13/11/2017

ENGINEERING MATHS (15 Marks)

MATRIX :

$$A = [a_{ij}]$$

Square matrix : $[A_{ij}]_{n \times n}$

Rectangular matrix : $[A_{ij}]_{m \times n}$ $m \neq n$

I. Linear Algebra : Linear algebra deals with linear system or linear transformation and also application.

Every linear system and every linear transformation can be expressed in terms of matrices :

Linear Algebra — Linear sys. / Linear transformation $\boxed{X' = AX}$

Operations on matrices :

$$A_{m \times n} \cdot B_{p \times q} \Rightarrow (AB)_{m \times q} \quad \text{if } n = p$$

NOTE

1. $AB \neq BA$
2. $A(BC) = (AB)C$
3. The no. of multiplications required to find $(AB)_{m \times q} = mpq$
4. The no. of additions required to fin $(AB)_{m \times q} = m\underbrace{(p-1)}_{\text{no. of rows} - 1}q$

Q. Let $P_{4 \times 2}, Q_{2 \times 4}, R_{4 \times 1}$ be '3' matrices. Find the minimum no. of multiplications required to find PQR.

$(PQ)R$

$(4 \times 2 \times 4)R$

$(32) PQ_{4 \times 4} \cdot R_{4 \times 1}$

$32 + 4 \times 4 \times 1$

48

$P(QR)$

$P(2 \times 4 \times 1)$

$P_{4 \times 2} \quad QR_{2 \times 1} \, (8)$

$PQR_{4 \times 1} = 8 + 8 = 16$

		Multiplications	Additions
$P(QR) \Rightarrow$	$Q_{2 \times 4} \cdot R_{4 \times 1} \Rightarrow$	8	6
	$P_{4 \times 2} \, (QR)_{2 \times 1} \Rightarrow$	8	4
		16	10
$(PQ)R \Rightarrow$	$P_{4 \times 2} \, Q_{2 \times 4} \Rightarrow$	32	16
	$PQ_{4 \times 4} \cdot R_{4 \times 1} \Rightarrow$	16	12
		48	28

Trace of matrix : $= a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$

- Sum of the principal diagonal elements

Transpose of matrix :

- If $A = [a_{ij}]_{m \times n}$, then $A^T = [a_{ji}]_{n \times m}$

E.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 6 & -1 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix}$

* Properties:

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(AB)^T = B^T . A^T$
4. $(A^n)^T = (A^T)^n$
5. $\text{Trace}(A) = \text{Trace}(A^T)$
6. $\text{Trace}(A \pm B) = \text{Trace}(A) \pm \text{Trace}(B)$

Symmetric Matrix :

→ $A = [a_{ij}]_{n \times n}$ is symmetric matrix, iff $\boxed{A^T = A}$ (or) $\boxed{a_{ij} = a_{ji}}$.

* Properties: If $A^T = A$, $B^T = B$, then:

1. A^n is symmetric matrix.
2. "$\lambda_1 A + \lambda_2 B$" is symmetric → Linear combination = scalar. vector + scalar. vector
3. $AB + BA$ is symmetric matrix.

1. $(A^n)^T = (A^T)^n = A^n$
3. $(AB + BA)^T = (AB)^T + (BA)^T = B^T A^T + A^T B^T$
   $= BA + AB.$
4. 'AB' is symmetric matrix, iff, $AB = B.A.$

E.g. $(AB)^T = B^T . A^T = B.A.$
NOT symmetric

Skew-Symmetric Matrix :

- $A = [a_{ij}]_{n \times n}$ is skew-symmetric matrix iff:

$\boxed{A^T = -A}$ (or) $\boxed{a_{ij} = -a_{ji} \ \& \ a_{ii} = 0}$

* Properties: If $A^T = -A$, $B^T = -B$, then:

1. A^n is symmetric matrix if, n is even.
2. A^n is skew-symmetric matrix if n is odd.
3. "$\lambda_1 A + \lambda_2 B$" is skew symmetric matrix.
4. $AB + BA$ is symmetric matrix.
5. "AB" is skew symmetric matrix if $AB + BA = 0$

Complex Matrix : $A = [a_{ij}]_{m \times n}$

a_{ij} → complex number

Conjugate of matrix :

$A = [a_{ij}]_{m \times n}$, then $\bar{A} = [\bar{a}_{ij}]_{m \times n}$

E.g. $A = \begin{bmatrix} i & 1+i \\ 2 & 2i \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} -i & 1-i \\ 2 & -2i \end{bmatrix}$

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Hermitian Matrix :

→ $A = [a_{ij}]_{n\times n}$ is Hermitian matrix if :

$\boxed{\bar{A} = A^T}$ (or) $\boxed{(\bar{A})^T = A}$ ⇒ $\boxed{A^{\theta} = A}$ and $\boxed{a_{ii} = \text{real} \;\&\; [\overline{a_{ij}}] = [a_{ji}]}$

θ → Transpose of conjugate

* Properties : if $A^{\theta} = A$ and $B^{\theta} = B$, then :

① A^n is Hermitian matrix.
② "$\lambda_1 A + \lambda_2 B$" is Hermitian matrix.
③ $AB + BA$ is Hermitian matrix.
④ "AB" is Hermitian matrix iff $AB = BA$.

· If A is a real matrix, ~~then~~ & $A^{\theta} = A$, then it is symmetric.
· So, similar properties to symmetric matrices.

Skew - Hermitian Matrix :

$A = [a_{ij}]_{n\times n}$ is skew-hermitian matrix if

$\boxed{\bar{A} = -A^T}$ (or) $\boxed{(\bar{A})^T = -A}$ ⇒ $\boxed{A^{\theta} = -A}$ and $\boxed{a_{ii} = \text{zero or purely imaginary and } \overline{a_{ij}} = a_{ji}}$

*Properties : if $A^{\theta} = -A$, $B^{\theta} = -B$, then :

① A^n is Hermitian matrix if "n" is even.
② A^n is skew-hermitian matrix, if "n" is odd.
③ ~~AB+B~~ $\lambda_1 A + \lambda_2 B$ is skew Hermitian matrix
④ $AB + BA$ is Hermitian matrix
⑤ "AB" is skew Hermitian matrix, iff $AB + BA = 0$

NOTE

↳ Every square matrix can be expressed as sum of symmetric and skew symmetric matrices.

$$A_{n\times n} = \underbrace{\boxed{\frac{1}{2}[A + A^T]}}_{\text{Sym. matrix}} + \underbrace{\boxed{\frac{1}{2}[A - A^T]}}_{\text{sk-sym. matrix}}$$

→ Every square matrix can be expressed as sum of Hermitian and skew-Hermitian matrix.

$$A_{n\times n} = \underbrace{\boxed{\frac{1}{2}[A + A^{\theta}]}}_{\text{H. matrix}} + \underbrace{\boxed{\frac{1}{2}[A - A^{\theta}]}}_{\text{sk-H. matrix}}$$

→ A, B are symmetric or skew symmetric, but $AB - BA$ is skew symmetric.

→ If A is Hermitian matrix, then "iA" is skew Hermitian matrix and vice versa.

Orthogonal Matrix :

$A = [a_{ij}]_{n\times n}$ is orthogonal matrix iff $\boxed{A \cdot A^T = A^T \cdot A = I \text{ (or) } A^{-1} = A^T}$

Unitary Matrix :

$A = [a_{ij}]_{n\times n}$ is unitary matrix iff $\boxed{A \cdot A^{\theta} = A^{\theta} \cdot A = I \text{ (or) } A^{-1} = A^{\theta}}$

NOTE

↳ If A, B are orthogonal/unitary, then A^{-1}, B^{-1}, AB, BA are orthogonal/unitary matrices.

Q. $M = \begin{bmatrix} 3/5 & 4/5 \\ x & 3/5 \end{bmatrix}$ & $M^{-1} = M^T$, $x = ?$

$\because M^{-1} = M^T \Rightarrow M.M^T = I$

$\Rightarrow \begin{bmatrix} 3/5 & 4/5 \\ x & 3/5 \end{bmatrix} \begin{bmatrix} 3/5 & x \\ 4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow 3/5\, x + \left(\frac{4}{5} \times \frac{3}{5}\right) = 0$

$\Rightarrow \boxed{x = -\frac{4}{5}}$

Idempotent Matrix :

→ If $A^2 = A$, then, A is idempotent matrix.

* Properties : If $A^2 = A$, $B^2 = B$, then

① "$A+B$" is Idempotent iff $AB + BA = 0$

Eg $(A+B)^2 = A^2 + B^2 + AB + BA$
$= A + B$

② 'AB' is idempotent iff $AB = BA$

Eg $(AB)^2 = B^2 A^2 = BA$

③ $A^n = A$ & $B^n = B$

Eg $A^2 = A$
$A^3 = A$
$A^4 = A$
$A^n = A$

NOTE

① If $AB = B$ and $BA = A$, then $A^2 = A$, $B^2 = B$.

② If $AB = A$ and $BA = B$, then $A^2 = A$, $B^2 = B$.

Eg $A^2 = A.A$
$= A.BA$
$= (AB) A$
$= B.A$
$= A$

WB

Q1 Given $xy = y$ and $yx = x$, then $x^2 + y^2 = ?$

$\cancel{x^2+y^2}$ $x^2 = x$ $\quad y^2 = y$

$x^2 + y^2 = x + y$ (or) $x^5 + y^3 = x + y$ $\quad [\because A^n = A]$

$x^n + y^n = x + y$

Q5. $AB = B$ and $BA = A$, A, B are of same size, then $(A+B)^5 = ?$

(a) 2^3 (b) 2^4 (c) $2^3(A+B)$ (d) $2^4(A+B)$

$\Rightarrow A^2 = A$, $B^2 = B$:] → ①

$(A+B)^2 = A^2 + B^2 + AB + BA$

$\boxed{(A+B)^2 = A + B + B + A = 2(A+B)}$

$(A+B)^3 = 2(A+B)^2$
$(A+B)^3 = 2^2(A+B)$
$(A+B)^4 = 2^2(A+B)^2$
$= 2^3(A+B)$
$(A+B)^5 = 2^3(A+B)^2$
$= 2^4(A+B)$

Involutary Matrix :

→ If $A^2 = I \Rightarrow A$ is involutary

$A \cdot A = I$

$\Rightarrow \boxed{A = A^{-1}}$

Eg $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Involuatary Matrix

Determinant of Matrix :

for $i = 1$

$$|A_{n\times n}| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij}\, \delta_{ij}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}_{2\times 2} = ad - bc$$

where δ_{ij} = minor of an element a_{ij}.

Eg $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = (-1)^{1+1} a_{11}\delta_{11} + (-1)^{1+2} a_{12}\delta_{12} + (-1)^{1+3} a_{13}\delta_{13}$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

~~Properties~~ **NOTE**

→ The no. of terms in the determinant expansion of $n\times n$ matrix = $n!$

Properties :

1. $|A| = |A^T|$
2. $|AB| = |A|\cdot|B| = |B|\cdot|A| = |B\cdot A|$
3. $|A^n| = |A|^n$
4. $|A^{-1}| = \frac{1}{|A|}$
5. $\begin{vmatrix} k\cdot a & k\cdot b \\ c & d \end{vmatrix} = k\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} ka & b \\ kc & d \end{vmatrix}$
6. $|k A_{n\times n}| = k^n \cdot |A|$
7. $|\text{Adj } A_n| = |A|^{n-1}$
8. *** $\text{Adj}(\text{Adj } A_n) = |A|^{n-2} \cdot A$
9. $|\text{Adj}(\text{Adj } A_n)| = |A|^{(n-1)^2}$

$|\text{Adj}(\text{Adj } A_n)| = \left||A|^{n-2} \cdot A_n\right|$

$= \left(|A|^{n-2}\right)^n \cdot |A|$

$\Rightarrow |A|^{n^2-2n+1} = |A|^{(n-1)^2}$

Q. $|A_{3\times 3}| = 6$. Find :

(a) $|3A| = 3^3 \cdot |A| = 27 \times 6 = 162$

(b) $|(2A)^{-1}| = \frac{1}{|2A|} = \frac{1}{2^3 \cdot |A|} = \frac{1}{2^3 \cdot 6} = \frac{1}{48}$

(c) $\text{Adj}(\text{Adj } A_3) = |A|^{3-2} \cdot A = |A| \cdot A = 6 \cdot A$

(d) $|\text{Adj } A_3| = |A|^{3-1} = 36$

(e) $|\cancel{\text{Adj}}\,(A^{-1})^T| = |A^{-1}| = \frac{1}{|A|} = \frac{1}{6}$

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(10) The determinant of a Lower triangular, upper triangular and diagonal matrix is the product of its principal diagonal elements.

Eg. $|L| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 1 * 3 * 1 = 3$

(11) The determinant of skew symmetric matrix of odd order is zero(0). and of even order is a perfect square.

Eg. $\begin{vmatrix} 0 & 2 & 1 \\ -2 & 0 & -1 \\ -1 & 1 & 0 \end{vmatrix}_{3\times 3} = 0$ *Because it is a skew symmetric matrix of odd order.

(12) The determinant of orthogonal/unitary is ± 1.

$|A^{-1}| = |A^T|$

$\frac{1}{|A|} = |A| \Rightarrow |A|^2 = 1 \Rightarrow \pm 1$

(13) The determinant of involutary matrix is ± 1.

$|A| = |A^{-1}|$

$\Rightarrow \frac{1}{|A|} = |A| \Rightarrow |A|^2 = 1 \Rightarrow \pm 1$

(14) The determinant of idempotent matrix is 0 (or) 1.

$|A^2| = |A|$

$|A|^2 = |A|$

$|A|^2 - |A| = 0$

$|A|(|A| - 1) = 0$

$\boxed{|A| = 0 \text{ (or) } 1}$

Linear Dependency:

→ A set of vectors ~~x_1 and x_2~~ $x_1, x_2 \dots x_n$ are ~~set~~ said to be linearly dependent, if their linear combination

$a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$,

for not all $a_i = 0$.

else for all $a_i = 0$,

$x_1, x_2 \dots x_n$ are linearly independent.

Eg $\sin^2 x, \cos^2 x, \cos 2x$

$\Rightarrow \cos 2x = 1.\cos^2 x - 1.\sin^2 x$

$f_1 = 1.f_2 - 1.f_3$

∴ Linearly Dependent.

Eg. $\sin x, \cos x, \tan x$.

$\because \sin x \neq c_1.\cos x + c_2.\tan x$

∴ They are not linearly dependent and they are linearly independent.

*
(1) $R_i = kR_j$
(2) $R_i = \lambda_1 R_j + \lambda_2 R_k + \dots$

* If any 2 rows/columns are linearly dependent, then the determinant is zero(0).

Elementary Transformations: R.T. / C.T.

$A \underset{(E.T.)}{\cong} B$

Rule 1: If $R_i \longleftrightarrow R_j$, then $|A| = -|B|$

Rule 2: If $R_i \to kR_i$, then $|B| = k|A|$

Rule 3: If $R_i \to R_i + kR_j$, then $|B| = |A|$.

(a) $\begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 2 & 1 & 3 \end{vmatrix}$ $\because R_3 = R_1 + R_2$ Linear dependency. $\therefore$ Determinant $= 0$ | $C_3 = C_1 + C_2$

* No. of independent rows = no. of dependent columns.

Q. $\begin{vmatrix} 2 & 1 & 1 & 2 \\ -1 & -1 & 1 & 1 \\ 1 & 0 & 2 & 3 \\ 0 & -1 & 3 & 4 \end{vmatrix} = 0$ $\because R_3 = R_1 + R_2$, $R_4 = R_2 + R_3$

Q56. $A = \begin{bmatrix} 3 & 4 & 45 \\ 7 & 8 & 105 \\ 13 & 2 & 195 \end{bmatrix}$ $\boxed{\begin{array}{l} ① R_1 \to R_1 + R_2 \\ ② C_1 \to C_1 - C_3 \end{array}}$ → Rule ③ of E.T.

$A \cong B$ then $|B| = ?$

① $\begin{bmatrix} 10 & 12 & 150 \\ 7 & 8 & 105 \\ 13 & 2 & 195 \end{bmatrix}$ $\xrightarrow{②}$ $\begin{bmatrix} -140 & 12 & 150 \\ -98 & 8 & 105 \\ -182 & 2 & 195 \end{bmatrix}$ ~~No need~~

$\therefore |B| = |A| = \begin{vmatrix} 3 & 4 & 45 \\ 7 & 8 & 105 \\ 13 & 2 & 195 \end{vmatrix} = 15 \begin{vmatrix} 3 & 4 & 3 \\ 7 & 8 & 7 \\ 13 & 2 & 13 \end{vmatrix} = 0$

$\because C_3 = 15 C_1$ $\therefore |B| = 0$.

