Discrete Mathematics: (10-15 Marks)

① No. of binary strings of length 2.

Each position can be filled in 2 ways.

$$2^2 = 4$$

0 — 0
0 — 1
1 — 0
1 — 1

00
01
10
11

② No. of binary strings of length 100: 2^{100}

1 2 . . . 100 ; each position 2 ways

Result: $\boxed{\text{No. of binary string of length } n = 2^n}$

E.g. No. B.S. of length 8 which start with 0.

1	2	3	4	5	6	7	8
0							
1	2	2	2	2	2	2	2

$= 2^7$

E.g. IPv4 (32-bit address)

class A: | 0 (net id) | host id | | |

class B: | 10 (net id) | | host id | |

class C: | 110 (net id) | | | host id |

net ids	host ids
$2^7 - 2$	$2^{24} - 2$
2^{14}	$2^{16} - 2$
2^{21}	$2^8 - 2$

Eg. No. of positive integers of length 4.
(No. of 4 digit +ve integers)

9 100 10 10 $= 9\times10^3 = 9000$

[lb, ub]
ub − lb + 1
[1000, 9999]
9999 − 1000 + 1
= 9000

Eg. ③ No. of 4-digit even nos. (+ve integers)

9 10 10 5 → 0, 2, 4, 6, 8 $= 4500$

Eg. ④ No. of 4-digit odd nos.

9 10 10 5 → 1, 3, 5, 7, 9 $= 4500$

OR : Total − even ⇒ 9000 − 4500
= 4500

Q. No. of 4 digit even no. having all 4 digits distinct.

(a) 2240 (b) 2296 (c) 2620 (d) 4536

~~9×9×8×7 = 81×56 = 4536~~ ⇒ 9 9 7

No. of 4 digit integers with distinct digits = 9×9×8×7
= 4536

No. of 4-digit odd ints. have all digits distinct

= 8 8 7 5 $= 2240$

Even = Total − odd = 4536 − 2240
= 2296

Direct:

0
9, 8, 7, 1

or

2
8, 8, 7, 1

or 4

or 6

or 8

2, 4, 6, 8
8 8 7 4

⇒ ~~9.8.7~~ 9.8.7 + 8.8.7.4
= 2296

10000
99999

Eg. No. of non-negative integers $< 10^5 =$

10 10 10 10 10 $= 10^5$

By allowing leading zeroes we get 4-digit, 3 digit, 2 digit & 1-digit nos

Q. No. of non-negative integers $< 10^5$; contain the digit 1.

Contain 1 = Total − (do not contain 1)
$= 10^5 - 9^5$

Q. 7 letter palindromes using english alphabets.

26 26 26 26 1 1 1 $= 26^4$

Eg. No. of 5 digit palindromes.

9 10 10 1 1 $= 9\times10^2 = 900$

(a) How many even.

4 10 10 1 1 $= 400$

(b) odd

5 10 10 1 1 $= 500$

Eg. No. of 5-digit no's. with repetition but not consecutive repetitive.

$\boxed{9\ 9\ 9\ 9\ 9} = 9^5$

Eg. No. of 5-letter words with repetition but not consecutive repetition. $\Rightarrow 26 \times 25^4$

GATE 2016: No. of 4-digit nos. having their digits in non-decreasing order (from left to right) constructed by using the digits belonging to the set {1,2,3}?

~~(a) 36~~ ~~(b) 27~~ Ans. 15

Non-Decreasing: $\boxed{3\ |\ \ |\ \ |\ \ }$ (1 … 3)

3 3 3 3 — 1

$2 <$ 2, 3 … — 4

1 < … — 10

$= 15$

Q'n' couples are invited to a party with the condition that
GATE: every husband should be accompanied by his wife. However wife ~~may~~ need not be accompanied by her husband. What is the no. of different gatherings possible?

(a) ${}^{2n}C_n \cdot 2^n$ (b) 3^n (c) $\frac{(2n)!}{2^n}$ (d) ${}^{2n}C_n$

Soln:

$\boxed{1\ |\ 2\ |\ -\ \text{———}\ |\ n}$

3 3 … 3

$= 3^n$

H	W
✓	✓
×	×
×	✓
✓	×

Definition: Subsequence of a string is formed by deleting 0 or more symbols.

Def.: Substring of a string is formed by deleting 0 or more symbols from the beginning or end or both.

COMPUTER
MUTE → Subsequence
MPUTE → Substring

Result: No. of subsequences of a string of length n.

$\boxed{1\ |\ 2\ |\ -\ -\ -\ |\ n}$ (2 2 … 2)

Each symbol has 2 choices Present or Absent.

$\boxed{= 2^n}$

Result: No. of substrings of a string of length n =

(1) length 1 = n
(2) length 2 = n-1
length n = (n-(n-1))

$$\boxed{\text{Total} = 1+2+3 \ldots + n = \frac{n(n+1)}{2}}$$

Results:

① $1+2+3+\dots+n = \frac{n(n+1)}{2}$

$$\sum_{i=1}^{n} i = \Sigma n = \frac{n(n+1)}{2}$$

② $1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

③ $1^3+2^3+3^3+\dots+n^3 = \frac{n^2(n+1)^2}{4}$

④ $\Sigma 1 \cdot \sum_{i=1}^{n} i = n$

Arithmetic Sequence: [a, a+d, a+2d, ...]

* first term = a
* Common difference = d
* n^{th} term (t_n) = $a+(n-1)d$
* Sum (S_n) = sum of first n terms

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$\boxed{S_n = \frac{n}{2}[t_1 + \boxed{t_n}]}$$

→ last term or n^{th} term

Geometric Sequence: $[a, ar, ar^2, \dots]$

* first term : a
* common ratio = r

n^{th} term $(t_n) = ar^{n-1}$

Sum to first n terms

$$S_n = \begin{cases} a\frac{(1-r^n)}{1-r} & ; \quad r<1 \\ \frac{a(r^n-1)}{r-1} & ; \quad r>1 \\ n.a & ; \quad r=1 \end{cases}$$

* Sum to infinite terms:

$a+ar+ar^2+\dots$

$$\boxed{= \frac{a}{1-r} \quad [-1<r<1]}$$

Q. 4, 5, 6, 8, 9, 12, 14, 16, 27, 28, 34, 35 of workbook.

Q E.g. No. of ~~divisors of 100~~ positive integers ≤100 divisible by 2

$\left\lfloor \frac{100}{x} \right\rfloor = 50$

Floor x

$\lfloor x \rfloor$ = ~~smallest~~ ~~is~~ greatest integer less than or equal to x.

Ceiling of x:

$\lceil x \rceil$ = Smallest integer greater than or equal to x.

E.g. $\lfloor 3.5 \rfloor = 3$ $\quad \lceil 2.3 \rceil = 3$

$\lfloor 4 \rfloor = 4$ $\quad \lceil 3 \rceil = 3$

E.g. No. of +ve integers ≤100 and divisible by 3.

$\left\lfloor \frac{100}{3} \right\rfloor = 33$

E.g. No. of positive integers ~~less than~~ ≤100, divisible by 2 & 3.

$\left\lfloor \frac{100}{LCM\ of(2,3)} \right\rfloor \Rightarrow \frac{100}{6} = 16$

E.g. No. of positive integers ≤100, divisible by 4 & 6.

$\left\lfloor \frac{100}{LCM(4,6)} \right\rfloor \Rightarrow \frac{100}{12} = 8$

Prime factorization:

$30 = 2 \times 3 \times 5$

$40 = 2 \times 2 \times 2 \times 5 = 2^3 \times 5$

Power of 2 in prime factorization of 40 = 3

GATE Q. What is the exponent of 11 in the prime factorization of 300!. $\boxed{29}$

$$\left\lfloor \frac{300}{11} \right\rfloor + \left\lfloor \frac{300}{11^2} \right\rfloor = 27+2 = 29$$

Eg Exponent of 13 in prime factorization of 300!.

$$\left\lfloor \frac{300}{13} \right\rfloor + \left\lfloor \frac{300}{13^2} \right\rfloor = 23+1 = 24$$

Eg. Exponent of 3 in prime factorization of 300!

$$\left\lfloor \frac{300}{3} \right\rfloor + \left\lfloor \frac{300}{3^2} \right\rfloor + \left\lfloor \frac{300}{3^3} \right\rfloor + \left\lfloor \frac{300}{3^4} \right\rfloor + \left\lfloor \frac{300}{3^5} \right\rfloor = 100+33+11+3+1$$

Eg. No. of divisors of 100.

$100 = 2^2 \times 5^2$

No. of divisors $= (2+1)\times(2+1)$
$= 9$

2	100
2	50
5	25
	5

Eg No. of divisors of 300.

$300 = 2^2 \times 3 \times 5^2$

No. of divisors $= (2+1) \times (1+1) \times (2+1)$
$= 3 \times 2 \times 3$
$= 18$

GATE Q No. of divisors of: 2010 & 2014

2010

$2010 = 2 \times 5 \times 3 \times 67$
$= (1+1) \times (1+1) \times (1+1) \times (1+1)$
$= 2 \times 2 \times 2 \times 2$
$= 16$

2014

$= 2 \times 19 \times 53$
$= (1+1) \times (1+1) \times (1+1)$
$= 8$

Divisors of 100:

$2^a 5^b \mid 100$ iff $0 \le a \le 2$, $0 \le b \le 2$

$2 \times 5 \mid 100$
$2^2 \times 5^0 \mid 100$
$2^0 \times 5^2 \mid 100$
$2^3 \times 5 \mid 100$

a b
$3 \cdot 3 = 9$

Result. If 'p' is prime no. and 'a' is any integer not divisible by 'p'. Then $a^{p-1} \bmod p \equiv 1$. [Fermat's Theorem]

[Fermat's Little Theorem]

E.g. ① 7 is prime
2 is not divisible by 7.

$a^{7-1} \bmod 7 = 1$

② $2^{340} \bmod 11$
$= 1$

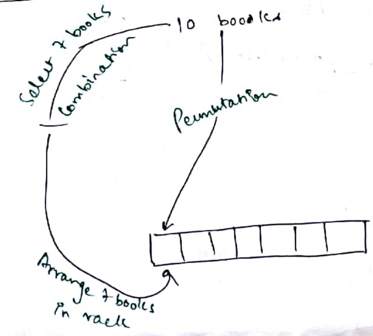
2 is not divisible by 11.
$2^{11-1} \bmod 11 = 1$
$\boxed{2^{10} \bmod 11 = 1}$
$(2^{10})^{34} \bmod 11 = (1)^{34}$
$= 1$

③ $3^{302} \bmod 5$ | $3^4 \bmod 5 = 1$
$= \cancel{0}\ 0$ ✗

$3^{302} = 3^{300} \cdot 3^2$
$= (3^4)^{75} \cdot 3^2 \bmod 5$
$= (1)^{75} \cdot 3^2 \bmod 5$
$= 4$

GATE ④ $13^{99} \bmod 17$
$\Rightarrow 13^{16} \bmod 17 = 1$
$(13^{16})^6 \cdot 13^3 \bmod 17$
$= (1)^6 \cdot 13^3 \bmod 17$
$= 4$

E.g.



Permutations & Combinations:

① 10 people attend a party and ~~10~~ ① shake hands with each other. No. of handshakes.

$A \to B$

= No. of 2-combination of 10 people

$= {}^{10}C_2 = 45$

and ② Kick each other, No. of kicks.

$A \rightleftarrows B$

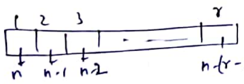
= No. of 2-permutation of 10 people

$= {}^{10}P_2$

Permutations without repetitions:

${}^nP_r$ = r-permutation of n-objects without repetition.

n-objects :

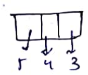


$${}^nP_r = n(n-1).(n-2)\dots.(n-r+1)$$

$$= \frac{n(n-1).(n-2)\dots(n-r+1).(n-r)!}{(n-r)!}$$

$$\boxed{{}^nP_r = \frac{n!}{(n-r)!}}$$

Ex. 3-permutation of 5



$$= \frac{5.4.3.2!}{2!}$$

$$= \frac{5!}{2!}$$

① $${}^nP_r = \frac{n!}{(n-r)!}$$

② ${}^nP_n$ = No. of permutation of n-objects = $n!$

③ No. of circular permutations of n-objects = $(n-1)!$

E.g. A, B, C

Linear Permutation :

ABC
ACB
BAC
BCA
CAB
CBA

Idea: Fix one object and arrange remaining objects linearly.

A
x
C B

A
x
B C

So ⇒ for 3 objects $= (n-1)!$
$= 2! = 2$

$\boxed{\text{for } n\text{-objects} = (n-1)!}$

OR

n Linear Permutation give → 1 circular permutation

$n!$ Linear permutation → ?

$\boxed{\text{No. of circular permutation} = n! \times \frac{1}{n} = (n-1)!}$

Q① No. of ways 10 people can be arranged in a line so that ① certain pair is always together.
② Certain pair never together.

① certain pair

$\boxed{AB}$

1 unit + 8 units = 9 units

9 units $\xrightarrow{\text{in line}}$ 9!

for each of these arrangements.

$AB \xrightarrow{\text{arranged}} 2!$

$\boxed{\boxed{AB} = 2! \times 9!}$

② $\cancel{\boxed{AB}}$ = Total $-$ $\boxed{AB}$

$= 10! - 2! \times 9!$

$= 9!(10-2)$

$\boxed{\cancel{\boxed{AB}} = 8 \times 9!}$

NOTE: No. of ways n-people arranged so that:

① $\boxed{AB} = 2!.(n-1)!$

② $\cancel{\boxed{AB}} = (n-2).(n-1)!$

Q.② No. of ways 10 people can be arranged in a circle so that ⓐ Certain pair always together
ⓑ Certain pair never together.

Solⁿ:

10 people $\xrightarrow{\text{Linear}}$ 10! ways.

10 people $\xrightarrow{\text{In circle}}$ 9! ways.

① Certain pair $\boxed{AB}$

1 unit + 8 units = 9 units

9 units $\xrightarrow{\text{In circle}}$ 8!

For each of these arrangements

AB $\xrightarrow{\text{arranged}}$ 2!

$\boxed{\boxed{AB} = 2! \times 8!}$

② $\cancel{\boxed{AB}}$ = Total $-$ $\boxed{AB}$

$= 9! - 2! \times 8!$

$= 8!(9-2)$

$\boxed{\cancel{\boxed{AB}} = 8! \times 7}$

NOTE No. of ways n-people arranged so that:

① $\boxed{AB} = 2!(n-2)!$

② $\cancel{\boxed{AB}} = (n-3)(n-2)!$

Q③. 5 men, 4 women arranged.

No. of arrangements such that

① All ~~members~~ women together

② No two women together

③ Men and women arranged alternatively.

ⓐ Total No. of ways 5M & 4W $\xrightarrow{\text{in line}}$ 9!

① $\boxed{4W}$ $\boxed{5M}$

1 unit + 5 units $\longrightarrow$ 6 units

6 units $\xrightarrow{\text{in line}}$ 6!

For each of these arrangements:

4W $\xrightarrow{\text{arranged}}$ 4! ways

No. of way $\boxed{4W}$ together $\longrightarrow$ $4! \times 6!$

② 5M $\xrightarrow{\text{in line}}$ 5!

(No restriction)

(One such arrangement):

$\boxed{\times M_1 \times M_2 \times M_3 \times M_4 \times M_5 \times}$

4W in 6 gaps = ${}^6P_4$

No two women together = $5! \times {}^6P_4$

③ 5M $\xrightarrow{\text{in line}}$ 5!

One such arrangement:

$\boxed{M_1 \times M_2 \times M_3 \times M_4 \times M_5}$

4W in 4 gaps = ${}^4P_4 = 4!$

No. of ways = $5! \times 4!$

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Q(4). No. of ways 5 M and 4 W can be arranged in a circle:
So that: (a) 4 W together
(b) No 2 W together
(c) Men and women alternatively arranged.

(a) Total no. of ways 5 M & 4 W $\xrightarrow{\text{in circle}}$ 8!

$\boxed{4\ W}$ together

1 unit + 5 unit = 6 units

6 units $\xrightarrow{\text{in circle}}$ 5!

For each of these arrangement

4W $\xrightarrow{\text{arranged}}$ 4!

$\boxed{\boxed{4W} = 5! \times 4!}$

(b) 5M $\xrightarrow{\text{circle}}$ 4!
(no restriction)

One such arrangement

$M_5 \times M_1 \times M_2 \times M_3 \times M_4$ 4 W in 5 gaps = $^5P_4$

No two women together = $4! * {^5P_4}$

(3) 5M

M_5 M_1 M_2 M_3 M_4

Not possible

Since, if we arrange 4W in 5 gaps. Two men are still together.

Combinations without repetitions: [^nC_r (or) $\binom{n}{r}$ (or) $C(n,r)$]

^nC_r = r combinations of n-objects without repetitions.

(1) $^nP_r = \underbrace{^nC_r}_{\text{selection of r-objects}} * \underbrace{r!}_{\text{arrangement of r-objects}}$

(2) $^nC_r = \dfrac{^nP_r}{r!}$

(3) $\boxed{^nC_r = \dfrac{n!}{r!\,(n-r)!}}$

(4) $^nC_0 = {^nC_n} = 1$

(5) $^nC_r = {^nC_{n-r}}$ [one to one correspondence]

r objects selected ↑ ↑ n-r objects selected for rejection

(6) Recurrence Relations: (Problem in terms of subproblems)
↳ Not solving it but expressing problem in terms of subproblems.

$\boxed{^nC_r = {^{n-1}C_r} + {^{n-1}C_{r-1}}}$

^nC_r:
- first object not selected → r-selection from remaining (n-1) objects. $\longrightarrow {^{n-1}C_r}$

OR

- first object selected → r-1 selection from remaining (n-1) objects $\longrightarrow {^{n-1}C_{r-1}}$

Q. ① No. of 3 member committee can be selected from 4M and 5W. So that the committee has:

ⓐ Exactly 2 women

ⓑ Atleast 2 women

ⓐ ~~Exactly 2 W~~ Total = 3 members from 4M & 5W

$= {}^9C_3$

M	W
3	0
2	1
$\boxed{1}$	$\boxed{2}$
0	3

$$\begin{aligned} &{}^4C_3 * {}^5C_0 \\ +\ &{}^4C_2 * {}^5C_1 \\ +\ &{}^4C_1 * {}^5C_2 \\ +\ &{}^4C_0 * {}^5C_3 \\ \hline &{}^9C_3 \end{aligned}$$

Exactly 2W $= {}^4C_1 \times {}^5C_2$

ⓑ Atleast 2W:

M	W
1	2
0	3

$= {}^4C_1 * {}^5C_2 + {}^4C_0 * {}^5C_3$

ⓒ Atleast 1W = Total − atmost 0

$= {}^9C_3$ − No W

$= {}^9C_3 - {}^5C_0 \times {}^4C_3$

Atleast K

= Total − Atmost (k−1)

Atleast → ≤

At most → ≥

Q.2 & Q.2:

Eg. 8 letter words formed from 3 vowels and 5 consonants without repetitions of letters.

① No. of such words can be formed.

② " " " " " contain a

③ " " " " " start with a.

① $^5C_3 * {}^{21}C_5 * 8!$

Select 3 vowels, Select 5 consonants, arrange 8 letters.

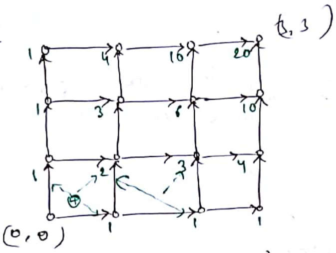
② $\underbrace{1 * {}^4C_2 * {}^{21}C_5}_{\text{8 letters selected}} * 8!$

1: a selected; $^4C_2$: 2 vowels from 4 vowels; $^{21}C_5$: Select 5 Consonants

③ | a | | | | | | | |

$^4C_2 * {}^{21}C_5 * 7!$

Q.3)



No. of paths from (0,0) to (3,3) in such a way that at each step, we can move one unit right (or) one unit up only.

In a path,

No. of horizontal moves = 3

No. of vertical moves = 3

Total moves = 6

No. of paths $= {}^6C_3 \cdot {}^3C_3 = {}^6C_3 = \boxed{20}$

In a path,

No. of horizontal moves $= x_2 - x_1$

No. of vertical moves $= y_2 - y_1$

Total moves $= (x_2 - x_1) + (y_2 - y_1)$

No. of paths = Selecting H moves or V-moves from total moves

$$= {}^{(x_2-x_1)+(y_2-y_1)}C_{(x_2-x_1) \text{ or } (y_2-y_1)}$$

Increasing Path Problem:

→ No. of paths (x_1, y_1) to (x_2, y_2) in cartesian plane such that at each step, we can move one unit up or one unit right only.

E.g. No. of increasing path from:

1. $(0,0)$ to $(10,10)$ — 1. ${}^{20}C_{10}$
2. $(0,0)$ to $(4,4)$ — 2. ${}^{8}C_{4}$
3. $(5,4)$ to $(10,10)$ — 3. ${}^{11}C_{5}$ or ${}^{11}C_{6}$

Q GATE: Suppose a robot is placed in cartesian plane. At each step it is allowed to move either one unit up or one unit right.

1. No. of distinct paths from $(0,0)$ to $(10,10)$
2. Suppose that robot is not allowed to traverse the line segment from $(4,4)$ to $(5,4)$ then no. of paths from $(0,0)$ to $(10,10)$.

1. ${}^{20}C_{10}$
2. Not using $(4,4)$ to $(5,4)$ = Total − Paths using $(4,4)$ to $(5,4)$

$$= {}^{20}C_{10} - \left[\underset{\text{to }(4,4)}{(0,0)} \cdot \underset{\text{to }(5,4)}{(4,4)} \cdot \underset{\text{to }(10,10)}{(5,4)}\right]$$

$$= {}^{20}C_{10} - \left[{}^{8}C_{4} \cdot 1 \cdot {}^{11}C_{5}\right]$$

$$= \boxed{{}^{20}C_{10} - \left[{}^{8}C_{4} \cdot {}^{11}C_{5}\right]}$$

Q1. If there are n-points in a plane:
1. No. of lines segments can be drawn.

Q2. If there are n-points, out of which m-points are collinear then no of line segments drawn = ____.

1. ${}^{n}C_{2}$
2. ${}^{n}C_{2} - {}^{m}C_{2} + 1$

Q3. No. of diagonals in a regular polygon with n-sides.

Eg

No. of lines = ${}^{5}C_{2}$ (with 5 points)

No. of ~~side~~ lines that are not diagonals = 5

No. of diagonals = ${}^{5}C_{2} - 5$

For n-sides:

$= {}^{n}C_{2} - n$

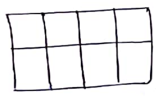
Q4. No. of triangles that can be formed with n-points.

${}^{n}C_{3}$.

Q5. No. of triangles with n-points out of which m are collinear.

${}^{n}C_{3} - {}^{m}C_{3}$

GATE-2016:



No. of rectangles that can be observed in 2*4 grid.

Rectangle is 2 horizontal lines followed by 2 vertical lines.

No. of rectangles = 2-comb. of Hori. lines followed by 2-comb. of vertical lines.

$= {}^{3}C_{2} * {}^{5}C_{2}$

$= 3 * 10 = \boxed{30}$

No. of squares can be observed in 2*4 grid

= No. of 1-squares + No. of 2 squares = $2 \times 4 + 1 \times 3$

$= 8 + 3 = 11$

Results:

① No. of rectangles that can be observed in $m \times n$ grid : ${}^{m+1}C_2 * {}^{n+1}C_2$

② No. of squares that can be observed in a $m \times n$ grid $= \sum_{r=1}^{\min(m,n)} (m-r+1)(n-r+1)$

③ No. of rectangles that can be observed in $n \times n$ grid $= {}^{n+1}C_2 * {}^{n+1}C_2 = \frac{n^2(n+1)^2}{4}$

④ No. of squares that can be observed in $n \times n$ grid.
$= 1^2 + 2^2 + 3^2 + \dots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$

Binomial Expansion:

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r \, x^r \, y^{n-r}$$

Results:

① ${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$ $[x=1; y=1]$

② ${}^nC_0 + {}^nC_2 + {}^nC_4 + \dots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \dots = 2^{n-1}$

| Workbook | Pg-15 | Q.3 |

Soln: No. of ways unsuccessful = fail in 9 or 8 or 7 or 6 or 5 subjects
$= {}^9C_9 + {}^9C_8 + {}^9C_7 + {}^9C_6 + {}^9C_5$
$= {}^9C_0 + {}^9C_8 + {}^9C_2 + {}^9C_6 + {}^9C_4$
$= 2^{9-1} = 2^8$

Problems cannot do: (Chapter-2)
11, 18, 19, 20, 23, 25, 26, 32, 36, 37, 38, 40, 41.

Problems to do:
43, 45, 46, 47, 49, 50

Sets :

→ Set is an unordered collection of well defined objects.

E.g. $A = \{1, 2, 3\}$
$= \{3, 1, 2\}$

Subset :

→ $A \subseteq B$ means if $x \in A$, then $x \in B$.

Equal :

→ $A = B$ means $A \subseteq B$ and $B \subseteq A$

Proper subset :

→ $A \subset B$ means $A \subseteq B$ but $A \neq B$

NOTE : $\boxed{A \subseteq B \text{ , then } \underbrace{B \supseteq A}_{\text{superset}}}$

Empty set :

→ ϕ = set with no elements. $= \{\}$

Universal set : [U]

→ A set containing all the elements.

Result:

1. $\phi \subseteq A$
2. $A \subseteq U$ Every set is subset of universal set.
3. $A \subseteq A$

Cardinality :

→ $|A|$ = No. of elements in the set.

E.g: $A = \{1, 2, 3\}$

$|A| = 3$

E.g I. $\phi \in A$
II. $\phi \subseteq A$] which is always true.

$\phi \subseteq A$ ✓

ϕ is not an element of A.

Q. Write all subsets of $A = \{a, b\}$

$\{a\}, \{b\}, \{a, b\}, \phi$

binary strings of length 2.

$\{\}$ ← 00
$\{b\}$ ← 01
$\{a\}$ ← 10
$\{a, b\}$ ← 11

Q. Write all the subsets of $A = \{1, 2, 3\}$

$= \{\}, \{c\}, \{b\}, \{b, c\},$
$\{a\}, \{a, c\}, \{a, b\}, \{a, b, c\}$

$= \{\}, \{3\}, \{2\}, \{2, 3\}, \{1\}$
$\{1, 3\}, \{1, 2\}, \{1, 2, 3\}$

000
001
010
011
100
101
110
111

Powerset of A : $P(A)$ = set of all subsets of A.

E.g. $A = \{a, b\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

$P(P(A)) = \{\phi, \{\{a\}\}, \{\{a\}, \{b\}\}, \{\{b\}\}, \{\{\}, \{a, b\}\}$

$A = \{a, b, c\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$

Result :

1. $|A| = n$
Number of subsets of A
= Number of binary strings of length $n = 2^n$

2. $|A| = n$
$|P(A)| = 2^n$

01/07/2017 :

Results:

③ $|A| = n$

① No. of subsets having 0 elements $= {}^nC_0 = 1$

② " " " 1 element $= {}^nC_1$

③ " " " 2 element $= {}^nC_2$

④ No. of subsets having n elements $= {}^nC_n$

Total no. of subsets $= {}^nC_0 + {}^nC_1 + \dots + {}^nC_n = 2^n$

④ (i) No. of subsets of A having 3 elements $= {}^nC_3$

(ii) No. of subsets of A having 3 elements and

(i) containing 1 $= {}^{n-1}C_2$

$\{1, \_, \_\}$

(ii) containing 2 $= {}^{n-1}C_2$

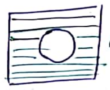
(iii) Containing i $= {}^{n-1}C_2$

(iv) Containing 1, 2 : ${}^{n-2}C_2$

Algebra	Set algebra	Propositional algebra
Objects + Operations	Sets + $C, \cup, \cap$	Propositions + $\neg, \wedge, \vee, \rightleftarrows$

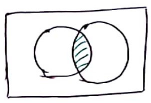
Set Operations:

① $A^c = \{x \mid x \notin A$ but $x \in U\}$

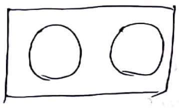


A^c ▤

② $A \cap B = \{x \mid x \in A$ and $x \in B\}$



$A \cap B$ ▨



$A \cap B = \phi$

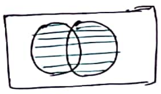
A & B are disjoint

NOTE:

$x \in A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n$ means

$x \in A_i$, for all $i = 1, 2, 3, \dots, n$

③ $A \cup B = \{x \mid x \in A$ or $x \in B$ or $x \in A \cap B\}$



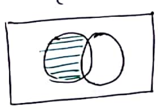
▤ $A \cup B$

NOTE

$x \in A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n$ means

$x \in A_i$ for at least one $i = 1, 2, 3, \dots$ or n.

