$$y(1)' = 0 \Rightarrow \quad -12 C_1 + C_2 = 0 \quad \ldots \ldots (ii)$$

Solving (i) and (ii) we get $C_1 = -4, C_2 = -48$

$$y(x) = -4x^{-12} - 48x^{-12} lnx$$