Q CS. $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 8 & 1 & 7 & 2 \\ 2 & 0 & 2 & 0 \\ 9 & 0 & 6 & 1 \end{vmatrix} = +2 \begin{vmatrix} 1 & 7 & 2 \\ 0 & 2 & 0 \\ 0 & 6 & 1 \end{vmatrix}$

$\Rightarrow 2\left[1 \begin{vmatrix} 2 & 0 \\ 6 & 1 \end{vmatrix}\right] = 2 * 2 = 4$

Q EC. $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$

$C_3 \to C_3 - 3C_1$

$\Rightarrow \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 3 & -6 & 1 \\ 3 & 0 & -8 & 2 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 2 & 3 \\ 3 & -6 & 1 \\ 0 & -8 & 2 \end{vmatrix} = (-1)[8(-8) + 2(-12)] = 88.$

Shortcuts :

① $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left[1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right]$

② $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

③ $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Q. $\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = 1[1+1+1+1+1] = 5$

Q. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 3.1.2.3 - 1^3 - 2^3 - 3^3$

$= 18 - 1 - 8 - 27$

$= -18$

Q. If $\begin{vmatrix} a & a^2 & \boxed{a^3+1} \\ b & b^2 & \boxed{b^3+1} \\ c & c^2 & \boxed{c^3+1} \end{vmatrix} = 0$ for $a \neq b \neq c$, then $abc = ?$

Sol^n. Given determinant can be written as: Rule 1

$\Rightarrow \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$

$\Rightarrow abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

$\Rightarrow (abc+1) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

from ③ $\neq 0$ for $a \neq b \neq c$

$\therefore abc + 1 = 0$

$\therefore \boxed{abc = -1}$

Inverse of Matrix:

Definition: $A^{-1} = \dfrac{\text{adj } A}{|A|}$; $|A| \neq 0$

where $\text{Adj}(A) = (\text{Cofactor matrix})^T$

$= [(-1)^{i+j} \delta_{ij}]^T$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

WB Q4 $A = [a_{ij}]_{3\times3}$ & $a_{ij} = i^2 - j^2$ $\forall i, j$, then $A^{-1} = ?$

$a_{11} = 1^2 - 1^2 = 0$ $\quad a_{21} = 2^2 - 1^2 = 3$ $\quad a_{31} = 3^2 - 1^2 = 8$

$a_{12} = 1^2 - 2^2 = -3$ $\quad a_{22} = 2^2 - 2^2 = 0$ $\quad a_{32} = 3^2 - 2^2 = 5$

$a_{13} = 1^2 - 3^2 = -8$ $\quad a_{23} = 2^2 - 3^2 = -5$ $\quad a_{33} = 3^2 - 3^2 = 0$

$A = \begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{bmatrix}_{3\times3}$ $\Rightarrow$ Sk-symmetric of odd order.

$\therefore |A| = 0$

$\therefore A^{-1}$ does not exist. (d)

WB Q3. A, B, C, D, E are non-singular

$A^{-1}, B^{-1}, C^{-1}, D^{-1}, E^{-1}$ exist.

given, $DABEC = I$, $B^{-1} = ?$ $\quad \boxed{\because (AB)^{-1} = B^{-1}.A^{-1}}$

Sol^n. Take inverse on both sides:

$\Rightarrow (DABEC)^{-1} = (I)^{-1} \Rightarrow C^{-1}E^{-1}B^{-1}A^{-1}D^{-1} = I$

Pre-multiplied by C & post multiplied by 'D' on both sides.

$E^{-1}B^{-1}A^{-1} = CD$

Pre $\Rightarrow$ E Post $\Rightarrow$ A on both sides.

$\boxed{B^{-1} = ECDA}$

Rank of Matrix : (ρ)

→ The no. of independent rows/columns in a matrix is called rank of matrix.

→ The highest order of non-zero minor of a matrix is called rank of matrix.

Let $A_{3\times3}$

if $|A_{3\times3}| \neq 0 \Rightarrow \rho(A) = \text{order} = 3$

if $|A_{3\times3}| = 0 \Rightarrow \rho(A) < \text{order}$

if $\exists$ atleast one non-zero minor of order 2×2

$\rho(A) = 2$

else $\rho(A) = 1$

Q1 Find Rank of :

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix} \text{dependent}$$

$R_3 = R_1 + R_2$

R_1 & R_2 are independent & minor $\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} \neq 0$

$\therefore \rho(A) =$ no. of independent rows/columns $= 2$

(2) $$\begin{bmatrix} 2 & 1 & 1 & 2 \\ -1 & -1 & 1 & 1 \\ 1 & 0 & 2 & 3 \\ 0 & -1 & 3 & 4 \end{bmatrix} \text{dependent}$$

$\therefore R_1$ & R_2 are independent

$\rho(A) =$ no. of independent rows/columns $= 2$

$R_3 = R_1 + R_2$

$R_4 = R_2 + R_3$

& minor $\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} \neq 0$

(3) $$\begin{bmatrix} 6 & 0 & 4 & 14 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix} \rightarrow \text{dependent}$$

$R_3 = 2R_1 - R_2$

$\rho(A) =$ highest order of non-zero minor (or) no. of ind. rows/column

$\rho(A) = 2$

(47) Find rank of $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \rightarrow \text{dependent}$

$R_4 = R_1 + R_2 + R_3$ & minor $\begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{vmatrix}_{3\times3}$

$= 2[-1] - 3[9] - 1[4] \neq 0$

$\Rightarrow R_1, R_2, R_3 \neq 0$, $\therefore \rho(A) = 3$

(48) Rank of $A = \begin{bmatrix} 4 & 7 & 2 \\ 3 & -1 & 5 \\ 7 & 6 & x \end{bmatrix}_{3\times3}$ is '2', then, $x = ?$

given $\rho(A_{3\times3}) = 2 <$ order $\Rightarrow |A_3| = 0$

$\rho(A_{3\times3}) = 2$ | $\because R_3 = R_1 + R_2$

$x = 2 + 5$

$x = 7$

(35) $A = [a_{ij}]_{n\times n}$, $a_{ij} = i \cdot j \ \forall i,j$, $n \geq 3$; $\rho(A) = ?$

$a_{11} = 1$ $a_{12} = 2$ $a_{13} = 3$

$a_{21} = 2$ $a_{22} = 4$ $a_{23} = 6$

$a_{31} = 3$ $a_{32} = 6$ $a_{33} = 9$

$$A = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 4 & 6 & \cdots & 2n \\ 3 & 6 & 9 & \cdots & 3n \\ \vdots \\ n & 2n & 3n & \cdots & n^2 \end{bmatrix}$$

$R_2 = 2R_1$

$R_3 = 3R_1$

$\vdots$

$R_n = nR_1$

R_1 is the only independent row.

$\rho(A) =$ no. of independent rows $= 1$

(49) $A = [a_{ij}]_{m \times n}$, $a_{ij} = 5 \ \forall \ i, j$; $\rho(A) = ?$

Sol^n. $A = \begin{bmatrix} 5 & 5 & 5 & \cdots & 5 \\ 5 & 5 & 5 & \cdots & 5 \\ \vdots & & & & \vdots \\ 5 & 5 & 5 & \cdots & 5 \end{bmatrix}$ $\because R_2 = R_3 = R_4 = R_5 = \cdots = R_m = R_1 \rightarrow$ Ind. row

$\therefore \rho(A) = 1$

Echelon Form:

$\underset{\text{R.T.}}{A \cong} \boxed{U} \rightarrow$ Echelon form of 'A'

* change into upper triangular matrix using row trans.

Definition:
$\rho(A) =$ ~~non ze~~ no. of non-zero rows in Echelon form of 'A'.

Q. Find rank of $\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

* At any stage of the Echelon form, dependency method can be applied.

By Echelon form,

$R_2 \rightarrow 2R_2 + R_1$ & $R_3 = 2R_3 + R_1$

$\begin{array}{l} \\ \text{dep.} \leftarrow \\ \text{dep.} \leftarrow \\ \\ \end{array} \begin{bmatrix} -2 & -1 & -3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \xrightarrow[R_4 = 3R_4 - R_2]{R_3 = 3R_3 + R_2} \begin{bmatrix} -2 & -1 & -3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

It is in Echelon form

$R_2 = 3R_4$ & $R_3 = -R_4$ $\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \neq 0 \therefore \rho(A) = 2$

$\therefore \rho(A) =$ No. of non-zero rows in Echelon form $= 2$

Properties of Rank:

① $\rho(A) = \rho(A^T)$

② $\rho(A \cdot A^T) = \rho(A)$

③ $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$

④ $\rho(A_{m \times n}) \leq \min\{m, n\}$

⑤ $\rho(I_n) = n$

⑥ $\rho(O_{m \times n}) = 0$

⑦ $\rho(\text{Diagonal matrix}) =$ no. of non-zero principal diagonal elements.

Q (1m) EE

$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ $B = \begin{bmatrix} p^2 + q^2 & pr + qs \\ rp + sq & r^2 + s^2 \end{bmatrix}$

If $\rho(A) = N$, then, $\rho(B) = ?$

$\underset{A}{\begin{bmatrix} p & q \\ r & s \end{bmatrix}} \underset{A^T}{\begin{bmatrix} p & r \\ q & s \end{bmatrix}} = \underset{B}{\begin{bmatrix} p^2 + q^2 & pr + qs \\ rp + sq & r^2 + s^2 \end{bmatrix}}$

$\Rightarrow \rho(B) = \rho(A \cdot A^T) = \rho(A) = N.$

(50) $X_{n \times 1} \neq 0$, $\rho(XX^T) = ?$

We know that (WKT) $\rho(XX^T) = \rho(X_{n \times 1}) \leq \min\{1, n\}$

≤ 1

$\because X_{n \times 1} \neq 0$ $\therefore \rho(XX^T) = \rho(X) = 1$

System of linear equations:

$AX = B$

If $B \neq 0 \Rightarrow AX = B$ is Non-Homogeneous system.

If $B = 0 \Rightarrow AX = 0$ is Homogeneous system.

$\hookrightarrow$ degree of every term in eqⁿ is same.

* Solving $AX = B$:

① Write Augmented matrix $C = [A \mid B]$

② Reduce 'C' into Echelon form.

we get $\rho(A)$, $\rho(C)$

③ If $\rho(A) < \rho(C)$, then system is Inconsistent $\rightarrow$ No solution

④ If $\rho(A) = \rho(C) =$ no. of unknowns, then, system possesses unique solution. $|A| \neq 0$

⑤ If $\rho(A) = \rho(C) <$ no. of unknowns, then, system possesses infinitely many solutions. $|A| = 0$

NOTE

① $A.X = B$ possesses unique solution iff $|A| \neq 0$.

② If $A.X = B$ possesses infinitely many solutions, then, $|A| = 0$.

③ If $AX = B$ is inconsistent, then, $|A| = 0$.

Solving $AX = 0$:

(i) Reduce 'A' into Echelon form, we get $\rho(A)$

If $\rho(A) =$ no. of unknowns, then, system possesses unique solⁿs.

$A \neq 0$ solution, $\boxed{X = 0,}$ trivial or zero solution.

If $\rho(A) <$ no. of unknowns, then system possesses infinitely many solⁿ $\Rightarrow \boxed{X \neq 0}$ (non-trivial or non-zero solⁿ)

$|A| = 0$

NOTE

$\rightarrow$ $AX = 0$ possesses trivial solution, iff $|A| \neq 0$.

$\rightarrow$ $AX = 0$ possesses non-trivial ~~soat~~ solutions, iff $|A| = 0$.

Q7. Given system has infinitely many solⁿ.

$\Rightarrow |A| = 0$ $\quad \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$

$R_1 \rightarrow R_1 + R_2 + R_3$ $\quad \begin{vmatrix} a+2 & a+2 & a+2 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} \Rightarrow (a+2) \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$ $\quad = 0$ for $a = 1$

$\Rightarrow \boxed{a = -2, \text{(or) } 1}$

Q.33. $px + qy + rz = 0$
$qx + ry + pz = 0$
$rx + py + qz = 0$
has non-trivial solution.

$\therefore |A| = 0$ $\quad \begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0 \Rightarrow (p+q+r) \begin{vmatrix} 1 & 1 & 1 \\ q & r & p \\ r & p & q \end{vmatrix} = 0$ for $p = q = r$

$p + q + r = 0$ (or) $p = q = r$. (C)

Q.51. System has ∞ly many solutions $\Rightarrow |A| = 0$

$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 1 & 2 & -k \end{vmatrix} = 0$ $\quad \because R_3 = R_2 - R_1$

$\therefore -k = -1 - 1$

$-k = -2$

$\boxed{k = 2}$

Q. $(a+r)x + by + cz = 0$

$ax + (b+s)y + cz = 0$

$ax + by + (t+c)z = 0$ has non-trivial solutions,

for $rst \neq 0$, $\frac{a}{r} + \frac{b}{s} + \frac{c}{t} = ?$

$\therefore |A| = 0$

Sol^n: $\begin{vmatrix} a+r & b & c \\ a & b+s & c \\ a & b & t+c \end{vmatrix} = 0 \Rightarrow abc \begin{vmatrix} 1+\frac{r}{a} & 1 & 1 \\ 1 & 1+\frac{s}{b} & 1 \\ 1 & 1 & 1+\frac{t}{c} \end{vmatrix} = 0$

$abc \cdot \left[\cancel{abc} \frac{r \cdot s \cdot t}{\cancel{a\, b\, c}} \left(1 + \frac{a}{r} + \frac{b}{s} + \frac{c}{t}\right)\right] = 0$

$rst \left(1 + \frac{a}{r} + \frac{b}{s} + \frac{c}{t}\right) = 0 \Rightarrow 1 + \frac{a}{r} + \frac{b}{s} + \frac{c}{t} = 0$

$rst \Downarrow \neq 0$

$\therefore \boxed{\frac{a}{r} + \frac{b}{s} + \frac{c}{t} = -1}$

(10) $AX = B$ such that $A^2 = I$

$A^2 = I \Rightarrow A \cdot A = I$ (b)

$\Rightarrow A = A^{-1} \Rightarrow |A| \neq 0$ $\therefore$ The system has unique solutions.

(9) $x + y + z = 6$

$x + 4y + 6z = 20$

$x + 4y + \lambda z = \mu$ has no solutions for $\lambda = ?$ $\mu = ?$

Write augmented matrix:

$C = [A \mid B]$

$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{array}\right]$

for $\lambda = 6 \Rightarrow \rho(A) = 2$

$\mu \neq 20 \Rightarrow \rho(C) = 3$ $\therefore \rho(A) < \rho(C)$ (d)

Hence, the system has no solutions.

(2) ∞ly many solutions:

for $\lambda = 6$, $\mu = 20 \Rightarrow \rho(A) = \rho(C) = 2 <$ no. of unknowns $= 3$

Hence, system has ∞ly many solutions.

(3) Unique solutions:

for $\lambda \neq 6 \Rightarrow \rho(A) = 3 = \rho(C) =$ no. of unknowns.

Hence, system has unique solution.

$(\mu = 20$ (or) $\mu \neq 20)$

(or) by Echelon form.

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38) $C = [A \mid B]$

$$C = \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 3 & 3 & b \\ 5 & 9 & -6 & c \end{array}\right] \Rightarrow R_3 = 3R_1 + R_2 \quad \therefore \boxed{c = 3a+b}$$

reduce into Echelon form:

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 5R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & -1 & 9 & c-5a \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & 0 & 0 & c-3a-b \end{array}\right]$$

$\because$ Given system is consistent $\therefore \rho(A) = \rho(C)$

$\because \rho(A) = 2$, then $\rho(C) = 2$

if $c - 3a - b = 0 \Rightarrow \boxed{c = 3a+b}$

NULLITY:

$AX = 0$

$|A| = 0 \Rightarrow$ System has ∞ly many solutions.

→ Nullity of a matrix A = the no. of independent solutions of the system "$AX = B$"

= the dimension of NULL space of the solution "$AX = B$"

= No. of unknowns − rank of A

= No. of columns of A − rank of A

Eg $x+y+z=0 \Rightarrow AX = 0$

$\Rightarrow \rho(A) = 1$, $n = 3$, $X = ?$

let $y = k_1$, $z = k_2$

$\Rightarrow x = -y-z = -k_1 - k_2$

Hence, the solution space $AX = 0$,

$$'X' = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 \\ k_1 + 0 k_2 \\ 0k_1 + k_2 \end{bmatrix}$$

$$\boxed{'X' = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

Space$(X) = \{(x,y,z) \mid x+y+z=0\}$

Null space $= \{(-1,1,0), (-1,0,1)\}$ generates Space (X).

$\therefore$ (Nullity of $AX=0$) = the no. of ind. soln in $AX=0$ = 2

Nullity of $AX=0$

= No. of unknown − rank of A

$\Rightarrow 3-1 = 2$

Basis:

→ A set of independent vectors which generates the entire vector space is called basis of the vector space.

→ In the above example NULL space generates the entire space of the solutions $AX=0$. Hence, $\{(-1,1,0), (-1,0,1)\}$ is ~~the~~ a basis here.

E.g. $S = \{\overset{X_1}{(2,1,-2)}, \overset{X_2}{(-2,-1,2)}, \overset{X_3}{(4,2,-4)}\}$

$\boxed{X_1 = -X_2}$ $\therefore$ Linearly dependent.

$\therefore$ S cannot be a basis

A subspace ~~or a set of vectors~~ spanned by a set of vectors :

→ A subspace which is set of all linear combination of any given set of vectors is said to be spanned by given set of vectors.

Q. $P = \begin{bmatrix} -10 \\ -1 \\ 3 \end{bmatrix}^T$ $Q = \begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix}^T$ $R = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix}^T$

① an orthogonal set of vectors having a span that contains P, Q, R.