④ $A - B = \{x \mid x \in A$ but $x \notin B\}$



$A - B$ ▤ $\boxed{A - B = A \cap B^c}$

Set Identities:

① **Idempotent:**

$A \cup A = A$
$A \cap A = A$

② **Identity:**

$A \cup \phi = A$
$A \cap U = A$

③ **Domination:**

$A \cup U = U$ → with r.t. Union, U dominates
$A \cap \phi = \phi$ → w.r.t. $\cap$, ϕ dominates

④ **Complementation:**

$A \cup A^c = U$
$A \cap A^c = \phi$

⑤ **Double Complement:**

$(A^c)^c = A$

⑥ **Commutative:**

$A \cup B = B \cup A$
$A \cap B = B \cap A$

⑦ **Associative:**

$(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$

⑧ **Absorption:**

$A \cup (A \cap B) = A$ → A absorbs $A \cap B$ w.r.t. union
$A \cap (A \cup B) = A$ → A absorbs $A \cup B$ w.r.t. intersection

⑨ **Distributive:**

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap C) \cup (A \cap C)$

⑩ **De Morgan's Law:**

$(A \cup B)^c = A^c \cap B^c$ | $(A \cap B)^c = A^c \cup B^c$

E.g.

① $A = A \cup A$
$= A \cap A$
$= A \cup (A \cap B)$
$= A \cap (A \cup B)$
$= A \cup \phi$
$= A \cap U$

② $(K \cup X) \cap (K \cup Y) = K \cup (X \cap Y)$

③ $A^c \cap (B \cap A^c) = A^c \cap (A^c \cap B)$
$= A^c$

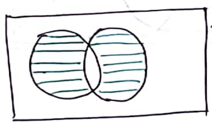
④ $A \cup (\underbrace{A \cap A^c}_{\phi})$
$= A \cup \phi = A$

⑤ $(A \cup B) \cap A^c \cap B^c$
$= (A \cup B) \cap (A \cup B)^c = \phi$

⑥ $(P^c \cup Q \cup R) \cap (P \cup Q \cup R) \cap Q^c \cap R^c$
$= (P^c \cup Q \cup R) \cap (P \cup Q \cup R) \cap (Q \cup R)^c$
$= \cancel{Q \cup R}\,(P^c \cap P) \cup (Q \cup R) \cap (Q \cup R)^c$
$= [\phi \cup (Q \cup R)] \cap (Q \cup R)^c$
$= (Q \cup R) \cap (Q \cup R)^c = \phi$

Symmetric Difference: (Δ or $\oplus$) → Exclusive OR

$A \Delta B = \{x \mid x \in A \text{ or } x \in B \text{ but } x \notin A \cap B\}$



$A \Delta B$

Results:

① $A \Delta B = (A - B) \cup (B - A)$
② $A \Delta B = (A \cup B) - (A \cap B)$
③ $A \Delta \phi = \cancel{\phi}\, A$
④ $A \Delta U = \cancel{\phi}\, A^c$
⑤ $A \Delta A = \phi$
⑥ $A \Delta A^c = U$

Results :

1. $A - (B \cup C) = (A - B) \cap (A - C)$ (b)
2. $A - (B \cap C) = (A - B) \cup (A - C)$ (a)

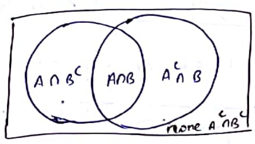
(a) $(A - B) \cup (A - C)$
(b) $(A - B) \cap (A - C)$
(c) $(A - B)$
(d) $(A - C)$

1. $A - (B \cup C)$
$= A \cap (B \cup C)^c$
$= A \cap B^c \cap C^c$
$= A \cap A \cap B^c \cap C^c$
$= A \cap B^c \cap A \cap C^c$
$= \cancel{A \cap} (A - B) \cap (A - C)$

Principle of inclusion-exclusion: *

I. For any sets A and B.

1. $|A \cup B| = |A| + |B| - |A \cap B|$



2. $|A^c \cap B^c| = |U| - |A \cup B|$

II. For any set A, B and C.

1. $|A \cup B \cup C| = |A| + |B| + |C|$
$- |A \cap B| - |A \cap C| - |B \cap C|$
$+ |A \cap B \cap C|$

2. $|A^c \cap B^c \cap C^c| = |U| - |A \cup B \cup C|$

H.W.

III For any sets A, B, C and D.

1. $|A \cup B \cup C \cup D|$
$= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |B$

W.B.

(41) How many positive integers not exceeding 1000 are div. by 7 or 11.

positive ints. ≤ 1000 $|U| = 1000$

Div. by 7 $|A| = \left\lfloor \frac{1000}{7} \right\rfloor = 142$

Div. by 11 $|B| = \left\lfloor \frac{1000}{11} \right\rfloor = 90$

Div. by 7 & 11 $|A \cap B| = \left\lfloor \frac{1000}{77} \right\rfloor = 12$

Div. by 7 (or) 11 $= |A \cup B| = 142 + 90 - 12$
$= 220$

Q. How many +ve integers ≤ 1000 are not divisible by 7 or 11.

Not div by 7 or 11 $= 1000 - 220$
$= 780$

$|A^c \cap B^c| = |(A \cup B)^c|$
$= |U| - |A \cup B|$
$= 1000 - 220 = 780$

Q. How many positive Int. not exceeding 1000 are not divisible by 2, 3, or 5 ?

$|U| = 1000$

$|A| = \left\lfloor \frac{1000}{2} \right\rfloor = 500$

$|B| = \left\lfloor \frac{1000}{3} \right\rfloor = 333$

$|C| = \left\lfloor \frac{1000}{5} \right\rfloor = 200$

$|A \cap B \cap C| = \left\lfloor \frac{1000}{30} \right\rfloor = 33$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$= 1033 - \left\lfloor \frac{1000}{6} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{1000}{10} \right\rfloor + 33$

$= 1033 + 33 - 100 - 66 - 166$

$= \cancel{1066}\ 900 - 166 = 734$

Not div. by 2, 3 or 5

$|(A \cup B \cup C)^c| = |A^c \cap B^c \cap C^c| = |U| - |A \cup B \cup C|$

$= 1000 - 734 = 266$

W.B. Q.40 Pg - 14.

binary strings of length 'n' start with 1 or end with 00.

b.s. of length 8

start with 1 $|A| = 2^7$

end with 00 $|B| = 2^6$

start with 1 and end with 00 $|A \cap B| = 2^5$

$|A \cup B| = 2^7 + 2^6 - 2^5$

$= 2^5(4 + 2 - 1)$

$= 2^5(5)$

$= 160$

GATE: Among 150 faculty:

55 are connected with each other through facebook.

85 whatsapp.

30 faculty member do not have facebook or whatsapp.

No. of faculty connected only through facebook.

(a) 35 (b) 45 (c) 65 (d) 90

$|U| = 150$

~~$|A| = 55$~~

~~$|B| = 85$~~

~~$|(A \cup B)^c| = F^c$~~

$|F| = 55$

$|W| = 85$

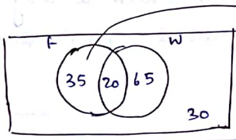
$|F^c \cap W^c| = U - |F \cup W|$

$30 = 150 - |F \cup W|$

$|F \cup W| = 150 - 30 = \cancel{120}\ 120$

$|F \cap W| = |F| + |W| - |F \cup W|$

$= 55 + 85 - 120 = 20$



Ans. 35

Q GATE: In a class there are 200 students.

125 taken Prog

85 " DS

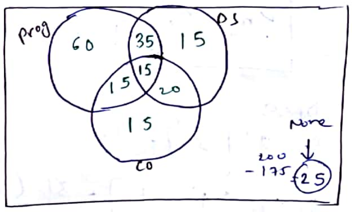
65 " CO

50 " DS & P

35 " DS & CO

30 " P & CO

15 taken all 3.



(1) None = 25

(2) Atleast one = $|A \cup B \cup C| = 175$

(3) Atleast two = 85 = 35 + 15 + 15 + 20

(4) At most one = 60 + 15 + 15 + 25

= 115

(5) At most two = Total - all 3

= 200 - 15 = 185

(6) Only P = 60

(7) Only DS = 15

(8) Only CO = 15

(9) Exactly one = 60 + 15 + 15

= 90

(10) Only P & DS but not CO = 35

(11) Only P & CO but not DS = 15

(12) Only DS & CO but not P: 20

(13) Exactly two = 35 + 15 + 20 = 70

Derangements:

→ Arrangement of 'n' objects in such a way that none of the objects occupy its natural position, is called Derangement.

No. of derangements of n-objects:

$$\boxed{D_n = n!\left[\frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right]}$$

$D_1 = 0$,

$D_2 = 2!\left[\frac{1}{2!}\right] = 1$

$D_3 = 3!\left[\frac{1}{2!} - \frac{1}{3!}\right] = \cancel{3!}\,\frac{(3-1)}{\cancel{3!}} = 2$

$D_4 = 4!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right] = \cancel{4!}\,\frac{(4\times3 - 4 + 1)}{\cancel{4!}} = 13 - 4 = 9$

$D_5 = 5!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right] = \cancel{5!}\left[\frac{60 - 20 + 5 - 1}{\cancel{5!}}\right]$

$= 44.$

GATE Five balls b_1, b_2, b_3, b_4, b_5 are to be kept in 5 cells, c_1, c_2, c_3, c_4, c_5, such that each cell can take exactly one ball. How many ways this can be done if ball b_i is not in cell c_i ($i = 1, 2, 3, 4, 5$).

$\Rightarrow D_5 = 44$

NOTE

$\boxed{GCD(a, b) = 1, \text{ then } a \text{ and } b \text{ are relatively prime.}}$

Euler ϕ-Function:

$\phi(n)$ = No. of positive integers less than and relatively prime to n.

E.g. $\phi(8) = 4$

1, 3, 5, 7

Euler ϕ-Function:

$$\phi(n) = n\left(1 - \frac{1}{P_1}\right)\left(1 - \frac{1}{P_2}\right)\dots\left(1 - \frac{1}{P_K}\right)$$

where $P_1, P_2, \dots P_K$ are distinct prime factors of n.

E.g. $\phi(8)$

$8 = 2^3$

$P_1 = 2$

$\phi(8) = 8\left(1 - \frac{1}{2}\right)$

$= 8 \times \frac{1}{2} = 4$

$\phi(32)$

$32 = 2^5$

$P_1 = 2$

$\phi(32) = 32\left(1 - \frac{1}{2}\right)$

$= 32 \times \frac{1}{2}$

$= 16$

Eg $\phi(40)$

$40 = 2^3 \times 5$

$p_1 = 2 \quad p_2 = 5$

$\phi(40) = 40\left(1 - \frac{1}{2}\right)\cdot\left(1 - \frac{1}{5}\right)$

$= 40 \cdot \frac{1}{2} \cdot \frac{4}{5}$

$= 16$

Eg $7 = 6$

Result: If p is prime, $\phi(p)$ is $p-1$ | $\phi(p) = p-1$

Eg $13 = 12$

Eg 91

$91 = 13 \times 7$

$\phi(91) = 91 \times 1 - \frac{1}{13} \times 1 - \frac{1}{7}$

$= \cancel{91} \times \frac{12}{\cancel{13}} \times \frac{6}{\cancel{7}} = 72$

$\boxed{\text{NOTE}}$ p, q are prime.

$$\boxed{\phi(p^2 q) = p.(p-1).(q-1)}$$

Logic:

→ Logic provides rules to verify validity of 2-valued arguments.

Proposition (STATEMENT):

→ Declarative sentence which is either true (or) false but not both.

Eg $p: 2+2=5$ [F]

q: Lucknow is a city. [T]

r: It is raining → It may be true at the moment and false later. But only one value at a time. It is a proposition.

E.g.: This sentence is false (T/F) or (F/T)

$\boxed{\text{NOTE}}$ → Self referential sentences cannot be propositions.

~~Proportional~~ Propositional Variable:

→ A variable representing proposition.

p
T
F

p	q
T	T
T	F
F	T
F	F

Truth Combinations

→ If there are n - propositional variables then the number of truth combinations (No. of lines of truth tables) = 2^n

$p_1 \quad p_2 \quad \ldots \quad p_n \Rightarrow 2^n$

$\downarrow \quad\; \downarrow \qquad\quad\; \downarrow$

$2 \quad\;\; 2 \qquad\quad\; 2$

Five Basic Connectives : ($\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$)
$\neg$ ↓ negation

Connectives (logical operators) :

① Negation ($\neg$ or $\sim$):
"Not p"

p	$\neg p$
T	F
F	T

E.g.:
p : Einstein is genius
$\neg p$: Einstein is not genius.

② Conjunction (AND, $\wedge$) :
" $p \wedge q$ is true only when both p and q are true ".

p	q	$p \wedge q$
T	T	T
T	F	
F	T	
F	F	

E.g. Jack and Jill went up the hill.
p: Jack went up the hill.
q: Jill went up the hill.
$\boxed{p \wedge q}$

E.g. I like tea but not coffee.
p: I like tea
q: I like coffe
$\boxed{p \wedge \neg q}$

No. of lines in which the following proposition is true.
$p \wedge q \wedge r \wedge s \wedge t$ = 1.

p	q	r	s	t	$p \wedge q \wedge r \wedge s \wedge t$
T	T	T	T	T	T

③ Disjunction: (OR, $\vee$) Inclusive OR.
→ " $p \vee q$ is false only when both p and q are false.

p	q	$p \vee q$
T	T	
T	F	
F	T	
F	F	F

E.g. I like Algo or DM.
p: I like Algo q: I like DM
$p \vee q$

E.g. No. of lines in which .
$p \vee q \vee r \vee s \vee t$ is true .

p	q	r	s	t	$p \vee q \vee r \vee s \vee t$
F	F	F	F	F	F

$\}$ 1, F
$\}$ 31 T

Implication : (Conditional, $\longrightarrow$) (**) Imp.

"A true statement cannot imply false."

p	q	$p \rightarrow q$
T	T	
T	F	F
F	T	
F	F	

E.g. If you get GATE rank, the you will get chocolate.

→ Expressing Implication:

$p \rightarrow q$

- p implies q $\longrightarrow$ q follows p.
- if p then q $\longrightarrow$ q if p.
- p only if q $\longrightarrow$ q unless $\neg p$

Eg If you go then I stay

p: you go , q: I stay $\boxed{p \rightarrow q}$

You go implies I stay — I stay follows you go

You go only if I stay — I stay if you go

I stay unless you don't go.

$\boxed{p \rightarrow q}$

p is sufficient condition for q.

q is necessary condition for p.

E.g If $\underbrace{\delta \geq \frac{n}{2}}_{\text{(sufficient cond.)}}$, then, G is Hamiltonian.

E.g G has $\underbrace{\text{perfect matching}}_{\text{(necessary cond.)}}$ only if n is even.

E.g. ABC in Δle, the $\angle A + \angle B + \angle C = 180^\circ$

NOTE $p \rightarrow q$

p: Hypothesis (premise) or antecedent

q: Conclusion or consequent

Converse, Inverse and Contrapositive:

Implication : $p \rightarrow q$

Converse : $q \rightarrow p$

Inverse : $\neg p \rightarrow \neg q$

Contrapositive : $\neg q \rightarrow \neg p$

Eg. Write converse, inverse & contrapositive.

If you go then I stay

$p \rightarrow q$

Converse: $q \rightarrow p$

~~If you don't go then I don't stay.~~ If I stay then you go.

Inverse: $\neg p \rightarrow \neg q$

~~If I don't stay then you don't go.~~

If you don't go then I don't stay.

Contrapositive: $\neg q \rightarrow \neg p$

~~If you don't go then I~~ If I don't stay then you don't go.

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Q. The no. of lines in which the following proposition is true. $(\sim p \wedge q \wedge r \wedge s) \to t$

(a) 1 (b) 15 (c) 17 (d) 31.

p	q	r	s	t
T	T	T	T	F $\to$ F

$\Big]$ 31 - T.

NOTE:

(1)

p	q	$p \to q$
T	F	F

(2)

p	q	$p \to q$
F		T

(3)

p	q	$p \to q$
	T	T

Eg No. of lines in which the following proposition is false $(p \vee q \vee r \vee s) \to t$:

(a) 1 (b) 15 (c) 17 (d) 31

p	q	r	s	t
F	F	F	F	

p	q	r	s	t	$(p \vee q \vee r \vee s) \to t$
				T	
				⋮	$\to$ T 16 lines
				T	
F	F	F	F		$\to$ T 1 lines
					$\to$ F 15 lines

(5) Bi implication (Biconditional, $\leftrightarrow$):

" $p \leftrightarrow q$ is true only when both p and q have the same truth value."

p	q	$p \leftrightarrow q$
T	T	T
T	F	
F	T	
F	F	T

$p \leftrightarrow q$

p if and only if q

p iff q

$(p \to q)$ and $(q \to p)$

~~# Wff~~ Well formed Formula (WFF):

(1) p is WFF (Any proposition is WFF)

(2) If p is WFF then $\sim p$ is WFF

(3) If p and q are WFF then. (1) $(p \wedge q)$ (2) $(p \vee q)$ (3) $(p \to q)$ (4) $(p \leftrightarrow q)$ are also WFFs.

(4) only those formulas obtained by application of (1) (2) & (3) are WFFs.

E.g $(p \wedge (q \vee r))$ is WFF?

(1) q WFF

(2) r WFF

(3) $(q \vee r)$ WFF

(4) p WFF

(5) $p \wedge (q \vee r)$ WFF

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NOTE: Using precedence of ~~opre~~ operator, we can reduce excessive parenthesis.

Operator Precedence:

1. $\neg$ — high
2. $\wedge$
3. $\vee$
4. $\rightarrow$
5. $\leftrightarrow$ — lower

E.g. ① $p \wedge q \rightarrow r$

$\equiv (p \wedge q) \rightarrow r$

② $p \wedge (q \rightarrow r)$

parenthesis required.

Tautology (T):

→ "A proposition which is always true is called tautology."

E.g. $p \vee \neg p \equiv T$

Contradiction (F):

→ "A proposition which is always false."

E.g. $p \wedge \neg p \equiv F$

Satisfiable:

→ "A proposition which is true for atleast one truth combination."

NOTE
① Every tautology is satisfiable.
But every satisfiable need not be tautology.

Contingency:

→ A proposition which is neither T nor F.

E.g. $p \vee q$

E.g.

p	q	$\neg p$	$\neg q$	$\neg(p \vee q)$ (x)	$\neg p \wedge \neg q$ (y)	$x \leftrightarrow y$
T	T	F	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

Tautology

Definition:
P is equivalent to Q $(P \equiv Q)$ if: they have the same truth table.

Result:
$P \equiv Q$ iff $P \leftrightarrow Q$ is tautology.

Dual: The dual of a compound proposition involving only the connectives $\neg, \wedge$ and $\vee$ is obtained by replacing every
$\wedge$ with $\vee$, ~~$\vee$ with $\wedge$~~
$\vee$ with $\wedge$
T with F
F with T

Eg ① Write Dual:
$P: (p \vee q) \wedge r$
$P^d: (p \wedge q) \vee r$

② $P: \neg p \vee (q \wedge \neg r)$
$P^d: \neg p \wedge (q \vee \neg r)$

RESULTS:

① $(P^d)^d = P$

② $P \equiv Q$ iff $P^d \equiv Q^d$

③ $\neg P(P_1, P_2 \ldots P_n) \equiv P^d(\neg P_1, \neg P_2, \ldots \neg P_n)$

De Morgan's rule in terms of dual.

E.g. $\neg(p \vee q)$

$P: (p \vee q)$

$\neg P : \neg(p \vee q)$

$= \neg p \wedge \neg q$

$P = (\neg p, \neg q) : \neg p \vee \neg q$

$P^d(\neg p, \neg q) : \neg p \wedge \neg q$

Equivalences - I : (Involving $\neg, \wedge, \vee$)

① **Idempotent:** $p \vee p \equiv p$, $p \wedge p \equiv p$

② **Identity:** $p \vee F \equiv p$, $p \wedge T \equiv p$

③ **Domination:** $p \vee T \equiv T$, $p \wedge F \equiv F$

④ **Negation:** $p \vee \neg p \equiv T$, $p \wedge \neg p \equiv F$

⑤ **Double Negation:** $\neg(\neg p) = p$

⑥ **Commutative:** $p \vee q \equiv q \vee p$, $p \wedge q \equiv q \wedge p$

⑦ **Associative:** $p \vee (q \vee r) \equiv (p \vee q) \vee r$, $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

⑧ **Absorption:** $p \vee (p \wedge q) \equiv p$, $p \wedge (p \vee q) \equiv p$

⑨ **Distributive:**

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

⑩ **DeMorgan's:**

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\boxed{a \longleftrightarrow c \text{ holds}}$

$a \equiv c$

Lunch Work:

① $p \wedge (p \vee \neg p)$

② $(p \wedge q) \vee \neg p \vee \neg q$

③ $(p \vee q) \vee (p \vee r)$

④ $\neg p \wedge (q \vee \neg p)$

⑤ $(\neg p \wedge q \wedge r) \vee (p \wedge q \wedge r) \vee \neg q \vee \neg r$

① $p \wedge T$

$= p$

② $(p \wedge q) \vee \neg(p \wedge q)$

$= T$

⑤ $\cancel{(p \wedge q \wedge r) \vee (\neg}$

$(q \wedge r) \wedge (\neg p \vee p) \vee \neg q \vee \neg r$

$= (q \wedge r) \wedge T \vee \neg q \vee \neg r$

$= (q \wedge r) \vee \neg q \vee \neg r$

$= (q \wedge r) \vee \neg(q \wedge r)$

$= T$

③ $(p \vee q) \vee (p \vee r)$

$= p \vee (q \vee r)$

④ $(\neg p \wedge q) \vee (\neg p \wedge \neg p)$

$= (\neg p \wedge q) \vee \neg p$

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Imp.

Equivalences - 2 (involving ~~indications~~ implications)

(1) **Law of implication:**

$$\boxed{p \to q \equiv \neg p \vee q}$$

p	q	$p \to q$	$\neg p \vee q$
T	F	F	F

Results:

1. $p \to q \equiv \neg p \vee q$
2. $\neg p \to q \equiv p \vee q$
3. $p \to \neg q \equiv \neg p \vee \neg q$
4. $\neg p \to \neg q \equiv p \vee \neg q$
5. $p \vee q \equiv \neg p \to q$
6. $\neg p \vee q \equiv p \to q$
7. $p \vee \neg q \equiv \neg p \to \neg q$
8. $\neg p \vee \neg q \equiv p \to \neg q$

Eg (1) $p \wedge q \equiv \neg(p \to \neg q)$

$p \wedge q \equiv \neg(\neg(p \wedge q))$

$= \neg(\neg p \vee q)$

$= \neg(p \to \neg q)$

(2) $p \wedge \neg q \equiv \neg(p \to q)$

(2) **Law of Contrapositive:**

$$\boxed{p \to q \equiv \neg q \to \neg p}$$

p	q	$p \to q$	$\neg q \to \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

(1) $\boxed{p \to q \equiv \cancel{\neg p \to \neg q}\ \neg q \to \neg p}$

(2) $\neg p \to q \equiv \cancel{p \to q}\ \neg q \to p$

(3) $p \to \neg q \equiv \cancel{\neg p \to q}\ q \to \neg p$

(4) $\boxed{\neg p \to \neg q \equiv \cancel{p \to q}\ q \to p}$

(3) **Exportation Law:**

$$p \to (q \to r) \equiv (p \wedge q) \to r$$

1. $p \to (q \to r)$
2. $p \to (\neg q \vee r)$
3. $\neg p \vee (\neg q \vee r)$
4. $(\neg p \vee \neg q) \vee r$
5. $\neg(p \wedge q) \vee r$
6. $\boxed{(p \wedge q) \to r}$ ⇒ RHS.

(4).

1. $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$
2. $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

Rules

(a) $(p \vee q) \to r$
(b) $(p \wedge q) \to r$
(c) $p \to r$
(d) $q \to r$

(1) $(\neg p \vee r) \vee (\neg q \vee r)$ (Law of implication)

$= (\neg p \vee \neg q) \vee r$

$= (\neg(p \wedge q)) \vee r$

$= (p \wedge q) \to r$

(5) -

1. $(p \to q) \vee (p \to r) \equiv p$.
2. $(p \to q) \wedge (p \to r) \equiv$

(1) $(\neg p \vee q) \vee (\neg p \vee r)$

$\boxed{\neg p \vee (q \vee r)}$

(a) $p \to (q \vee r)$

(2) $(p \to q) \wedge (p \to r)$

$= (\neg p \vee q) \wedge (\neg p \vee r)$

$= \neg p \vee (q \wedge r)$

(b) $p \to (q \wedge r)$

Equivalences - 3 : (Involving Biimplication)

1. $\boxed{p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)}$ Both true or both false.
2. $\boxed{p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)}$
3. $\boxed{p \leftrightarrow q \equiv q \leftrightarrow p \equiv \neg q \leftrightarrow \neg p \equiv \neg p \leftrightarrow \neg q}$

Q Chp-1 Q-4 x is even iff x is divisible by 2.

$\longrightarrow (p \leftrightarrow q)$

$\equiv$ (b)

Q5. x is even iff x is divisible by 2.

$p \leftrightarrow q$ (d)

Q42. I stay only if you go.

p: I stay q: you go

$p \to q$

if I stay then you go

Conv. $p \to q$

If you go then I stay

I stay if you go (a)

1. x if y
   $y \to x$
2. x only if p
   $x \to p$
3. z follows $\neg p$
   $\neg p \to z$
4. l unless $\neg z$
   $z \to l$
5. p unless q
   $\neg q \to p$
6. If x then y unless $\neg z$.
   $\equiv z \to (x \to y)$
   $\equiv (z \wedge x) \to y$
   $\equiv (x \wedge z) \to y$

q unless $\neg z$

~~p only if~~

q if not $\neg z$

q if z

$z \to q$

Logical Implication : (One-sided condition)

→ An implication which is tautology is called logical implication.

E.g. $p \to (p \vee q)$ — logical implication

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

→ Tautology

Method to verify given implication is tautology or not:

Method - I :

Try to make T

$\overset{T}{P} \longrightarrow \overset{\text{fix as } F}{Q}$
If (possible) Not Tautology
If (not possible) Tautology

Method - II :

Try to make F

$\overset{\text{fix } T}{P} \longrightarrow \overset{F}{Q}$
If (possible) Not Tautology
If (not possible) Tautology

Abbreviated Group Table method

E.g. ① $p \to (p \vee q)$

Try T ✗   fix F

p	q	$p \to (p \vee q)$
F	F	∴ Tautology

② $p \to (p \vee q)$

Try F ✗   fix T

p	q	$p \to (p \vee q)$
T		∴ Tautology

③ $[p \wedge (p \to q)] \to q$

try to make T   fix F

p	q	$[\underset{F}{p} \wedge (\underset{F}{p} \to \underset{F}{q})] \to q$
F	F	Tautology

④ $[(p \to q) \wedge \neg p] \to \neg q$

Try to make T   fix F

p	q	$(p \to \overset{T}{q}) \wedge \neg p$
F	T	Not Tautology

E.g. $[p \wedge (p \to r) \wedge (q \to r) \wedge (p \vee q)] \to r$

Try T.   fix F

p	q	r	$[p \wedge (p \to r) \wedge (q \to r) \wedge (p \vee q)] \to r$
F	F	F	Tautology

Eg- $[(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)] \to (\neg p \vee \neg r)$

Try T   Fix F

p	q	r	s	$(\overset{T}{p} \to q) \wedge (\overset{T}{r} \to \overset{T}{s}) \wedge (\overset{F}{\neg q} \vee \overset{F}{\neg s})$	$(\neg p \vee \neg r)$
T	T	T	T	Tautology	

E.g. $[(p \to r) \wedge (q \to r)] \to (r \to p)$

Try T — Fix F (r: T, p: F)

p	q	r	$[(p \to r) \wedge (q \to r)] \to (r \to p)$
F	T	T	Not tautology

GATE: $[p \to (q \vee r)] \to [(p \wedge q) \to r]$

(a) Satisfiable but not valid
(b) Valid
(c) F
(d) none.

Try T — Fin F

p	q	r	$p \to (q \vee r) \to (p \wedge q) \to r$
T	T	F	Not Tautology
		T	T

(a) Satisfiable but not valid.

GATE: which of the following are valid:

P: $[(\neg p \vee q) \wedge (r \to s) \wedge (p \vee r)] \to [\neg s \to q]$
Q: $[(\neg p \vee q) \wedge (q \to (p \to r))] \to \neg r$
R: $[[(q \wedge r) \to p] \wedge (\neg q \vee p)] \to r$
S: $[p \wedge (p \to r) \wedge (q \vee \neg r)] \to q$

P & S only

(a) P & Q only (b) P & R only (c) P & S only. (d) P, Q, R, S.

Q: Try T — Fin F

p	q	r	$(\neg p \vee q) \wedge (q \to (p \to r))$	$\neg r$
	T	T	Not tautology	

R: Try T — Fin F

p	q	r	$[(q \wedge r) \to p] \wedge (\neg q \vee p)$	r
T		F	Not Tautology	

GATE: A logical binary relation $\odot$ is defined as:

$A \wedge B =$ ______ .