(a) $\overset{X_1}{\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}}$ & $\overset{X_2}{\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}}$ (c) $\begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ -11 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \\ -3 \end{bmatrix}$

(a) ~~For ①~~ Sol^n. The linear combination of set of vectors in option (a) are generating P, Q, R as :

(i) $P = 1 \cdot X_1 - X_2$

(ii) $Q = 1 \cdot X_1 + 1 \cdot X_2$

(iii) $R = 1 \cdot X_2 + 2 \cdot X_2$; Hence option (a) is the set of vectors having a span contains P, Q, R.

NOTE

→ Dimension of a vector space = no. of independent vectors in it

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Q.6 . $A = (a_{ij})_{n \times n}$ s.t. $a_{ij} = 3 \ \forall i, j$, then the nullity of A is.

n-1. (a) : No. of columns − rank.

Q.8 $\rho(A_{m \times n}) = r$ n-r (b)

Eigen Values and Eigen Vectors:

Let $A_{n \times n}$

for any scalar λ, (there exists some non-zero vector X)

$\exists \ X \neq 0$,

such that $\boxed{AX = \lambda X}$

λ → Eigen value, X → Eigen vector.

→ "λ" is called an Eigen value of the matrix "A" and $X \neq 0$ is called Eigen vector corresponding to Eigen value "λ".

* Find Eigen value :

$AX = \lambda X$

$AX - \lambda I X = 0$

$\boxed{(A - \lambda I) X = 0}$ → Homogeneous system have non-zero solutions

⇒ $\boxed{|A - \lambda I| = 0}$ → Characteristic equation of $A_{n \times n}$

Solving $|A - \lambda I| = 0$, we get $\lambda_1, \lambda_2, \ldots, \lambda_n$

* Find Eigen vector :

Substitute "λ" in $(A - \lambda I)X = 0$

Solving the equations, we get $X \neq 0$

Q. Find Eigen value and Eigen vector of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Find Eigen values:

Characteristic equation of $A \Rightarrow |A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$= (3-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \boxed{\lambda = 3, 0, 0}$$

$\begin{vmatrix} \cdots \end{vmatrix} = 0$ for $\lambda = 0$

Find Eigen vector: $(X \neq 0)$

Sub. $\lambda = 3$ in $(A - \lambda I)X = 0$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x + y + z = 0$

$x - 2y + z = 0$

$$\begin{array}{ccc} x & y & z \\ \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} \end{array}$$

$\Rightarrow \dfrac{x}{3} = \dfrac{y}{3} = \dfrac{z}{3} = k$

$x = 3k, \ y = 3k, \ z = 3k$

$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}; \ k \in R - \{0\}$

for $\lambda = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + y + z = 0$

$y = k_1 \quad z = k_2$

$x = -k_1 - k_2$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -k_1 \\ -k_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -k_2 \\ 0 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

General Characteristic equation of $A_{n \times n}$:

$$|A_{n\times n} - \lambda I| = (-1)^n \lambda^n + (-1)^{n-1} [\text{Trace}(A)] \lambda^{n-1} + \cdots + |A| = 0$$

Constant term in characteristic polynomial of $A = |A|$.

Q.55: $A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ -1 & 3 & 2 & -5 \\ 2 & 4 & 12 & 10 \\ 3 & -2 & 1 & -4 \end{bmatrix}$

$\because R_3 = 2R_1$

Const. term of characteristic polynomial $= |A| = \underline{\underline{0}}$

Q. If $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$ is characteristic equation of $A_{3\times 3}$, then $\text{adj}(\text{adj}(A)) = ?$

(a) $3A^2$ (b) $-45A^4$ (c) $-45A$ (d) $45A$

Converting the characteristic equation into standard form by x'lying $(-1)^3$ on both sides.

$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$; $\therefore$ Const term in char poly$^l = |A| = 45$

We know that: $\text{adj}(\text{adj } A_3) = |A|^{3-2} \cdot A$

$= \underline{\underline{45 \cdot A}}$

Q 29. $f(t) =$

Multiply by $(-1)^n$ on both sides.

---

* Properties of Eigen value:

① $\lambda_1 + \lambda_2 + \dots + \lambda_n = a_{11} + a_{22} + a_{33} + \dots + a_{nn}$

② $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \dots \lambda_n = |A|$

③ A is a singular matrix, iff atleast one of the Eigen value is zero(0).

④ A is non-singular ($|A| \neq 0$), then, none of the Eigen values is zero(0).

⑤ A and A^T have the same Eigen values. [same char. eq^n]

⑥ The Eigen values of lower triangular, upper triangular and diagonal matrix are its principal diagonal elements.

e.g. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} \rightarrow$ L.T.M.

$\therefore \lambda = a_{11}, a_{22}, a_{33} = 1, 2, 3$

⑦ The Eigen values of real symmetric matrix | Hermitian are real numbers.

Q.27. $M = \begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$ ⇒ Hermitian matrix ⇒ λ's are real

$\because a_{ii} =$ real

& $\overline{a_{ij}} = a_{ji}$

if "M" is H.matrix, then "iM" is sk-H.M.

(b) Q and R only.

⑧ The Eigen values of skew symmetric | skew Hermitian matrix are zeroes or purely imaginary.

Q.20. (c) 0.

$\because$ ~~Sk~~ the determinant of sk-symmetric matrix of odd order = 0

$\therefore \boxed{\lambda = 0}$

⑨ The Eigen values of unitary | orthogonal or unit modulus $[|\lambda| = 1]$ are $1, -1, i, -i$.

$1, -1, i, -i$

⑩ Let "λ" be the Eigen value of "A", i.e., $AX = \lambda X$, then,

(i) "λ^n" is the Eigen value of A^n.

(ii) "$k\lambda$" is the Eigen value of $k.A$.

(iii) "$\lambda + k$" is the Eigen value of $A + kI$.

(iv) "$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0$" is the Eigen value of

"$a_n A^n + a_{n-1} A^{n-1} + \dots + a_1 A + a_0 I$" → Polynomial in A.

NOTE

→ The Eigen vector "X" corresponding to "λ" of matrix "A" is same for the matrices A^n, $k.A$, $A+kI$ and polynomial in A.

Q.26. (b) The Eigen vector of A and A^3 are same.

$\therefore \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

⑪ Let "λ" be the Eigen value of non-singular matrix "A", then,

(i) $\frac{1}{\lambda}$ is the Eigen value of A^{-1}.

(ii) $\frac{|A|}{\lambda}$ is the Eigen value of adj A.

NOTE

→ The Eigen vector 'x' of "A", "A^{-1}", 'adj A' is same.

Q.17. second matrix is the inverse of first matrix.

$\therefore \frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ are the Eigen values.

(12) The Eigen values of Idempotent matrix are 0, 1.

$A^2 = A$
$\lambda_A^2 = \lambda_A$
$\lambda_A^2 - \lambda_A = 0$
$\lambda_A(\lambda_A - 1) = 0$
$\lambda = 0$ (or) $\lambda = 1$

(13) The Eigen value of Involutary matrix are

$A^2 = I \Rightarrow \lambda_A^2 = 1$

$\therefore \lambda_A = \pm 1$

Q.39. $A^3 = A$

$\lambda^3 = \lambda$

$\lambda(\lambda^2 - 1) = 0$

$\lambda = 0, 1, -1$

Q.4. $\lambda_1 \lambda_2 \lambda_3 = |A|$

$x(a^2-1) - (a) = 0$

$a^3 - 2a = 0$ (d)

Q.12. $A_2 \begin{vmatrix} \lambda-1 & 1 & 1 \\ 1 & \lambda-1 & 1 \\ 1 & 1 & \lambda-1 \end{vmatrix}$

NOTE → Since, the rank(A) = 1

$\therefore$ the non-zero Eigen values = 1

(d)

K-B

Q.12, 14, 32, 31, 37.

(13) $\frac{1}{\cancel{2/3}}$ $\frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{x} = \frac{1}{n!}$ (c) (c)

(32) Since Eigen values are real, A is Hermitian.

Then, $a_{21} = \overline{a_{12}} \Rightarrow \overline{5+j} = 5-j$

$x = 5-j$ (b)

(31) $\lambda_1^2 + \lambda_2^2 + \lambda_3^3$ $\lambda_1 = 1$

$\lambda_1 + \lambda_2 + \lambda_3 = 13$ $\lambda_1\lambda_2\lambda_3 = 32$

$\lambda_2 + \lambda_3 = 12$ $\lambda_2\lambda_3 = 32$

$= 1 + 64 + 16$

$= 81$ (d)

(37) $k + 2 > 0$ $A = \begin{bmatrix} 2 & 1 \\ 1 & k \end{bmatrix}$

NOTE

→ If the leading minors of a real symmetric matrix are positive then the Eigen values are ~~to~~ positive.

Leading minor $(1\times1) = 2 \neq 0 > 0$

Leading minor $(2\times2) = 2k - 1 > 0$

$k > \frac{1}{2}$ (a)

Q.57. $A = \begin{bmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{bmatrix}$ $|A| = 100$

Trace$(A) = 14$ $|a-b| = ?$

$$\left|\begin{bmatrix} a-\lambda & 0 & 3 & 7 \\ 2 & 5-\lambda & 1 & 3 \\ 0 & 0 & 2-\lambda & 4 \\ 0 & 0 & 0 & b-\lambda \end{bmatrix}\right| \Rightarrow (b-\lambda)(2-\lambda)\left[(a-\lambda)(5-\lambda)\right] = 0$$

$\Rightarrow \lambda = b, a, 2, 5$

$a+b+2+5 = 14 \Rightarrow \boxed{a+b=7}$ $a=5$, $b=2$

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4 = |A| = 100$

$10ab = 100 \Rightarrow \boxed{ab=10}$

* If in any row/column, if all elements are "0", except principal diagonal elements, then, the principal diagonal element is an Eigen value.

Q.54. $\boxed{A_{3\times3}}$ $\lambda_A = 1, -1, 0$ $|A^{100}+I| = ?$

$\lambda_{A=1}$ $= 1^{100} + 1 = 2$

Let $B = A^{100} + I$

Eigen value of $B = \lambda_B^{100} + 1$

$\lambda_A = 1$ " " " " $= 1^{100} + 1 = 2$

$\lambda_A = -1$ " " " " $= (-1)^{100} + 1 = 2$

$\lambda_A = 0$ " " " " $= (0)^{100} + 1 = 1$

$\therefore |B| = |A^{100} + I| = |2 \times 2 \times 1| = \underline{4}$

Q18. $\lambda_A = 3, 2, -1$ $B = A^2 - A$

find ① $|B|$
② $Tr(A+B)$

$\lambda_A = 3$ Eigen value of $B = \lambda^2 - \lambda = 6$

$\lambda_A = 2$ " " " " $= 4 - 2 = 2$

$\lambda_A = -1$ " " " " $= 1 + 1 = 2$

$|B| = 6 \times 2 \times 2$
$= 24$

$Tr(A+B) = Tr(A) + Tr(B)$
$= (3+2-1) + (6+2+2)$
$\Rightarrow 4 + 10 = 14$

Q.53. $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$ $\lambda = 4, 8$ $x = ?$ $y = ?$

$2+y = 12 \Rightarrow \boxed{y=10}$ | $2y - 3x = 32$
$3x = 20 - 32$
$3x = -12$
$\boxed{x = -4}$

Q.41. $3x_1 + 2x_2 = c_1$
$4x_1 + x_2 = c_2$

$\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$Tr = 4$ $|A| = -5$

Characteristic Eq^n $(-1)^2\lambda^2 + (-1)^1(4)\lambda - 5 = 0$

$\lambda^2 - 4\lambda - 5 = 0$ (a)

36. (c) ~~21. (a)~~ 22. (a)

$M^{4} = I$ $\Rightarrow M^{4K} = I$

$M^3 \cdot M = I$ $\therefore M^{-1} = M^{4K} \cdot M^3$

$M^{-1} = M^3$ $= M^{4K+3}$

Properties of Eigen Vectors :

① The Eigen vector corresponding to an Eigen value is not unique.

② The number of linearly independent Eigen vectors of a matrix "A" = the no. of distinct Eigen values.

= dimension of Eigen space of the matrix "A"

Eg $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$. Find no. of independent Eigen vectors.

LTM

$\lambda = 2, 2$. | No. of independent Eig. vectors = no. of distinct 'λ'

= 1.

Q16. $A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

Eigen value

~~$\lambda = 4, 3, (1-\lambda), (2-\lambda)$~~

~~$(2-\lambda).(1-\lambda).3.4 = 0$~~

~~$(2-\lambda).(1-\lambda) = 0$~~

~~$\lambda = 1, 2, 3, 4$~~

No. of independent Eigen vectors = no. of distinct "λ"

= 4 ⓓ

$\lambda = 3, 4$

Remaining given by: $\begin{vmatrix} 2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$

$\Rightarrow \lambda^2 - 3\lambda - 2 = 0$

$\lambda = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2}$

* Do not apply Elementary Transformations to find the Eigen values.

③ Two vectors X, Y are orthogonal vectors if $\boxed{X^T . Y = 0}$.

inner product (or) dot product

④ X, Y, are called orthonormal vectors, if $X^T . Y = 0$ & $\|X\| = \|Y\| = 1$. [length of X and length of Y is 1]

Length of a vector (or) norm of a vector $= \sqrt{a^2 + b^2 + c^2 \ldots}$

* Orthonormal vectors are orthogonal vectors whose length is 1.

Normalized Vector :

→ Normalized Vector of "X" or unit vector of "X"

Unit vector = (or) Normalized Vector	$\frac{\text{Vector} = X}{\text{Length} \ \|X\|}$

*** ⑤ The Eigen vectors of a real symmetric matrix are orthogonal to each other.

Q. 2017. $A = \begin{bmatrix} 3/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix}$ & $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ of 'A'

Then one of the Eigen vector of 'A' -

ⓐ $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ ⓑ $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ ⓒ $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ ⓓ $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Since, "A" is a real symmetric matrix, then, the eigen vectors of A must be orthogonal to each other.

$\therefore X^T . Y = 0$

ⓐ $X^T Y = (1 \times 2) + (1 \times 1) + (1 \times 1) \neq 0$

ⓑ $X^T Y \Rightarrow (1 \times 1) + (1 \times -1) + (1 \times 0) = 0$

Q.21. $\lambda_{M_1} = 1$ $\quad X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \lambda_{M_2} = 4$ $\quad X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$A_{2\times 2} = ?$

$AX_1 = \lambda_1 X_1$ $\qquad$ $AX_2 = \lambda_2 X_2$

(c) $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ — (1) $\qquad$ $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ — (2)

Q.23. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Find the normalized Eigen vector, corresponding to largest Eigen value.

λ_1

Rem. λ given by: $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$

$\lambda = 1, 3$ $\qquad \boxed{\lambda_{max} = 3}$

find $X \neq 0$ corr to $\lambda_{max} = 3$

Substitute $\lambda_{max} = 3$ in $(A - \lambda I)X = 0$

$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\Rightarrow z = 0$

$-x + y + z = 0 \Rightarrow -x + y = 0$

$\Rightarrow \frac{x}{1} = \frac{y}{1} = k$ $\qquad \Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Normalised vector of "X" $= \frac{X}{\|X\|} = \frac{1}{\sqrt{k^2 + k^2 + 0^2}}\begin{bmatrix} k \\ k \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

(OR) $AX = 3X$

$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Q.34. $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ $\qquad \lambda_1 = 1$

(b)

$AX = \lambda X$

$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix} = 1\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$

Caley-Hamilton Theorem:

→ Every square matrix satisfies its characteristic equation.

→ It is used to find the higher powers of "A" and to write a matrix as a matrix polynomial.

Q.28. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if $a + d = ad - bc = 1$, then $A^3 = ?$

Characteristic Eqn: $\quad (a-\lambda)\cdot(d-\lambda) = 0$

$(-1)^2\lambda^2 + (-1)^1 Tr(A)\lambda + |A| = 0$ $\qquad ad - a\lambda - d\lambda + \lambda^2 = 0$

$\lambda^2 - \lambda(a+d) + ad = 0$

$\Rightarrow \lambda^2 - \underset{1}{(a+d)}\lambda + 1 = 0$

$\Rightarrow$ By C.H. Thm:

$A^2 - \cancel{(a+d)}A + I = 0$

$A^2 = A - I \quad \Rightarrow A^3 = A^2 - A$

$= \cancel{A} - I - \cancel{A}$

$\boxed{A^3 = -I}$ (c)

Q. $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$ ① find A^{-1}, $A^3 = ?$ ② $A^9 = ?$

ⓐ $511A + 510I$
ⓑ $304A + 104I$
ⓒ $154A + 155I$
ⓓ $\exp(9A)$

C.E: $(-1)^2\lambda^2 + (-1)^1(-3)\lambda + 2 = 0$

$\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \boxed{\lambda = -1, -2}$

1) By C.H:

$\Rightarrow A^2 + 3A + 2I = 0 \Rightarrow A^2 = -3A - 2I$

x'ying by A: $A^3 = -3A^2 - 2A$

$A^3 = -3(-3A - 2) - 2A$

$= 9A + 6I - 2A$

$= \cancel{11A + 6}\ \boxed{7A + 6I}$

By A^{-1}:

$A = -3I - 2A^{-1}$

$2A^{-1} = -3I - A$

$A^{-1} = \boxed{-\frac{1}{2}\left[A + 3I\right]}$

② $A^9 \Rightarrow (A^3)^3 = (7A + 6I)^3$

(or) Shortcut:

$\lambda_A = -1, -2$

$\lambda_{A^9} = (-1)^9, (-2)^9 = (-1), (-512)$

ⓐ Eigen values of $511A + 510I = 511\lambda_A + 510$

$\Rightarrow 511(-1) + 510 = -1 = \lambda_{A^9}$

$\Rightarrow (511)(-2) + 510 = -512 = \lambda_{A^9}$

$\therefore \lambda_A^9 = 511\lambda_A + 510$

$\boxed{\therefore A^9 = 511A + 510I}$

Q. If $\lambda = 1, 2, 3$ of $A_{3\times3}$, Then 'A' satisfies:

ⓐ $A^3 - 6A^2 + 11A - 6I = 0$
ⓑ $A^3 - 6A^2 - 11A + 6I = 0$
ⓒ $A^3 + 6A^2 - 11A + 6I = 0$
ⓓ $A^3 + 6A^2 - 6I = 0$

~~A~~ ⓑ $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

✓ $1 - 6 + 11 - 6 = 0$

✓ $8 - 24 + 22 - 6 = 0$

✓ $27 - 54 + 33 - 6 = 0$

Characteristic eqn of $A_{3\times3}$ with give $\lambda_1 \lambda_2 \lambda_3$:

$|A_{3\times3} - \lambda I| = (-1)^3\lambda^3 + (-1)^2[\lambda_1 + \lambda_2 + \lambda_3]\lambda^2 + (-1)^1[\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1]\cdot\lambda$

$+ (\lambda_1 \lambda_2 \lambda_3) = 0$

$\boxed{x^2 - x(\alpha + \beta) + (\alpha\beta) = 0}$

$-x^3 + (\alpha + \beta + \gamma)x^2 - (\alpha\beta + \beta\gamma + \alpha\gamma)x + \alpha\beta\gamma = 0$

$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

By CH: $-A^3 + 6A^2 - 11A + 6I = 0$

NOTE

→ Replacing "λ" by "$\frac{1}{\lambda}$" in the characteristic equation of A, we get characteristic equation of A^{-1}.

→ Similarly, replacing "λ" by "$k\lambda$", we get characteristic equation of kA.

→ Replacing "λ" by $\frac{|A|}{\lambda}$, we get characteristic eqn of adj (A).

X

PROBABILITY :

Random Experiment :

→ An experiment whose result is unpredictable is called random experiment.

→ The happening of each outcome of ~~exp~~ an experiment is called an event.

Types of Events :

① Exhaustive events
② Equally likely events
③ Mutually Exclusive events
④ Independent events.

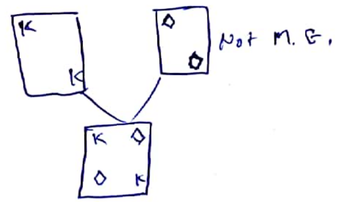
* Exhaustive Events : All possible outcomes of an experiment called exhaustive events.

E.g. ① head and tail in tossing a coin.
② getting 1, 2, 3, 4, 5, 6 in throwing a dice.

* Equally likely Events : The events having the same chance of occurence are called equally likely (or) equiprobable events. [$P(A) = P(B)$]

* Mutually Exclusive Events : Happening of an events, prevents the happening of other events, i.e., the events do not occur simultaneously. [$A \cap B = \phi$]

E.g.



* **Independent Event**: Happening of an event does not influence the happening of other events.

→ The probability of an event does not affect the probability of other event.

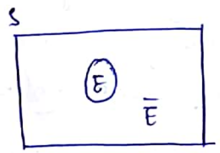
Probability:

Probability (P) $= \frac{m}{n} = \frac{\text{No. of favorable outcomes}}{\text{No. of exhaustive events.}}$

Set definition:

Sample ~~set~~ space = S

$$\boxed{P(E) = \frac{n(E)}{n(S)}}$$



Sample space: Set of all exhaustive events of experiment.

$n(E) + n(\bar{E}) = n(S)$

Divided by $n(S)$:

$\Rightarrow \frac{n(E)}{n(S)} + \frac{n(\bar{E})}{n(S)} = \frac{n(S)}{n(S)} \Rightarrow P(E) + P(\bar{E}) = 1$

$\Rightarrow \boxed{P(\bar{E}) = 1 - P(E)}$

Axioms of Probability:

① $P(S) = 1$

② $P(\phi) = 0$ (Probability of impossible event)

③ $0 \le P(E) \le 1$

④ $P(E) + P(\bar{E}) = 1$

⑤ If $A \subseteq B$, then $P(A) \le P(B)$

⑥ If A, B are mutually exclusive, then $P(A \cap B) = P(\phi) = 0$

⑦ If A, B are independent

$P(A \cap B) = P(A) \cdot P(B)$

⑧ Addition Theorem:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

⑨ Odds in favour of an event 'E' $= P(E) : P(\bar{E})$

Odd against an event 'E' $= P(\bar{E}) : P(E)$

Q.12 Exp. drawing '4' nos. from 14 numbers and ×ed.

$\therefore n(S) = {}^{14}C_4 = \frac{14 \times 13 \times 12 \times 11}{4 \times 3 \times 2 \times 1} = 1001$

E = Product to be +ve

$E = \{4\text{+ve}, 4\text{−ve}, (2\text{+ve} \,\&\, 2\text{−ve})\}$

$n(E) = {}^6C_4 + {}^8C_4 + ({}^6C_2 \cdot {}^8C_2)\}$

$\Rightarrow \frac{3 \times 5}{2} + \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} + \frac{6 \times 5}{2} \times \frac{8 \times 7}{2} = 15 + 70 + 420$

$= 505$

$P(E) = \frac{n(E)}{n(S)}$

$= \frac{505}{1001}$

(a)

Q.6.

B1	B2
3, 6, 9, 12, 15	6, 11, 16, 21, 26

Exp: drawing a chip from each box and multiplied.

$\therefore n(S) = {}^5C_1 * {}^5C_1 = 25$ ways.

let

E = product to be even

$E = \{(e,e), (o,e), (e,o)\}$; $\bar{E} = \{(o,o)\}$

$E = \cancel{{}^5C_1 * {}^5C_3 + {}^5C_3 \cdot {}^5C_2 + {}^5C_2 \cdot {}^5C_2}$

$n(E) = {}^2C_1 \cdot {}^3C_1 + {}^3C_1 \cdot {}^3C_1 + {}^2C_1 \cdot {}^2C_1$

$= 6 + 9 + 4$

$n(E) = 19$

$P(E) = \frac{n(E)}{n(S)} = \frac{19}{25}$ (d)

$n(\bar{E}) = {}^3C_1 * {}^2C_1 = 6$

$P(E) = 1 - P(\bar{E})$

$= 1 - \frac{6}{25}$

$= \frac{19}{25}$

Q.7.

Container 1	Container 2
4-R	3-B
3-G	4-G

Exp: Drawing a ball from each container.

$\therefore n(S) = {}^7C_1 \cdot {}^7C_1 = 49$ ways

let.

E = drawing 1 Red & 1 Blue ball.

$E = \{(R, B)\}$

$n(E) = {}^4C_1 * {}^3C_1$

$n(E) = 12$

$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{12}{49}$

(c)

Q.8.

4-R
6-B

~~Exp: Drawing 3 balls from box one after~~

Event: Drawing one Red & 2 black balls one after the other without replacement.

$E = \{(R,B,B), (B,R,B), (B,B,R)\}$

$P(E) = P(R,B,B) + P(B,R,B) + P(B,B,R)$

$= \left(\frac{4}{10} * \frac{6}{9} * \frac{5}{8}\right) + \left(\frac{6}{10} * \frac{4}{9} * \frac{5}{8}\right) + \left(\frac{6}{10} * \frac{5}{9} * \frac{4}{8}\right)$

$\Rightarrow 3\left(\frac{4 \times 6 \times 5}{10 \times 9 \times 8}\right) = \frac{1}{2}$ (d)

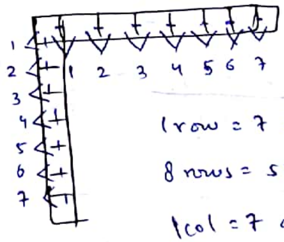
$n(S) = 10 \times 9 \times 8 = 720$

~~$n(S) = {}^{10}C_3 = \frac{10 \times 9 \times 8}{3 \times 2}$~~ $= 120$

Q9. Exp: drawing two squares from chess board.

$n(S) = {}^{64}C_2 = \frac{64 \times 63}{2}$

Let. E = no. of pairs of adjacent squares in a chess board.



$n(E) = 56 + 56$

$= 112$

1 row = 7 adj pairs

8 rows = 56 adj pairs

1 col = 7 adj pairs

8 col = 56 adj pairs.

$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{112 \times 2}{64 \times 63}$

$= \frac{1}{18}$

Q.10. Exp: The arrangement of all the letters of PROBABILITY

$\therefore n(S) = \frac{11!}{2!\,2!}$

Let E = Arrangements of letters of words PROBABILITY in which 2B's and 2I's to be together.

Consider 2B's ⟶ single letters

2I's ⟶ single letter

$n(E) = 9!$

$P(E) = \frac{n(E)}{n(S)} = \frac{9! \times 2! \times 2!}{11!} \Rightarrow \frac{\cancel{9!} \times 2 \times 2}{11 \times \cancel{10} \times \cancel{9!}} = \frac{2}{55}$ (b)

Q.11. Exp. = Arrangement of 'n' persons at round table.

$n(S) = (n-1)!$

E = 2 specified person do not sit together

$\bar{E}$ = 2 specified person sit together.

$n(\bar{E}) = (n-2)! \times 2$

$P(\bar{E}) = \frac{2(n-2)!}{(n-1)!} = \frac{2}{n-1}$ (b)

$P(E) = 1 - \frac{2}{n-1} = \frac{n-3}{n-1}$

Odd ~~agn~~ against $E = \frac{n-1-(n-3)}{n-3} = \frac{2}{n-3}$

Q.45 X, Y are independent.

$X, \bar{Y}$ are also independent.

$P(X) = 0.4 \cdot P(X \cup \bar{X}) = 0.7$

$P(X \cup \bar{Y}) = P(X) + P(\bar{X}) - P(X) \cdot P(\bar{Y})$

$0.7 = 0.4 + P(\bar{Y})(0.6)$

$P(\bar{Y}) = \frac{1}{2}$ (a)

Q.48. Exp. = drawing or item from bin.

$\boxed{\frac{125}{200}}$

$n(S) = {}^{325}C_1 = 325$

E = drawing a nut as defective

$E = n \cup d$

$\therefore P(E) = P(n \cup d) = P(n) + P(d) - P(n \cap d)$

$= \frac{200}{325} + \frac{175}{325} - \frac{150}{325} = \frac{9}{13}$

$P(E) = 1 - P(\text{no defect bolt})$

$= 1 - \frac{100}{325} = 1 - \frac{4}{13} = \frac{9}{13}$

Q.50. Given L, W are independent $P(W) = \frac{1}{2}$ $P(L) = \frac{1}{2}$

$E = 2^{nd}$ wind in 3^{rd} match

$E = \{(L,W,W), (W,L,W)\}$

$= \frac{1}{2^3} + \frac{1}{2^3} = \frac{2}{2^3} = \frac{1}{4}$

→ Exp: Throwing a dice '2' times

$n(S) = 36$

Since, it is symmetric probability curve at $X = 7$.

∴ It is called symmetrical probability distribution with

mean = mode = median ~~at~~

Q1. $P(\text{sum}>8) = P(9)+P(10)+P(11)+P(12)$

$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36}$

Q.13. $P(E) = P(4)+P(6)+P(8)+P(12)$

$= \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{1}{36}$

$= \frac{14}{36} = \frac{7}{18}$

Q.14. $P(\overline{8\cup 9}) = 1 - P(8\cup 9)$

$= 1 - [P(8)+P(9)-P(8\cap 9)]$

$= 1 - \left[\frac{5}{36} + \frac{4}{36} - 0\right]$

$= 1 - \frac{9}{36} = \frac{3}{4}$

Q.18. E = 1st head occuring in odd tosses.

$E = \{H, TTH, TTTTH, \dots\}$

$P(E) = P(H) + P(TTH) + \dots$

$= \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \dots$

$= \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}$

Q.19. $P(W) = \frac{1}{5}$ $\quad P(L) = \frac{5}{6}$

$E = \{A_W, A_L B_L A_W, \dots\}$

$P(E) = P(A_W) + P(A_L B_L A_W)$

$= \frac{1}{6} + \left(\frac{5}{6}\right)^2 \frac{1}{6} + \dots$

$= \frac{\frac{1}{6}}{1-\frac{25}{36}} = \frac{1}{6} \times \frac{36}{11} = \frac{6}{11}$

Q.20. $E = \{5, (\overline{5\cup 7}, 5), (\overline{5\cup 7}, \overline{5\cup 7}, 5), \dots\}$

$P(E) = P(5) + P(\overline{5\cup 7}, 5) + P(\overline{5\cup 7}, \overline{5\cup 7}, 5) \dots$

$P(\overline{5\cup 7}) = 1 - P(5\cup 7)$

$= 1 - \left[\frac{4}{36} + \frac{6}{36}\right] = \frac{26}{36}$

$P(E) = \frac{4}{36} + \frac{26}{36} \times \frac{4}{36} + \dots$

$\frac{\frac{4}{36}}{\frac{12}{36}} = \frac{4}{10} = \frac{2}{5}$

Q. 4 dice are rolled. The probability that sum of the results being 22 is $X/1296$. Then $X = ?$

sum = 22 . $\{(6,6,6,4) \times 4, (6,6,5,5) \times 6\}$

$P(E) = P(6,6,6,4) \times 4 + P(6,6,5,5) \times 6$

$= \frac{1}{6^2} \times 4 + \frac{1}{6^4} \times 6 = \frac{10}{1296} \Rightarrow \boxed{X = 10}$

Q. S be a sample space and 2 mutually exclusive events A and B, such that $(A\cup B) = S$, the max value of $P(A).P(B)$.

$A, B \longrightarrow$ M. exclusive $\Rightarrow A\cap B = \phi$

$P(A\cap B) = 0$

given $A\cup B = S$

$P(A\cup B) \Rightarrow P(A) + P(B) = 1$

Max = $P(A).P(B) \Rightarrow P(A).(1-P(A))$

$= P(A) - (P(A))^2$

$= 1 - 2P(A) = 0 \Rightarrow P(A) = \frac{1}{2}$ $\quad P(B) = \frac{1}{2}$

$\Rightarrow \frac{1}{4}$

NOTE $\Rightarrow$ If $a+b = k$, then $\max(a*b)$ is at $a=b$

15 # Conditional Probability:

→ If $P(A)$ is given, then, probability of 'B' is called conditional probability of B given that 'A' happened denoted by $P(B/A)$.

→ If $P(B)$ is given, the probability of A is called conditional probability, given, that B happend denoted by $P(A/B)$.

Multiplication Theorem: A, B are events happening sequentially, then,

$P(A\cap B) = P(A).\,P(B/A) = P(B).\,P(A/B)$

$$\boxed{P(B/A) = \frac{P(B\cap A)}{P(A)} \quad ; \quad P(A/B) = \frac{P(A\cap B)}{P(B)}}$$

Q.15. $P\left(\frac{\text{Sum}=7}{\mathrm{I}d > \mathrm{II}d}\right) = \frac{P(\text{Sum}=7 \ \& \ \mathrm{I}d > \mathrm{II}d)}{P(\mathrm{I}d > \mathrm{II}d)}$

$= \frac{\frac{3}{36}}{\frac{15}{30}} = \frac{1}{5}$

Q.16. $P(P_1) = 0.3$ $\quad P(P_2) = 0.2$

$P\left(\frac{P_1}{P_2}\right) = 0.6$ $\quad P(P_1\cap P_2) = ?$

$P(P_1\cap P_2) = P(P_2).\,P\left(\frac{P_1}{P_2}\right) = 0.2\times 0.6$

$= 0.12.$

By Multi. Thm

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Q.47. $P(P) = \frac{1}{4}$ $\quad P\left(\frac{P}{Q}\right) = \frac{1}{2}$ $\quad P\left(\frac{Q}{P}\right) = \frac{1}{3}$

$P\left(\frac{\bar{P}}{\bar{Q}}\right) = \frac{P(\bar{P}\cap\bar{Q})}{P(\bar{Q})} = \frac{P(\overline{P\cup Q})}{P(\bar{Q})} = \frac{1-P(P\cup Q)}{1-P(Q)}$ — (1)

By multiplication thm:

$P(P\cap Q) = P(P).\,P\left(\frac{Q}{P}\right) = P(Q).\,P(P/Q)$

$\therefore \frac{1}{4}\times\frac{1}{3} = P(Q).\frac{1}{2} \Rightarrow P(Q) = \frac{1}{6}$

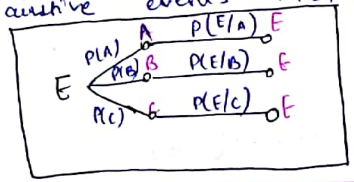
(1) → $\Rightarrow 1 - \frac{[P(P)+P(Q)-P(P\cap Q)]}{1-P(Q)} = \frac{1-\left[\frac{1}{4}+\frac{1}{6}-\frac{1}{12}\right]}{1-\frac{1}{6}}$

$= 4/5.$ (a)

Total Probability:

→ Let 'E' be an event corresponds to more than one mutually exclusive and exhaustive events of all experiment, then $P(E)$ is Total probability.