A	B	$A \odot B$
T	T	T
T	F	T
F	T	F
F	F	T

(a) $\sim A \odot B$
(b) $\sim (A \odot \cancel{\sim B)}\ B)$
(c) $\sim (\sim A \odot \sim B)$
(d) $\sim (\sim A \odot B)$

A	B	$A \odot B$	$\sim (A \odot B)$	$A \wedge B$	$\sim A \odot B$	$\sim (\sim A \odot B)$
T	T	T	F	T	F	T
T	F	T	F	F	T	F
F	T	F	T	F	T	F
F	F	T	F	F	T	F

$A \odot B \equiv B \to A$
$A \wedge B \equiv \neg (A \to \neg B)$
$\equiv \neg (B \to \neg A)$
$\equiv \neg (\neg A \odot B)$
] Law of contrapositive

Q: $\neg p \wedge q$
$q \to (p \to r)$
$\therefore \neg r$

1. $q \to (p \to r)$
2. q — Simp (1)
3. $p \to r$
4. $\neg p$ — Simp (1)

] fallacy of denying antecedent.

$\therefore \neg r$

S: p
$p \to r$
$q \vee \neg r$
$\therefore q$

$p \to r$
p
r
] Modus Ponens

$q \vee \neg r$
$r \to q$
] Implication

r
q
] M. P.

GATE - In a room, there are only 2 types of people (Type1, Type2). Type 1 people always tell the truth and Type 2 people always ~~tell~~ lie.
You give a fair coin to a person in that room, w/o knowing which type he is from and tell him to toss it and hide the result from you till you ask for it.
Upon asking, the person replies the following:
"The result of the toss is head iff I am telling the truth."
Which of the following option is correct.

ⓐ Head ⓑ Tail ⓒ Type 2, result tail
ⓓ Type 1, result is tail.

ⓒ ~~Tail~~ Type 2 / Tail	$F \leftrightarrow F$	T ⊗
ⓓ Type 1 / ~~Head~~ Tail	$F \leftrightarrow T$	F ⊗
Type 1 / Head	$T \leftrightarrow T$	T ✓
Type 2 / Head	$T \leftrightarrow F$	F ✓

2019. Let p, q, and r be propositions and the expression $(p \rightarrow q) \rightarrow r$ be a contradiction, then $(r \rightarrow p) \rightarrow q$ is:

ⓐ T ⓑ F ⓒ Always true when p is false
✓ⓓ Always ~~false~~ true when q is true

p	q	r	$(p \rightarrow q) \rightarrow r$
T	T	F	F
F			
F	F		

$\underbrace{(r \rightarrow p)}_{\text{will be T if r is F}} \rightarrow q$ → Depends on q. So if q is T then it is always T

2017: p: It is raining
q: It is cold
r: it is pleasant, Then the statement:

"It is not raining and pleasant, and it is not pleasant only if it is raining and it is cold."

$[(\neg p \wedge r) \wedge (\neg r \rightarrow (p \wedge q))]$

ⓐ $[(\neg p \wedge r) \wedge (\neg r \rightarrow \neg(p \wedge q))]$ ✓

ⓑ $(\neg p \wedge r) \wedge ((p \wedge q) \rightarrow \neg r)$

ⓒ $(\neg p \wedge r) \vee ((p \wedge q) \rightarrow \neg r)$

ⓓ $(\neg p \wedge r) \vee (r \rightarrow (p \wedge q))$

General form of an argument:

[Conjunction of premises] → Conclusion.

$[P_1 \wedge P_2 \wedge \dots \wedge P_n] \rightarrow q$

→ An inference which is tautology is called valid inference. Otherwise, it is invalid inference.

→ Any valid inference is called rule of inference.

Some important rules of inference:

Name / Form	Tautological Form
(1) Conjunction p q $\therefore p \wedge q$	$p \wedge q \Rightarrow (p \wedge q)$ To say that it is tautology
(2) Simplification $p \wedge q$ $\therefore p$	$p \wedge q \Rightarrow p$
(3) Addition: p $\therefore p \vee q$	$p \Rightarrow (p \vee q)$
(4) Modus Ponens : (Rule of detachments) $p \rightarrow q$ p $\therefore q$	$[p \wedge (p \rightarrow q)] \Rightarrow q$
(5) Hypothetical Syllogism (Transitive Rule) $p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \Rightarrow (p \rightarrow r)$
(6) Modus Tollens $p \rightarrow q$ $\neg q$ $\therefore \neg p$	$[(p \rightarrow q) \wedge \neg q] \longrightarrow \neg p$

GATE: I_1 : If it rains, then cricket match will not be played.
The cricket match was played.
∴ There was no rain.

I_2 : If it rains then cricket match will not be played.
It did not rain
∴ The cricket match was played.

(a) only I_1 (b) only I_2 (c) both valid (d) None.

~~I_1: $p \rightarrow q$~~
~~$\neg q$~~
~~$\neg p$~~

I_2: $p \rightarrow q$
$\neg p$

p: it rains
q : cricket match played.

I_1 :
$$\begin{array}{l} p \rightarrow \neg q \\ q \\ \hline \therefore \neg p \end{array}$$
Modus Tollens.

I_2 :
$$\begin{array}{l} p \rightarrow \neg q \\ \neg p \\ \hline \therefore q \end{array}$$
(Fallacy)

An invalid inference which resembles a valid inference is called Fallacy.

<u>Fallacy of affirming Consequence</u>:

$p \to q$
q
$\therefore p$

Invalid

<u>Fallacy of denying antecedent</u>:

$p \to q$
$\neg p$
$\therefore \neg q$

Invalid

<u>Non-sequitur fallacy</u>:

p
$\therefore q$

(No - sequence)

<u>Rules</u>: (Continued). [03/07/2017]

Name / Form	Tautological Form
⑦ <u>Disjunctive Syllogism</u>:	$[(p \vee q) \wedge \neg p] \Rightarrow q$
$p \vee q$ $\neg p$ $\therefore q$	Pf. 1. $p \vee q$ given 2. $\neg p \to q$ implication 3. $\neg p$ given 4. q. MP. 2, 3
$p \vee q,\ \neg p \vdash q$	
⑧ <u>Conditional Proof</u>:	$[[p \to (q \to r)] \wedge (p \wedge q)] \Rightarrow r$
$p \to (q \to r)$ $p \wedge q$ $\therefore r$	Pf. 1. $p \wedge q$ given 2. p Simplification ① 3. q Simplification ① 4. $p \to (q \to r)$ given 5. $q \to r$ 2, 4, M.P. 6. r M.P. 3, 5

⑨ <u>Disjunctive Elimination</u>:

$p \to r$
$q \to r$
$p \vee q$
$\therefore r$

⑩ <u>Resolution</u>:

$p \vee q$
$\neg p \vee r$
$\therefore q \vee r$

⑪ <u>Construction Dilemma</u>:

$p \to q$
$r \to s$
$p \vee r$
$\therefore q \vee s$

$[(p \to r) \wedge (q \to r) \wedge (p \vee q)] \Rightarrow r$

<u>P.f.</u>: 1. $p \to r$ given
2. $q \to r$ given
3. $(p \to r) \wedge (q \to r)$ conjunction
4. $(p \vee q) \to r$ implic.
5. $(p \vee q)$ given
6. $\therefore r$ M.P. 4, 5

$[(p \vee q) \wedge (\neg p \vee r)] \Rightarrow (q \vee r)$

<u>Proof</u>: 1. $q \vee p$ ~~$\neg(p \wedge r)$~~ given
2. $\neg q \to p$ implic.
3. $\neg p \vee r$ given
4. $p \to r$ implication
5. $\neg q \to r$ Transitive 2, 4.
6. $q \vee r$ implication

$[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \Rightarrow q \vee s$

<u>Proof</u>: 1. $r \vee p$
2. $\neg r \to p$
3. $r \to s$
4. $\neg s \to \neg r$
5. $\neg s \to p$
6. $p \to q$
7. $\neg s \to q$
8. $s \vee q$
9. $q \vee s$

$\neg p \vee q$
$p \vee r$
$q \vee r$
$r \vee q$
$\neg r \vee s$
$q \vee s$

Resolution

(12) Destructive Dilemma :

$p \to q$
$r \to s$
$\neg q \vee \neg s$
$\therefore \neg p \vee \neg r$

Pf: $(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s) \Rightarrow (\neg p \vee \neg r)$

1. $s \to \neg q \to \begin{bmatrix} \neg q \vee \neg s \\ \neg s \vee \neg q \\ s \to \neg q \end{bmatrix}$
2. $r \to s$
3. $r \to \neg q$
4. $\neg q \to \neg p$
5. $r \to \neg p$
6. $\neg r \vee \neg p$

Q. which of the following is not T.

(a) $(p \wedge q) \to (p \vee q)$ ×
(b) $\neg p \to (p \to q)$ ×
(c) $p \to (p \wedge q)$ ✓
(d) $q \to (p \vee q)$
(e) $q \to (p \to q)$
(f) $p \vee (q \to p)$ ✓

(a) $\dfrac{p \wedge q}{\therefore p \vee q}$ 1. $p \wedge q$ 2. p 3. $p \vee q$

(b) $\dfrac{\neg p}{\therefore p \to q}$ 1. $\neg p$ 2. $\neg p \vee q$ 3. $p \to q$

(c) $\dfrac{p}{\therefore p \wedge q}$ Fallacy

(d) Addition

(e) $\dfrac{q}{\therefore p \to q}$ 1. q 2. $\neg p \vee q$ 3. $p \to q$

(f) $p \vee (q \to p) \equiv \neg p \to (q \to p)$

$\dfrac{\neg p}{\therefore (q \to p)}$ Not Tautology.

p	q	$p \vee (q \to p)$
F	T	Not T.

GATE:

Q. S1: If a candidate is corrupt, then he will not be elected.
S2: If a candidate is kind, then he will be elected.
Which of the following statement follows from S1 & S2.

(a) If person is corrupt, he is kind.
(b) If a person is not corrupt, he is not kind.
(c) If a person is kind, he is not corrupt. ✓
(d) If a person is not kind, he is not corrupt.

p: candidate is corrupt.
q: candidate is kind

S1: $c \to \neg e$
S2: $k \to e$

$k \to e$
$e \to \neg e$

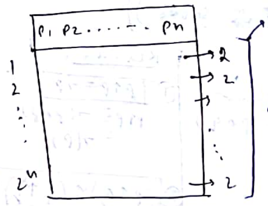
$\boxed{k \to \neg e}$ ✓

Transitive Rule

RESULT:

→ No. of propositional functions using n propositional variables
$= 2^{2^n}$

(No. of boolean functions using n boolean variables $= 2^{2^n}$

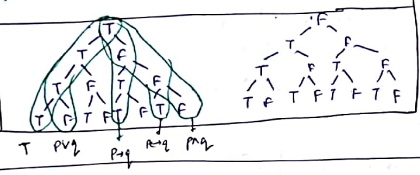


Each result will have 2 values

2^n times $= 2^n \times 2^n$
$= 2^{2^n}$

Eg No. of propositional ~~fn~~ fn. using 2 propositional variables $= 2^{2^2} = 16$

p	q
T	T
T	F
F	T
F	F



Additional Connectives :

① **Exclusive OR** : (XOR, $\bar{\vee}$ or $\oplus$)

$p \bar{\vee} q$ is true only when p is true or q is true but not both.

p	q	$p \bar{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

Result :

① $p \bar{\vee} q \equiv \neg(p \leftrightarrow q)$

② **NAND** : [$\uparrow$ or $|$ (Sheffer's Stroke)]

NAND ≡ Not AND

$p \uparrow q \equiv \neg(p \wedge q)$

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

$(p \uparrow p) \uparrow (q \uparrow q)$
$= (\neg p \uparrow \neg q)$
$= \neg(\neg p \wedge \neg q) = p \vee q$

Results :

① $\boxed{p \uparrow p \equiv \neg p}$

$p \uparrow p \equiv \neg(p \wedge p)$
$\equiv \neg(p)$

② $\boxed{(p \uparrow q) \uparrow (p \uparrow q) \equiv p \wedge q}$

$(p \uparrow q) \uparrow (p \uparrow q) = \neg(p \uparrow q)$
$= \neg(\neg(p \wedge q))$
$= p \wedge q$

③ $\boxed{(p \uparrow p) \uparrow (q \uparrow q) \equiv p \vee q}$

③ **NOR** ($\downarrow$ Pierce arrow) :

~~$p \downarrow$~~ $p \downarrow q = \neg(p \vee q)$

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Results:

① $p \downarrow p \equiv \neg p$
② $(p \downarrow q) \downarrow (p \downarrow q) \equiv p \vee q$
③ $(p \downarrow p) \downarrow (q \downarrow q) \equiv p \wedge q$

→ A set of connectives is said to be functionally complete if every compound proposition can be expressed as a proposition involving only the connectives in the given set.

Eg $\{\neg, \wedge, \vee, \to, \leftrightarrow\}$ F.C.
$\{\neg, \wedge, \vee\}$ F.C.

$\{\neg, \wedge\}$, $\{\neg, \vee\}$ → Minimal functionally complete (F.C.)
No proper subset is F.C.

$\{\uparrow\}$, $\{\downarrow\}$ → Smallest minimal functionally complete set. (Universal gates)

$\{\wedge, \vee\}$ → Not F.C.

Predicates:

→ An open (predicate) is a proposition except for the fact that it contains variables whose values are to be taken from some universe of discourse.

E.g. $E(x) : x+3 = 7$ | $L(x, y)$: x ~~toy~~ loves y

$G(x, y) : x+y \geq 8$ | $F(x, y, t)$: x can fool y at time t.

Greater

Def. A predicate involve n-variables is called n-place predicate.

Subject: Abt which assertion is made

Predicate: Property the sub. has

Predicates to Propositions:

① Substitution

② Quantification

Quantifiers:

① Universal Quantifier ($\forall$) for all

② Existential Quantifier ($\exists$) There exists | For some

E.g. $U = \{1, 2, 3, 4\}$

$P(x) : x + 4 \leq 7$ $\forall x\, P(x)$

For all x, P(x) is true (false)

For some x, P(x) is true (True) → $\exists x\, P(x)$

Quantification.

E.g. $U\{1, 2, 3\}$

$P(x) : x + 4 \leq 7$

$\forall x\, P(x)$ [T]

$\exists x\, P(x)$ [T]

Form	Meaning
① $\forall x\, P(x)$	All True
② $\exists x\, P(x)$	Some True (atleast one true)
③ $\neg \forall x\, P(x)$	Not all true
④ $\neg \exists x\, P(x)$	None True
⑤ $\forall x\, \neg P(x)$	All False
⑥ $\exists x\, \neg P(x)$	Some False (atleast one false)
⑦ $\neg \forall x\, \neg P(x)$	Not all False
⑧ $\neg \exists x\, \neg P(x)$	None False

Equivalences:

① $\forall x\, P(x) \equiv \neg \exists x\, \neg P(x)$

② $\exists x\, P(x) \equiv \neg \forall x\, \neg P(x)$

③ $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

④ $\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$

Negating Quantified Predicates:

$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

Q. Problems:

① $\neg \forall x\,[P(x) \rightarrow Q(x)]$

$\equiv \neg \forall x\,[\neg P(x) \vee Q(x)]$

$\equiv \exists x\, \neg[\neg P(x) \vee Q(x)]$

$\equiv \exists x\,(P(x) \wedge \neg Q(x))$

② $\neg \exists x\,[P(x) \wedge Q(x)]$

$\equiv \forall x\, \neg[P(x) \wedge Q(x)]$

$\equiv \forall x\,[\neg P(x) \vee \neg Q(x)]$

$\equiv \forall x\,[P(x) \rightarrow \neg Q(x)]$

③ $\neg[\forall x, \alpha \rightarrow (\exists y, \beta \rightarrow (\forall u, \exists v, r))]$

$\equiv$

$\equiv \exists x \neg[\alpha \to (\exists y, \beta \to (\forall u, \exists v, r))]$

$\equiv \exists x [\alpha \wedge \neg(\exists y, \beta \to (\forall u, \exists v, r))]$

$\equiv \exists x [\alpha \wedge \forall y \neg(\beta \to (\forall u, \exists v, r))]$

$\equiv \exists x [\alpha \wedge \forall y [\beta \wedge \neg(\forall u, \exists v, r)]]$

$\equiv \exists x [\alpha \wedge \forall y [\beta \wedge (\exists u, \forall v, \neg r)]]$

$\equiv \exists x, \alpha \wedge \forall y, \beta \wedge (\exists u, \forall v, \neg r)$

Statements into symbolic form:

Every student in this class studied DM.

Case1: U = students in this class.

$D(x)$: x studied D.M.

$\forall x\, D(x)$

Case2: U = set of all people

$S(x)$: x is a student in this class.

$D(x)$: x studied DM.

For every x, if x is student in this class then x studied D.M..

$\forall x\, [S(x) \to D(x)]$

Aristotle form:

1. All P's are Q's — $\forall x\, [P(x) \to Q(x)]$
2. Some P's are Q's — $\exists x\, [P(x) \wedge Q(x)]$
3. Not all P's are Q's $[\neg\forall x (P(x) \to Q(x))]$ $\exists x\, [P(x) \wedge \neg Q(x)]$ → Some P's are not Q's.
4. No P's are Q's $\equiv \forall x [P(x) \to \neg Q(x)]$
   $\neg \exists x [P(x) \wedge Q(x)]$

Q. 1. Some real nos. are rational $\quad \exists x [re(x) \wedge ra(x)]$

~~2~~ (a) $\exists x [re(x) \vee ra(x)]$
(b) $\forall x [re(x) \to ra(x)]$
(c) $\exists x [re(x) \wedge va(x)]$
(d) $\forall x [ra(x) \to re(x)]$

2. Not all rainy days are cold. $\quad \exists d [R(d) \wedge \neg C(d)]$

$\sim \forall d [(R(d) \to C(d))]$

(a) $\forall d [R(d) \wedge \sim C(d)]$
(b) $\forall d [\sim R(d) \to C(d)]$
(c) $\exists d [\sim R(d) \to C(d)]$
(d) $\exists d [R(d) \wedge \sim C(d)]$

3. Not all that glitters is gold

$\neg \forall x [g(x) \to go(x)]$

$\exists x [g(x) \wedge \neg go(x)]$

(a) $\forall x : g(x) \Rightarrow \neg go(x)$
(b) $\forall x : go(x) \Rightarrow gl(x)$
(c) $\exists x : go(x) \wedge \neg g(x)$
(d) $\exists x : g(x) \wedge \neg go(x)$

4. None of my friends are perfect.

$\forall x [F(x) \to \neg P(x)]$

$\equiv$

$\neg \exists x [F(x) \wedge P(x)]$

(a) $\exists x (F(x) \wedge P(x))$
(b) $\neg \exists x (F(x) \wedge P(x))$
(c) $\exists x [\neg F(x) \wedge P(x)]$
(d) $\exists x [\neg F(x) \wedge \neg P(x)]$

5. Gold and silver ~~are~~ ornaments are precious. [d]

(a) $\forall x [P(x) \to [G(x) \wedge S(x)]]$
(b) $\forall x [[G(x) \wedge S(x)] \to P(x)]$
(c) $\exists x [(G(x) \wedge S(x)) \to P(x)]$
(d) $\forall x [(G(x) \vee S(x)) \to P(x)]$

Ex:
Gold ornaments are precious and silver ornaments are precious

$(p \to r) \wedge (q \to r)$

$(p \vee q) \to r$ → when separable "$\vee$" is used

6. All purple mushrooms are poisonous. [b] → when not separable "$\wedge$" is used.

$G(x)$: x is purple

$S(x)$: x is mushroom

$P(x)$: x is poisonous.

⑦ Gate and CAT exams need preparation.

$(g \to p) \wedge (c \to p)$

$(g \vee c) \to p$

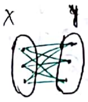
⑧ Connected and acyclic graphs are trees.

$(c \wedge a) \to t$

Two-Place Predicates:

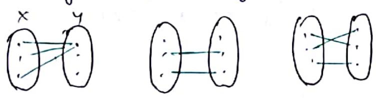
$L(x,y)$: x ~~toyes~~ loves y

J. x y



① $\forall x \forall y\, L(x,y)$: Every one ~~toyes~~ loves everybody.

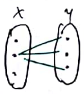
② $\forall x\, \exists y\, L(x,y)$: Every one loves some body.



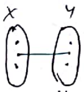
$\forall x \exists y [x+y=0]$ T

$\forall x \exists y [x+y=x]$ T

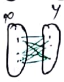
③ $\exists x \forall y\, L(x,y)$: Someone loves everybody



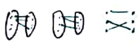
④ $\exists x \exists y\, L(x,y)$: Some one loves somebody.



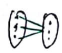
⑤ $\forall y \forall x\, L(x,y)$: Everybody loved by everyone.



⑥ $\forall y\, \exists x\, L(x,y)$: Everybody loved by someone.



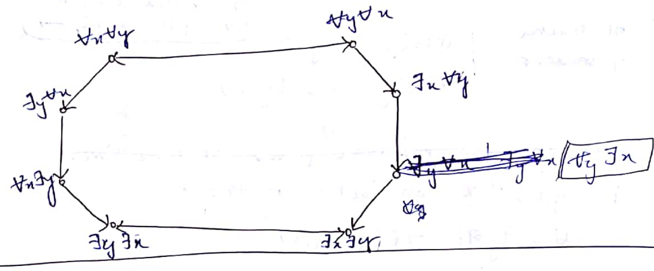
⑦ $\exists y \forall x\, L(x,y)$: Somebody loved by everyone.



⑧ $\exists y \exists x\, L(x,y)$: Somebody loved by someone.

$\boxed{\exists y \forall x \Rightarrow \forall x \exists y}$

Relation between Quantified 2-place predicate:



GATE: ① $F : \forall x [\exists y\, R(x,y)]$

Which of the following is implied by F.

✓I. $\exists y (\exists x\, R(x,y))$

II. $\exists y [\forall x\, R(x,y))$

III. $\forall y (\exists x\, R(x,y))$

✓IV. $\neg \exists x (\forall y\, \neg R(x,y))$

ⓐ IV only

✓ⓑ I and IV only

ⓒ II only

ⓓ II and III only

Q ② Some boys in class are taller than all the girls.

$\exists$

(a) $\exists x [B(x) \to \forall y\, (G(y) \wedge taller(x,y))]$

✓(b) $\exists x [B(x) \wedge \forall y\, (G(y) \to taller(x,y))]$

(c) $\exists x [B(x) \wedge \forall y [G(y) \wedge taller(x,y))]$

(d) $\exists x [B(x) \to \forall y\, (G(y) \to taller(x,y))]$

Q ③ Every teacher is liked by some student.

(a) $\forall x [T(x) \to \exists y [S(y) \to likes(y,x)]]$

(b) $\forall x [T(x) \wedge \exists y [S(y) \to likes(y,x)]]$

✓(c) $\forall x [T(x) \to \exists y [S(y) \wedge likes(y,x)]]$

(d) $\exists y \forall x [T(x) \to [S(y) \wedge likes(y,x)]]$

(2) x: Boys y: girls taller(x,y) : x is taller than y.

$\exists x \forall y$ taller (x,y) → Aristotle form: $\exists x$ followed by $\wedge$, $\forall y$ followed by $\rightarrow$

(3) x: teacher y: student likes(y, x) : y likes x

$\forall x \exists y$ likes (y,x)

Q.4. ~~Every one can fool some person at some time.~~

$F(x,y,t)$: x can fool y at time t

$\forall x \exists y \exists t \; \neg F(x,y,t)$

(a) Every one can fool some person at some time.
(b) No one can fool everyone at all the time.
(c) Every one cannot fool some person all the time.
(d) No one can fool some person at some time.

$\underbrace{\forall x}_{\text{Everyone}} \exists y \exists t \; \underbrace{\neg F(x,y,t)}_{\text{cannot fool}}$ some person at some time.

Not present in this form.

$$\boxed{\forall x \exists y \exists t \; \neg F(x,y,t) \equiv \neg \exists x \forall y \forall t \; F(x,y,t)}$$

No one can fool everyone at all time.

(d) $\neg \exists x \exists y \exists t \; F(x,y,t)$ ✗

(a) $\forall x \exists y \exists t \; F(x,y,t)$

(c) $\forall x \exists y \forall t \; \neg F(x,y,t)$

W.B. Pg-8

(54) $B(x,y)$: y is best friend of x

Everyone has exactly one best friend.

Everyone has some best friend.

$\forall x \exists y \, B(x,y)$

Everyone has exactly one B.F.

$\forall x \exists y [B(x,y) \wedge \forall z [(z \neq y) \rightarrow \neg B(x,z)]]$

(or)

$\forall x \exists y [B(x,y) \wedge \forall z (B(x,z) \rightarrow (z=y))]$

Pg.9. T.11.

D. There is exactly one apple.

$\exists x [A(x) \wedge \forall y (A(y) \Rightarrow (y = x))]$

OR

$\exists x [A(x) \wedge \forall y [y \neq x \Rightarrow \neg A(y)]]$

B. There are exactly two apples.

$\exists x \exists y [A(x) \wedge A(y) \wedge (x \neq y) \wedge \forall z (A(z) \Rightarrow (z = x) \vee (z = y))]$

C. There is at most one apple.

$\forall x \forall y [[A(x) \wedge A(y)] \Rightarrow (x = y \vee y = x)]$

We cannot have 2 apples.

⇒ max 1 apples ⇒ at most 1 apple

A. There are atmost two apples.

$\forall x \forall y \forall z [[A(x) \wedge A(y) \wedge A(z)] \Rightarrow (x = y \vee x = z \vee y = z)]$

Quantifiers and Connectives:

Let $U = \{1, 2\}$

- $\forall x\, P(x) \equiv P(1) \wedge P(2)$
- $\exists x\, P(x) \equiv P(1) \vee P(2)$

RESULTS:

1. $\boxed{\forall x\, [P(x) \wedge Q(x)] \equiv \forall x\, P(x) \wedge \forall x\, Q(x)}$

$$\forall x\, [P(x) \wedge Q(x)] \equiv [P(1) \wedge Q(1)] \wedge [P(2) \wedge Q(2)]$$
$$\equiv [P(1) \wedge P(2)] \wedge [Q(1) \wedge Q(2)]$$
$$\equiv \{\forall x\, P(x)\} \wedge \forall x\, Q(x)$$

2. $\boxed{\exists x\, [P(x) \vee Q(x)] \equiv \exists x\, P(x) \vee \exists x\, Q(x)}$

E.g. $P(1)$ T, $P(2)$ F; $Q(1)$ F, $Q(2)$ ~~F~~ T

$\forall x\, [P(x) \vee Q(x)]$

$[P(1) \vee Q(1)] \wedge [P(2) \vee Q(2)]$

T F F T

T $\wedge$ T

T

$\forall x\, P(x) \vee \forall x\, Q(x)$

$[P(1) \wedge P(2)] \vee [Q(1) \wedge Q(2)]$

T F T F

F $\vee$ F

F

3. $\forall x\, P(x) \vee \forall x\, Q(x) \Rightarrow \forall x\, [P(x) \vee Q(x)]$
4. $\exists x\, [P(x) \wedge Q(x)] \Rightarrow \exists x\, P(x) \wedge \exists x\, Q(x)$
5. $\forall x\, [P(x) \wedge Q] \equiv \forall x\, P(x) \wedge Q$
6. $\exists x\, [P(x) \vee Q] \equiv \exists x\, P(x) \vee Q$
7. $\forall x\, [P(x) \vee Q] \equiv \forall x\, P(x) \vee Q$
8. $\exists x\, [P(x) \wedge Q] \equiv \exists x\, P(x) \wedge Q$

E.g. $P(1)$ T, $P(2)$ F; $Q(1)$ F, $Q(2)$ T

$\forall x\, [P(x) \rightarrow Q(x)]$

$[P(1) \rightarrow Q(1)] \wedge [P(2) \rightarrow Q(2)]$

T F F T

F $\wedge$ T

F

$\forall x\, P(x) \rightarrow \forall x\, Q(x)$

$[P(1) \wedge P(2)] \rightarrow [Q(1) \wedge Q(2)]$

T F T F

F $\rightarrow$ F

T

9. $\boxed{\forall x\, [P(x) \rightarrow Q(x)] \Rightarrow \forall x\, P(x) \rightarrow \forall x\, Q(x)}$
10. $\boxed{\forall x\, [P \rightarrow Q(x)] \equiv P \rightarrow \forall x\, Q(x)}$

- $\forall x\, [P \rightarrow Q(x)]$

$\equiv \forall x\, [\neg P \vee Q(x)]$

$\equiv \neg P \vee \forall x\, Q(x)$

11. $\exists x\, [P \rightarrow Q(x)] \equiv P \rightarrow \exists x\, Q(x)$
12. $\boxed{\forall x\, [P(x) \rightarrow Q] \equiv \exists x\, P(x) \rightarrow Q}$

$\forall x\, [P(x) \rightarrow Q]$

$\equiv \forall x\, [\neg P(x) \vee Q]$

$\equiv \neg[\exists x\, P(x)] \vee Q$

$\equiv \exists x\, P(x) \rightarrow Q$

13. $\boxed{\exists x\, [P(x) \rightarrow Q] \equiv \forall x\, P(x) \rightarrow Q}$

Scope of a Quantifier:

→ The extent upto which a quantifier is applicable.