→ Let 'E' be an event occuring in mutually exclusive and exhaustive events A, B, C of all experiments.



$$\boxed{\therefore P(E) = P(A).\,P\left(\frac{E}{A}\right) + P(B).\,P\left(\frac{E}{B}\right) + P(C).\,P\left(\frac{E}{C}\right)}$$

Baye's.

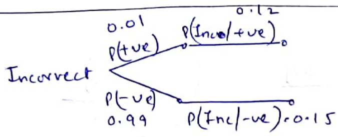
Posterior Probabilities:

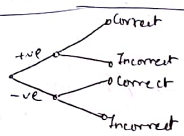
$$P(A/E) = \frac{P(A \cap E)}{P(E)} = \frac{P(A).\,P(\frac{E}{A})}{P(E)}$$ ← from Total probability

$$P(\frac{B}{E}) = \frac{P(B \cap E)}{P(E)} = \frac{P(B).\,P(\frac{E}{A})}{P(E)}$$

$$P(\frac{C}{E}) = \frac{P(C \cap E)}{P(E)} = \frac{P(C).\,P(\frac{E}{C})}{P(E)}$$

Q.23.





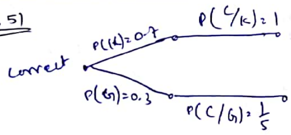
$$P(\text{Incorrect}) = P(+ve).\,P(\frac{Inc}{+ve}) + P(-ve).\,P(\frac{Inc}{-ve})$$

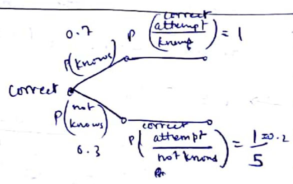
$= 0.01 * 0.12 + 0.99 * 0.15$

$= 0.0012 + 0.1485$

$= 0.1497$

Q.51





$$P(\text{correct}) = P(K).\,P(C/K) + P(G).\,P(C/G)$$

$= 0.7*1 + 0.3*0.2$

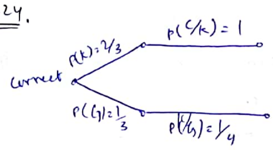
$= 0.7 + 0.06$

$\boxed{= 0.76}$

Q.24.

$P(\frac{K}{C}) = ?$

Baye's



$$P(\text{correct}) = P(K).\,P(C/K) + P(G).\,P(C/G)$$

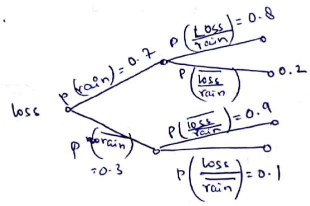
$$= \left(\frac{2}{3}*1\right) + \left(\frac{1}{3}*\frac{1}{4}\right)$$

$$P(\frac{K}{C}) = \frac{P(K \cap C)}{P(C)} = \frac{P(K).\,P(\frac{C}{K})}{P(C)}$$

$$\Rightarrow \frac{\frac{2}{3}*1}{\frac{2}{3}+\frac{1}{12}} = \frac{8}{9}$$

Q.25.

$P(\frac{\bar{R}}{\bar{L}})$



$$P(L) = P(R) * P(\frac{L}{R}) + P(\bar{R}).\,P(\frac{L}{\bar{R}})$$

$$P(\frac{\bar{R}}{\bar{L}}) = \frac{P(\bar{R} \cap \bar{L})}{P(\bar{L})} = \frac{P(\bar{R}).\,P(\frac{\bar{L}}{\bar{R}})}{1 - P(L)}$$

$$= \frac{0.3 * 0.9}{1 - [(0.7 * 0.8) + 0.3 * 0.1]}$$

$$\Rightarrow \frac{0.27}{1 - [0.56 + 0.03]} = \frac{0.27}{0.41}$$

$$= \frac{27}{41}$$

Random Variable:

→ The variable "X" is a function from sample space of an experiment to real number set.

$X: S \rightarrow R$ → Integer → ~~DRV~~ 'X' is DRV
→ $(-\infty, +\infty)$ → ~~CRV~~ 'X' is CRV

* Discrete Random Variable:

→ If "X" is ~~not~~ integer, then, "X" is called discrete random variable (DRV).

Eg ~~Probability of mass function~~ Tossing coin 2 times.

$S = \{(H,H), (H,T), (T,H), (T,T)\}$

Let X = no. of heads.

$X(H,H) = 2$	$P(X=2) = 1/4$
$X(H,T) = 1$	$P(X=1) = 2/4$
$X(T,H) = 1$	~~$P(X=1)$~~
$X(T,T) = 0$	$P(X=0) = 1/4$

$\boxed{\Sigma P_i = 1}$

① Probability mass function $= f(X_k) = P(X = k)$

② $\sum_{i=0}^{n} P(X = x_i) = 1$

③ Cumulative Probability distribution:

$F_X(X_k) = P(X \leq X_k) = \sum_{i=0}^{k} P(X = x_i)$

④ Expectation of 'X' or Mean of Distribution:

$$\boxed{E(X) = \sum_{i=0}^{n} x_i \, P(X = x_i)}$$

$\frac{\Sigma x_i P_i}{\Sigma P_i = 1}$

Continuous Random Variable:

→ 'X' is infinite set of values.

① Probability Density function:

$$f(x) = \begin{cases} f_1(x), & x \leq a \\ f_2(x), & x > a \end{cases}$$

② $\int_{-\infty}^{+\infty} f(x)\,dx = 1$ [valid probability density func^n should be 1]

③ Cumulative distribution function:

$F_X(X_k) = P(X \leq X_k) = \int_{-\infty}^{x_k} f(x)\,dx$

$P(a \leq x \leq b) = \int_a^b f(x)\,dx$

④ Expectation of 'X' (or) mean of Distribution:

$E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx$

⑤ Variance $(X) = E[X - E(X)]^2$

$= E[x^2] - [E[x]]^2$

Standard deviation: $\sigma_x = \sqrt{var(x)} > 0$

* Properties:

① $E[\text{const}] = \text{const}$

② $E[ax+b] = aE(x) + b$

③ $var(\text{const}) = 0$

④ $var[ax+b] = a^2 var(x)$

⑤ $\sigma[ax+b] = \sqrt{var(ax+b)} = \sqrt{a^2\, var(x)}$

$= a\sqrt{var(x)} = \underline{a\sigma_x}$

⑥ $E[XY] = E(X) \cdot E\left[\frac{Y}{X}\right]$

$= E(Y) \cdot E\left[\frac{X}{Y}\right]$

Q.27. $f(x) = e^{-x}$ ① $P(X>1) = ?$ ② $P(|X-2|<1) = ?$

① $P(X>1) = \int_1^{\infty} f(x)\cdot dx = \int_1^{\infty} e^{-x}\cdot dx$

$= \left[\frac{e^{-x}}{-1}\right]_1^{\infty} = \frac{1}{e}$

② $|x-2| < 1$

$= -1 \leq (x-2) \leq 1$

$= 1 < x < 3$

$$\boxed{\begin{array}{l} |x-a| < b \\ = -b \leq x-a \leq b \end{array}}$$

$\therefore P(|X-2|<1) = P(1<x<3)$

$= \int_1^3 e^{-x} dx = \left[\frac{e^{-x}}{-1}\right]_1^3 = -\left[e^{-x}\right]_1^3$

$= \cancel{\frac{1}{e}-\frac{1}{e^3}}\ e^{-1} - e^{-3}$

Q. If $f(x) = k|x|e^{-|x|}$ is a valid probability density function in $(-\infty, \infty)$. find :

① k ② $E(x)$ ③ $E(x^2)$ ④ $var(x)$
⑤ σ_x ⑥ $E(2x+3)$ ⑦ $E(x+2)^2$ ⑧ $\sigma(2x+3)$

Sol^n.

① $\because f(x)$ is a pdf $\Rightarrow \int_{-\infty}^{\infty} f(x)\cdot dx = 1$

$\Rightarrow \int_{-\infty}^{\infty} k|x|e^{-|x|}\, dx = 1$ even

+ve interval || (mod can be removed)

$\Rightarrow 2\int_0^{\infty} kx\, e^{-x}\cdot dx = 1$

$\Rightarrow 2\int_0^{\infty} k\, x^{2-1} e^{-x}\cdot dx = 1$

$\Rightarrow 2k\, \Gamma 2 = 1$

$\Rightarrow 2k \cdot 1! = 1 \Rightarrow k = 1/2$

② $E(x) = \int_{-\infty}^{\infty} x\cdot f(x)\cdot dx$

$= \int_{-\infty}^{\infty} x\cdot \frac{1}{2}\cdot |x|\cdot e^{-|x|}\cdot dx$ odd

$= 0$

③ $E(x^2) = \int_{-\infty}^{\infty} x^2\cdot f(x)\, dx = \int_{-\infty}^{\infty} x^2\cdot\frac{1}{2}\cdot|x|\cdot e^{-|x|}\cdot dx$ even

$= \frac{1}{2}\times 2\times \int_0^{\infty} e^{-x} x^3\cdot dx$

$\Rightarrow \int_0^{\infty} e^{-x} x^{4-1} dx = \Gamma 4 = 3! = 6$

$\int_{-a}^{a} f(x)\, dx$
↓
Even (or) odd
↓
$f(-x) = f(x)$
$= 2\int_0^{\infty} f(x)\, dx$

$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\, dx.$ Gamma f

$\Gamma n+1 = n!$

$\int_{-a}^{a} f(x)\, dx = 0$ if $f(x)$ is odd
↓
odd
↓
$f(-x) \neq f(x)$

$= 0$

(4) $Var(x) = E(x^2) - (E(x))^2 = 6 - 0^2 = 6$

(5) $\sigma_x = \sqrt{Var(x)} = \sqrt{6}$

(6) $E[2x+3] = 2E(x) + 3 = 2(0)+3$
$= 3$

(7) $E((x+2)^2) = E[x^2 + 4x + 4] = E[x^2] + 4E[x] + 4$
$= 6 + 4(0) + 4 = \underline{10}$

(8) $\sigma(2x+3) = \sqrt{Var(2x+3)} = \sqrt{2^2(Var(x))}$
$= 2\sqrt{Var(x)} = 2\sigma_x = 2\sqrt{6}$

Q29 Find (1) P (2) E(x) (3) $E(x^2)$ (4) var(x)
(5) σ_x (6) $E[2x+3]$ (7) var[2x+3] (8) Mode (9) $P(X \le 2)$

X	1	2	3
P(x)	$\frac{2+5P}{5}$ (0.45)	$\frac{1+3P}{5}$ (0.23)	$\frac{1.5+2P}{5}$ (0.32)

(1) $\Sigma P_i = 1 \Rightarrow \frac{2+5P+1.3P+1.5+2P}{5} = 1 \Rightarrow \boxed{P = 0.05}$

(2) $E(x) = \Sigma x_i P_i = 1 \times 0.45 + 2 \times 0.23 + 3 \times 0.32$
$= 1.87$

(3) $E(x^2) = \Sigma x^2 P_i = 1 \times 0.45 + 4 \times 0.23 + 9 \times 0.32$
$= 4.25$

(4) $Var(x) = E(x^2) - [E(x)]^2 = (4.25) - (1.87)^2 = 0.753$

(5) $\sigma_x = \sqrt{Var(x)} = 0.867$

(6) $E[2x+3] = 2E(x) + 3 = 2(1.87) + 3 = 6.74$

(7) $var(2x+3) = 2^2 var(x) = 4 \times 0.753 = 3.012$

~~(8) Mode = The value~~

(8) ~~The value~~ Mode = The value of 'x' at which P(x) is max
$= 1$

(9) $P(X \le 2) = P(X=1) + P(X=2)$
$= 0.68$

Q54.

X	3	4	5
P(x)	4/9	2/9	3/9

$E(x) = \Sigma x . P(x) = 3 \times \frac{4}{9} + 4 \times \frac{2}{9} + 5 \times \frac{3}{9}$

$= \frac{12+8+15}{9} = \frac{35}{9} = 3.88 \approx 3.9$

Q.53. X = odd numbers less than 100

X =	1	3	5	7	---	99
P(x)	$\frac{1}{50}$	$\frac{1}{50}$	$\frac{1}{50}$	$\frac{1}{50}$		$\frac{1}{50}$

$E(x) = \Sigma x . P(x) = 50^2 \times \frac{1}{50} = 50$

Odd no. sum $= n^2$

$= \frac{1}{50}[1 + 3 + 5 + - - - + 99]$

$= \frac{1}{50}\left[S_n = \frac{n}{2}(a+l) = \frac{50}{2}(1+99)\right] = 50$

Q. A coin is tossed till a first head appears with the probability of head is "p". Then, find the expected no. of tosses required.

X	1	2	3	......	∞
$P(x)$	p	$(1-p)p$	$(1-p)^2 p$		
		TH	TTH		

$E[x] = 1 \cdot p + 2 \cdot (1-p) \cdot p + 3(1-p)^2 \cdot p + \dots$

$= p[1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \dots]$

$= p \cdot \frac{1}{p^2} = \frac{1}{p}$

$F = 1 + 2(1-p) + 3(1-p)^2 + \dots$

$F(1-p) = (1-p) + 2(1-p)^2 + 3(1-p)^3$

$F(1-1+p)$

$Fp = 1 + (1-p) + (1-p)^2 + (1-p)^3 \dots$

$= \frac{1}{1-1+p} = \frac{1}{p}$

$= \frac{1}{p^2}$

Probability Distribution:

* **Binomial Distribution**: $B(n, p)$ [Two parameters $- n \& p$]

$X \rightarrow DRV$, $n =$ no. of times experiment is repeated.

$p =$ probability of success

$q =$ probability of failure

Prob. Mass func^n (PMF): Prob. of 'r' success in 'n' trials is given by $\boxed{P(X=r) = {}^nC_r\, p^r q^{n-r}}$

Mean $= np$; variance $= npq$

* **Poisson Distribution**: [One parameter $- \lambda$]

$X \rightarrow DRV$, $n \rightarrow$ largest, $p \rightarrow$ least

$p =$ prob. of success

$\lambda =$ mean of dist. $=$ variance of 'x' $= np$

Then, proof: prob of 'r' success

$= \boxed{P(X=r) = \frac{e^{-\lambda}\lambda^r}{\lfloor r}}$

Q. A fair coin is tossed 4 times. Find the probability of getting atleast 2 heads.

$p = \frac{1}{2}$, $P=2$, $n=4$

$P(X > 1) = P(X=2) + P(X=3) + P(X=4)$

$= {}^4C_2\left(\frac{1}{2}\right)^4 + {}^4C_3\left(\frac{1}{2}\right)^4 + {}^4C_4\left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^4(6+4+1) = \frac{11}{16}$

Q.33. $n=5$, $p = \frac{2}{6}$, $q = \frac{4}{6}$, $r=4$

$P(X=4) = {}^5C_4\left(\frac{2}{6}\right)^4\left(\frac{4}{6}\right)^1$ (a)

Q.32. $p = \frac{1}{2}$, $q = \frac{1}{2}$, $n=10$, $x=4$

$P(4^{th}$ H in 10^{th} toss$) = P(3H$ in 9 tosses$) * P(H)$

$= {}^9C_3 * \left(\frac{1}{2}\right)^9 * \frac{1}{2}$

$= \frac{{}^{3}\cancel{9} * {}^{4}\cancel{8} * 7}{\cancel{3} \times \cancel{2}} * \left(\frac{1}{2}\right)^{10} = 0.082$

Q.37. $X \rightarrow$ follows Poisson Distribution.

Let $x =$ no. of events per 2 year.

$P(X \le 2) = ?$

$\lambda = 2 * 3 = 6$ per 2yr.

$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= \frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!}$

$\Rightarrow e^{-6}[1+6+18] = 25e^{-6} = 0.0619$. (b)

Q.38. P(defective) = 1%

Let X = no. of defective condensors per box.

$P(X>3) = ?$

$P(X \geq 3) = 1 - P(X \leq 2)$

$= 1 - [P(0) + P(X=1) + P(X=2)]$

$= 1 - \left[\frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!}\right]$

$= 1 - e^{-1}\left[2 + \frac{1}{2}\right] = \boxed{1 - \frac{5}{2}e^{-1}}$ (b)

Q. 'x' follows P.D. with second moment = 2.

Find variance(x)?

$\boxed{E[X^r] = r\text{th moment}}$

$E[X^2] = 2$, var(x) = $E[x] = np$

$var[x] = E[x^2] - [E(x)]^2$

$v = 2 - v^2$

$= v^2 + v - 2 = 0$ ⇒ $v = -2, 1$

$v = 1$

var(x) = 1

Normal Distribution: $n \to \infty$. (2 parameters = μ, σ)

$X \to$ CRV with given μ, σ

Pdf:

$f(X_k) = P(X = k) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{X_k - \mu}{\sigma}\right)^2 / 2}$

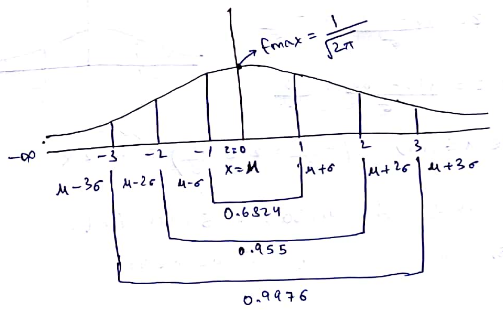
Let $z = \frac{X - \mu}{\sigma}$

Then, standard normal p.d.f:

$\phi(z) = \left.\frac{1}{\sqrt{2\pi}} e^{-z^2/2}\right]_{\sigma = 1}$

Cumulative Density f^n $= f(z_k) = \int_{-\infty}^{z_k} \phi(z)\,dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_k} e^{-z^2/2}\,dz$

Standard normal curve: $N(\underset{\text{mean}}{0}, \underset{\sigma}{1})$



NOTE ⇒ since, it is a symmetric curve, at z=0;

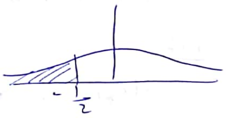
mean = mode = median = 0

Q.43. $X \to NRV$

$\mu = 1 \quad \sigma^2 = 4$

$P(X<0) = ?$

w.r.t. $\quad z = \dfrac{X-\mu}{\sigma}$



for $x = 0 \Rightarrow \quad z = \dfrac{0-1}{2} = -\dfrac{1}{2}$

$P(X<0) = P\left(z < -\frac{1}{2}\right) = 0.5 - P\left(0<z<\frac{1}{2}\right)$

$= <0.5 \;\&\; 0>$ (b)

Q.44. $X \to NRV$

$\mu = 1000 \quad \sigma = 200 \quad P(X>1200) = ?$

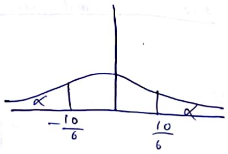
$z = \dfrac{X-\mu}{\sigma} = \dfrac{1200-1000}{200} \Rightarrow P(X>1200) = P(z>1)$

>0.5 from graph (a)

Q.42. $\mu = 100 \quad P(X)>110) = \alpha$

$P(90 \le X \le 110) = ?$

$X=90 \Rightarrow z = \dfrac{90-100}{\sigma} = \dfrac{-10}{\sigma}$



$X = 110 \Rightarrow z = \dfrac{110-100}{\sigma} = \dfrac{10}{\sigma}$

$X(90 \le X \le 110) = P\left(\dfrac{-10}{\sigma} \le z \le \dfrac{10}{\sigma}\right) = 1-2\alpha$

Q. If X is NRV with $\mu = 30$, $\sigma = 5$, then, find:

(i) $P(X \ge 45)$ $\qquad$ (ii) $P(30 \le X \le 40)$

Q→ If 'x' is NRV with $\mu = 30$, and $\sigma_x = 5$, then, find:

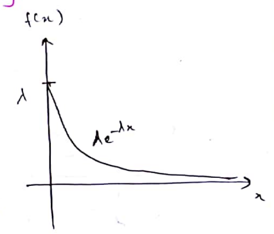
① ~~Probability of~~ $P(x) \ge 45)$

② $P(30 \le x \le 40)$

Exponential Distribution: [1 parameter - λ]

$X \to CRV$

Pdf: $f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \ge 0 \\ 0 & , \text{else} \end{cases}$



Mean $= \dfrac{1}{\lambda} \qquad var(X) = \dfrac{1}{\lambda^2}$

$$\boxed{E[x^r] = \int_0^\infty x^r f(x)\,dx}$$

Q.41. $f(t) = \alpha e^{-\alpha t}, \quad 0 \le t \le \infty$

$\Rightarrow \displaystyle\int_{100}^{200} \alpha . e^{-\alpha t} . dt = \alpha . \left[\frac{e^{-\alpha t}}{-\alpha}\right]_{100}^{200}$

$= e^{-100\alpha} - e^{-200\alpha}$ (a)

Q.40. ① $P(X>5) = \displaystyle\int_5^\infty f(x)\,dx$

$\Rightarrow \displaystyle\int_5^\infty \frac{1}{5} e^{-x/5} . dx = \frac{1}{5} . \left[\frac{e^{-x/5}}{-1/5}\right]_5^\infty$

$= \dfrac{1}{e}$ (a)

② Mean $= \dfrac{1}{\lambda}$

$= 5$

③ Variance $= \dfrac{1}{\lambda^2} = 25$

④ $E[x^3] = \displaystyle\int_0^\infty x^3 . \frac{1}{5} e^{-x/5} . dx$

$= \dfrac{x}{5} = t \quad \Rightarrow dx = 5dt$

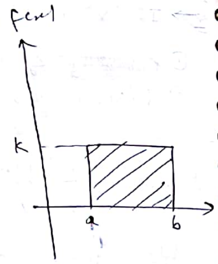
$= \dfrac{1}{5} \cdot 5 \displaystyle\int_0^\infty 5^3 . t^3 e^{-t} dt$

$\Rightarrow 5^3 \displaystyle\int_0^\infty e^{-t} t^{4-1} dt$

$\Rightarrow 5^3 \Gamma 4 = 5^3 \times 3!$

Uniform Distribution :

$X \to [a,b]$



Pdf :

$$f(x) = \begin{cases} k = \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{else} \end{cases}$$

Mean $= \dfrac{a+b}{2}$, $var(x) = \dfrac{(b-a)^2}{12}$

$$\boxed{E[x^r] = \int_a^b x^r f(x)\,dx}$$

NOTE Q. "x" follows uniform distribution, in the interval $[0,1]$. Then, find, :

① mean ② variance ③ $\cancel{P(x)}$ $E[x^3]$ ④ $E[(x+2)^2]$

① Mean $= \dfrac{0+1}{2} = \dfrac{1}{2}$ ② variance $= \dfrac{(1-0)^2}{12} = \dfrac{1}{12}$

③ $\int_0^1 x^3 \cdot \dfrac{1}{1-0}\,dx \Rightarrow \left[\dfrac{x^4}{4}\right]_0^1 = \dfrac{1}{4}$

④ $E[(x+2)^2] = \int_0^1 (x+2)^2 \cdot \dfrac{1}{1-0}\cdot dx = \int_0^1 (x+2)^2 dx$

$\Rightarrow \int_0^1 (x^2 + 2x + 2)\, dx = \dfrac{x^3}{3} + x^2 + 2x$

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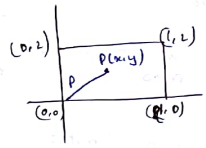
Q. A point $P(x,y)$ drawn at random from a rectangle with vertices $(0,0), (1,0), (1,2), (0,2)$ in xy plane with uniform probability. P is the length of the position vector of point $P(x,y)$, then, find expectation of P^2.



Position vector of any pt $P(x,y)$

$\vec{r} = xi + yj = \vec{P}$

$\therefore P = |\vec{P}| = \sqrt{x^2+y^2}$

$\therefore P^2 = x^2 + y^2$

$\Rightarrow E[P^2] = E[x^2 + y^2]$

$= E[x^2] + E[y^2]$

$= \dfrac{1}{3} + \dfrac{4}{3} = 5/3$

$X : [0,1]$

$Y : [0,2]$

$E[x^2] = \int_0^1 x^2 \cdot \dfrac{1}{1-0} \cdot dx$

$= \dfrac{1}{3}$

$E[y^2] = \int_0^2 y^2 \cdot \dfrac{1}{2-0}\, dy$

$= \dfrac{8}{6} = \dfrac{4}{3}$

Differential Calculus:

→ **Limit of a function f(x) at x = a :**

$\lim_{x\to a} f(x) = l$ exists iff

$\lim_{h\to 0} f(a-h) = \lim_{h\to 0} f(a+h)$

Standard Limits:

① $\lim_{\theta\to 0} \frac{\sin k\theta}{\theta} = \lim_{\theta\to 0} \frac{\tan k\theta}{\theta} = k$

② $\lim_{n\to\infty} \left(1+\frac{p}{n}\right)^n = e^p$

③ $\lim_{n\to 0} [1+np]^{1/n} = e^p$

④ $\lim_{x\to 0} \frac{a^x-1}{x} = \log a$

⑤ $\lim_{x\to 0} \frac{e^{ax}-1}{x} = a$

⑥ $I =$ If $\lim_{x\to a} [f(x)]^{g(x)} = 1^\infty$

then, $\boxed{I = e^{\lim_{x\to a} g(x)[f(x)-1]}}$

⑦ If $I = \lim_{x\to a} [f(x)]^{g(x)} = 0^0$

then, $\boxed{I = e^{\lim_{x\to a} g(x)\log f(x)}}$

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Q. $\lim_{x\to 0} \frac{e^{4x}-1}{\sin 2x} = \frac{1-1}{0} = \frac{0}{0}$

L-Hospital :

$\lim_{x\to 0} \frac{4e^{4x}}{2\cos 2x} = \frac{4.1}{2.1} = 2$

Q. $\lim_{x\to \pi/2} \frac{\sec x}{\sec 3x} = \frac{\infty}{\infty}$

$\lim_{x\to\pi/2} \frac{\cos 3x}{\cos x} = \frac{0}{0}$ ⇒ L Hospital :

⇒ $\lim_{x\to\pi/2} \frac{\cancel{-}3\sin 3x}{\cancel{-}\sin x} = \frac{3\sin(3\pi/2)}{\sin(\pi/2)} = \frac{3(-1)}{1} = -3$

Q. $\lim_{x\to 0} \frac{2^x-3^x}{2^x-4^x}$ $\cancel{\frac{0}{0}}$

~~L Hospital $\lim_{x\to 0} 2^x$~~

$\lim_{x\to 0} \frac{2^x\left[1-\left(\frac{3}{2}\right)^x\right]}{2^x\left[1-\left(\frac{4}{2}\right)^x\right]} * \frac{x}{x}$

$\lim_{x\to 0} \frac{-\frac{\left[\left(\frac{3}{2}\right)^x-1\right]}{x}}{-\frac{\left[\left(\frac{4}{2}\right)^x-1\right]}{x}} = \frac{\log 3/2}{\log 4/2}$ from ⑤

Q6 $\lim_{x\to 0} \dfrac{\sin 2x + a\sin x}{x^3} = b$ "b" is finite find a, b = ?

$\lim_{x\to 0} \dfrac{\sin 2x + a\sin x}{x^3} = \dfrac{0}{0}$

L-Hosp. $\lim_{x\to 0} \dfrac{2\cos 2x + a\cos x}{3x^2} = \dfrac{a+2}{0}$

$\because$ Limit is finite $\Rightarrow$ $a+2=0$ $\boxed{a=-2}$

$\lim_{x\to 0} \dfrac{2\cos 2x - 2\cos x}{3x^2} = \dfrac{0}{0}$

L-Hosp. $\lim_{x\to 0} \dfrac{-2^2\sin 2x + 2\sin x}{6x} = \dfrac{0}{0}$

L-Hosp. $\lim_{x\to 0} \dfrac{-2^3\cos 2x + 2\cos x}{6} = \dfrac{-6}{6}$

$= -1 = b$

$\therefore a = -2, b = -1$ (a)

Q. 5. $\lim_{x\to 0} e^x (\cos x)^{\frac{1}{\sin^2 x}} = 1. \, 1^{\infty} \rightarrow$ Indeterminate

from (b) $= 1. \, e^{\lim_{x\to 0} \frac{1}{\sin^2 x}(\cos x - 1)} = \dfrac{0}{0}$ $=$

L-Hosp. $= e^{\lim_{x\to 0} \frac{-\sin x}{2\sin x\cos x}} = e^{-1/2}$ (b)

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Q $\lim_{x\to\infty} \left(e^{\frac{1}{5x}} - 1\right)\left[5x + \dfrac{x}{5}\sin\left(\dfrac{1}{x}\right)\right] : \lim_{x\to\infty} \dfrac{\left(e^{\frac{1}{5x}}-1\right)\left[5 + \frac{1}{5}\sin\left(\frac{1}{x}\right)\right]}{\left(\frac{1}{x}\right)}$

let $y = \frac{1}{x} \rightarrow 0$

$\Rightarrow \lim_{y\to 0} \dfrac{e^{\frac{1}{5}y} - 1}{y} \cdot \lim_{y\to 0}\left[5 + \frac{1}{5}\sin y\right] = \frac{1}{5}\times[5+0] = 1$

Q $\lim_{x\to 0} \dfrac{1-\cos x^2}{2x^4}$

let $y = x^2 \rightarrow 0 \Rightarrow \lim_{y\to 0} \dfrac{1-\cos y}{2y^2} = \dfrac{0}{0}$

Q. $\lim_{x\to\infty} (x^2+1)^{e^{-x}} = \infty^{e^{-\infty}}$
$= \infty^0 \rightarrow$ Indeterminate

take log on both sides:

$\log_e I = \lim_{x\to\infty} e^{-x}\log(1+x^2) = 0.\infty = \dfrac{\infty}{1/0} = \dfrac{\infty}{\infty}$

~~L-Hosp.~~

$\log_e I = \cancel{\lim_{x\to\infty}\left[e^{-x}(\log(1+x^2)) + \frac{e^{-x}}{(1+x^2)}\right]}$

$\log_e I = \lim_{x\to 0} \dfrac{\log(1+x^2)}{e^x} = \dfrac{\infty}{\infty}$

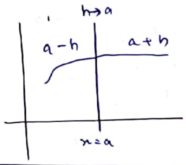
L-Hosp. $\log_e I = \lim_{x\to 0} \dfrac{2x}{(1+x^2).e^x} = 0$

$I = e^0 = 1$

Continuity of f(x) at $x = a$:

If $\lim_{x \to a} f(x) = f(a)$, then, f(x) is continuous at $x = a$

L.H.L = R.H.L. = f(a)



Q.1. Given f(x) is continuous

f(x) cont. at $x = 1, 3$

at $x=1$: ~~f(1+h)~~ $f(1^-) = f(1^+)$

$$2x+1\Big|_{x=1} = ax^2+b\Big|_{x=1}$$

$\boxed{3 = a+b}$ — ①

at $x=3$: $f(3^-) = f(3^+)$

$$ax^2+b\Big|_{x=3} = 5x+2a\Big|_{x=3} \Rightarrow 9a+b = 15+2a$$

$\boxed{7a+15 = 15}$ — ②

from ① & ② $\boxed{a = 2 \quad b = 1}$ (a)

Q.2. ∵ ② $f(3) = f(3^-) = f(3^+)$

$2 = 2 = 2$

f(x) is continuous.

Differentiability of f(x) at $x = a$:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists}$$

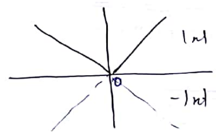
iff: L.H.D. = R.H.D.

i.e., $\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

① f(x) is not differentiable at the corner points, where, f(x) is having ~~shaed~~ sharp edge or any sudden change.

② Every differentiable function is continuous ~~and~~ but every continuous need not be differentiable.

E.g $f(x) = |x|$



③ f(x) has either maxima or minima where it is not differentiable.