Def: "A variable within scope of its quantifier is called bound variable."

Other variables are called free variables.

NOTE A quantified predicate is a proposition only when all variables are bound variables.

E.g.

$\underbrace{\forall x\, P(x)}_{\text{scope}} \vee Q(x)$ → Predicate (x in $P(x)$: bound; x in $Q(x)$: free)

$\underbrace{\forall x\, [P(x) \vee Q(x)]}_{\text{scope}}$ → Proposition (bound, bound)

$\underbrace{\forall x\, [P(x) \vee Q(y)]}_{\text{scope}}$ → Predicate (x: bound; y: free [should be in y's scope])

RELATIONS:

$A \times B = \{(a,b) \mid a \in A;\ b \in B\}$

E.g.: $A = \{1,2,3\}$ $B = \{2,3,4\}$

$A \times B = \{(1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,2), (3,3), (3,4)\}$

"$=$" $= \{(2,2), (3,3)\}$

"$<$" $= \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

"brother of" $= \{\ \}$

"$x+y \geq 8$" $= \{\ \}$

Definition: A relation R from A to B is a subset of $A \times B$.

i.e., $R \subseteq A \times B$

NOTE: Relation from A to A is called Relation on A.

Operations: R, R_1, R_2 are relation from A to B.

1. $R^c = \{(a,b) \mid (a,b) \notin R$ but $(a,b) \in A \times B\}$
2. $R_1 \cap R_2 = \{(a,b) \mid (a,b) \in R_1$ and $(a,b) \in R_2\}$
3. ~~$R_1 \cap R_2$~~ $R_1 \cup R_2 = \{(a,b) \mid (a,b) \in R_1$ or $(a,b) \in R_2$ or both$\}$
4. $R_1 - R_2 = \{(a,b) \mid (a,b) \in R_1$ but $(a,b) \notin R_2\}$

Inverse:

$R \subseteq A \times B$

then $R^{-1} \subseteq B \times A$ defined as

$R^{-1} = \{(b,a) \mid (a,b) \in R\}$

NOTE:
* Imp:

R	R^{-1}
$=$	$=$
$<$	$>$
$\leq$	$\geq$
(divides) $\mid$	Multiple of
$\subseteq$	$\supseteq$ (superset)

For every relation, we have an inverse relation.

Composite Relations:

Let $R \subseteq A \times B$
$S \subseteq B \times C$

then $RS \subseteq A \times C$ defined

$RS = \{(a,c) \mid \exists\, b \in B$ such that $(a,b) \in R$ and $(b,c) \in S\}$

E.g. $A = \{1,3,5\}$ $B = \{2,3,4\}$ $C = \{3,5\}$

$R \subseteq A \times B$ $S \subseteq B \times C$

defined ① xRy iff $y = x+1$
xSy iff $y = x+2$. Find RS.

Soln.
$R = \{(1,2), (3,4)\}$
$S = \{(3,5)\}$

$1 \to 2 \to \times$
$3 \to 4 \to \times$

$RS = \{\}$

② xRy iff ~~$x \leq y$~~ $y = x+1$
xSy iff $x \leq y$
$R = \{(1,2), (3,4)\}$
$S = \{(2,3), (2,5), (3,3), (3,5), (4,5)\}$

$1 \to 2 \to 3, 5$
$3 \to 4 \to 5$

$\boxed{RS = \{(1,3), (1,5), (3,5)\}}$

RESULT:

① $|A| = m$ $|B| = n$
$|A \times B| = m.n$

② $|A| = m$ $|B| = n$
$|P(A \times B)| = 2^{mn}$] Powerset of $A \times B$

③ $|A| = m$ $|B| = n$
No. of relations from A to B = No. of subsets of $A \times B$
$= 2^{mn}$

④ $|A| = n$
No. of relations on $A = 2^{n^2}$
$(R \subseteq A \times A)$

04/07/2017

Eg. ① Let $A = \{1,2,3\}$
$A \times A = \{(1,1), (2,2), (3,3), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)$ Universal relation on A

$\phi = \{\}$
$\Delta = \{(1,1), (2,2), (3,3)\}$
$< = \{(1,2), (1,3), (2,3)\}$
$> = \{(2,1), (3,1), (3,2)\}$
$\leq = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$
$\geq = \{(1,1), (2,2), (2,1), (3,1), (3,2), (3,3)\}$
$\mid = \{(1,1), (1,2), (1,3), (2,2), (3,3)\}$
m of $= \{(1,1), (2,1), (2,2), (3,1), (3,3)\}$

Definition: A relation R on A, is ~~reflexive, if~~ :

① **Reflexive** if $(a,a) \in R \quad \forall a \in A$

i.e. aRa, $\forall a \in A$

Eg ① Yes → $A \times A$, Δ, $\leq$, $\geq$, l.m. of
No → ϕ, $<$, $>$.

$R_{11} = \{(1,1), (2,2), (2,3)\}$ on $A = \{1,2,3\}$
Not reflexive

② **Irreflexive** if $(a,a) \notin R \quad \forall a \in A$

i.e., $a \not R a$, $\forall a \in A$

E.g. ①. Yes → ϕ, $<$, $>$
No → $A \times A$, Δ, $\leq$, $\geq$, l, m. of

$R_1 = \{(1,1),(2,2),(2,3)\}$ on $A = \{1,2,3\}$
Not reflexive
Not irreflexive.

③ **Symmetric** if $(a,b) \in R$, then, $(b,a) \in R$ where $a,b \in A$.

i.e. aRb, then, bRa,

Eg ① Yes → $A \times A$, ϕ, Δ
No → ~~ϕ~~ $<$, $>$, $\leq$, $\geq$, l, m. of

$R_{23} = \{(1,2), (2,3), (3,2)\}$
Not symmetric

④ **Asymmetric** if $(a,b) \in R$, then, $(b,a) \notin R$ where $a, b, \in A$
i.e., if aRb then $b \not R a$. (Doesn't allow reflexive elements)

E.g.① Yes → ϕ, $<$, $>$,
No → $A \times A$, Δ, $\leq$, $\geq$, l, m. of

$R_{23} = \{(1,2),(2,3),(3,2)\}$ Not asymmetric	$R_{41} = \{(2,3)\}$ Asymmetric

⑤ **Anti-symmetric** if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$ where $a, b \in A$

i.e., if aRb and bRa then $a = b$.

Every asymmetric is antisymmetric

Eg ①: Yes → ϕ, Δ, $<$, $>$, $\geq$, $\leq$, l, m. of
No → $A \times A$,

$R_{23} = \{(1,2),(2,3),(3,2)\}$ Not anti symmetric	$R_{21} = \{(1,1)\}$ Not ~~as~~ asymmetric, ~~An~~ Symmetric, Anti symmetric

$R_{a1} = \{(1,1),(1,2)\}$
Not symmetric
Not asymmetric, Anti symmetric.

⑥ **Transitive** if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.
i.e., if aRb and bRc then aRc.

E.g. ①
Yes → $A \times A$, $\{\}$, Δ, $<$, $>$, $\leq$, $\geq$, l, m. of
No →

$R_{210} = \{(1,3),(3,1)\}$ Not transitive	$R_{420} = \{(1,3),(3,1),(1,1)\}$ Not Transitive	$R_{211} = \{(1,1)\}$ Transitive; $R_{212} = \{(1,2)\}$ Transitive

Representation:

Matrix Relation: $M_R = [a_{ij}]_{n \times n}$

where $a_{ij} = \begin{cases} 0 & (i,j) \notin R \\ 1, & (i,j) \in R \end{cases}$

Digraph: Each element in A is ~~r~~ represented by a vertex.

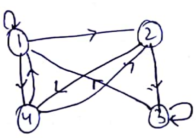
$(i,j) \in R \qquad \underset{i}{\circ} \longrightarrow \underset{j}{\circ}$

E.g. $A = \{1,2,3,4\}$

$R = \{(1,1),(1,2),(1,4),(2,3),(2,4),(3,1),(3,3),(4,1),(4,2)\}$

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 \end{array}$$

Digraph:



Results

Let ~~R1 R2~~ R, R_1 and R_2 ~~ar~~ be relations on A.

R, R_1 and R_2 are	R^{-1}	$R_1 \cap R_2$	$R_1 \cup R_2$
① Reflexive	✓	✓	✓
② Irreflexive	✓	✓	✓
③ Symmetric	✓	✓	✓
④ Asymmetric	✓	✓	✗
⑤ Antisymmetric	✓	✓	✗
⑥ Transitive	✓	✓	✗

✗ → Need not be

E.g. $A = \{1,2,3\}$

$R_1 = \{(1,2)\}$	Asym	Antisym	Transitive
$R_2 = \{(2,1)\}$	Asym	Antisym	Transitive
$R_1 \cup R_2 = \{(1,2),(2,1)\}$	Not asymm.	Not Antisym.	Not Transitive

Closures:

① **Reflexive Closure of R (R_r):**

Smallest reflexive relation containing R.

E.g. $A = \{1,2,3\}$

$R = \{(1,1),(2,2),(2,3)\}$

$R_r = \{(1,1),(2,2),(2,3),(3,3)\}$

Results:

① $R_r = R \cup \Delta$

② R is reflexive iff $R = R_r$.

② **Symmetric Closure of R (R_s):**

Smallest symmetric relation containing R.

E.g. $A = \{1,2,3\}$

$R = \{(1,2),(2,1),(2,3)\}$

$R_s = \{(1,2),(2,1),(2,3),(3,2)\}$

Results:

① $R_s = R \cup R^{-1}$

② R is symmetric iff $R = R_s$

③ **Transitive Closure of R (R^+):**

Smallest transitive relation containing R.

Result:

① R is transitive iff $R = R^+$

② $R^+ = R \cup R^2 \cup R^3 \cup \dots \cup R^n$

when $R^2 = RR$

$R^3 = R^2 R$

Warshall's Algorithm: [Complexity: $\Theta(n^3)$]

[To find transitive closure of R]

E.g. $A = \{1,2,3,4\}$

$R = \{(1,2), (2,1), (3,4), (4,3)\}$, find R^{+} (Transitive closure of R)

$$M_R^{(0)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & \boxed{0} & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

→ cross the columns corresponding to 0 in the selected row.

→ Uncrossed element

If
0 → 1] conversion
1 → 1]

cross the rows corresponding to 0 in the selected column

$$M_R^{(1)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \boxed{0} & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

→ Convert to 1.

$$M_R^{(2)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & \boxed{0} \to 1 \end{array}$$

$$M_R^{(3)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & \boxed{0} & 1 \\ 4 & 0 & 0 & 1 & 1 \end{array}$$

$$M_R^{(4)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 1 \\ 4 & 0 & 0 & 1 & 1 \end{array}$$

Ans. Transitive Closure

$R^{+} = \{(1,1), (1,2), (2,1), (3,2), (3,3), (3,4), (4,3), (4,4)\}$

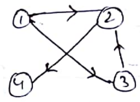
Q. H.W.

Find transitive closure:

① $A = \{1,2,3,4\}$

$R = \{(1,2), (2,3), (3,4), (4,1)\}$,

②



Counting: Let $|A| = n$:

① $|A \times A| = n^2$

② No. of relations on $A = 2^{n^2}$

③ No. of reflexive relations on $A = 2^{n(n-1)}$

④ No of irreflexive relations on $A = 2^{n(n-1)}$

⑤ No. of symmetric relations on $A = 2^{\frac{n(n+1)}{2}}$

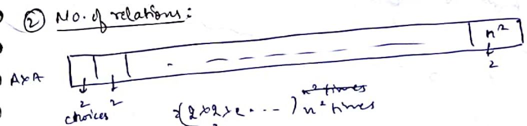
⑥ No. of asymmetric relations on $A = 3^{\frac{n(n-1)}{2}}$

⑦ No. of antisymmetric relations on $A = 2^n \cdot 3^{\frac{n(n-1)}{2}}$

⑧ No. of transitive relations on A = No closed formula.

⑨ No. of Symmetric and reflexive relations on $A = 2^{\frac{n(n-1)}{2}}$

② No. of relations:

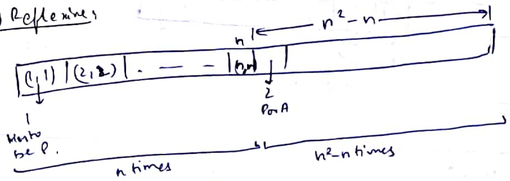


Each element in A×A has 2 choices (Present or Absent in relation)

$= (2 \times 2 \times 2 \cdots)$ ~~n² times~~ n² times

$= 2^{n^2}$

③ Reflexive:

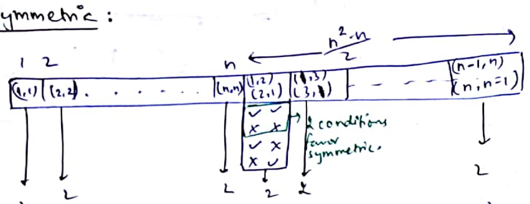


$\Rightarrow 1^n \cdot 2^{n^2-n} = 2^{n^2-n}$

$= 2^{n(n-1)}$

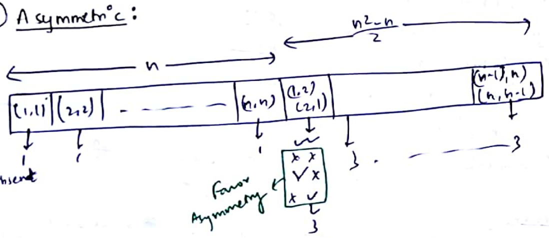
④ Irreflexive: Reflexive elements should be absent.

⑤ Symmetric:



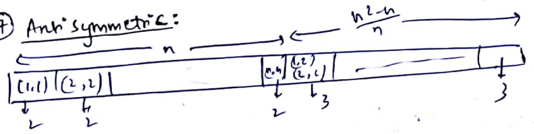
$\Rightarrow 2^n \cdot 2^{\left(\frac{n^2-n}{2}\right)} = 2^{n+\frac{n(n-1)}{2}} \Rightarrow 2^{\frac{n^2+n}{2}} = 2^{\frac{n(n+1)}{2}}$

⑥ Asymmetric:



$\Rightarrow 1^n \cdot 3^{\frac{n^2-n}{2}} = 3^{\frac{n(n-1)}{2}}$

⑦ Antisymmetric:



$\Rightarrow 2^n \cdot 3^{\frac{n(n-1)}{2}}$

Eg-1 $A = \{1,2,3,4,5\}$

① 2^{25}

②

③

④

⑤

⑥

⑦

Partition of a Set:

→ A non-empty collection of non-empty subsets.
$P = \{S_1, S_2, \dots, S_n\}$ of a set 'S' such that:

① $S_1 \cup S_2 \cup \dots \cup S_n = S$ (Collectively exhaustive)

② $S_i \cap S_j = \phi$ (Mutually exclusive)

is called partition of S.

Eg. $S = \{1,2,3,4\}$

$P_1 = \{\{1\},\{2\},\{3\},\{4\}\}$ → 4-part partition.

$P_2 = \{\{1,2\},\{3\},\{4\}\}$ → 3-part partition.

$P_3 = \{\{1,2,3\},\{4\}\}$ → 2-part partition

$P_4 = \{\{1,2,3,4\}\}$ → 1-part partition

$P_5 = \{\{1,2\},\{4\}\}$ → Not a partition

$P_6 = \{\{1,3,4\},\{2,4\}\}$ → Not a partition.

Definition: A relation R on A which is
① Reflexive
② Symmetric
and ③ Transitive
is called an equivalence relation.

Def.: Let R be an equivalence relation on A and $a \in A$. The equivalence class of a is defined as:

$[a] = \bar{a} = \{b \mid (a,b) \in R\}$

Eg. $A = \{1,2,3,4\}$
$R = \{(1,1),(2,2),(3,3),(4,4)\}$

Reflexive
Symmetric
Transitive
→ E.R. on A.

Equivalence Class,

Eg. [1] who are related to 1 {1}

[2] = {2}

[3] = {3}

[4] = {4}

$P = \{\{1\}, \{2\}, \{3\}, \{4\}\} \Rightarrow$ partition of A.

* Parts of the partition are distinct equivalence classes.

(2) $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$

Ref. / Sym / Tran.] E.R. ✓

Eq class:

$[1] = \{1,2\}$

$[2] = \{1,2\}$

$[3] = \{3,4\}$

$[4] = \{3,4\}$

Any two eq. classes are either completely same or completely different (distinct)

$P = \underbrace{\{\{1,2\},\{3,4\}\}}_{\text{Partition of } A}$

* For every eq. relation we can make a partition and vice versa

Result: Let R be an E.R on A and $a, b \in A$.

1. $a \in [a]$
2. $a \in [b]$ then $b \in [a]$
3. $a \in [b]$ then $[a] = [b]$
4. $[a] = [b]$ (or) $[a] \cap [b] = \phi$

Eg. Find E.R. for partition $P = \{\{1,3\}, \{2\}, \{4\}\}$ of $A = \{1,2,3,4\}$

$R = \{1,3\} \times \{1,3\} \cup \{2\} \times \{2\} \cup \{4\} \times \{4\}$

$= \{(1,1), (1,3), (3,1), (3,3), (2,2), (4,4)\}$

Eg. Find E.R for partition

$P = \{\{1,3,4\} \{2\}\}$ of $A = \{1,2,3,4\}$

$R = \{1,3,4\} \times \{1,3,4\} \cup \{2\} \times \{2\}$

$= \{(1,1), (1,3), (1,4), (3,1), (3,3), (3,4), (4,1), (4,3), (4,4), (2,2)\}$

Result:

1. There is one to one (1-1) correspondence between no. of E.R. on A and no. of partitions of A.
2. $|A| = n$

   No. of E.R. on A = No. of partitions of A = B_n (Bell No.)

3. Bell No.:

$$\boxed{B_n = \sum_{k=0}^{n-1} {}^{n-1}C_k B_k}$$

$B_0 = 1 \quad B_1 = 1$

$B_2 = {}^1C_0 B_0 + {}^1C_1 B_1$

$= 1 + 1$

$= 2$

$B_3 = {}^2C_0 B_0 + {}^2C_1 B_1 + {}^2C_2 B_2$

$= 1 + 2 + 1(2)$

$= 5$

$B_4 = {}^3C_0 B_0 + {}^3C_1 B_1 + {}^3C_2 B_2 + {}^3C_3 B_3$

$= 1 + 3 + 3(2) + 1.(5)$

$B_4 = 15$

$B_5 = {}^4C_0 B_0 + {}^4C_1 B_1 + {}^4C_2 B_2 + {}^4C_3 B_3 + {}^4C_4 B_4$

$= 1 + 4 + 6(2) + 4(5) + 1.(15)$

$= 17 + 20 + 15$

$= \boxed{52}$

E.g. 1. No. of partitions of $A = \{a, b, c, d\} = B_4 = 15$

2. No. of E.R. on $A = \{1, 2, 3\} = B_3 = 5$

Result

1.

R, R_1, R_2	R^{-1}	$R_1 \cap R_2$	$R_1 \cup R_2$
ref	ref	ref	ref.
sym	sym	sym	sym
tran.	tran.	trans	× need not be
E.R	E.R.	E.R.	×

2. $A = \{1, 2, 3, \ldots, n\}$
$|A| = n$
1. Smallest E.R. on $A = \Delta$.
2. Largest E.R. on $A = A \times A$

3. $|A| = n$
No. of elements in
1. Smallest E.R. on $A = n$
2. Largest E.R on $A = n^2$

Chp 3. WB

24. R on Z

xRy iff $x+y$ is even

$x+x = 2x$ even
$\Rightarrow xRx$ Ref.

$x+y$ = even
$y+x$ = even. $\Rightarrow$ sym

$x+y = 2l$
$y+z = 2m$
$x+z+2y = 2(l+m)$
$x+z = 2(l+m-y)$
even
$xRz \Rightarrow$ trans.

$\therefore$ E.R.

Now. Equi class.

$[0] = \{\ldots, -2, 0, 2, 4, 6, 8 \ldots\}$

$[1] = \{\ldots, -3, -1, 1, 3, 5, 7, 9 \ldots\}$

No of distinct eq. classes = 2

Partition $P = \{[0], [1]\}$

Def. A relation R on A which is
1. Reflexive
2. Anti symmetric
and 3. Transitive
is called partial ordered relation.

E.g. 1. $\leq$ on Z
(a) only reflexive (b) only anti (c) only trans (d) All

$a \leq a$ ref.
$a \leq b$ and $b \leq a \Rightarrow a = b$ Anti
$a \leq b$ and $b \leq c \Rightarrow a \leq c$ Trans.

Partial ordered rel.

2. ~~only~~ | on Z
only transitive

Not partial ordered. ~~All~~ $\Rightarrow$ of o

$a | a$; ~~ref~~ Not ref
$a | b$ and $b | a \Rightarrow a = b$ ~~Anti~~ Not anti $\Rightarrow -1 | 1$ & $1 | -1 \Rightarrow -1 \neq 1$
$a | b$ and $b | c \Rightarrow a | c$ Trans

3. | on Z^+
(a) only reflexive (b) only anti sym (c) only trans. (d) All

Partial ordered relation

4. $\subseteq$ on $P(A)$
(a) only ref (b) only anti (c) only trans. (d) All

$A \subseteq A \Rightarrow$ ref.
$A \subseteq B$ & $B \subseteq A \Rightarrow B = A$; anti sym
$A \subseteq B$ & $B \subseteq C \Rightarrow A \subseteq C$; Trans.

Partial ordered.

Def. A set P together with a partial ordered relation R, $\langle P, R \rangle$, is called partial ordered set (poset)

E.g. $\langle Z, \leq \rangle$ is poset.

RESULT:

①

R	R^{-1}
ref.	ref.
anti	anti
trans	trans.
P.O.R	POR
$\langle P, R \rangle$ poset	$\langle P, R^{-1} \rangle$ is also poset

$\boxed{\text{NOTE}}$ The poset $\langle P, R \rangle$ and $\langle P, R^{-1} \rangle$ are called **duals**.

Def. D_n = set of positive ~~integ~~ divisors of n.

Ex. $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

Result: ① $\langle D_n, | \rangle$ is poset

②

$\langle P, R \rangle$	$\langle P, R^{-1} \rangle$
① $\langle Z, \leq \rangle$ is poset	① $\langle Z, \geq \rangle$ is poset
② $A \subseteq Z$ $\langle A, \leq \rangle$ is poset	② $A \subseteq Z$ $\langle A, \geq \rangle$ is poset
③ $\langle Z^+, | \rangle$ is poset	③ $\langle Z^+$, m. of $\rangle$ is poset
④ $A \subseteq Z^+$ $\langle A, | \rangle$ is poset	$A \subseteq Z^+$ $\langle A$, m. of $\rangle$ ... is poset
⑤ $\langle D_n, | \rangle$ is poset	⑤ $\langle D_n$, m. of $\rangle$ is poset
⑥ $\langle P(A), \subseteq \rangle$ is poset	⑥ $\langle P(A), \supseteq \rangle$ is poset

Notation: Partially ordered relation $\rightarrow \preccurlyeq$

Definition: Let $\langle P, \preccurlyeq \rangle$ be a poset.

$\rightarrow$ Two elements a and b are said to be comparable, if either $a \preccurlyeq b$ or $b \preccurlyeq a$.

E.g. $A = \{1, 2, 3, 4\}$

① $\langle A, \leq \rangle$ is poset	① $\langle A, | \rangle$ is poset
1, 3 are comparable	2, 4 comparable
$1 \leq 3$	$2	4$
4, 2 comparable	2, 3 not comparable
$4 \nleq 2$ but $2 \leq 4$	$2 \nmid 3$, $3 \nmid 2$
Toset $(1 \leq 2 \leq 3 \leq 4)$	Poset

$\boxed{\text{Def. A poset } \langle P, \preccurlyeq \rangle \text{ in which every pair of elements is comparable is called Totally ordered set (Toset) or chain.}}$

Hasse diag. is like a chain

Def. Let $\langle P, \preccurlyeq \rangle$ be a poset

Associated Relation:

$x \prec y$ means $x \preccurlyeq y$ but $x \neq y$.

Covering:

y cover x means $[x \prec y \wedge [x \preccurlyeq z \prec y \Rightarrow x = z \vee y = z]]$

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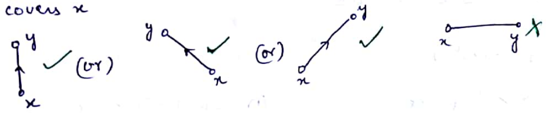
Hasse Diagram (Poset Diagram)

Let $\langle P_1, \preceq \rangle$ be a poset

① Every element in P is denoted by small circle [o].

② y covers x

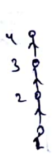


③ $x \preceq y$ but y does not cover x.

There will be n-edges between y and x.

E.g. $A = \{1, 2, 3, 4\}$

$\langle A, \leq \rangle$ is poset



Toset (chain)

E.g. $A = \{1, 2, 3, 4, 5, 6\}$; $\langle A, | \rangle$

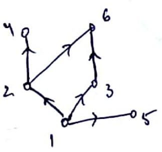
1, 2

~~1~~, 1, 3

~~1~~, 2, 4

1, 5

~~1~~, 2, 3, 6



E.g. $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$\langle A, | \rangle$

1, 2

1, 3

~~1~~, 2, 4

1, 5

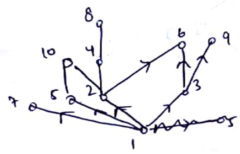
1, 2, 3, 6

1, 7.

~~1~~, ~~2~~, 4, 8

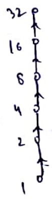
~~1~~, 3, 9

1, ~~2~~, 5, 10



E.g. $\langle D_{32}, | \rangle$

$D_{32} =$ 1, 2, 4, 8, 16, 32



$\langle D_{24}, | \rangle$

$D_{24} =$ 1, 2, 3, 4, 6, 8, 12, 24

1, 2

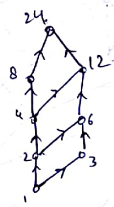
1, 2, 3, 4

~~1~~, 2, 6

1, 2, 4, 8

~~1~~, ~~2~~, 4, ~~12~~, 6, 12

~~1~~, ~~2~~, ~~4~~, ~~6~~, 8, 12, 24



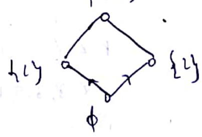
E.g. $A = \{1, 2\}$ | $P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

$\langle P(A), \subseteq \rangle$

ϕ, {1}

ϕ, {2}

ϕ, {1}, {2}, {1, 2}



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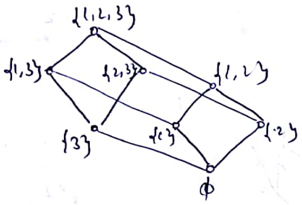
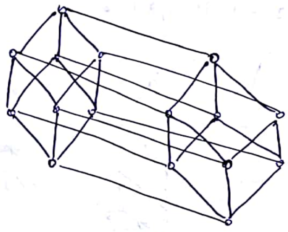
E.g. $A = \{1,2,3\}$

$\langle P(A), \subseteq \rangle$



No. of lines = $4+4+4 = 12$

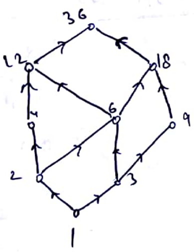
E.g. $A = \{1,2,3,4\}$



$12+12+8 = 32$ lines.

E.g. $\langle ~~36~~ \, D_{36}, | \rangle$

$36 = \{1,2,3,4,6,9,12,18,36\}$



1, 2

1, 3

~~1~~, 2, 4

~~1~~, 2, 3 6

~~1~~, 3, 9

~~1~~, ~~2~~, ~~3~~, 4, 6, 12

~~1~~, ~~2~~, ~~3~~, 6, 9, 18

1, .

18 lines

Special Element:

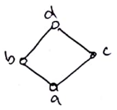
Def. Let $\langle P, \preceq \rangle$ be a poset.

A element $l \in P$ is maximal if There is no $x \in P$ such that $l \preceq x$ ⟹ Elements not having horns

An element $m \in P$ is minimal if there is no $x \in P$ such that $x \preceq m$ ⟹ Elements not having tails.

Q. Find maximal and minimal elements.

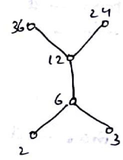
E.g.



Maxi – d

Min – a

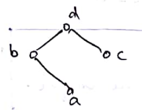
E.g.



Max — 24, 36

Min — 2, 3

E.g.