Q3. Given f(x) is differentiable everywhere at $x = 1$,

f(x) is piecewise.

$$f'(1^-) = f'(1^+)$$

$$\Rightarrow 2x+3\Big|_{x=1} = b\Big|_{x=1} \Rightarrow \boxed{b = 5}$$

∵ Every differentiable f^n is continuous.

$$\therefore f(1^-) = f(1^+) \Rightarrow 1^2 + 3(1) + a = b(1) + 2 \Rightarrow \boxed{a = 3}$$

Q.4. $\frac{d^2(y)}{dx^2} = ?$, $x = a(\theta - \sin\theta)$; $y = a(1-\cos\theta)$

$\frac{dx}{d\theta} = (a - a\cos\theta) = a(1-\cos\theta)$ $\frac{dy}{d\theta} = a(\sin\theta)$

$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{a(\sin\theta)}{a(1-\cos\theta)} = \frac{2\sin\theta/2 \cos\theta/2}{2\sin^2\theta/2} = \cot\theta/2$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx}$

$= \frac{\frac{d}{d\theta}(\cot\theta/2) \times 1}{a(1-\cos\theta)} = \frac{-\frac{1}{2}\operatorname{cosec}^2\theta/2}{a\sin^2\theta/2}$

$= -\frac{1}{4a}\operatorname{cosec}^4\theta/2$

Q.16. $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots}}}$ then $\frac{dy}{dx} = ?$

$y = \sqrt{f(x) + \boxed{\sqrt{f(x) + \sqrt{f(x) \dots \infty}}}}$, $\frac{dy}{dx} = ?$

$y \longrightarrow$ Implicit form

$\Rightarrow y = \sqrt{f(x) + y}$

$\Rightarrow y^2 = f(x) + y$ $\Rightarrow \boxed{y^2 - y = f(x)} \to$ Explicit form

diff. w.r.t "x":

$2y \cdot \frac{dy}{dx} - \frac{dy}{dx} = f'(x)$

$$\boxed{\frac{dy}{dx} = \frac{f'(x)}{2y-1}}$$

$\Rightarrow$ ~~tan x~~ $\frac{dy}{dx} = \frac{\sec^2 x}{\cancel{2\tan x - 1 \, dy} \; 2y-1}$ ✓ (d)

Partial Differentiation:

$\to$ If $u = f(x,y)$

$\frac{\partial u}{\partial x} = p$, $\frac{\partial u}{\partial y} = q$, $\frac{\partial^2 u}{\partial x^2} = r$, $\frac{\partial^2 u}{\partial x \partial y} = s$, $\frac{\partial^2 u}{\partial y^2} = t$

Q.17 If $f = y^x$, $\frac{\partial^2 f}{\partial x \partial y} = ?$

$\frac{\partial f}{\partial x} = y^x \cdot \log y$ $\left[\because \frac{d}{dx}(a^x) = a^x \log a\right]$

$\frac{\partial^2 f}{\partial x \partial y} = y^x \cdot \frac{1}{y} + \log y \cdot x \cdot y^{x-1}$ $\left[\because \frac{d}{dy}(y^n) = ny^{n-1}\right]$

$= y^{x-1}\left[1 + \log y\right]$

Homogeneous Function:

$u = f(x,y)$ is homogeneous function of degree 'n' iff:

$$\boxed{f(kx, ky) = k^n f(x,y)}$$

E.g. $f(x,y) = \frac{x^3 + y^3}{x - y} \begin{matrix} \to 3 \\ \to 1 \end{matrix}$

It is homogeneous of degree:

$(n) =$ degree (N_r) − degree (D_r)

$= 3 - 1$

$= 2$

Euler Theorems:

Type I : If $u = f(x,y)$ is homogeneous f^n of degree 'n',

then, $xu_x + yu_y = n.u$

$x^2u_{xx} + y^2u_{yy} + 2xyu_{xy} = n(n-1)u$

Type II : If $u = f+g$, where f, g are homogeneous funcⁿs of degree n_1 and n_2 respectively.

then, ① $xu_x + yu_y = n_1 f + n_2 g$

② $x^2u_{xx} + y^2u_{yy} + 2xyu_{xy} = n_1(n_1-1)f + n_2(n_2-1)g$

Type III : If $\phi(u) = F(x,y)$, where $F(x,y)$ is homogeneous f^n of degree 'n'.

then, ① $xu_x + yu_y = \dfrac{n.\phi(u)}{\phi'(u)} = F(u)$

② $x^2u_{xx} + y^2u_{yy} + 2xyu_{xy} = F(u)[F'(u) - 1]$

Q. 18. $u = \tan^{-1}\left[\dfrac{x^3+y^3}{x-y}\right]$, then, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = ?$

~~$xu_x + yu_y$~~ $\tan u = \dfrac{x^3+y^3}{x-y}$ → Homg f^n of $(n) = 2$

$\Downarrow$

$\phi(u)$

$\Rightarrow \dfrac{n.\phi(u)}{\phi'(u)} = 2.\dfrac{\tan u}{\sec^2 u} = 2\sin u\cos u = \sin 2u = F(u)$ (a)

Q19 $(n) = -7$ $\phi(u) = \sin u$

$\therefore \dfrac{n.\phi(u)}{\phi'(u)} \Rightarrow -7.\dfrac{\sin u}{\cos u} = -7\tan u$ (b)

Q. 20. $u = \dfrac{x^3+y^3}{x-y} + x\sin\left(\dfrac{x}{y}\right)$, then

$x^2.u_{xx} + 2xyu_{xy} + y^2u_{yy} = ?$

Type II. $n_1(n_1-1)f + n_2(n_2-1)g$

$= 2(2-1)f + 1(1-1)g$

$= 2f$. (b)

Mean Value Theorems:

* Rolle's Theorem:

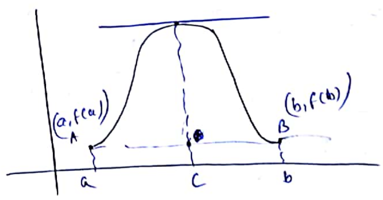
→ If $f: [a,b] \to R$ such that

① $f(x)$ is continuous in $[a,b]$

② $f(x)$ is differentiable in (a,b)

③ $f(a) = f(b)$

then, ∃ a mean value $c \in (a,b)$ such that $\boxed{f'(c) = 0}$



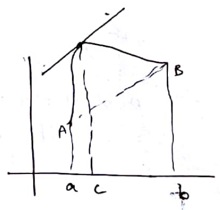
draw a tangent parallel to $\overline{AB}$

$\Rightarrow$ (Slope of tangent at $x=c$) = slope of $(\overline{AB})$

$f'(c) = \dfrac{f(b)-f(a)}{b-a} = 0$

slope of tangent $\quad \longrightarrow$ slope of $(\overline{AB})$

Lagrange's Mean Value Theorem:



If $f:[a,b]\to R$ such that

① $f(x)$ is constant in $[a,b]$

② $f(x)$ is diff in (a,b)

Then $\exists$ a mean value $c\in(a,b)$ such that $f'(c) = \boxed{\dfrac{f(b)-f(a)}{b-a}}$

* used to find $\frac{dy}{dx}$ at certain 'x' in $[a,b]$.

(slope of tangent at $x=c$) = slope of $(\overline{AB})$

$f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Q50. $f(x) = \dfrac{\sin x}{e^x}$ in $[0,\pi]$, $c = ?$.

By Rolle's Thm: $f'(c) = 0$

$\cancel{e^x \sin x - \sin}\left[\dfrac{e^x\cos x - \sin x\, e^x}{e^{2x}}\right]_{x=c} = 0$

$\Rightarrow \dfrac{\cos c - \sin c}{e^c} = 0 \Rightarrow \cos c - \sin c = 0 \Rightarrow \cos c = \sin c$

$\Rightarrow c = \dfrac{\pi}{4} \in (0,\pi)$

Q10. $f(x) = (1+x)\log(1+x)$ on $[0,1]$

by LMVT: $f'(c) = \dfrac{f(1)-f(0)}{1-0}$

$\left[(1+x)\cdot\dfrac{1}{(1+x)} + \log(1+x)\right]_{x=c} = \dfrac{2\log 2 - 1\log 1}{1-0}$

$1+\log(1+c) = \log 4 \Rightarrow \log(1+c) = \log 4 + \log e$
$= \log(4/e)$

$1+c = \dfrac{4}{e} \Rightarrow \boxed{c = \dfrac{4-e}{e} \in (0,1)}$

Cauchy's Mean Value Theorem:

$\rightarrow$ If $f,g : [a,b] \to R$, such that

① f,g are continuous in

Q. $y = 5x^2 + 10x \quad \in (1,2)$

find $\frac{dy}{dx}$ at atleast one value of x in $(1,2)$.

ⓐ 15 ⓑ 25 ⓒ 35 ⓓ 45

By LMVT.

$\left.\dfrac{dy}{dx}\right|_c = \dfrac{f(2)-f(1)}{2-1} = \dfrac{(20+20)-(15)}{1} = 25$

Cauchy's Mean Value Theorem:

→ If $f, g : [a,b] \to R$, such that

1. f, g are continuous in $[a,b]$
2. f, g are differentiable in (a,b)
3. $g'(x) \neq 0 \ \forall x \in (a,b)$

then $\exists$ a mean value, $c \in (a,b)$, such that $\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$

Q.52. $f(x) = x^2$ $g(x) = x^3$ over $[1,2]$ $c(1,2) = ?$

$\frac{f'(c)}{g'(c)} = \frac{(4-1)}{(8-1)} \Rightarrow \frac{2x}{3x^2} = \frac{3}{7} \Rightarrow \boxed{x = \frac{14}{9}}$

Taylor's Series:

→ Expansion of $f(x)$ at $x = a$ (or) in the powers of $(x-a)$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \dots$$

$$+ \dots + \frac{(x-a)^n}{\lfloor n} f^n(a) + \dots$$

$$\boxed{\text{Coefficient of } (x-a)^n = \frac{f^n(a)}{\lfloor n}}$$

Mc Laurence Series: [put $a=0$ in Taylor's]

we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \dots + \frac{x^n}{\lfloor n} f^n(0) + \dots$$

Applications of Taylor:

1. It is used to expand $f(x)$ as a polynomial.
2. It is used to approximate a function.
3. It is used to solve the differential equation.

Q. Find expansion of $\sin x$ at $x = \frac{\pi}{6}$.

$f(x) = \sin x$ $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ $f''(x) = -\sin\frac{\pi}{6} = -\frac{1}{2}$

$f'(x) = \cos x$ $\cos\left(\frac{\pi}{2}\right) = \frac{\sqrt{3}}{2}$ $f'''(x) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$$f(x) = \frac{1}{2} + \left(x - \frac{\pi}{6}\right)\frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{6}\right)\left(-\frac{1}{2}\right) + \frac{1}{3!}\left(x - \frac{\pi}{6}\right)^3\left(-\frac{\sqrt{3}}{2}\right) + \dots$$

Q13. $f(x) = \frac{\sin x}{x - \pi}$ $\sin(\pi) = 0$ $\cos(\pi) = -1$, $-\sin(\pi) = 0$ (b)

$f(x) = -(x-\pi) + \frac{(x-\pi)^3}{3!} \dots f(x) = -1 + \frac{(x-\pi)^3}{3!} + \dots$

Q. Find the linear approximation of e^{-x} at $x = 2$.

e^{-2} $f'(2) = -e^{-2}$, $f''(2) = e^{-2}$ $f'''(2) = -e^{-2}$

$$f(x) = e^{-2} - (x-2)e^{-2} + \frac{(x-2)^2 e^{-2}}{2!} - \frac{(x-2)^3 e^{-2}}{3!}$$

Linear approximation of $f(x)$ at $x = a \Rightarrow \boxed{f(x) = f(a) + (x-a) f'(a)}$

$f(x) = e^{-2}(1 - x + 2)$

$f(x) = e^{-2}(3 - x)$

Q Find quadratic expansion of $x^3 - 3x^2 - 5$ at $x = 0$

$f(0) = -5$ $f'(0) = 0$ $f''(x) = 12x - 6$

$f(x) = -5 - \frac{6x^2}{2!}$, $-5 - 3x^2$ $-3x^2 - 5$

Q.11. e^x x^2

(b) $\frac{e^x}{4!} = \frac{e^2}{4!}$

Q. $e^{x+x^2} = e^x \cdot e^{x^2}$

e^{x+x^2}
$= e^{(x+x^2)} \cdot (1+2x)$

Standard series expansions:

$$e^x = 1 + \frac{x^1}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$$

$$\sin x = 0 + \frac{x^1}{\lfloor 1} \cdot (1) - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

Q.14. $\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$

$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots$

$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right]$ (a)

Increasing/Decreasing Function:

→ $y = f(x)$ is increasing, if $\frac{dy}{dx} > 0$.

→ $y = f(x)$ is decreasing, if $\frac{dy}{dx} < 0$.

→ $y = f(x)$ is strictly/ monotonically decreasing if $\frac{dy}{dx} < 0, \forall x \in R$

→ $y = f(x)$ is strictly/monotonically increasing if $\frac{dy}{dx} > 0, \forall x \in R$.

E.g. $f(x) = \frac{e^x}{e^x+1}$ $\forall x \in R$

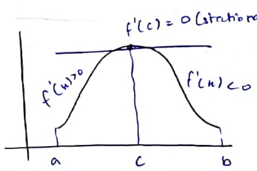
$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}}$$

$f'(x) = \frac{(e^x+1)e^x - e^x \cdot e^x}{(e^x+1)^2}$

$= \frac{e^x}{(e^x+1)^2}, > 0 \; \forall x \in R$

$\therefore f(x)$ is strictly increasing.

Ques.



(a, c) → interval of incr.

(c, b] → interval of dec.

Q.45. $f(x) = x^3 - 3x^2 + 1$

find stationary pt:

$f'(x) = 3x^2 - 6x = 0 \Rightarrow x = 0, 2 \in [-1, +3]$

$[-1, 3] = [-1, 0) \cup [0, 2] \cup (2, 3]$

In $[-1, 0)$: at $x = -1$

$f'(-1) = 3(-1)^2 - 6(-) = 9 > 0$

$\therefore$ Increasing

In $(0,2)$: open interval

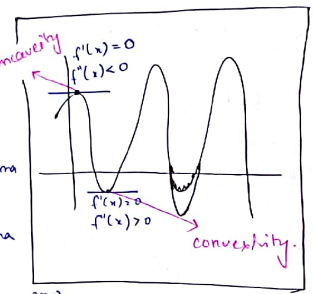
at $x=1 \Rightarrow f'(1) = 3(1)^2 - 6(1)$
$= -3 < 0 \Rightarrow$ decreasing.

In $(2,3)$:

at $x=3 \Rightarrow f'(3) = 3(3^2) - 6(3)$
$= 9 > 0 \Rightarrow$ increasing.

(b)

Maxima and Minima :



→ An n^{th} degree polynomial can have $(n-1)$ bends [extreme].

→ The greatest among all local maxima is called global maxima.

→ The least among all the local minima is called global minima.

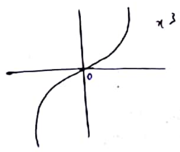
⇒ If $f'(a) = f''(a) = f'''(a) = \dots - f^{2n-2}(a) = 0$ and $f^{2n-1}(a) \neq 0$, then $x=a$ is called saddle point (or) point of inflexion, where $f(x)$ has neither maxima nor minima.

E.g. $f(x) = x^3$

$f'(x) = 3x^2 = 0 \Rightarrow x = 0$ station. point

$f''(0) = 6x = 0$ ~~$\neq 0$~~

$f'''(x) = 6 \neq 0 \quad \therefore x=0$ is saddle pt.



NOTE
→ The point at which the curve is changing from concavity to convexity is k/a saddle pt.

Q. $e^y = x^{1/x}$, y has max at $x = ?$

log on both sides:

$y = \frac{1}{x}\log x$

Find stationary pt:

$y' = \frac{1}{x}\cdot\frac{1}{x} - \frac{1}{x^2}\cdot\log x = 0$

$\Rightarrow \frac{1}{x^2}\left[1 - \log x\right] = 0$

$\Rightarrow \log_e x - 1 = 0$

$\Rightarrow \log_e x = 1$

$\boxed{x = e}$

diff wrt 'x' again:

$y'' = \frac{1}{x^2}\cdot\left[-\frac{1}{x}\right] + \frac{2}{x^3}\left[1 - \log x\right] = 0.$

$y''_{x=e} = \frac{-1}{e^3} < 0$

$\therefore y' = 0$ & $y'' < 0$ at $x = e$

$\therefore y$ is maximum at $x = e$

Q. $y = x^{1/x}$

log on b.s.:

$\log y = \frac{1}{x}\log x$

diff. wrt x & equate to 0:

$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x}\cdot\frac{1}{x} - \frac{1}{x^2}\log x$

$\frac{dy}{dx} = x^{1/x}\left[\frac{1}{x^2} - \frac{1}{x^2}\log x\right]$

$= \frac{(x^{1/x})}{x^2}\left[1 - \log x\right]$ (y)

Find stationary pt: $\frac{dy}{dx} = 0$

$\Rightarrow 1 - \log_e x = 0$

$\Rightarrow \log_e x = 1 \quad \Rightarrow \boxed{x = 1}$

diff. wrt 'x' again:

$\frac{d^2y}{dx^2} = y\left[-\frac{2}{x^3}(1-\log x) + \frac{1}{x^2}\left(-\frac{1}{x}\right)\right]$

$+ \frac{dy}{dx}\left[\frac{1-\log x}{x^2}\right]$

at $x = e$:

$y'' = y_e\left(-\frac{1}{e^3}\right) = e^{1/e}\cdot\left(-\frac{1}{e^3}\right) < 0 \Rightarrow$ max.

$\therefore f(\cdot)_{max} = y_{max} = \left(x^{1/x}\right)_{x=e} = e^{1/e}$

Shortcuts:

1. If $y = a\cos\theta + b\sin\theta + c$
then $y_{max} = c + \sqrt{a^2+b^2}$
$y_{min} = c - \sqrt{a^2+b^2}$

2. $f(x) = x^n e^{-x}$ is max at $x = n$

3. $y = a\cos^2\theta + b\sin^2\theta$, then:
$\left.\begin{aligned} y_{max} &= a \\ y_{min} &= b \end{aligned}\right]$ if $a > b$

4. $y = a\tan x + b\cot x$ has $y_{min} = 2\sqrt{ab}$
at $x = \tan^{-1}(b/a)$

Q. $f(x) = (x-1)^{2/3} \Rightarrow \because f(x) = \left[(x-1)^{1/3}\right]^2 \geq 0, \forall x \in R$

$\therefore f_{min} = 0$ at $x = 1$.

---

5. Minimum value of $a^2\sec^2 x + b^2\operatorname{cosec}^2 x$ is
$(a+b)^2$ at $x = \tan^{-1}(b/a)$

6. $f(x) = \dfrac{x}{(x+a)(x+b)}$ is maximum at $x = \sqrt{ab}$

Q.22. Find max value of $\dfrac{e^{\sin x}}{e^{\cos x}}$, $\forall x \in R$?