Max – d

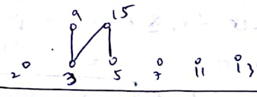
Min – a, c

E.g. $\langle P, | \rangle$

$P = \{2,3,5,7,9,11,13,15\}$

Max – 2, 9, 15, 7, 11, 13

Min – 2, 3, 5, 7, 11, 13



RESULT

① ~~for a~~ finite posets always have maximal & minimal elements.

② Infinite posets need not have maximal and minimal elements.

E.g. $\langle Z, \leq \rangle$ is poset

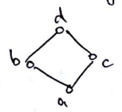
Max –
Min –] Does not exist

Def. Let $\langle P, \preceq \rangle$ be a poset.
An element $g \in P$ is <u>greatest</u> if $\boxed{x \preceq g \ \forall x \in P}$

An element $l \in P$ is <u>least</u> if $\boxed{l \preceq x \ \forall x \in P}$

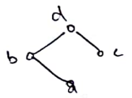
<u>Notation</u>: Greatest $\longrightarrow 1$
Least $\longrightarrow 0$

Eg. Find greatest and least.



max $\to$ d min $\to$ a

$1 \to d$
$0 \to a$



max $\to$ d min $\to$ a, c

$1 \to d$
$0 \to$ Does not exist

<u>RESULT</u>:
① Greatest and least elements, if exists, are unique.

Def. Let $\langle P, \preceq \rangle$ be a poset and $A \subseteq P$.
An element $l \in P$ is lower bound (lb) of A if $\boxed{l \preceq x \ \forall x \in A}$

An element $u \in P$ is upper bound (ub) of A if $\boxed{x \preceq u \ \forall x \in A}$

Eg. $P = \{1, 2, 3, 4, 5, 6\}$
$\langle P, \leq \rangle$ is a poset.
$A = \{2, 5\}$
lb's of $A = 1, 2$
ub's of $A = 5, 6$

Least Upper bound (LUB):
lub $\leq$ ub

Greatest lower bound (GLB)
lb $\leq$ glb

lub$\{2,5\} = 5$
glb$\{2,5\} = 2$

lub$\{3,4\} = 4$
glb$\{3,4\} = 3$

lub$\{1,6\} = 6$
glb$\{1,6\} = 1$

Def. A poset $\langle P, \preceq \rangle$ in which every two element subset (every pair of elements) having lub and glb is called <u>lattice</u>. [Every toset is a lattice]

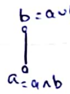
Def. A poset $\langle P, \preceq \rangle$ in which every subset having lub and glb is called <u>complete lattice</u>.

<u>Notation</u>: lub$\{a,b\} = a \vee b$ a join b
glb$\{a,b\} = a \wedge b$ a meet b

NOTE

	$\leq$	$\mid$	$\subseteq$
lub$\{A,B\}$	max$\{A,B\}$	LCM$\{A,B\}$	$A \cup B$
glb$\{A,B\}$	min$\{A,B\}$	GCD$\{A,B\}$	$A \cap B$

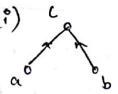
<u>Consistency</u>:
Let $\langle P, \preceq \rangle$ be a poset then $a \preceq b$ iff $a \wedge b = a$ iff $a \vee b = b$



[If $a \preceq b$ then lub and glb exist]

Find lub and glb of non-comparable elements:

① (i)



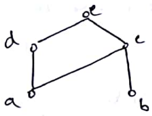
→ Walk along their horns.
→ ~~Where~~ The point where they meet is the lub.
$a \vee b = c$

(ii)

→ Walk along their tails to find glb.

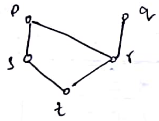
$p \wedge q = r$

② (i)



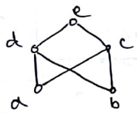
$a \vee b = lub = c$ → c, e (ub's)

(ii)



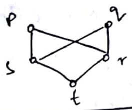
$p \wedge q = glb = r$ → r, t (glb's)

③ (i)



$a \vee b$ = Does not exist → c, d, e ub's

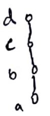
(ii)



$p \wedge q$ = Does not exist. → r, s, t lb's

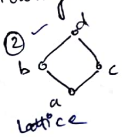
Q. Which of the following is/are lattice(s):

①



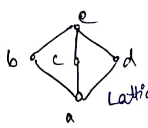
Lattice

②



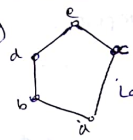
Lattice

③



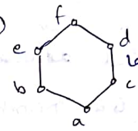
Lattice

④



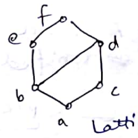
Lattice

⑤



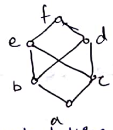
Lattice

⑥



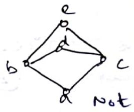
Lattice

⑦



Not lattice

⑧



Not lattice

Results

① In any lattice $\langle L, \preceq \rangle$, the following properties always hold good:

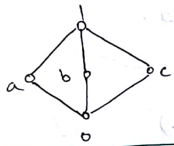
① Idempotent
$a \vee a = a$
$a \wedge a = a$

② Commutative:
$a \vee b = b \vee a$
$a \wedge b = b \wedge a$

③ Associative
$a \vee (b \vee c) = (a \vee b) \vee c$
$a \wedge (b \wedge c) = (a \wedge b) \wedge c$

④ Absorption:
$a \vee (a \wedge b) = a$
$a \wedge (a \vee b) = a$

E.g.



$a \vee (b \wedge c)$
$= a \vee 0$
$= a$

$(a \vee b) \wedge (a \vee c)$
$= 1 \wedge 1$
$= 1$

Hence distributive not applicable.

Results

② In any lattice $\langle L, \preceq \rangle$ the following distributive inequalities satisfied.

① $a \vee (b \wedge c) \preceq (a \vee b) \wedge (a \vee c)$

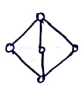
② $(a \wedge b) \vee (a \wedge c) \preceq a \wedge (b \vee c)$

Def. A lattice $\langle L, \preccurlyeq \rangle$ in which distributive properties ~~st~~ satisfied is called distributive lattice.

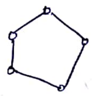
E.g. ① $\langle \underbrace{P(A)}_{\text{Powerset}}, \underbrace{\subseteq}_{\text{subset}} \rangle$ is a Distributive lattice

Famous non-distributive lattices :

① Diamond Lattice



② Pentagon Lattice



05/07/2017 :

Bounded Lattice :

→ A lattice $\langle L, \preccurlyeq \rangle$ in which greatest and least elements exist.

i.e., $\boxed{0 \le x \le 1 \ \forall x \in L}$

NOTE

① Every finite lattice is bounded.

② Infinite lattices need not be bounded.

E.g. $\langle Z, \le \rangle$ is a lattice.

It is not bounded.

(No greatest & no least elements)

Definition: Let $\langle L, \preccurlyeq \rangle$ be a bounded lattice.

$b \in L$ is complement of $a \in L$.

if : $a \vee b = 1$

$a \wedge b = 0$

NOTE

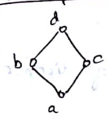
① If b is complement of a then a is complement of b.

② $\left.\begin{array}{l} 1 \vee 0 = 1 \\ 1 \wedge 0 = 0 \end{array}\right\}$ "1" and "0" are complements of each other.

Definition: A bounded lattice $\langle L, \preccurlyeq \rangle$ in which every element having complement is called complemented lattice.

Find complements :

①



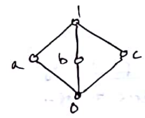
complemented lattice

element	complement
a	d
d	a
b	c
c	b

$b \vee c = 1$

$b \wedge c = 0$

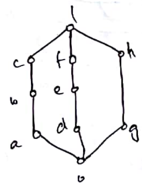
②



Complemented lattice

element	complement
a	b, c
b	a, c
c	~~b~~ a, b
0	1
1	0

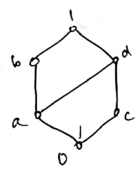
③



Complemented lattice

element	complement
a	d, g, e, f, h
b	d, e, f, g, h
c	d, e, f, h, g
d	a, b, c, g, h
e	a, b, c, g, h
f	a, b, c, g, h
g	a, b, c, d, e, f
h	a, b, c, d, e, f

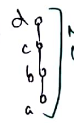
④



Not a complemented lattice

element	complement
a	~~d~~ Does not exist
b	~~c, d~~ c
c	~~a, b~~ b
d	~~b, a, b~~ Does not exist

⑤



Not complemented lattice

$\boxed{\text{* Every chain with 3 or more elements is not complemented}}$

RESULT :

In a distributive lattice, complements, if exist, are unique.

> **Definition:**
> Bounded, Distributive and Complemented lattice is called Boolean Algebra.

E.g. ① $\langle P(A), \subseteq \rangle$ is boolean algebra

RESULT

$\langle D_n, | \rangle$ is Boolean algebra <u>iff</u> n is product of distinct prime factors.

E.g. ① $\langle D_{30}, | \rangle$

$30 = 2 \times 3 \times 5$

Product of distinct prime factors

$\therefore \langle D_{30}, | \rangle$ is B.A.

② $\langle D_8, | \rangle$

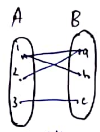
$8 = 2^3$, It is <u>not</u> product of distinct prime factors.

$\therefore$ Not a BA.

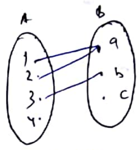
Functions :

→ A fn "f" from A to B is a rule which assign every element in A, to a unique element in B.

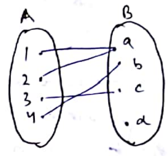
E.g.



Not f^n

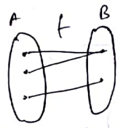


Not f^n



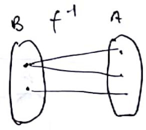
f^n (Into)

Not all elements of B are used.

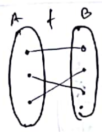


fn (onto)

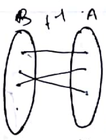
All elements of B are used.



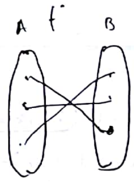
Not f^n



f^n (one-one)

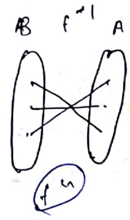


Not f^n



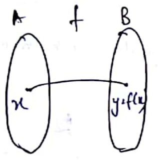
f^n

(1-1 and onto)



f^n

Terminology:



$f: A \to B$ is a function

A : Domain

B : Co-domain

$f(A)$: Range

$y = f(x)$: image of x

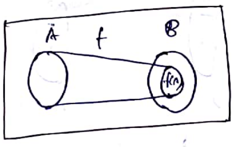
x : preimage

Note

$f(A) \subseteq B$

Onto f^n :

$f(A) = B$



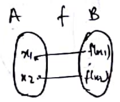
Definition

→ $f: A \to B$ is 1-1 (injection).

→ if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

(or)

→ if $f(x_1) = f(x_2)$ then $x_1 = x_2$



Definition

$f: A \to B$ is onto (surjection).

if $\boxed{\forall y \in B \; \exists x \in A \; [f(x) = y]}$

Definition

→ one-one and onto f^n is called bijection.

Result : $\boxed{f^{-1} \text{ exist iff } f \text{ is bijection (1-1 \& onto)}}$

Eg. ① $f: Z^+ \to Z^+$

$f(x) = x - 3$

ⓐ only 1-1 ⓑ only onto ⓒ both ✓ⓓ none.

$f(1) = 1 - 3 = -2 \notin Z^+$ (codomain) ∴ Not a fun^n.

② $f: Z \to Z$ $f(x) = 2x + 3$

✓ⓐ only 1-1 ⓑ only onto ⓒ both ⓓ none

1-1

$f(x_1) = f(x_2)$

$2x_1 + 3 = 2x_2 + 3$

$x_1 = x_2$

∴ 1-1

onto:

$y = f(x) = 2x + 3$

y in terms of x

write x in terms of y

$x = \frac{y-3}{2} \notin Z$ should be in co domain

should be in domain

$y = 2 \Rightarrow \frac{2-3}{2} = -\frac{1}{2} \notin Z$ Not onto

③ $f: R \to R$

$f(x) = 2x + 3$

ⓐ only 1-1 ⓑ only onto ✓ⓒ both ⓓ none

④ $f(x) = 2x + 3$. find $f^{-1}(x)$

$y = 2x + 3$

write in terms of x

$f^{-1}(y) = x = \frac{y-3}{2}$

$\Rightarrow \boxed{f^{-1}(x) = \frac{x-3}{2}}$

⑤ $f(x) = \dfrac{x+5}{2x-4}$, find $f^{-1}(x)$.

$y = \dfrac{x+5}{2x-4} \Rightarrow (2x-4)y = x+5$

$\Rightarrow 2xy - x = 5 + 4y \Rightarrow x(2y-1) = 5+4y$

$\Rightarrow \boxed{x = \dfrac{5+4y}{2y-1}} \Rightarrow f^{-1}(x) = \dfrac{4x+5}{2x-1}$

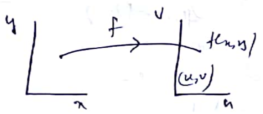
⑥ $f(x) = (5x+1)^2$

$\Rightarrow y = (5x+1)^2 \Rightarrow \sqrt{y} = 5x+1 \Rightarrow \sqrt{y} - 1 = 5x$

$\Rightarrow \dfrac{\sqrt{y}-1}{5} = x \Rightarrow \boxed{f^{-1}(x) = \dfrac{\sqrt{x}-1}{5}}$

⑦ $f(x,y) = (x+y, x-y)$ $f^{-1}(x,y) = ?$

$\Rightarrow (u,v) = (x+y, x-y)$



$x + y = u$
$-x \mp y = v$
$2y = u - v$
$y = \dfrac{u-v}{2}$

$x + y = u$
$x - y = v$
$2x = u + v$
$x = \dfrac{u+v}{2}$

$f^{-1}(u,v) = (x,y) = \left(\dfrac{u+v}{2}, \dfrac{u-v}{2}\right)$

$\boxed{f^{-1}(x,y) = \left(\dfrac{x+y}{2}, \dfrac{x-y}{2}\right)}$

⑧ $f(x,y) = (x+2y, 2x-y)$

$x + 2y = u$
$2\times$ $2x - y = v$
$5x = u + 2v$
$x = \dfrac{u+2v}{5}$

$2x + 2y = u$
$2x - y = v$
$5y = 2u - v$
$y = \dfrac{2u-v}{5}$

$f^{-1}(x,y) = \left(\dfrac{x+2y}{5}, \dfrac{2x-y}{5}\right)$

Counting:

① $f: A \to B$ is one-one f^n, then $|A| \le |B|$.

ⓐ $|A| \le |B|$ ⓑ $|A| \ge |B|$ ⓒ $|A| = |B|$ ⓓ none

② $f: A \to B$ is onto f^n, then $|A| \ge |B|$

ⓐ $|A| \le |B|$ ⓑ $|A| \ge |B|$ ⓒ $|A| = |B|$ ⓓ none

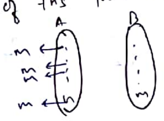
③ $f: A \to B$ is bijection (1-1 & onto) then $|A| = |B|$

ⓐ $|A| \le |B|$ ⓑ $|A| \ge |B|$ ⓒ $|A| = |B|$ ⓓ none

RESULTS:

① $|A| = n$ $|B| = m$

No. of fns from A to B $= |B|^{|A|}$



m^n

Q GATE: There are 97 fns from X to Y $|Y|^{|X|} = 97^1$

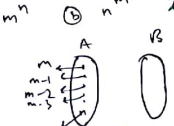
ⓐ $|X| = 1$ $|Y| = 97$ ⓑ $|X| = 97$ $|Y| = 1$ $= 97$

ⓒ $|X| = 97$ $|Y| = 97$ ⓓ None

② $|A| = n$ $|B| = m$

$\boxed{\text{No. of 1-1 fns from A to B} = {}^{m}P_{n}}$ ${}^{|B|}P_{|A|}$

ⓐ m^n ⓑ n^m ⓒ ${}^{m}P_{n}$ ⓓ ${}^{n}P_{m}$



: ${}^{m}P_{n}$

③ No. of ~~fns~~ 1-1 function ~~from~~ on A = $n!$; $|A| = n$

GATE: $A = \{1,2,3\}$ $B = \{a,b,c,d\}$

No. of 1-1 fns from A to B = ${}^4P_3 = 24$

No. of 1-1 fns ~~fro~~ on $A = \{1,2,3,4,5\} = 5! = 120$

Important

④ $|A| = n$ $|B| = m$

No. of onto fns from A to B

$$= \sum_{i=0}^{m} {}^mC_i (-1)^i (m-i)^n$$

E.g. **GATE:** $|A| = 4$ (n) $|B| = 3$ (m)

No. of onto functions from A to B.

$= {}^3C_0 (-1)^0 (3)^4 + {}^3C_1 (-1)^1 (3-1)^4 + {}^3C_2 (-1)^2 (3-2)^4 + {}^3C_3 (-1)^3 (3-3)^4$

$= 3^4 - 3 \cdot 2^4 + 3 -$

$= 81 - 48 + 3 \Rightarrow 36$

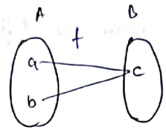
GATE: $|A| = n$ $|B| = 2$ (m)

No. of onto fns from A to B = ______

$= {}^2C_0 (2)^n - {}^2C_1 (1)^n + {}^2C_2 (0)^n$

$= \boxed{2^n - 2}$

E.g.



$C_1 = \{a\} \subseteq A$

$C_2 = \{b\} \subseteq A$

$C_1 = \{a\}$ $f(C_1) = \{c\}$

$C_2 = \{b\}$ $f(C_2) = \{c\}$

$C_1 \cup C_2 = \{a,b\}$ $f(C_1 \cup C_2) = \{c\}$

$C_1 \cap C_2 = \{\}$ $f(C_1 \cap C_2) = \{\}$

$f(C_1) \cup f(C_2) = \{c\}$

$f(C_1) \cap f(C_2) = \{c\}$

Result:

If $C_1 \subseteq A$ and $C_2 \subseteq A$ and $f: A \to B$ is a f^n, then

① $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$

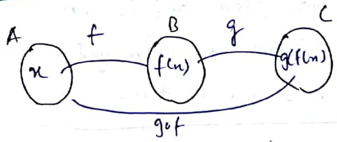
② $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$

Composite Function:

→ Let $f: A \to B$ and $g: B \to C$

then composite f^n. $gof: A \to C$ defined as

$$\boxed{gof(x) = g[f(x)]}$$



NOTE

① If gof is defined the fog ~~is~~ need not be defined.

E.g. $f: A \to B$ $g: B \to C$

gof is defined

$g: B \to C$ $f: A \to C$

fog is defined if $\boxed{C = A}$

② fog need not be equal to gof when both are defined.

E.g. $f(x) = x^2$ $\qquad$ $g(x) = x+2$

$gof(x)$
$= g[f(x)]$
$= g[f(x)]$
$= g[x^2]$
$= x^2+2$

$fog(x)$
$= f[g(x)]$
$= f[x+2]$
$= (x+2)^2$

$fog \neq gof$

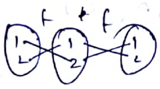
③ $fog = gof$ if, ~~for~~ f or g is identity.

$\boxed{f(x) = x} \to$ Identity.

④ $fof(x) = x$

Case1: $f(x) = x$

Case2: $f = f^{-1}$

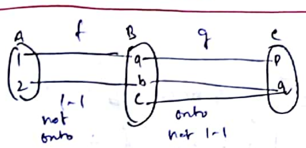


~~for~~ $f(1) = 1$
$f(2) = 2$

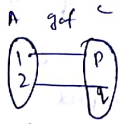
RESULTS:

I. ① If f and g are one-one, then gof is 1-1.
② If f and g are onto, then gof is onto
③ If f and g are 1-1 and onto then gof is 1-1 and onto.
But the converse need not be true.

Eg



$gof(1) = g(f(1)) = g(a) = p$
$gof(2) = g(f(2)) = g(b) = q$

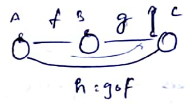


1-1 and onto.

II. ① If gof is 1-1 then f is 1-1.
② If gof is onto then g is onto.
③ If gof is 1-1 and onto, then, f is 1-1 and g is onto.

GATE: $f: A \to B$ · $g: B \to C$ $\quad h: A \to C$; such that $h(a) = g[f(a)]$ for all $a \in A$. Which of the following is always true.

a) g onto ⇒ h onto
ⓑ h onto ⇒ f onto
ⓒ h onto ⇒ g onto
ⓓ h onto ⇒ f and g onto.



GATE: Let X, Y, Z be ~~such~~ sets of sizes x, y, z respectively.
$W = X \times Y$ and E = set of all subsets of w.
No. of fns from z to E =

ⓐ $z^{2^{xy}}$ $\quad$ ⓑ $z \cdot 2^{xy}$ $\quad$ ⓒ $z^{2^{x+y}}$ $\quad$ ⓓ 2^{xyz}

$|w| = xy$ $\quad$ $E = 2^{xy}$ $\quad$ $f: z \to E$

$|E|^{|z|} = (2^{xy})^z = 2^{xyz}$

GATE: Let S denote set of all fns $f: \{0,1\}^4 \to \{0,1\}$
Denote by N the no. of fns from S to the set $\{0,1\}$.
The value of $\log_2 \log_2 N = 16$

$|S| = 2^{16}$ $\qquad$ $f: S \to \{0,1\}$

$= 2^{2^{16}}$

$\log_2 \log_2 2^{2^{16}} = \underline{16}$

$\{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\}$
$= \{0,1\}^4$

GATE: Let $S = \{1, 2, 3, \ldots, m\}$, $m > 3$

Let $X_1, X_2, \ldots, X_n$ be subsets of S each of size 3.

Define a fn f from S to N as:

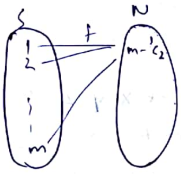
$f(i)$ = No. of sets X_j that contain i

i.e., $f(i) = |\{j \mid i \in X_j\}|$, then $\sum_{i=1}^{m} f(i)$:

(a) $3m$ (b) $3n$ (c) $2m+1$ (d) $2n+1$

$f: S \to N$

No. of subsets of S of size $3 = {}^{m}C_3 = n$



$f(i)$ = No. of subsets of S of size 3 containing i (X_j)

$= {}^{m-1}C_2$

$\sum_{i=1}^{m} f(i) = \sum_{i=1}^{m} {}^{m-1}C_2 = {}^{m-1}C_2 + {}^{m-1}C_2 + \cdots + {}^{m-1}C_2$

$\Rightarrow m \cdot {}^{m-1}C_2 = \dfrac{m \cdot (m-1)!}{2!\,(m-3)!}$

$\Rightarrow \dfrac{3 \cdot m\,(m-1)!}{3 \cdot 2!\,(m-3)!} = \dfrac{3m\,(m-1)!}{3!\,(m-3)!}$

$\Rightarrow \dfrac{3 \cdot m!}{3!\,(m-3)!} \Rightarrow 3 \cdot {}^{m}C_3 = \boxed{3 \cdot n}$

ALGEBRAIC STRUCTURES:

→ Let S be a non-empty set and $*$ be an operation.

(i) Closure: (S is closed w.r.t. $*$)

$a * b \in S \quad \forall a, b \in S$ $[\forall a \forall b\, [a*b \in S]]$

(ii) Associative: (S is associative w.r.t. $*$)

$a * (b * c) = (a * b) * c \quad \forall a, b, c \in S$

(iii) Existence of identity: (Identity exist in S w.r.t. $*$)

$\exists e \in S\ \forall a \in S\ [a * e = a]$

(iv) Existence of inverse: (Inverses exist in S w.r.t. $*$)

$\forall a \in S\ \exists b \in S\ [a * b = e]$

(v) Commutative (S is commutative w.r.t. $*$)

$a * b = b * a \quad \forall a, b \in S$.

Def. $(S, *)$ is

1. Closure — $(S, *)$ is an algebraic structure. or $*$ is binary operation on S.
2. Closure + associative — $(S, *)$ is semigroup.
3. Closure + associative + identity exist — $(S, *)$ is monoid.
4. Closure + associative + identity exist + inverses exist — Group.
5. Group + commutative — $(S, *)$ is an abelian group. (commutative group)

Scanned by CamScanner

$(S, *)$	Closure	Associative	Identity	Inverse	Commutative	Nature
① $(Z, +)$	✓	✓	✓ $0 \in Z$ $a+0=a$	$-a \in Z$ $a+(-a)=0$	✓	Abelian group
② $(Z, \times)$	✓	✓	$1 \in Z$ ✓ $a * 1 = a$	$1/a \notin Z$ ✗		Monoid
③ $(R, \times)$	✓	✓	$1 \in R$ ✓	$1/a \in R$ $0 \in R$ $\frac{1}{0} \notin R$ ✗		Monoid
④ $(R-\{0\}, \times)$	✓	✓	✓	✓	✓	Abelian group
⑤ $(M_{n \times n}, \times)$ $n \times n$, Matrix w.r.t. Multiplication	✓	✓ Matrix mul. is associative	$I_n \in M_{n \times n}$ $AI_n = A$ ✓	If $\lvert A \rvert = 0$ (singular) A^{-1} does not exists		Monoid
⑥ $(M_{n \times n}, \times)$ Set of non-singular matrices w.r.t. multiplication	✓	✓	✓	$\lvert A \rvert \neq 0$ non singular A^{-1} exists ✓	Matrix multi. need not be Commutative ✗	Group
⑦ Set of bijective func^n w.r.t composition of function.	✓	$(f \circ g) \circ h$ $= f \circ (g \circ h)$ ✓	$I(x) = x$ $f \circ I = f(x)$	f^{-1} exists for bijective fn^s ✓	✗	Group
⑧ $(Z, *)$ $n * m = \max\{n, m\}$	✓	✓	$n * e = n$ $\max\{n, e\} = n$ Does not exist ✗			Semi group

E.g. $x \circ y = x^2 + y^2$

~~(a)~~ Commutative only

(b) Associative only.

(c) both

(d) none

$(x \circ y) \circ z$ | $x \circ (y \circ z)$

$(x^2 + y^2) \circ z$ | $x^2 \circ (y^2 + z^2)$

$(x^2+y^2)^2 + z^2$ | $x^2 + (y^2+z^2)^2$

Only commutative

E.g. $(R - \{-1\}, \circ)$

$a \circ b = a + b + a \cdot b$.

① Closure: ✓

② Associative ✓

$(a \circ b) \circ c$	$a \circ (b \circ c)$
$(a+b+ab) \circ c$	$a \circ (b+c+bc)$
$(a+b+ab) + c + (a+b+ab) \cdot c$	$a + (b+c+bc) + a \cdot (b+c+bc)$
$= a+b+ab+c+ac+bc+abc$	$= a+b+c+bc+ab+ac+abc$

③ Identity: ✓

$a \circ e = a$

$a + e + a \cdot e = a$

$e(1+a) = 0$

$\boxed{e = 0}$

④ Inverse: ✓

$a \circ b = e$

$a + b + ab = 0$

$b(1+a) = -a$

$b = \frac{-a}{1+a}$

⑤ Commutative:

$a \circ b = b \circ a$

Abelian group

E.g. $(R - \{0\}, *)$ $\quad a * b = \frac{ab}{5}$

Identity → ✓ 5

Inverse → ✓ $b = \frac{25}{a}$

$a * e = a$

$\cancel{a * e} = \frac{a \cdot e}{5} = a$

$\boxed{e = 5}$

$\frac{ab}{5} = 5 \quad b = \frac{25}{a}$ ✓

E.g. $G = \{1, -1, i, -i\}$ fourth roots of unity. $\boxed{i = \sqrt{-1}}$

$(G, *)$ multiplication

Composition Table:

×	1	−1	i	−i
1	1	−1	i	−i
−1	−1	1	−i	i
i	i	−i	−1	1
−i	−i	i	1	−1

Closure: elements of table are elements of set.

Associative: Multiplication of complex no. is associative

Identity: 1 is identity.
The row in which elements are same as column

Inverse: Identity exists in every row.

Commutative:
$\boxed{A = A^T}$ ✓ $(G = \{1, -1, i, -i\}, *)$ is a abelian group.

Modulo m :

Definition:

$a +_m b$ = Remainder when m divides $a+b$ [addition modulo]

$a \times_m b$ = Remainder when m divides $a \times b$

$3 +_4 7 = 2$ $\quad$ $3 \times_5 7 = 1$

$2 +_3 5 = 1$ $\quad$ $3 \times_2 4 = 0$

E.g. $(G = \{0,1,2,3\}, +_4)$

Composition Table:

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Closure ✓
Associative ✓ $\boxed{\text{(modulo - m is associate)}}$
Identity = 0
Inverse ✓
Commutative ✓

⇒ Abelian Group

E.g. $(G = \{0,1,2,3\}, \times_4)$

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Closure ✓
Associative ✓
Identity ✓ e1
Inverse ✗
Commutative

Ex $(G = \{1,2,3\}, \times_4)$

$\times_4$	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

Closure ✗
Associative
~~Identity~~
∴ Inverse
Commutative

E.g. $(G = \{1,2,3,4\}, \times_5)$

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Closure ✓
Associative ✓
Identity = 1
Inverse ✓
Commutative ✓

Abelian

RESULT:

① $(G = \{0,1,2,3,\ldots, m-1\}, +_m)$ is abelian group.

Identity → 0

Inverse of $a \to m-a$ mod m

② $(G = \{1,2,\ldots, p-1\}, \times_p)$ is abelian group.

Group when p is prime.

identity → 1

E.g $(G = \{0,1,2,\ldots,32\}, +_{33})$ is abelian group

identity = 0

inverse of $11 \to 33-11 = 22$

inverse of $22 \to 33-22 = 11$

inverse of $0 \to 33 - 0 = 0$

Def. A non-empty subset H of G ($\phi \neq H \subseteq G$) is subgroup of $(G, *)$ if $(H, \times)$ itself is a group.

E.g. $(G = \{1,-1,i,-i\}, \times)$ is abelian group.

$H = \{1, i\}$

$\times$	1	i
1	1	i
i	i	-1

Not closed

$(H, \times)$ is not group

it is not a subgroup.

Result

① $\phi \neq H \subseteq G$ is a subgroup of $(G, *)$ iff $\boxed{a * b^{-1} \in H \ \forall a, b \in H.}$

② $\phi \neq H \subseteq G$ is a subgroup of finite group $(G, *)$ iff $\boxed{a * b \in H \quad \forall a, b \in H}$

③ If H_1 and H_2 are subgroups; then:
ⓐ $H_1 \cap H_2$ is subgroup.
ⓑ $H_1 \cup H_2$ need not be subgroup.

$p \to q \to$ necess. cond.
$p \to$ sufficient cond.

Order:

$O(G)$ = No. of elements in G.

④ Lagrange's Theorem: If H is subgroup of G, then
(Necessary Condition) $O(H) \mid O(G)$ [$O(H)$ divides $O(G)$]

→ Condition is not sufficient

E.g. $(G = \{1, -1, i, -i\}, \times)$ Abelian

$O(G) = 4$ $\quad H = \{1, i\}$ $O(H) = 2$

$O(H) \mid O(G)$ but H is not subgroup.

E.g. which of the following cannot be order of subgroup of a group of order 15.

(a) 3 ✓(b) 4 (c) 5 (d) 15

Definition: Let $(G, +)$ be an additive group and e is identity. The smallest positive integer 'm' such that $\boxed{m \cdot a = e}$ is called order of a.

Def: Let $(G, \cdot)$ be a multiplicative group and e is identity. The smallest positive integer m such that $\boxed{a^m = e}$ is called order of a.

We write $\boxed{O(a) = m.}$

NOTE

① $\underbrace{a + a + \cdots + a}_{m \text{ times}} = m \cdot a$

② $\underbrace{a \cdot a \cdot a \cdots\cdots a}_{m \text{ times}} = a^m$

Def: An element $a \in G$ such that $O(a) = O(G)$ is called generator of group $(G, *)$.

Def: A group having at least one generator is called cyclic group.

Result ⇒ No. of generators in cyclic group of order $n = \phi(n)$

E.g. $(G = \{1, -1, i, -i\}, \times)$ is abelian group

identity → 1

$1 = 1$	$(1)^1 = 1$	⇒ $O(1) = 1$	$O(G) = 4$
$(-1) \times (-1) = 1$	$(-1)^2 = 1$	⇒ $O(-1) = 2$	
$i \times i \times i \times i = 1$	$(i)^4 = 1$	⇒ $O(i) = 4$	
$(-i) \times (-i) \times (-i) \times (-i) = 1$	$(-i)^4 = 1$	⇒ $O(-i) = 4$	

$O(i) = O(-i) = O(G)$

⇒ $i, -i$ are generator of G.

∴ G is cyclic group.

No. of generators $= \phi(4)$
$= 2$

i is generator

$i = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$

All the elements of group can be generated by i.

② $(G = \{0,1,2,3\}, +_4)$

Identity = 0

$0:0$

$1 +_4 1 +_4 1 +_4 1 = 0$

$2 +_4 2 = 0$

$3 +_4 3 +_4 3 +_4 3 = 0$

$o(0) = 1$
$o(1) = 4$
$o(2) = 2$
$o(3) = 4$

1 and 3 are generators of G.
∴ G is cyclic.

E.g. $O(G) = 15$
If G is cyclic then
No. of generators $= \phi(15) = 8$

$15 = 3 \times 5$

$\phi(15) = 15\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$

$= 15 \times \frac{2}{3} \times \frac{4}{5} = 8$

WB Pg 21. (46) $\frac{1+i}{\sqrt{2}}$ identity = 1

$\left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{1+i}{\sqrt{2}}\right) = \frac{1-1+2i}{2}$

$= i$

$(i)^4 = \left[\left(\frac{1+i}{\sqrt{2}}\right)^2\right]^4 = 1 \Rightarrow \left(\frac{1+i}{\sqrt{2}}\right)^8 = 1 \Rightarrow o\left(\frac{1+i}{\sqrt{2}}\right) = 8$

6.

(48) $o(A) = 4$
$o(B) = 5$

$A \cap B$ is a subgroup of A and B.

∴ $o(A\cap B) \mid o(A)$ [Lagrange's]
$o(A \cap B) \mid o(B)$

$\boxed{\therefore o(A\cap B) = 1}$

(49) $g^8 = 8$
$\phi(g) \neq 8 \Rightarrow o(g)$ should be some divisor of 8.
$o(g) \mid 8 \Rightarrow \cancel{1}, \cancel{2}, 4$
$\therefore o(g) = 4$

Compulsory (34)
33 & 37
Pg more

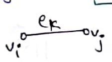
GRAPH THEORY:

→ Graph $G = (V, E)$

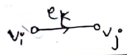
$V \rightarrow$ vertex set $= \{v_1, v_2, \ldots, v_m\}$

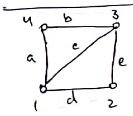
$E \rightarrow$ edge set $= \{e_1, e_2, \ldots, e_n\}$

→ If each edge $e_k = \underbrace{\{v_i, v_j\}}_{\text{unordered pair of vertices}}$, then G is undirected graph (Graph)

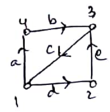


→ If each edge $e_k = \underbrace{\{v_i, v_j\}}_{\text{ordered pair of vertices.}}$, then G is directed graph (digraph).



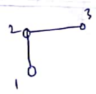


$V = \{1,2,3,4\}$
$E = \{\{1,4\},\{4,3\},\{4,3\},\{1,3\},\{1,2\},\{2,3\}\}$



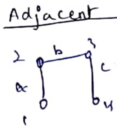
$V = \{1,2,3,4\}$
$E = \{\{1,2\},\{2,3\},\{4,3\},\{1,4\},\{3,1\}\}$

Adjacent ~~edges~~ vertices: Common edge



1, 3 are not adjacent vertex.

Adjacent edges: Common vertex



a, b & b, c are adjacent edges.

Self Loop :

- Vertex connected to itself
- Edge having same starting and end vertex.

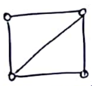
Multi-edges : (||el edges)

Multiples edges having same end vertex.

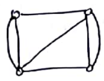
	Self loop	||el edges
simple graph	✗	✗
Multigraph	✗	✓
General graph or Pseudo graph	✓	✓

✗ → not allowed
✓ → allowed

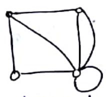
Eg.



Simple graph
Multi graph
General graph



Multigraph
General graph

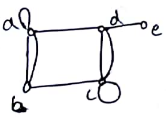


General graph

Degree of a vertex :

→ No. of edges incident on the vertex (counting loop twice).

E.g.



$V = 5$
$e = 10$

v	d(v)
a	5
b	4
c	5
d	5
e	1

No. of odd degree edges $= 4$ (even)

$\boxed{\Sigma(d(v)) = 20} = 2e$

Notation :

$\Delta =$ Max degree
$\delta =$ Min degree

First Theorem of Graph Theory (Hand shaking Lemma) :

In any graph. $G = (V, E)$, the sum of degrees of vertices is twice the number of edges.

i.e., $\boxed{\Sigma\, d(v) = 2|E|}$

Proof : Each edge contributes 2 degrees.

RESULT
"The number of odd degree vertices is always even."

Important :

↳ ③ In any graph :

always. $\boxed{\delta \leq \frac{2e}{v} \leq \Delta}$

e: No. of edges
v: No. of vertices

Proof :

$\Sigma d(v) = 2e$

Replace every degree with δ

$\underbrace{\delta + \delta + \cdots + \delta}_{v \text{ times}} \leq 2e$

$\boxed{v \cdot \delta \leq 2e}$ —— ①

Replace every degree with Δ

$\boxed{v \cdot \Delta \geq 2e}$ —— ②

① & ②

$$\boxed{\delta \leq \frac{2e}{v} \leq \Delta}$$

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E.g. Let G be a graph with 11 edges and min degree 5. Then max No. of vertices in G =

$\delta \le \frac{2e}{v} \Rightarrow v \le \frac{2e}{\delta} \Rightarrow v \le \left\lfloor \frac{2\times 11}{5} \right\rfloor \Rightarrow \boxed{v \le 4}$

E.g. Let G be a graph with 7 vertices and min degree 3 then min no. of edges in G =

$\delta \le \frac{2e}{v} \Rightarrow 7\times 3 \le 2e \Rightarrow \left\lceil \frac{21}{2} \right\rceil \le e \Rightarrow e \ge 11$

E.g. Let G be a graph with 13 edges and max degree 5, then min no. of vertices in G =

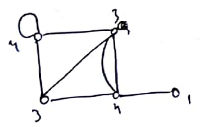
$\frac{2e}{v} \le \Delta \Rightarrow \left\lceil \frac{13\times 2}{5} \right\rceil \le v \Rightarrow v \ge \left\lceil \frac{26}{5} \right\rceil \Rightarrow v \ge 6$

E.g. Let G be a graph with 11 vertices and max degree of 4 then max no. of edges in G =

$\frac{2e}{v} \le \Delta \Rightarrow \frac{2e}{11} \le 4 \Rightarrow e \le \left\lfloor \frac{4\times 11}{2} \right\rfloor \Rightarrow e \le 22$

Degree Sequence:

→ Sequence of degrees in non-increasing order.



4, 4, 4, 3, 1

Def, A degree sequence is graphic if it corresponds to a simple graph.

Havel-Hakimi Algorithm:

(To verify a degree sequence is graphic or not).

① Arrange degrees in non-increasing order.

② Delete the highest degree (say k) and from the next k degrees subtract 1.

③ Continue the steps ① and ② till stop condition is reached.

Stop conditions:

① If all zero degree → Graphic

② If any negative degree → Not graphic

③ Not enough degrees → Not graphic

E.g. 3, 3, 2, 2

① $\cancel{3}, \underbrace{3, 2, 2}_{-1} \rightarrow 2, 1, 1$

$\cancel{2}, \underbrace{1, 1}_{-1} \rightarrow 0, 0$

graphic

E.g. 3, 3, 2, 1, 1

$\cancel{3}, \underbrace{3, 2, 1}_{-1}, 1 \rightarrow 2, 1, 0, 1$

~~1, 1, 0, 0 → 0, 1, 1~~

~~Not graphic~~

$\cancel{2}, \underbrace{1, 1}, 0 \rightarrow 0, 0, 0$

graphic

Q. Which of the following are not graphic.

I. 7, 6, 5, 4, 4, 3, 2, 1

II. 6, 6, 6, 6, 3, 3, 2, 2

III. 7, 6, 6, 4, 4, 3, 2, 2

IV. 8, 7, 7, 6, 4, 2, 1, 1

I. $\cancel{7}, \underbrace{6, 5, 4, 4, 3, 2, 1}_{-1} \rightarrow \cancel{5}, \underbrace{4, 3, 3, 2, 1}_{-1}, 0 \rightarrow 3, 2, 2, 1, 0, 0$

$\cancel{3}, \underbrace{2, 2, 1}, 0, 0 \rightarrow \cancel{1}, \underbrace{1}, 0, 0, 0 \rightarrow 0, 0, 0, 0$ graphic.

(II) $\cancel{7}, \underbrace{6,6,6,3,3,2}_{-1}, 2 \rightarrow \cancel{5}, \underbrace{5,5,2,2,1}_{-1}, 1 \rightarrow 4,4,1,1,0,1$

$\cancel{4}, \underbrace{4,1,1,1}_{-1}, 0 \rightarrow \cancel{3}, \underbrace{0,0,0}_{-1}, 0 \rightarrow \underbrace{-1,-1,-1}, 0$ Not graphic
Negative ~~entries~~ degrees.

(III) $\cancel{7}, \underbrace{6,6,4,4,3,2,2}_{-1} \rightarrow \cancel{5}, \underbrace{5,3,3,2,1}_{-1}, 1 \rightarrow 4,2,2,1,0,1$

$\cancel{4}, \underbrace{2,2,1,1}_{-1}, 0 \rightarrow \cancel{1}, \underbrace{1}_{-1}, 0,0,0 \rightarrow 0,0,0,0$ graphic.

(IV) 8, 7, 7, 6, 4, 2, 1, 1 Not enough degrees Not graphic

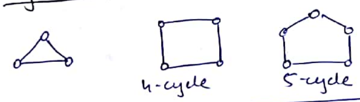
Special Graphs:

① Null Graph: (N_n) → Graph with no edges.

N_1 o
N_2 o o
N_3 o o o
N_4 o o o o

Note
$d(v) = 0 \Rightarrow v$ is isolated vertex
$d(v) = 1 \Rightarrow v$ is pendant vertex

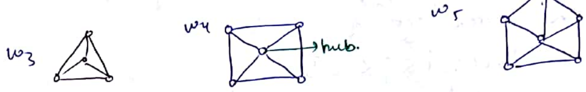
② Cycle Graph ($C_n, n \geq 3$)



A cycle graph C_n is a simple graph with n vertices $v_1, v_2, \ldots, v_n$ such that:

(i) v_i is adjacent to v_{i+1} $(i = 1, 2, \ldots, n-1)$
and
(ii) v_n is adjacent to v_1.

③ Wheel Graph: ($W_n, n \geq 3$)



→ W_n is a cycle graph C_n, with an additional vertex (hub) which is adjacent to all the vertices of cycles.

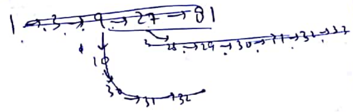
	V	e	d(v)
N_n	n	0	0
C_n	n	n	2
W_n	n+1	n+n = 2n	n → v is hub; 3 → v is not hub

Problems:

Q. G be a directed graph whose vertex set is no. from 1 to 100. There is an edge from i to j, iff $j = i+1$ (or) $j = 3i$

Min. no. of edges in a path in G from 1 to 100.

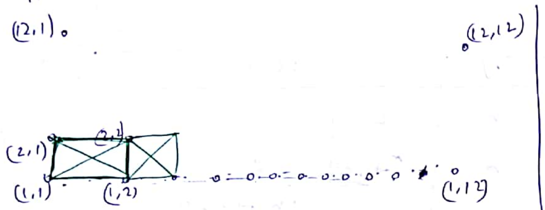
(a) 4 (b) 7 (c) 23 (d) 93

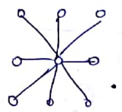


$1 \rightarrow 3 \rightarrow 9 \rightarrow 10 \rightarrow 11 \rightarrow 33 \rightarrow 99 \rightarrow 100$

7 edges.

Q5. Consider an undirected graph G, where self loops not allowed. The vertex set of G is
$\{(i,j): 1 \le i \le 12;\ 1 \le j \le 12\}$
There is an edge b/w (a,b) and (c,d) iff $|a-c| \le 1$ and $|b-d| \le 1$. No. of edges in G = ____





Horizontal lines = 11×12 = 132
Vertical lines = 12×11 = 132
Diagonal = 22×11 = 242

$\underline{\underline{506}}$

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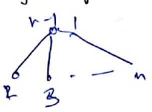
Result

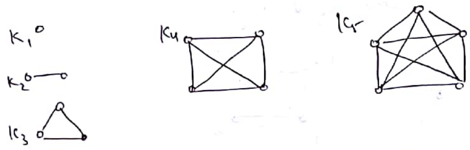
① Max no of edges in a simple graph with n-vertices $= {}^nC_2$
$= \frac{n(n-1)}{2}$

Max no. of edges = No. of 2-combination of n-vertices
$= {}^nC_2 = \frac{n(n-1)}{2}$

② Max degree of a vertex in a simple graph with n-vertices.
$= n-1$



④ Complete graph (K_n):
→ A simple graph in which every pair of vertices are adjacent.
(or)
Simple graph with max no. of edges.



	v	e	d(v)
K_n	n	${}^nC_2 = \frac{n(n-1)}{2}$	n-1

⑤ Regular Graph:
→ A graph in which all the vertices have same degree. i.e,
$\boxed{\forall v\ d(v) = k}$ → k-regular.
Need not be a simple graph.

E.g.:
N_n → 0-regular
C_n → 2-regular
K_n → $(n-1)$-regular
W_n → 3-regular when $n=3$

Result
→ No. of edges in a k-regular graph with n-vertices $= \frac{n.k}{2}$

$d(v) = k$
$\sum d(v) = 2e$
$\underbrace{k+k+k+\dots+k}_{n\ times} = 2e$

$n.k = 2e \Rightarrow$ $\boxed{e = \frac{n.k}{2}}$

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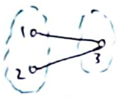
⑥ Bipartite Graph:

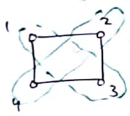
→ Bipartite graph $G = (V, E)$ is a simple graph in which the vertex set V can be partitioned into two sets V_1 and V_2 $\boxed{(V = V_1 \cup V_2,\ V_1 \cap V_2 = \phi)}$, such that every edge in G is b/w a vertex of V_1 to a vertex of V_2 only.

E.g.



Bipartite

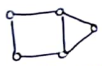
Not Bipartite



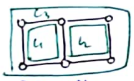
Bipartite

Result:

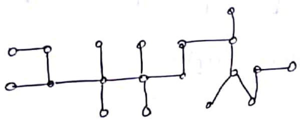
① → A simple graph is bipartite iff every cycle in G is even cycle.



Not bipartite



Bipartite



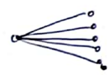
* Tree with 2 or more vertices is bipartite.

Every tree is 0-cycle (even cycle) (Tree) Bipartite.

② Maximum no. of edges in a bipartite graph with n-vertices $= \frac{n^2}{4}$

E.g. 6-vertices.

Max edges:



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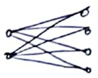

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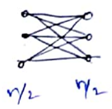


9

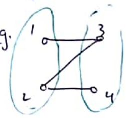
We have max^m edges, when half vertices are present in 1 part and other half present in other part.

$= \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4}$

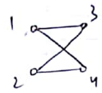
⑦ Complete Bipartite Graph: $(K_{m,n})$

→ $K_{m,n}$ is a bipartite graph $G = (V_1 \cup V_2, E)$, $|V_1| = m$, $|V_2| = n$, such that every vertex in V_1 is adjacent to every vertex in V_2.

E.g.



Bipartite
Not complete bipartite



$K_{2,2}$
Complete Bipartite

$K_{2,3}$

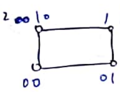
	V	e	d(v)
$K_{m,n}$	$m+n$	$(m \cdot n)$	$\begin{cases} n & v \in V_1 \\ m & v \in V_2 \end{cases}$

⑧ n-Cube (Q_n):

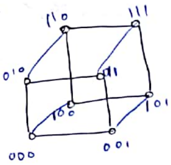
→ n-cube is a simple graph with 2^n vertices representing binary string of length n, such that two vertices are adjacent iff ~~there~~ their b.s. differ in exactly one bit position.

Q_1

Q_2



Q_3.



	V	e	d(v)
Q_n	2^n	$n \cdot 2^{n-1}$	n

$\sum d(v) = 2e$

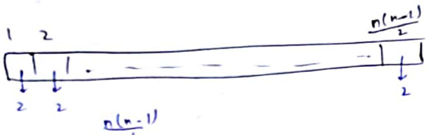
$\underbrace{n + n + \cdots + n}_{2^n \text{ times}} = 2e$

$\frac{n\, 2^n}{2} = e$

$n\, 2^{n-1}$

Result

① No. of distinct simple graphs with n- labelled vertices: $2^{\frac{n(n-1)}{2}}$



$= 2^{\frac{n(n-1)}{2}}$

Every edge has 2 choices, i.e., to be present or absent in the graph.

② No. of distinct simple graphs with n-labelled vertices.

No. of distinct simple graphs ~~havi~~ with n - labelled vertices having:

(i) 0 edges $\longrightarrow {}^{\frac{n(n-1)}{2}}C_0$

(ii) 1 edges $\longrightarrow {}^{\frac{n(n-1)}{2}}C_1$

⋮

(iii) k edges $\longrightarrow {}^{\frac{n(n-1)}{2}}C_k$

⋮

(iv) $\frac{n(n-1)}{2}$ edges $\longrightarrow {}^{\frac{n(n-1)}{2}}C_{\frac{n(n-1)}{2}}$

$$\boxed{\text{Total: } {}^{\frac{n(n-1)}{2}}C_0 + {}^{\frac{n(n-1)}{2}}C_1 + \dots + {}^{\frac{n(n-1)}{2}}C_{\frac{n(n-1)}{2}} = 2^{\frac{n(n-1)}{2}}}$$

E.g. No of distinct simple graphs with 5 labelled vertices having:

$\frac{5(5-1)}{2} = 10$ ~~tcount~~ total)

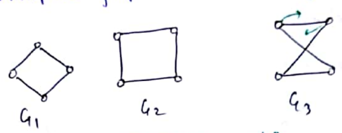
0 edges $\longrightarrow 1 = {}^{10}C_0 =$

3 edges $\longrightarrow {}^{10}C_3 = \frac{10!}{7! \times 3!} = \frac{10 \times 9 \times 8}{3 \times 2} = 120$

5 edges $\longrightarrow {}^{10}C_5 = \frac{10!}{5! \times 5!} = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2} = 252$

Isomorphism : (Adjacency preserving).

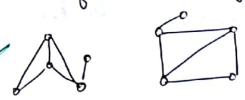
→ Isomorphic graphs are same graphs drawn differently.



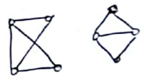
$G_1 \cong G_2 \cong G_3$ → Isomorphic

Q. Which of the following pairs are isomorphic.

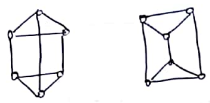
① ✓



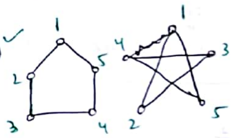
③ ✓



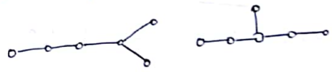
② ✓



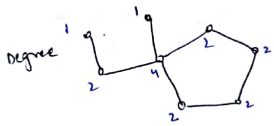
④ ✓



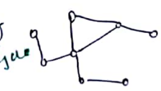
⑤ → adj



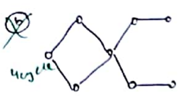
Q. Which of the following is isomorphic to:



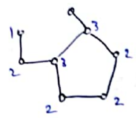
(a) 3 cycle



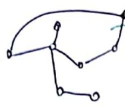
(b) 4 cycle



(c)



✓ (d)



Definition:

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exist a function.

$f : V_1 \longrightarrow V_2$ such that

① f is 1-1 } $\rightarrow$ Necessary condition $\left(\begin{matrix} |V_1| = |V_2| \\ |E_1| = |E_2| \end{matrix} \quad d(v_1) = d(v_2) \right)$
② f is onto }
③ f preserves adjacency

RESULT

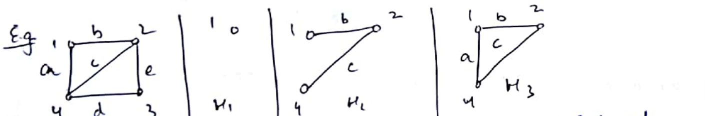
→ The necessary conditions for $G_1 = (V_1, E_1) \cong G_2 = (V_2, E_2)$

① $|V_1| = |V_2|$
② $|E_1| = |E_2|$
③ They have same degree sequence.

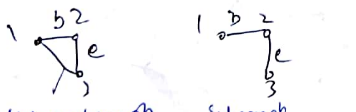
Subgraphs:

→ $H = (V_1, E_1)$ is subgraph of $G_1 = (V, E)$ if $V_1 \subseteq V$ & $E_1 \subseteq E$.

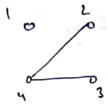
Eg



Subgraph | Subgraph formed by 1, 2, 4. | Subgraph induced by 1, 2, 4. (having all the edges b/w 1,2,4 as in the original graph).



Not subgraph | Subgraph induced by 1, 2, 3



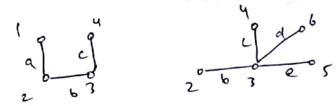
Subgraph containing all the vertices of a graph is called spanning subgraph.

Def: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$

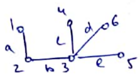
Union: $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$

Intersection: $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$

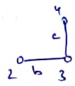
E.g.



$G_1 \cup G_2$



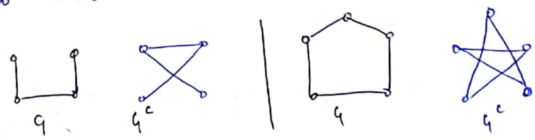
$G_1 \cap G_2$



Def: The complement of a simple graph $G = (V, E)$ is $G^c = (V, E^c)$ such that:

Two vertices in G^c are adjacent iff they are not adjacent in G.

E.g.



E.g. G is a simple graph with 13 ~~eto~~ edges and G^c is its complement with 15 ~~edg~~ edges, then, no. of vertices in G.

~~total~~ Max no. of edges = 28 = $\frac{n(n-1)}{2}$ $\boxed{n = 8}$

$\delta \le \frac{2e}{v} \le \Delta$

$56 = \cancel{n^2(n-1)}\ n^2 - n$

$n^2 - n - 56 = 0$

$n^2 - 8n + 7n - 56 = 0$

Scanned by CamScanner

Result G is a simple graph with n vertices and e edges and G^c is complement of G with e^c edges.

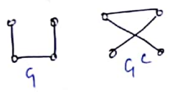
① $G \cup G^c = K_n$

② $e + e^c = \frac{n(n-1)}{2}$

Def: A simple graph which is isomorphic to its complement is called self-complemented graph.

i.e. $\boxed{G \cong G^c}$

E.g.



$G \cong G^c$

$\therefore G$ is self-complementary

E.g. Which of the following cannot be no. of vertices in self-complementary graph.

(a) 4 (b) 5 (c) 9 (d) 10 ✓

(a) Max $= n(n-1) = 12$

(b) Max $= 10$

(c) Max $= 36$

(d) Max $= 45$ — Cannot be divided between G & G^c

Result:

→ Let G be a self-complementary graph with n-vertices and e-edges. and G^c is complement of G with e^c-edges.

① $G \cup G^c = K_n$

② $e + e^c = \frac{n(n-1)}{2}$

③ $e = e^c$

④ $e = \frac{n(n-1)}{4}$

⑤ $\boxed{n \text{ is of the form } 4k \text{ or } 4k+1}$

$e = \frac{n(n-1)}{4}$ integer

$\Rightarrow 4/n$ or $4/n-1$

$\Rightarrow \frac{n}{4} = k$ or $\frac{n-1}{4} = k$

$\Rightarrow n = 4k$ or $n = 4k+1$

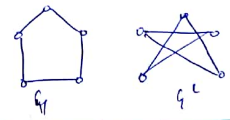
GATE, No. of vertices in a self-complementary cycle =

self-complementary: n vertices	cycles
$e = \frac{n(n-1)}{4}$ —①	$e = n$ —②

$\frac{n(n-1)}{4} = \cancel{n} \Rightarrow n-1 = 4$

$\boxed{n = 5}$

NOTE: C_5 is the only self-complementary cycle.



$G \cong G^c$

Walk, path, cycle:

→ An alternating sequence of vertices and edges is walk.

→ A walk which starts and end same vertex is closed walk.

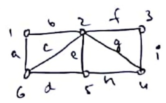
→ A walk in which no edge is repeated is called trail.

→ A walk in which no vertex is repeated is called path.

→ A closed walk in which no vertex except starting vertex is repeated is called cycle. (No edge should be repeated)

Definitions are ~~same~~ similar for directed graphs.

Eg.



1 b 2 b 1 b 2 f 3 → Walk

1 b 2 b 1 → Closed walk

1 b 2 e 5 d 6 c 2 f 3 → Trail

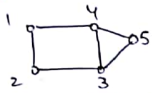
1 b 3 e 5 h 4 → Path

1 b 2 c b a 1 → Cycle

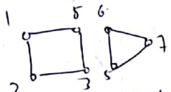
1 b 2 f 3 i 4 h 5 d 6 a 1 → Cycle

2 f 3 i 4 g 2 → Cycle

Def. Two vertices are connected if there exists atleast one path b/w them.



Connected graph
Component = 1



Disconnected graph.
Component = 2

Def: A graph in which every pair of vertices are connected ~~adjacent~~ is called connected graph.
otherwise, the graph is disconnected.

Def: Maximal connected subgraph of a graph is component.

Result

(1) If G is a simple graph with n-vertices, and k-components then maximum no. of edges in G is $\frac{(n-k)(n-k+1)}{2}$.

i.e., $\boxed{e \le \frac{(n-k)(n-k+1)}{2}}$ [condition is necessary]

Note The condition is not sufficient.

(2) If G is a simple graph with n-vertices and $\boxed{e > \frac{(n-1)(n-2)}{2}}$, then, G must be connected.

Note ⇒ Condition is not necessary.

if let G not connected:
$k \geq 2$
let $k=2$

$e \le \frac{(n-2)(n-2+1)}{2}$

$e \le \frac{(n-2)(n-1)}{2}$

But; $e > \frac{(n-1)(n-2)}{2}$

⇒ Contradiction ∴ Our assumption is wrong
∴ G must be connected.

E.g.

$n=4$
$e=4$
$e > \frac{(5-1)(5-2)}{2}$ $\left[e > \frac{(n-1)(n-2)}{2}\right]$

$e \not> 6$; But G is connected.

E.g. G is a simple graph with 8 vertices and 3-components
Max. no. of edges in G = ____

⇒ $e \le \frac{(n-k)(n-k+1)}{2}$ ⇒ $e \le \frac{(8-3)(8-3+1)}{2}$

⇒ $e \le \frac{5 \times 6}{2}$ ⇒ $e \le 15$

RESULT

(3) If G is a simple graph with $\delta \geq \left(\frac{n-1}{2}\right)$, then G is connected. (Sufficient).

Note: Condition is not necessary.

E.g. C_7 is connected.
$\delta = 2 \not\geq \frac{7-1}{2}$

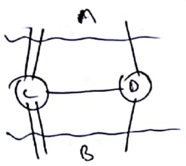
(4) G or G^c must be connected.

GATE: Which of the following is always true.

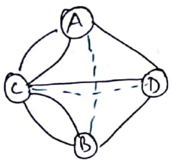
I. G connected then G^c is disconnected.
II. G disconnected then G^c is connected.

(a) only I (b) only II (c) both (d) None.

Euler Graph:



→ Start at any ~~p~~ land (A, B, C, D), cover ~~any~~ every bridge exactly once and come back to starting position.



→ Start at any vertex, cover every ~~bridge~~ edge exactly once & come back to starting ~~vate~~ vertex.

Euler Walk: [Unicursal Walk]
→ An open walk containing all the edges of connected graph in which no edge is repeated.

Euler Circuit,
→ A closed walk containing all the edges of a connected graph in which no edge is repeated.

Definition: A connected graph having atleast one Euler circuit is called Euler graph.

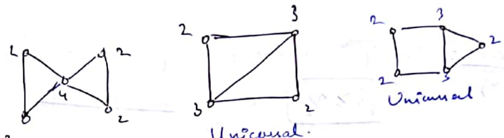
Definition: A connected graph having atleast one Euler walk is called Unicursal graph.

Result:

① A connected graph is Euler graph iff degree of every vertex is even.

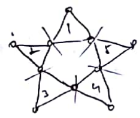
② A connected graph is Unicursal graph iff there are exactly two vertices of odd degree.

E.g.



Euler graph   Unicursal.   Unicursal

③ A connected graph is Euler graph iff it can be decomposed into edge disjoint cycles.

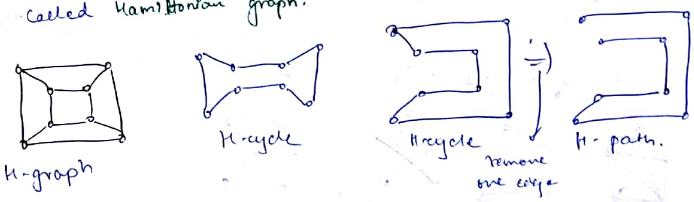


Hamiltonian Graph:

Def: An open path containing all the vertices of a connected graph in which no vertex is repeated is called Hamiltonian Path.

Def: Hamiltonian Cycle (Spanning Cycle)
→ A cycle containing all the vertices of a connected graph.

Def: A connected graph containing atleast one H-cycle is called Hamiltonian graph.



H-graph   H-cycle   H-cycle   H-path.

RESULT :

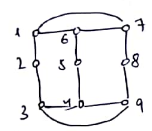
1. **Dirac's Theorem** : If G is a simple graph with n-vertices (Sufficient cond.ⁿ) and $\boxed{\delta \geq n/2}$ then G is Hamiltonian.

→ But condition is not necessary.

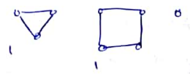
E.g. C_6 is hamiltonian, but $\delta = 2 \not\geq \frac{6}{2}$

2. **Ore's Theorem** : (Sufficient Condⁿ).
If $\boxed{deg(u) + deg(v) \geq n}$ for every pair of non-adjacent vertices in a simple graph with 'n' vertices, then G is Hamiltonian.

E.g.

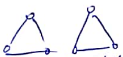


Hamiltonian Path present but no H-cycle. So, not H-graph.



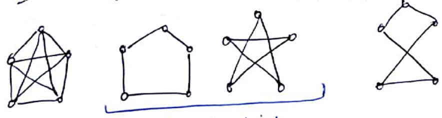
Result

1. No. of H-cycles in $K_n (n \geq 3) = \frac{(n-1)!}{2}$



2. C.P → 1 H cycle
$(n-1)!$ C.P → $\frac{(n-1)!}{2}$ H cycles

E.g. No. of H-cycles in $K_5 = \frac{4!}{2} = 12$



edge disjoint cycles.

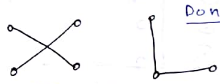
2. No. of edge disjoint H-cycles in $K_n (n \geq 3)$ with odd no. of vertices $= \frac{(n-1)}{2}$

E.g. No. edge disjoint H-cycles in $K_5 = \frac{5-1}{2} = 2$

Q. No. of circular arrangements in which each vertex has no two same neighbors $= \frac{n-1}{2}$

Planar Graph :

→ Drawing a graph in a plane without cross over is called planar representation.



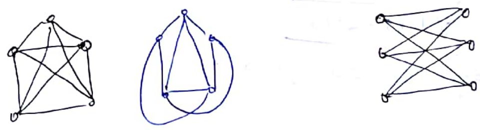
Crossover

Do not cross over.

→ A graph having planar representation is called planar graph.

E.g.

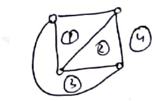
Planar Graph Planar Representation Planar graph Planar rep.



✓

Def: The planar representation of planar graph divides the plane into regions or faces.

WRONG



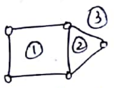
4-regions.

1, 2, 3 → interior regions

4 → exterior region.

Def: The number of ~~all~~ edges in the boundary of a region is called degree of region.

E.g.



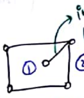
Planar Graph

$v = 5$
$e = 6$
$r = 3$

region(r)	d(r)
1	4
2	3
3	5
$\Sigma d(r) =$	$12 = 2(e)$

$\boxed{v - e + r = 2}$

E.g.



→ interior edge (Not counted in degree)

Planar Graph

$v = 5$
$e = 5$
$r = 2$

r	d(r)
1	4
2	4
$\Sigma d(r)$	$8 < 2(e)$

$\boxed{v - e + r = 2}$

Result → In a simple ~~graph~~ planar graph:

$$\boxed{\Sigma d(r) \leq 2(e)}$$

Sum of degrees of ~~regions~~ regions $\leq 2(e)$

Euler's formula:

→ In a connected planar graph with

v – vertices

e – edges

and r – regions

we have, $\boxed{v - e + r = 2}$

Result

→ G is connected planar graph with min degree of region = 3 having v-vertices, e-edges, r-regions, then;

1. $v - e + r = 2$
2. $3r \leq 2e$
3. $e \leq 3v - 6$

Proof:

2. $\Sigma d(r) \leq 2e$

Replace every degree with 3.

$\underbrace{3 + 3 + 3 + \dots + 3}_{r \text{ times}} \leq 2e \Rightarrow \boxed{3r \leq 2e}$

3. $2 = v - e + r$ $\qquad r \leq \frac{2e}{3}$

$\leq v - e + \frac{2e}{3}$

$= \frac{3v - e}{3}$

$6 \leq 3v - e$

$\boxed{e \leq 3v - 6}$

Result

G is connected planar graph with min degree of region = k having v - vertices
e - edges
r - regions

then: ① $v - e + r = 2$

② $kr \le 2e$

③ $e \le \frac{k(v-2)}{k-2}$

NOTE Min. degree of region = 4

① $v - e + r = 2$

② $4r \le 2e$

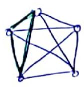
③ $e \le \frac{4(v-2)}{2} \Rightarrow e \le 2(v-2)$

Result

③ K_5 is ______

let K_5 is planar.

$v = 5$
$e = 10$



Min. degree of region = 3

① $v - e + r = 2$

$r = 7$

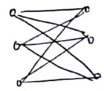
② $3r \le 2e$

$21 \not\le 20$

Contradiction ⇒ So, K_5 is non-planar.

NOTE: Min. degree of region in $K_{m,n} = 4$ complete Bipartite graph.

④ $K_{3,3}$ is non-planar



$v = 6$
$e = 9$

Let $K_{3,3}$ is planar

① $v - e + r = 2$

$r = 5$

min degree = 4

② $2r \le e$

$10 \not\le 9$

∴ Our assumption is wrong.

Results

① K_5 and $K_{3,3}$ are called Kuratowski's graphs.

② $K_5, K_{3,3}$ are non-planar graphs.

③ K_5 is non-planar graph with minimum no. of vertices.

④ $K_{3,3}$ is non-planar graph having minimum no. of edges.

Kuratowski's Theorem:

→ G is planar iff G does not contain any subgraph homeomorphic to:

K_5 or $K_{(3,3)}$

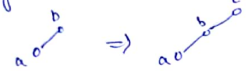
$\boxed{p \leftrightarrow q \\ \neg p \leftrightarrow \neg q}$

→ G is non-planar iff G contains any subgraph homeomorphic to:

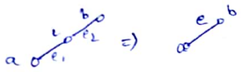
K_5 (or) $K_{3,3}$

Homeomorphic Operations:

① Adding a vertex in series



② Merging edges in series



E.g.

∴ Non-planar.

Q. Which of the following is non-planar graph,

① with min no. of vertices $= K_5$ (10e, 5v)

(a) 9e, 6v (b) 6e, 4v (c) 10e, 5v (d) 9e, 5v

② with min no. of edges. $K_{3,3}$ (9e, 6v)

II. G be a connected planar graph with 10 vertices. If no. of edges in each face is 3, then no. of edges in G is ______

GATE. Let δ denote the min degree of a vertex in graph. For all planar graphs on n-vertices with $\delta \geq 3$, which of the following is true.

(a) In any planar embedding, no. of faces $< \frac{n}{2}+2$

(b) In any planar embedding, no. of faces atleast $\frac{n}{2}+2$

(c) There is a planar embedding in which no. of faces $< \frac{n}{2}+2$

(d) There " " " " " " " atmost $(n/(\delta+1))$

$\boxed{\text{WB Pg-23} \quad 1, 2, 10, 15, 23}$

11) $\Sigma d(r) = 2e$

$2e = 3r$

$v-e+r=2$

$10 - e + \frac{2e}{3} = 2$

$\frac{1}{3}e = 8 \Rightarrow \boxed{e = 24}$

GATE soln $\delta = 3$

$v.\delta \leq 2e$

$\boxed{3v \leq 2e}$

$v-e+r=2$

$e = v+r-2$

$v+r-2 \geq \frac{3v}{2} \Rightarrow r-2 \geq \frac{v}{2}$

$\Rightarrow \boxed{r \geq \frac{v}{2}+2}$

15) W.B.

$v = 20$

$d(v) = 3$

$\Sigma d(v) = 2e$

$2e = 3(20)$

$\boxed{e = 30}$

$v-e+r=2$

$20-30+r=2$

$\boxed{r=12}$

23) $d(v) = 3$ (cubic)

$d(r) = 5$

$\Sigma d(v) = 2e$

$\boxed{3v = 2e}$

$\Sigma d(r) = 2e$

$5r = 2e$

$v-e+r=2$

$\frac{2}{3}e - e + \frac{2}{5}e = 2$

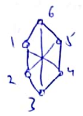
$\frac{10e - 15e + 6e}{15} = 2$

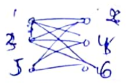
$\boxed{e = 30}$

① G_1 is $K_{3,3}$ Not planar

G_2 Planar

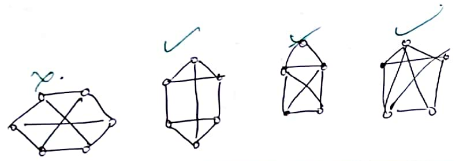
G_3





$K_{3,3}$
Not planar

② Q



Result .(Maybe important) *

→ In a planar graph with v-vertices
e-edges
r-regions
& k-components.

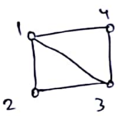
$$\boxed{v - e + r = k + 1}$$

Adjacency Matrix:

→ Let G be a graph with n-vertices and no parallel edges,
then, $A_G = [a_{ij}]_{n \times n}$

where $\boxed{a_{ij} = \begin{cases} 0 & (i,j) \notin E \\ 1 & (i,j) \in E \end{cases}}$

E.g.



$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

No self loops.

$$A^2_G = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

$$A^{(2)}_G = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \end{array}$$

Gives the no. of walks of length 2.

length of walk = No. of edges in the walk

Result

① The main diagonal elements of A_G are zero iff, there are no self loops in G.

② If G has no self loops and no parallel edges, then, sum of elements in i^{th} row (i^{th} column) = degree of a vertex corresponding to i^{th} row. (i^{th} column)

③ The $(i,j)^{th}$ element in A^K_G represents:
No. of walks of length K from vertex i to vertex j.

Pg-19
Q.33.

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ 3 & \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \times \begin{array}{c|c} & 5 \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \end{array}$$

$= 0+1+0+1+0+1$
$= 3$

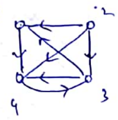
(a) 3, 2

37

$$A = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

$A^3 = ?$

$\boxed{\text{Directed graph } A \neq A^T}$



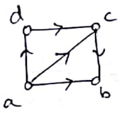
$$A^{3} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 2 & \\ 3 & & & & \\ 4 & & & & \end{array}$$

No. of walks of length 3.

(b) ✓

Directed graph (Digraph):

E.g.



Indegree (i_d): No. of edges incident in

Outdegree (o_d): No. of edges incident out

v	$i_d(v)$	$o_d(v)$
a	0	3
b	2	0
c	2	1
d	1	1

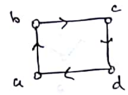
$\Sigma(\text{degree}) = 5 \quad = 5 \quad = |E|$

Result

→ In any digraph $G = (V, E)$, the $\boxed{\text{sum of in degrees} = \text{sum of out degrees} = |E|}$

 # Def: A digraph is strongly connected if b/w every pair of vertices u and v, there exists a directed path from u to v and v to u.

E.g.

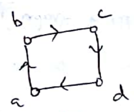


Strongly Connected

Can go from any vertex to any vertex in the graph.

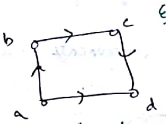
 # Def. A directed graph is unilaterally connected if b/w every pair of vertices u and v, there exist a directed path from $\boxed{\text{u to v or v to a}}$

E.g.



Strongly
Unilaterally

E.g.



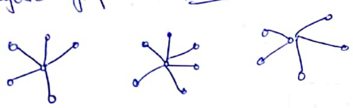
Not strongly
Unilaterally

Eg can go from a → c but not from c → a

Tree:

→ Connected and acyclic graph is tree.

→ Acyclic graph is forest.



Result.

① The following statements are equivalent for a graph T with n-vertices:

① T is a tree
② T is connected and has ~~no~~ n-1 edges
③ T is acyclic and has n-1 edges
④ T is minimally connected
⑤ There is exactly exactly one path b/w every pair of vertices in T.

② If T is tree with n-vertices then,
No. of edges in T = n-1

GATE.
Q. T be a tree with 10 vertices. Then sum of degrees of all vertices in T.

$\sum d(v) = 2e$ 10 vertices
$\sum d(v) = 2\times 9$ 9 edges
$= \underline{18}$

Eg. T is a tree with 7v of degree 2, 6 vertices of degree 3, 9 vertices of degree 4 and remaining vertices of degree 1. Find the number of vertices in T.

Eg.-

No. of v.	deg.
x	1
7	2
6	3
9	4

$\sum d(v) = x+14+18+36$
$= \boxed{x+68 = 2e}$ — ①

No. of vertices $= x+22$
No. of edges $= x+22-1 = x+21$
$\boxed{e = x+21}$ — ②

$x+68 = 2(x+21)$
$x+68-42 = 2x$
$\boxed{x = 26}$

$V = x+22$
$= 26+22 = \underline{48}$

GATE: G is a forest with n-vertices and k-components.
No. of edges in G = ________

ⓐ $\left\lfloor \frac{n}{k} \right\rfloor$ ⓑ $\left\lceil \frac{n}{k} \right\rceil$ ✓ⓒ $n-k$ ⓓ $n-k+1$

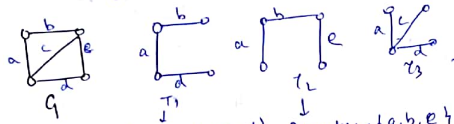
Tree
1 component n
$e = n-1$
$k=2$ $e = n-2$
⋮
$k=k$ $e = n-k$

1	2	...	k
n_1	n_2		n_k
n_1-1	n_2-1		n_k-1

$n_1+n_2+\dots+n_k = n$
$e = (n_1-1)+(n_2-1)+\dots+(n_k-1)$
$= (n_1+n_2+n_3+\dots+n_k) - k$
$= n-k$

Spanning Tree:

→ A spanning subgraph which is a tree.



→ $B = \{a, c, d\}$ $C = \{b, e\}$

T_1 → Branches $= \{a, b, d\}$ Chords $= \{c, e\}$

T_2 → Branches $= \{a, b, e\}$ Chords $= \{c, d\}$

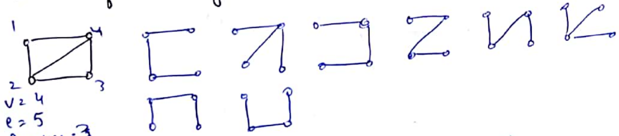
→ Result

→ In a connected graph with n- vertices and e edges.
or
In a spanning tree T,

$$\boxed{\begin{aligned} \text{No. of branches} &= n-1 \\ \text{No. of chords} &= e-(n-1) \\ &= e-n+1 \end{aligned}}$$

Q. Find no. of spanning trees:

①



$v = 4$
$e = 5$
Branches $= 3$
Chords $= 2$

No. of spanning trees $=$ ~~5C2 → choose two chords~~

$= {}^5C_2 -$ No. of 2-chords which disconnect the graph

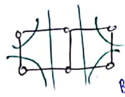
$= 10 - 2 = 8.$

Kirchoff's Matrix.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 2 & -1 & 0 & -1 \\ 2 & -1 & 3 & -1 & -1 \\ 3 & 0 & -1 & 2 & -1 \\ 4 & -1 & -1 & -1 & 3 \end{array}$$

$$\boxed{\text{No. of spanning trees} = {}^eC_{(e-n+1)} - \text{No. } \overset{\text{no. chord}}{(e-n+1)} \text{ combination of edges which disconnect the graph}}$$

②



$v = 6$
$e = 7$
Branches $= 5$
Chords $= 2$

No. of S.T. $= {}^7C_2 - 6$
$= 21 - 6$
$= 15$

Kirchoff's Matrix:

$K =$ Degree matrix $- A_G$

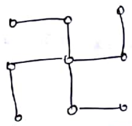
Cofactor of any element is No. of spanning trees.

Result: No. of spanning trees in $K_n = n^{n-2}$

Cut-edge (or) Bridge

→ Single edge whose removal disconnects the connected graph is cut-edge.

① No. of cut-edges: 8



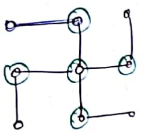
$v = 9$
$e = 8$

Result

→ In a tree every edge is a cut-edge.

Cut-Vertex (Articulation Point):

→ A single vertex whose removal disconnects the connected graph, is cut-vertex.

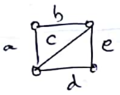


① No. of articulation point = 5

→ Connected graph having no articulation points is called Biconnected graph.

Cut-Set:

→ A set of edges whose removal disconnects the connected graph is called the cut set.



$C_1 = \{a, b, c, d, e\}$

$C_2 = \{a, c, e\}$
$C_3 = \{a, c, d\}$ → Min cut set — No proper subset is a ~~subs~~ cut set

$C_4 = \{a, b\}$
$C_5 = \{d, e\}$ → Smallest minimal cut-sets

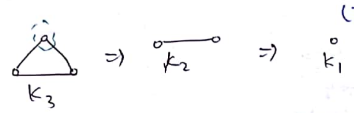
Edge Connectivity (λ):

→ The minimum no. of edges whose removal disconnects the connected graph.

Vertex Connectivity (κ - Kappa):

→ Minimum no. of vertices whose removal disconnects the connected graph or leaves ~~tri~~ trivial graph.

E.g.

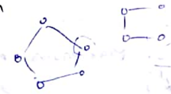


$\boxed{\kappa = 2}$ → Kappa

H.W

G	λ	κ (Kappa)
C_n	2	2
W_n	3	3
K_n	$n-1$	$n-1$
$K_{m,n}$	min $\{m, n\}$	min $\{m, n\}$
Q_n	n	n

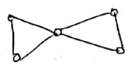
C_n



Whitney's Theorem:

$$\boxed{\kappa \leq \lambda \leq \delta}$$

E.g.



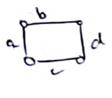
$\lambda = 2$

$\kappa = 1$

Scanned by CamScanner

Matching:

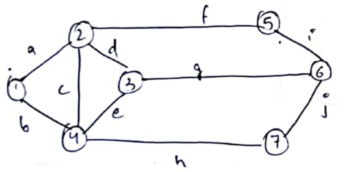
→ Set of non-adjacent edges.



$M_1 = \{a\}$
$M_2 = \{a, d\}$
$M_3 = \{b, c\}$

→ Matching No. (α') Maximum no. of non-adjacent edges.

Eg-



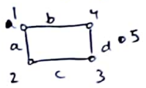
$M_1 = \{a, e, i\}$ $\alpha' = 3$

$M_2 = \{f, g, h\}$

$M_3 = \{c, g\}$

Edge Covering:

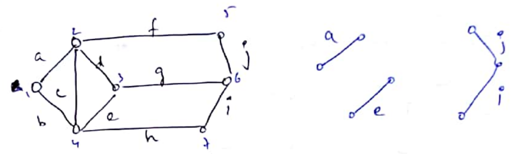
→ Set of edges which can cover all the vertices of positive degree.



$E_1 = \{a, b, c, d\}$
$E_2 = \{a, d\}$ → $\beta' = 2 + 1$ → isolated vertices.
$= 3$

→ Edge Covering No. (β'): Minimum no. of edges which can cover all the vertices of positive degree + no. of isolated vertices (if any).

Minimum $= \frac{n}{2}$



$E_1 = \{a, e, i, j\}$ $\beta' = 4$

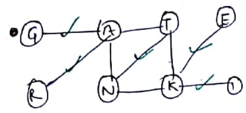
Result

In any simple graph with n-vertices

$$\boxed{\alpha' + \beta' = n}$$

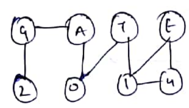
WB Pg 25

(19)



$\alpha' = 3$
$\beta' = 5$

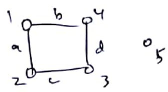
(20)



$\alpha' = 4$
$\beta' = 4$

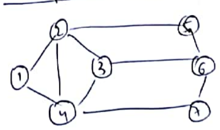
Independent Set:

→ Set of non-adjacent vertices. Look for maximum.



$I_1 = \{1\}$.
$I_2 = \{1, 3, 5\}$
$I_3 = \{2, 4, 5\}$

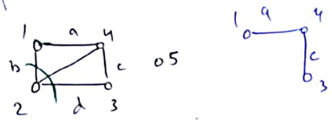
→ Independence No. (α): Maximum no. of non-adjacent vertices.



$I_1 = \{1, 3, 5, 7\}$
$I_2 = \{2, 6\}$ | $\alpha = 4$

Vertex Covering:

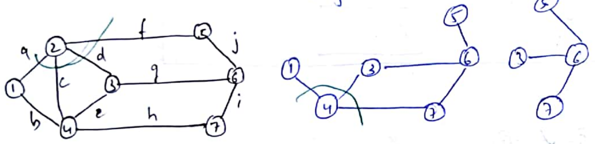
→ Set of vertices which can cover all the edges.



$V_1 = \{2, 4\}$

~~α~~ β = 2

→ Vertex Covering No. (β): Minimum no. of vertices which can cover all the edges.



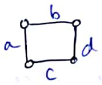
$V_1 = \{2, 4, 6\}$

Result

In any simple graph, with n-vertices:
$\alpha + \beta = n$

Perfect Matching:

→ A matching which can cover all the vertices.
There cannot be perfect matching in a graph with ~~isola~~ isolated vertices.



$M_1 = \{a, d\}$ covers all vertices.
∴ perfect match.

→ Same as edge covering in the case of no isolated vertices.

P only if Q
P → Q

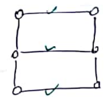
Result:

① A simple graph G with n vertices has perfect matching only if n is even (Necessary Condition)

even no. of vertices but no perfect matching ⇒ Cond. not sufficient.

② No. of perfect matching in $K_{2n} = \frac{(2n)!}{2^n . n!}$

K_6



5, 3, 1
= 15

K_{2n}

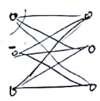
$= (2n-1)(2n-3)(2n-5) \ldots 3.1.$

$= \frac{2n(2n-1)(2n-2)(2n-3) \ldots 3.2.1}{2n.(2n-2).(2n-4) \ldots 2}$

$= \frac{(2n)!}{2^n . n!}$

③ No. of perfect matching in $K_{n,n} = n!$

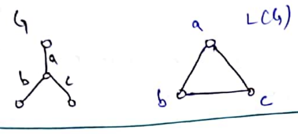
$K_{3,3}$



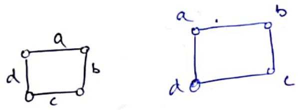
3.2.1
= 6

→ **Def.** Line graph $(L(G))$ of a graph G is constructed as follows:
① For every edge in G there is a vertex in L(G).
② Two vertices in L(G) are adjacent iff, their corresponding edges are adjacent in G.

E.g. G — L(G)



Result: Line graph of a cycle is cycle.

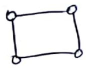


Clique: Maximal complete subgraph.

Clique is K_4

Coloring:

* Graph Coloring Problem: Coloring the vertices of a graph in such a way that no two adjacent vertices have the same color.



$\chi(G) = 2$

Chromatic Number: $[\chi(G)]$

→ The minimum no. of colors required to color the graph.

Results:

① If G is a graph with n vertices:

$\boxed{\chi(G) \leq n}$ $\chi(G) \geq 1$

②

G	$\chi(G)$
N_n	1
C_n	$\begin{cases} 2, & n \text{ even} \\ 3, & n \text{ odd} \end{cases}$
W_n	$\begin{cases} 3, & n \text{ even } (2+1) \\ 4, & n \text{ odd } (3+1) \end{cases}$
K_n	n
$K_{m,n}$	2
Q_n (Bipartite)	2

③ If K_n is subgraph of G, then $\boxed{\chi(G) \geq n}$

④ $\chi(G) \leq 1 + \Delta$

⑤ $\chi(G) \geq \dfrac{|V|}{|V| - \delta}$

⑥ The following statements are equivalent:

ⓐ G is Bipartite.

ⓑ G is 2-colorable.

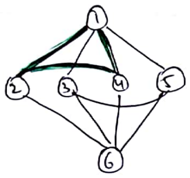
ⓒ Every cycle in G is even cycle.

⑦ Every tree with 2 or more vertices is 2 ~~colour~~ colourable. [since, every tree is bipartite (every cycle is 0-cycle, i.e., even cycle)]

⑧ Every planar graph is 4 colourable (maximum).

⑨ $\chi(G) = K$, iff G is k-partite, i.e., G has k-chromatic partitions.

find $\chi(G)$: (Welsch Powell Algorithm):



Arrange vertices in non-increasing order of degrees.

1	6	2	3	4	5
c_1	c_1	c_2	c_2	c_3	c_3
		$\cancel{c_1}$	$\cancel{c_1}$	$\cancel{c_1}$ $\cancel{c_2}$	$\cancel{c_1}$ $\cancel{c_2}$

So, $\boxed{\chi(G) \leq 3}$

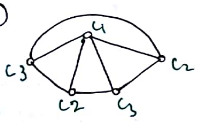
$\because K_3$ is a subgroup of G.

So, $\chi(G)$ ~~minimum~~ minimum 3

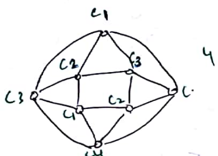
$\chi(G) \geq 3$

$\boxed{\therefore \chi(G) = 3}$

Q. Find $\chi(G)$:

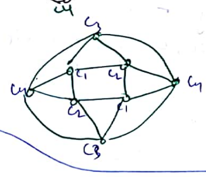
① 

$\chi(G) = 3$

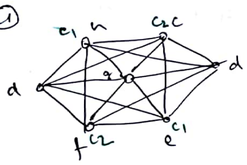
② 

4

$\chi(G) = 4$



2021 (21)



Atleast = 4

K_4

$\chi(G) = 4$

No. of edges in G^c =

(22) $e + e^c = \frac{n(n-1)}{2}$ $n = 7$ $e = 18$

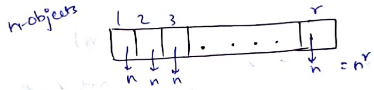
$18 + e^c = \frac{7 \times \cancel{6}^3}{\cancel{2}}$

$e^c = 21 - 18 = \boxed{3}$

Counting - 2 :

Permutations with repetitions:

r-permutation of n-objects with repetitions.



E.g. ① No. of outcomes when:

ⓐ 2 coins tossed $= 2^2 = 4$

ⓑ 10 coins tossed $= 2^{10} = 1024$

ⓒ 2 dice are rolled $= 6^2 = 36$

ⓓ 10 dice are rolled $= 6^{10}$.

② No. of ways to answer 25 T or F questions (assuming we cannot skip any question).

$= 2^{25}$

E.g. No. of arrangements of letters $\boxed{AAABC}$. Permutations with constrainted repetitions.

Let no. of arrangements $= x$

If 3 A's are differents A_1, A_2, A_3.

No. of arrangements $= x * 3!$

But A_1, A_2, A_3, B, C are different.

No. of arrangements $= 5!$

$x * 3! = 5! \Rightarrow x = \frac{5!}{3!}$

or

${}^5C_3 \times 1 \times {}^2C_1 \times {}^1C_1$

$\Rightarrow \frac{5!}{\cancel{2!}\,3!} \times \cancel{2!} = \frac{5!}{3!}$

E.g. AAAA BBB CCCCC

$x * 4! * 3! * 5! = 12!$ → Total arrangements.

No. of arrangements. $x = \frac{12!}{4!\,3!\,5!}$

|| ${}^{12}C_4\, {}^8C_3\, {}^5C_5$

Permutations with constrained repetitions:

→ No. of permutations of n-objects in which:

q_1 are same,
q_2 are same,
⋮
q_t are same (where $q_1+q_2+\dots+q_t = n$)

$$= C(n: q_1, q_2, \dots, q_t) = \frac{n!}{q_1!\,q_2!\dots q_t!} = {}^{n}C_{q_1}\,{}^{n-q_1}C_{q_2}\cdots{}^{n-q_1-q_2\dots q_{t-1}}C_{q_t}$$

Q5. No. of arrangements in MISSISSIPPI.

2 1 1

Total = 11!

$$\text{Arrangements} = \frac{11!}{4!\,4!\,2!} = \frac{11\times10\times9\times\cancel{8}\times7\times\cancel{6}\times5}{\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{2}}$$

$$= 34650.$$

(a) Such that 4 I's are together.

$\underset{\text{1 unit}}{\boxed{4I}}$ + 7 units = 8 units. ⇒ Total = 8!

For each I's → 1!

$$\text{Arrangements} = \frac{8!}{4!\,2!\,1!\,1!} \times 1!$$

$$= \frac{8\times7\times\overset{3}{\cancel{6}}\times5}{\cancel{2}} = \underline{840}$$

Multinomial Theorem:

→ The coefficient of $x_1^{q_1}, x_2^{q_2} \dots x_t^{q_t}$ in the expansion of $(x_1+x_2+\dots+x_t)^n$

$$\boxed{= \frac{n!}{q_1!\,q_2!\dots q_t!}}$$

E.g. ① Coefficient of $x^2y^3z^4$ in $(x+y+z)^9 = \dfrac{9!}{2!\,3!\,4!}$

② Coefficient of $x^2y^3z^2$ in $(2x-3y+z)^7 = (2)^2(-3)^3(1)^2 \cdot \dfrac{7!}{2!\,3!\,2!}$

(WB) Pg. 12

$$\boxed{(42)\quad (2)^3(-1)^2(3) \cdot \frac{6!}{(2)!\,1!\,3!}}$$

(11) 8 different shirts.
4 people.

$${}^{8}C_2 \cdot {}^{6}C_2 \cdot {}^{4}C_2 \cdot {}^{2}C_2 = \frac{8!}{2!\,2!\,2!\,2!}$$

$$= \frac{8\times7\times\overset{3}{\cancel{6}}\times5\times\cancel{4}\times3\times\cancel{2}}{\cancel{2}\times\cancel{2}\times\cancel{2}\times\cancel{2}} = 56\times45 = 2520$$

$$\begin{array}{r} 56 \\ \times\ 45 \\ \hline 280 \\ 2240 \\ \hline 2520 \end{array}$$

Result

→ No. of ordered partitions of type $(q_1, q_2, \dots q_t)$ of a set S with n elements $= \dfrac{n!}{q_1!\,q_2!\dots q_t!}$

E.g. No. of ordered partitions of type (2,1,1) of the set $S = \{a, b, c, d\}$ = ______

$$\frac{4!}{2!\,1!\,1!} = 4\times3 = 12$$

Unordered partition = 6.
(No. defined formula)

$\overline{ab}$	$\overline{d}$	$\overline{c}$	$\overline{ab}$	$\overline{c}$	$\overline{d}$
$\overline{ac}$	$\overline{b}$	$\overline{d}$	$\overline{ac}$	$\overline{d}$	$\overline{b}$
$\overline{ad}$	$\overline{b}$	$\overline{c}$	$\overline{ad}$	$\overline{c}$	$\overline{b}$
$\overline{bc}$	$\overline{a}$	$\overline{d}$	$\overline{bc}$	$\overline{d}$	$\overline{a}$
$\overline{bd}$	$\overline{a}$	$\overline{c}$	$\overline{bd}$	$\overline{c}$	$\overline{a}$
$\overline{cd}$	$\overline{a}$	$\overline{b}$	$\overline{cd}$	$\overline{b}$	$\overline{a}$

$\boxed{\text{Result}}$

→ No. of ~~undo~~ unordered partitions of equal cell type $(q, q, \dots q)$ where $q \cdot t = n$ of the set S with n elements $= \dfrac{1}{t!} \dfrac{n!}{(q!)^t}$

E.g. No. of ~~ordered~~ unordered partitions of type $\{3, 3, 3, 3\}$ for set $S = \{1, 2, 3, \dots, 12\}$

$= \cancel{\dfrac{12!}{4!}} \quad \dfrac{1}{4!} \cdot \dfrac{12!}{(3!)^4}$

Problem:

① No. of ways 12 people can be partitioned into 3 teams where 1st team has 5, 2nd has 4, 3rd team 3.

$(P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}) = \dfrac{12!}{5!\,4!\,3!}$

② No. of ways 10 of 14 people can be partitioned into 3 teams.

$1 \to 4,\ 2 \to 3,\ 3 \to 3$

${}^{14}C_{10} \cdot \dfrac{10!}{4!\,3!\,3!}$

③ No. of ways 9 of 12 can be ~~partition~~ partitioned in 3 teams with 3 members each.

$= {}^{12}C_9 \cdot \dfrac{1}{3!} \dfrac{9!}{(3!)^3}$

$\{a, b, c, d\}$ $\boxed{B_4 = 15}$ → No. of unordered partitions.

1 part $\bar{a}\ \bar{b}\ \bar{c}\ d$ — 1

2 part $\begin{cases} \overline{ab}\ \overline{cd} & — \dfrac{1}{2!} \cdot \dfrac{4!}{2!\,2!} = 3 \\ \overline{abc}\ \bar{d} & — \dfrac{4!}{3!} = 4 \end{cases}$

3 part (2 1 1) $\{\overline{ab}\ \bar{c}\ \bar{d}\} = \dfrac{4!}{2!\,1!\,1!} = 12$ ordered $= 6$ unordered.

4 part $\overline{abcd}\ \} = 1$ $= 15$

Combinations with repetitions:

$V(n, r)$ = r-combination of n-objects with repetitions.

= No. of non-negative integral solutions of

$x_1 + x_2 + \dots + x_n = r$

= No. of ways r-similar balls can be placed into n-distinct boxes.

□ □ □

3 3 4

6 + 1 + 3

000000 + 0 + 000

000 + 000 + 0000

$\boxed{\text{Ans.} = {}^{n-1+r}C_r = {}^{n-1+r}C_{n-1}}$

E.g. Select 10 fruits from 3 diff. fruits (M, O, A).

M O A

6 + 3 + 1 = 10

2 + 7 + 1 = 10

$x_1 + x_2 + x_3 = 10$

$\boxed{x_1 \geq 0\ x_2 \geq 0\ x_3 \geq 0}$

0 + 0 + 10 = 10

= No. of binary strings with r- 0's and (n-1) 1's.

E.g. No. of bs. with 10 0's and 2 1's

or

No. of bs ~~with~~ of length 12 with exactly 10 0's.

$= {}^{12}C_{10} * 1 * 1$

Select 10 places for 0s; Arrange 10 0's; 2 1's in 2 places.

$\Rightarrow {}^{12}C_{10} = \dfrac{12!}{2!\times 10!} = \dfrac{12 \times 11}{2} = 66$

Q. No of ways ~~for~~ 20 similar balls can be placed in 5 boxes.

$x_1 + x_2 + x_3 + x_4 + x_5 = 20$

$x_i \geq 0$

$n = 5 \quad r = 20$

${}^{n-1+r}C_{n-1} \Rightarrow {}^{24}C_4 \Rightarrow \frac{24!}{20! \times 4!} \Rightarrow \frac{24 \times 23 \times 22 \times 21}{4 \times 3 \times 2}$

Q. No. of ways 20 similar balls placed into 5 boxes, where.

(1) Each box is non-empty.

$x_1 + x_2 + x_3 + x_4 + x_5 = 20$

$x_i \geq 1$

$\boxed{1} \ \boxed{1} \ \boxed{1} \ \boxed{1} \ \boxed{1} = 20 - 5 = 15$

Now, 15 balls in 5 boxes:

$x_1 + x_2 + x_3 + x_4 + x_5 = 15$

$x_i \geq 0 \quad n = 5 \quad r = 15 \Rightarrow {}^{19}C_4 = \frac{19 \times \cancel{18}^{3} \times 17 \times \cancel{16}^{4}}{\cancel{4} \times \cancel{3} \times \cancel{2}} =$

(2) 1st box has at least 3

2nd " " " " 4

3rd " " " " 2

4th & 5th boxes are non-empty.

$\boxed{3} \ \boxed{4} \ \boxed{2} \ \boxed{1} \ \boxed{1}$ $\quad 20 - 11 = 9$

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

Now, $n = 5$; $r = 9$

${}^{13}C_4 \Rightarrow \frac{13 \times \cancel{12} \times 11 \times \cancel{10}^{5}}{\cancel{4} \times \cancel{3} \times \cancel{2}} =$

WB

Pg. 13

(26) colours = 5

$x_1 + x_2 + x_3 + x_4 + x_5 = 6$

$x_i \geq 0 \quad n = 5 \quad r = 6$

${}^{10}C_4 \Rightarrow \frac{10 \times \cancel{9}^{3} \times \cancel{8} \times 7}{\cancel{4} \times \cancel{3} \times \cancel{2}} \Rightarrow 210$

(37) $x + y + z + t = 20$

~~$n = 4 \quad r = 20$~~

$\boxed{1} \ \boxed{1} \ \boxed{1} \ \boxed{1} \Rightarrow 20 - 4 = 16$

$x + y + z + t = 16$

$n = 4 \quad r = 16 \quad {}^{19}C_3 = \frac{19 \times \cancel{18}^{3} \times 17}{\cancel{3} \times \cancel{2}}$

=

(44) a b c

$x_i \geq 1 \quad x_1 + x_2 + x_3 = 5$

$x_i \geq 0 \quad x_1 + x_2 + x_3 = 5 - 3 = 2$

$n = 3 \quad r = 2 \quad {}^{4}C_2 = \frac{\cancel{4}^{2} \times 3}{\cancel{2}} = 6$

abac aa, abc bb, abc cc $\Bigg] \ 3 \times \frac{5!}{3!1!1!} = 60$

abc ab, abc ac, abc bc $\Bigg] \ 3 \times \frac{5!}{2!2!} = ~~45~~ 90$

Total = 90 + 60 = 150

No. of outcomes when

(1) 2 similar coins are tossed.

H T

$x_1 + x_2 = 2$

$n = 2 \quad r = 2$

$\Rightarrow {}^{3}C_1 = 3$

H	T	
2	0	= 2
0	2	= 2
1	1	= 2

Blak	Red
H	T
T	H
H	H
T	T

Similar

Blak	Blak
H	T
T	H
T	T
H	H

→ Same

(2) 10 similar coins are tossed.

$x_1 + x_2 = 10$

$n = 2 \quad r = 10$

${}^{11}C_1 \Rightarrow 11$

Scanned by CamScanner

(3) 2 similar dice rolled.

1 2 3 4 5 6

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 2$

$x_i \geq 0$

$n = 6 \quad r = 2$

${}^{7}C_5 \Rightarrow \frac{7 \times 6^{3}}{2} = \underline{\underline{21}}$

(4) 10 similar dice rolled

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$

$n = 6 \quad r = 10$

$\Rightarrow {}^{15}C_5 = \frac{15 \times 14 \times 13 \times 12 \times 11}{5 \times 4 \times 3 \times 2} = 143 \times 21$

$= 2860 + 143$

$= \underline{\underline{3003}}$.

Q5. 3 identical dice are rolled. Probability that same no. will ~~no~~ appear on each of them. $n = 6 \quad r = 3$

Total $= {}^{8}C_3$

Favourable $= 6$ $\quad$ Prob of $\frac{\text{fav}}{\text{Total}} = \frac{6}{{}^{8}C_3}$

6 cases:
- 1, 1, 1
- 2, 2, 2
- 3, 3, 3
- ⋮
- 6, 6, 6

Generating Functions:

→ Gen. func. for the sequence $\{a_r\}_{r=0}^{\infty}$ is:

$$\boxed{A(x) = \sum_{r=0}^{\infty} a_r x^r}$$

$A(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots$

$= a_0 + a_1 x^1 + a_2 x^2 + \cdots$

E.g. $A(x) = 1 + 2x$ is G.F. for $1, 2, 0, 0, 0, 0 \ldots$

$A(x) = 1 + 2x^3 + 5x^5$. G.F. for $1, 0, 2, 0, 0, 5, 0, 0 \ldots$

$A(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

$= 1, 1, 1, 1, \ldots$

$A(x) = (1+x)^n =$ for $= {}^{n}C_0, {}^{n}C_1, {}^{n}C_2, \ldots, {}^{n}C_n, 0, 0, 0, \ldots$

Important Result

(1) $\frac{1}{1-x} = \sum_{r=0}^{\infty} x^r$

(2) $\frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} {}^{n-1+r}C_r \, x^r$

(3) $1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$

Pg-15

Q.T.10. Coeff of x^{12} in $(x^3+x^4+x^5+x^6+\dots)^3$ → Take common

$= $ Coeff. of x^{12} in $x^9(1+x+x^2+\dots)^3$

$= $ Coeff. of x^3 in $(1+x+x^2+\dots)^3$

$= $ Coeff. of x^3 in $\left(\frac{1}{1-x}\right)^3$

$\Rightarrow$ Coeff. of x^3 in $\frac{1}{(1-x)^3}$ $=$ Coeff. of x^3 in $\sum_{r=0}^{\infty} {}^{3-1+r}C_r x^r$

Put $n=3$ | $r=3$

$= {}^{2+3}C_3 \Rightarrow {}^5C_3 = \boxed{10}$

Pg-14

Q.51. $\frac{1+z}{(1+z)^3} \Rightarrow$ G.F. of $\{a_n\}_{n=0}^{\infty}$.

$\frac{1+z}{(1-z)^3} = 1+z \cdot \frac{1}{(1-z)^3}$ $n=3$

$= (1+z) \cdot \sum_{r=0}^{\infty} {}^{3-1+r}C_r z^r$

$= \sum_{r=0}^{\infty} {}^{2+r}C_r z^r + z \cdot \sum_{r=0}^{\infty} {}^{2+r}C_r z^r$

$a_3 = $ Coeff. $z^3 = {}^{2+3}C_3 + {}^{2+2}C_2$

$a_0 = $ Coeff. of $z^0 = {}^2C_0 = 1$

$a_3 - a_0 = {}^5C_3 + {}^4C_2 - 1$

$= 15$

E.g. Coeff. of x^8 in

$(1+x^3+x^5+x^8)(1+x^4+x^5+x^8)$

$\Rightarrow e_1+e_2 = 8$

$e_1 = 0,3,5,8$ $e_2 = 0,4,5,8$

$$\left.\begin{matrix} 0+8 \\ 3+5 \\ 8+0 \end{matrix}\right\} = \boxed{3}$$

E.g. Coeff. of x^r in

$(1+x)^n$

$= e_1+e_2+e_3+\dots+e_n = r$

$0 \le e_i \le 1$ $= {}^nC_r$

Important

(3) Coeff. of x^r in

$\frac{1}{(1-x)^n}$

$= $ coeff. of x^r in

$(1+x+x^2+x^3+\dots)^n$

$A_1(x)\, A_2(x) \dots A_n(x)$

$= e_1+e_2+\dots+e_n = r$

$$\boxed{= {}^{n-1+r}C_r}$$

Conditions of e_i are the powers of x_i in $A_i(x)$

~~E.g. Coeff. of x^8 in~~

Q. Coeff. of x^r

$(1+x+x^2+x^3)(1+x+x^2)(1+x+x^2+\dots)$

$\Rightarrow e_1 + e_2 + e_3 = r$

$0 \le e_1 \le 3 \quad 0 \le e_2 \le 2 \quad e_3 \ge 0$

Pg. 14

Q.48. No. of 7 digit integers.

Sum of digits = 11 Digits = {1, 2, 3}

$x_1+x_2+x_3+x_4+x_5+x_6+x_7 = 11 \quad n=7 \; r=11$

$\boxed{1 \le x_i \le 3}$

$=$ Coefficient of x^{11} in $(x+x^2+x^3)^7$

$=$ Coefficient of x^4 in $x^7(1+x+x^2)^7$

$=$ Coeff. of x^4 in $\left(\frac{1-x^3}{1-x}\right)^7$

$=$ Coeff of x^4 in $(1-x^3)^7 \times \frac{1}{(1-x)^7}$

$=$ Coeff. of x^4 in $\left[{}^7C_0 + {}^7C_1(-x^3)\right] \Rightarrow$ ignore higher power in binomial expansion.

Coeff. of x^4 in

$= [1-7x^3] \sum_{r=0}^{\infty} {}^{6+r}C_r x^r$

$=$ Coeff. of x^4 in $\sum_{r=0}^{\infty} {}^{6+r}C_r x^r - 7x^3 \sum_{r=0}^{\infty} {}^{6+r}C_r x^r$

$\Rightarrow {}^{10}C_4 - 7 \cdot {}^7C_1 = {}^{10}C_4 - 49$

Recurrence Relations: (Problem in terms of subproblem)

Pg. 14

36 T_n = population after n years $T(n)$

$T(n-1)$ = " " $(n-1)$ years

$\boxed{T_n = 2(2T_{n-1})}$

$\boxed{T_n = 4T_{n-1}}$

GATE'16: a_n = No. of n-bit strings that do not contain 2 consecutive 1's.

Recurrence relation for a_n.

(a) $a_n = a_{n-1} + 2a_{n-2}$

(b) $a_n = 2a_{n-1} + a_{n-2}$

(c) $a_n = a_{n-1} + a_{n-2}$

(d) $a_n = 2a_{n-1} + 2a_{n-2}$

b.s. of length $n-1$ not containing 11 — a_{n-1}

a_n — start with 0 → 0 | 0 0 …

start with 1 → 1 | 0 0 …

b.s. of length $n-1$ not containing 11 and not starting with 1

0 1

~~x0~~

b.s. of length $n-2$ not containing 11 & starting with 10.

$\boxed{a_n = a_{n-1} + a_{n-2}}$

$a_1 = 2$

$a_2 = 3$

$a_3 = 5$

Solving linear Recurrence Relation of order 2:

$K_0 a_n + K_1 a_{n-1} + K_2 a_{n-2} = 0$

order 2.

Characteristic Eq^n :

$$\boxed{C(t) = K_0 t^2 + K_1 t + K_2 = 0}$$

Then find root

Roots	General solution
① b_1, b_2 (Real distinct)	$a_n = \underbrace{C_1}_{\text{some coefficient}}(b_1)^n + C_2(b_2)^n$
② b, b (real & equal)	$a_n = C_1(b)^n + C_2\, n\, (b)^n$

(36) $T_n = 4T_{n-1}$

$T_n - 4T_{n-1} = 0$

$C(t)$: $t - 4 = 0$

$t = 4$

$\boxed{T_n = C(4)^n}$

$T_0 = 37$

$T_0 = C(4)^0$

$\Rightarrow C = 37$

$\boxed{T_n = 37(4)^n}$

$\boxed{T_5 = 37(4)^5}$

(32) $S(k) - 10S(k-1) + 9S(k-2) = 0$ | $S(0) = 3$; $S(1) = 11$.

(a) $1 + 2^k + 9^{k-1}$ $\quad S_0 = 1 + 1 + 9^{-1} \neq 3$

(b) $3 + 8^k$ $\quad S_0 = 3 + 8^0 = 4 \neq 3$

(c) $2 + 9^k$ $\quad S_0 = 2 + 9^0 = 3$ ✓

(d) $1 + 3^k + 11^{k-1}$

Q. $T_n - 7T_{n-1} + 12T_{n-2} = 0$ | $T_0 = 1$ $T_1 = 0$

(a) $T_n = 3(2)^n - 2(3)^n$ $\quad T_0 = 1$ $\quad T_1 = 0$ $\quad T_2 = 12 - 18 \times$

(b) $T_n = 7(6)^n - 6(7)^n$ $\quad T_0 = 1$ $\quad T_1 = 0$ $\quad T_2 =$

(c) $T_n = 4(3)^n - 3(4)^n$ $\quad T_0 = 1$ $\quad T_1 = 0$ $\quad T_2 =$ ✓

✗ (d) $T_n = 5(3)^n - 4(4)^n$ $\quad T_0 = 1$ $\quad T_1 = -1$

$C(t) = t^2 - 7t + 12 = 0$

$t = 3, 4$

$T_2 = 7T_1 - 12T_0$

$= 0 - 12$

$T_2 = -12$

(25) $T(2^k) = 3T(2^{k-1}) + 1$ $\quad T(1) = 1$

$S(k) = T(2^k)$

$S(0) = T(2^0) = T(1) = 1$

$S(k) - 3S(k-1) = 1$

$S(0) = 1$, $S(1) = 3 + 1 = 4$

Scanned by CamScanner

Pigeonhole Principle :

→ If there are n-pigeon holes and $\boxed{(n+1)}$ pigeons, then some pigeon hole contains at least 2 pigeons. (upto 2n)

→ If there are n-pigeon holes and $\boxed{(2n+1)}$ pigeons, then some pigeonholes contains at least 3 pigeons. (upto 3n)

⋮

→ If there are n-pigeons holes and $\boxed{(kn+1)}$ pigeons, then some pigeon hole contains atleast (k+1) pigeons.

Result

If there are n-pigeonholes and k-pigeons the some pigeon hole contains at least $\left\lceil \frac{k}{n} \right\rceil$ pigeons.

Q. In a class there are 75 students ① at least ___ students are born on the same day of a week.

② atleast ___ students are born on the same ~~day~~ month of a ~~month~~ year.

① $n = 7$ $k = 75$ (pigeons)

$\left\lceil \frac{75}{7} \right\rceil = 11$ students

② $n = 12$ $k = 75$

$\left\lceil \frac{75}{12} \right\rceil = 7$ students.

Q. At least 5 students are born in the same month of a year. The minimum no. of students which guarantees this result.

① ~~(a)~~ 60 (b) 61 (c) 72 (d) 73

② ~~(a)~~ 49 (b) 50 (c) 51 (d) 60

$n = 12$ For at least $k+1 = 5 \Rightarrow \boxed{k=4}$

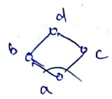
min $kn+1 \Rightarrow 4 \times 12 + 1 = 49$

(23) $k+1 \Rightarrow k = 2$ | $n = 4$

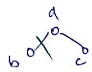
min $kn+1 \Rightarrow 2 \times 4 + 1 = 9$

Topological Sort: (Total order)

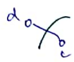
→ Total order containing given partial order.



Min: a — T. Sort: a



Min: b, c — $a \prec b$



$a \prec b \prec c$

Min: c

°d

$\boxed{a \prec b \prec c \prec d}$

→ a . 2! . c

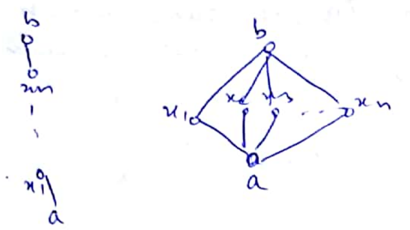
= (2)

Pg 48

(19)

⇒ ~~S = {a, x_1,~~

$S = \{x, a_1, a_2, \ldots, a_n, y\}$

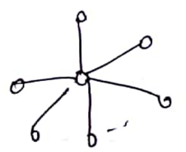


: a n! b

(32). n < d < c < t < s

Star Graph :

$K_{1,n}$



#Refinement

Let P_1 and P_2 be partitions of A.

P_1 is refinement of P_2 ($P_1 \leq P_2$)

if every part in P_1 is subset of somepart in P_2.

E.g. $A = \{1,2,3,4\}$

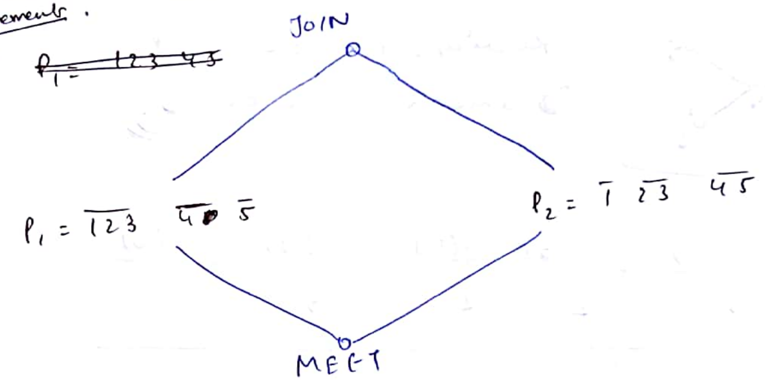
$P_1 = \{\{1\},\{2\},\{3\},\{4\}\}$ $P_2 = \{\overline{12}, \overline{34}\}$

P_1 is refinement of P_2.

$\boxed{\text{* JOIN and *MEET}}$

Refinements.

(39) ~~$P_1 = \overline{123}\ \overline{45}$~~

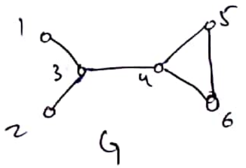


(a) $\overline{1}\ \overline{23}\ \overline{4}\ \overline{5}$ = MEET JOIN = $\overline{123}\ \overline{45}$

(b) $\overline{123}\ \overline{45}$ = MEET X JOIN = $\overline{12}\ \overline{3}\ \overline{45}$

Should be subset of both P_1 and P_2.

E.g.



Distance between u to v :

→ $d(u,v)$ = Length of shortest path from u to v.

$d(1,4) = 2$

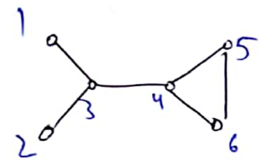
$d(1,5) = 3$

Eccentricity of v :

→ The greatest distance of any vertex u from v.

Radius : Minimum eccentricity

Diameter : Maximum ~~ecc~~ eccentricity



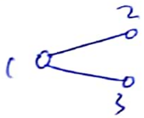
Radius = Min. ec = 2

Diameter = Max. ec = 3

ec(1)	ec(2)	ec(3)	ec(4)	ec(5)	ec(6)
3	3	2	2	3	
max. $\{d(1,3) = 1$, $d(1,2) = 2$, $d(1,4) = 2$, $d(1,5) = 3$, $d(1,6) = 3\ 4$	max $\{d(2,3) = 1$, $d(2,1) = 2$, $d(2,4) = 4$, $d(2,5) = 3$, $d(2,6) = 3$				

Chap 4 (WB)

Q(4). $K_{1,n}$ $n \geq 2$



ec(1)	ec(2)	ec(3)
1	2	2

Diameter = 2