$f(x) = e^{\sin x - \cos x}$

$\Rightarrow f_{max} = e^{max(\sin x - \cos x)} = e^{0 + [\sqrt{(-1)^2+(1)^2}]} = e^{\sqrt{2}}$. (a)

Q. Find the max value of $f(x) = \dfrac{x}{(x+1)(x+2)}$

(6) f_{max} at $x = \sqrt{ab} = \sqrt{2}$

$f_{max}(x=\sqrt{2}) = \dfrac{\sqrt{2}}{(1+\sqrt{2})(2+\sqrt{2})}$

Q. EE (M) .. Find the max value of $f(x) = x^2 e^{-x}$?

(2) f_{max} at $x = n \Rightarrow x = 2$

$f_{max}(2) = 4e^{-2}$.

Q. Find the min value of $y = 3\tan\theta + 4\cot\theta$?

$y_{min} = 2\sqrt{ab} = 2\sqrt{12}$ at $\tan^{-1}\left(\frac{4}{3}\right)$

Find global max (or) min of $f(x)$ in $[a,b]$

global max = max $[f(a), f(b), \text{local maximas}]$

global min = min $[f(a), f(b), \text{local minima}]$

Q. Find global minima of $f(x) = 2x^3 - 3x^2$ in $[-1, 2]$.

Sol^n:
$f(a) = f(-1) = 2(-1)^3 - 3(-1)^2 = -5$
$f(b) = f(2) = 2(2)^3 - 3(2)^2 = 4$

find local minima:

$f'(x) = 6x^2 - 6x = 0 \Rightarrow x = 0, 1 \in [-1, 2]$
$f''(x) = 12x - 6$

at $x=0 \Rightarrow -6 < 0 \Rightarrow$ maximum at $x = 0$.
at $x=1 \Rightarrow f''(1) = 6 > 0 \Rightarrow$ minimum at $x = 1$.
$\therefore$ The local minima at $x = 1$.
$f(1) = 2 - 3 = -1$

$\therefore$ global minima = min $[-1, -5, 4] = \boxed{-5}$

Q. $f(x) = x^2 - x - 2$ in $[-4, 4]$, find maximum value of $f(x) = ?$

$f(-4) = 18 \quad f(4) = 10$

$f'(x) = 2x - 1 = 0$

$\Rightarrow x = \frac{1}{2}$ $\boxed{\text{minima}}$

$f''(x) = 2 > 0$

W.K.T.:

Global max = $\max\{f(-4), f(4), \cancel{\text{local max}}\}$ doesn't exist here

$= 18$

Q. 61. $f(x) = x.(x-1)(x-2)$.
Find ~~minimum~~ maximum value of $f(x)$ in $[1, 2]$

$f(1) = 0 \quad f(2) = 0$

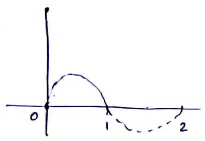
~~$f(x)$~~ In $[1,2]$

at $x = 3/2 \in [1,2]$

$f(3/2) = \frac{3}{2}\left(\frac{3}{2} - 1\right)\left(\frac{3}{2} - 2\right)$

Hence, $f(x) \le 0$ in $[1,2]$

Hence, $f_{max} = 0$



Constraint Maxima (or) Minima:

Let $u = f(x,y)$ to be extremized w.r.t. condition

$\phi(y\, x, y) = 0$

$\phi(x,y) = c \Rightarrow y = f(x)$

then, $u = f(x, f(x))$ to be extremized.

Short cuts:

① $\forall\, a, b \in R$, if $(a+b) =$ const., then, the max value of $a*b$ is when $a = b$.

Q. If the trace of a 2×2 real symmetric matrix is 14. Then find the maximum value of the determinant.

$(7+7) = 14 \quad \therefore \; | \; 7*7 = \underline{\underline{49}}$ = max value of determinant.

---

② The maximum area of a square inscribed in a circle of radius "R" is "$2R^2$".

Q. Find the max^m area of square when it is inscribed in a circle of radius = 2.

$\Delta_{max} = 2R^2 = 2*(2)^2 = 8$.

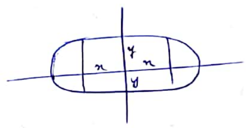
---

③ The maximum area of a rectangle inscribed in an ~~sphere~~ ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is $2ab$.

Q. Find the max^m area of rectangle whose vertices lies on the ellipse $x^2 + 4y^2 = 1$

$\frac{x^2}{1^2} + \frac{y^2}{(1/2)^2} = 1 \Rightarrow$ Max^m area of ▭ $= 2 * \frac{1}{2} * 1 = \underline{\underline{1}}$

(OR) $x = \cos\theta \quad y = \frac{1}{2}\sin\theta$



$\Delta = (2x)(2y)$

$= 4xy$

$\Delta = 4^2\cos\theta * \frac{1}{2}\sin\theta$

$\Delta = \sin 2\theta$

$\boxed{\therefore \Delta_{max} = 1}$

(4) In a right angled triangle, sum of length of a side and hypotenuse is a constant.
In order to have, the maximum area of a triangle, the angle between the side and hypotenuse is $\underline{60^\circ}$.

* Extremize $u = f(x,y)$

(1) Solve $p = 0$ $q = 0$
we get stationary pt. (x,y)

(2) Find r, s, t at (x,y)

(3) If $rt - s^2 > 0$ and $r > 0 \Rightarrow$ 'u' is min at (x,y)

(4) If $rt - s^2 > 0$ and $r < 0 \Rightarrow$ 'u' is max at (x,y)

(5) If $rt - s^2 < 0$ ~~and~~, then (x,y) is saddle point.

Q.25. $u = f(x,y) = 4x^2 + 6y^2 - 8x - 4y + 8$

$\frac{\partial u}{\partial x} = p$ $\quad \frac{\partial u}{\partial y} = q$ $\quad \frac{\partial^2 u}{\partial x^2} = r$ $\quad \frac{\partial^2 u}{\partial x \partial y} = s$ $\quad \frac{\partial^2 y}{\partial y^2} = t$

$p = u_x = 8x - 8 = 0 \Rightarrow x = 1$
$q = u_y = 12y - 4 = 0 \Rightarrow y = \frac{1}{3}$
$(1, \frac{1}{3}) \Rightarrow$ stationary point.

$r = u_{xx} = 8 > 0$, $s = u_{xy} = 0$, $t = u_{yy} = 12$

$rt - s^2 = 96 > 0$, $r = 8 > 0$

$\therefore$ u is min at $(1, \frac{1}{3}) \Rightarrow$ optimal value will be fmin.

Optimal value $\Rightarrow$ fmin $= f(1, \frac{1}{3}) = 4 + 6 \times \frac{1}{3} \times \frac{1}{3} - 8 - 4 \times \frac{1}{3} + 8$

$\Rightarrow 4 + \frac{2}{3} - \cancel{8} - \frac{4}{3} + \cancel{8} = 4 - \frac{2}{3}$

$\Rightarrow \frac{10}{3}$.

(a)

Q. Find the shortest distance from origin to any point on surface $z^2 = 1 + xy$?

At $(0,0,\pm 1)$, surface cuts the 'z' axis, then, the distance between $(0,0,0)$ and $(0,0,1)$ will be the $\perp^r$ distance from the origin.

Hence, the shortest distance $= \perp^r$ distance $= 1$.

Let 'd' be the distance b/w $O(0,0,0)$ & $P(x,y,z)$ on $z^2 = 1 + xy$.

$\Rightarrow d = \sqrt{x^2 + y^2 + z^2}$

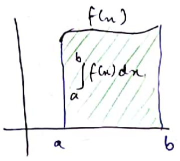
$\Rightarrow u = d^2 = \underbrace{x^2 + y^2 + 1 + xy}_{\text{to be minimized}}$ $[\because z^2 = 1 + xy]$

we get $P(0,0,\pm 1)$

i.e. $d_{min} = \sqrt{0^2 + 0^2 + 1^2} = 1$

INTEGRATION :

$I = \int_a^b f(x)\,dx$ gives the area under the curve $y = f(x)$ bounded by lines $x = a$, $x = b$ and x-axis.



* Properties:

① $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

② $\int_0^{2a} f(x)\,dx = \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \end{cases}$

③ $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$

④ $\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & \text{if } f(-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & \text{if } f(-x) = f(x) \end{cases}$

⑤ $\int_0^{\pi/2} \sin^n\theta\,d\theta = \int_0^{\pi/2} \cos^n\theta\,d\theta = \left[\frac{n-1}{n} \times \frac{n-3}{n-2} \times \dots \times \frac{2}{3} \text{ (or) } \frac{1}{2}\right] K$

where $K = \begin{cases} 1 & \text{if 'n' is odd} \\ \frac{\pi}{2} & \text{if 'n' is even} \end{cases}$

⑥ $\int_0^{\pi/2} \sin^m\theta \cos^n\theta\,d\theta = \left[\frac{[(m-1)(m-3)\dots 2 \text{ (or) } 1][(n-1)(n-3)\dots 2 \text{ (or) } 1]\,K}{(m+n)(m+n-2)\dots 2 \text{ (or) } 1}\right]$

$K = \begin{cases} \pi/2, & \text{if } m, n \text{ are even} \\ 1, & \text{else} \end{cases}$

⑦ $\int uv = uv_1 - u'v_2 + u''v_3 - \dots$

* Standard Integrals:

① $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ $(a>b)$

② $\int e^x(f(x)+f'(x))\,dx = e^x f(x)$

③ $\int_0^{\pi/2} \log\sin\theta\,d\theta = \int_0^{\pi/2} \log\cos\theta\,d\theta = -\frac{\pi}{2}\log 2$

Q. $\int_0^{\pi/2} \frac{dx}{1+\tan x} = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx = \frac{\pi/2 - 0}{2} = \frac{\pi}{4}$

$\cos(0+\frac{\pi}{2}-x)$ (by ③)

Q. $\int_0^{\pi} x \cdot \sin^6 x \cos^4 x\,dx = I$

by prop ①,

$I = \int_0^{\pi} (0+\pi-x) \sin^6(0+\pi-x) \cos^4(0+\pi-x)\,dx$

$I = \int_0^{\pi} (\pi-x) \sin^6 x \cdot \cos^4 x \cdot dx$

$\therefore I + I = \int_0^{\pi} (\cancel{x}+\pi-\cancel{x}) \sin^6 x \cos^4 x\,dx$

$2I = \pi \int_0^{\pi} \sin^6 x \cos^4 x\,dx$

By prop. ②: $\cancel{2}I = \cancel{2}\pi \int_0^{\pi/2} \sin^{6}x \cos^{4}x\, dx$

$I = \pi\left[\frac{(6-1)(6-3)(6-5)(4-1)(4-3)}{(10)(8)(6)(4)(2)}\right] * \frac{\pi}{2}$

$\Rightarrow \frac{\pi^2}{2}\left[\frac{\cancel{5}\times\cancel{3}\times 1\times 3\times 1}{\cancel{10}\times 8\times\cancel{6}\times 4\times 2}\right] = \frac{3}{512}\pi^2$

Q. $\int_0^2 \frac{(x-1)^2 \sin(x-1)}{(x-1)^4 + \cos(x-1)}\, dx$

$\boxed{\begin{matrix} + \text{ even} \\ - \text{ odd}\end{matrix}}$

$\Rightarrow$ $x-1 = t$ $\quad$ $x=0 \Rightarrow t=-1$

$\therefore dx = dt$ $\quad$ $x = 2 \Rightarrow t = 1$

$\Rightarrow \int_{-1}^{1} \frac{t^2 \sin t}{t^4 + \cos t}\, dt = 0$ odd

---

Q. $f(n) = \int_0^{\pi/4} \tan^n\theta\, d\theta$; $f(3)+f(1) = ?$

n=3

$f(3) = \int_0^{\pi/4} \tan^3\theta\, d\theta = \int_0^{\pi/4} \tan^2\theta . \tan\theta\, d\theta$

$= \int_0^{\pi/4} (\sec^2\theta - 1).(\tan\theta)\, d\theta$

$= \int_0^{\pi/4} \sec^2\theta . \tan\theta\, d\theta - \int_0^{\pi/4} \tan\theta\, d\theta$ $\quad$ $\tan\theta = t$, $\sec^2\theta\, d\theta = dt$

$f(3) = \left.\frac{\tan^2\theta}{2}\right]_0^{\pi/4} - f(1)$

$f(3) + f(1) = \frac{1}{2}$

Q. $\int_0^{\pi} \sin^5 x \cos^3 x\, dx$ By property ② $= 0$.

Q 26. $I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$

By property ①:

$I = \int_0^{\pi/4} \log[1 + \tan(\pi/4 - \theta)]\, d\theta$

$= \int_0^{\pi/4} \log\left[1 + \frac{1-\tan\theta}{1+1.\tan\theta}\right] d\theta$

$I \Rightarrow \int_0^{\pi/4} \log\left[\frac{2}{1+\tan\theta}\right] d\theta = \int_0^{\pi/4} \log 2\, d\theta - \underbrace{\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta}_{I}$

$2I = \int_0^{\pi/4} \log 2\, d\theta \Rightarrow 2I = \log_2 [\theta]_0^{\pi/4} \Rightarrow \boxed{I = \frac{\pi}{8}\log_2}$

Q. $I = \int_0^{\pi/2} \log\tan\theta\, d\theta = \int_0^{\pi/2} \log\sin\theta\, d\theta - \int_0^{\pi/2} \log\cos\theta\, d\theta$

$= -\frac{\pi}{\cancel{2}}\cancel{\log_2} - \left(-\frac{\pi}{2}\cancel{\log_2}\right) = 0$

Q. $I = \int_0^{\pi/2} \cot\theta\, d\theta$ = By pt ① $\int_0^{\pi/2} \cot(\pi/2 - \theta)\, d\theta = \int_0^{\pi/2} \tan\theta\, d\theta = 0$.

Q. $I = \int_{-1}^{2} 1 + |x|\, dx = \int_{-1}^{2} 1\, dx + \int_{-1}^{2} |x|\, dx$ $\quad$ $|x| = \begin{cases} -x, & x<0 \\ x, & x>0 \end{cases}$

$= [x]_{-1}^{2} + \int_{-1}^{0} (-x)\, dx + \int_0^2 (x)\, dx$

$\Rightarrow 2-(-1) + \left[\frac{-x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^2 = 3 + \left(+\frac{1}{2}\right) + 2$

$= \frac{11}{2}$

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Q. $I = \int_0^{2\pi} |x \sin x|\, dx$

$|x\sin x| = |x|\,|\sin x| = \begin{cases} x\sin x, & 0<x<\pi \\ -x\sin x, & \pi<x<2\pi \end{cases}$

$I = \int_0^{\pi} x \sin x\, dx + \int_{\pi}^{2\pi} -x\sin x\, dx$

$= \Big[(x)(-\cos x) - (1)(-\sin x)\Big]_0^{\pi} - \Big[(x)(-\cos x) - (1)(-\sin x)\Big]_{\pi}^{2\pi}$

$I = (\pi - 0) - (-2\pi - \pi) = \underline{4\pi}$

Improper Integrals:

Type-I: $I = \int_{-\infty}^{\infty} f(x)\,dx$ (or) $\int_0^{\infty} f(x)\,dx$

Type II, $I = \int_a^b f(x)\,dx$

where $f(x)$ is disc (or) undefined in (a,b).

NOTE
- If I = finite, it is said to be convergent.
- If I = infinite, then, I is said to be divergent (or) does not exist (or) unbounded.

Q. $\int_{-1}^{1} \frac{1}{x^2}\, dx$ $\quad f(x)$ is not defined at $x = 0 \in [-1,1]$

$I = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx$

$\Rightarrow \left[-\frac{1}{x}\right]_{-1}^{0} + \left[-\frac{1}{x}\right]_0^{1} = \underline{\infty}$ divergent

Q. $I = \int_0^1 \frac{1}{1-x}\,dx = \left[\frac{\log(1-x)}{-1}\right]_0^1 = \frac{-\infty}{-1} = \infty$ divergent

Q. $\int_0^{\infty} \frac{x}{x^2+4}\,dx = \lim_{x\to\infty} \int_0^{x} \frac{x}{x^2+4}\,dx$

let $x^2+4 = t \Rightarrow 2x\,dx = dt$ $\quad \begin{cases} x=0 \Rightarrow t=4 \\ x=\infty \Rightarrow t=\infty \end{cases}$

$\therefore I = \frac{1}{2}\int_4^{\infty} \frac{1}{t} = \frac{1}{2}\left[\log t\right]_4^{\infty} = \infty$

Hence, it is divergent.

Gamma function:

Definition: $\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\,dx$

Note
1. $\Gamma(n+1) = n\Gamma n = n!$
2. $\Gamma 2 = \Gamma 1 = 1$
3. $\Gamma \tfrac{1}{2} = \sqrt{\pi}$

* Standard Integral :

$$I = \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

Proof - $I = 2\int_0^{\infty} e^{-ax^2} dx$ $\qquad ax^2 = t$, $2ax\,dx = dt$

$$I = \cancel{2}\int_0^{\infty} e^{-t} \cdot \frac{1}{\cancel{2}a} \cdot \frac{dt}{\sqrt{t}} \cdot \sqrt{a}$$

$$\Rightarrow \frac{1}{\sqrt{a}} \int_0^{\infty} e^{-t} \cdot t^{-1/2} \cdot dt = \frac{1}{\sqrt{a}} \int_0^{\infty} e^{-t}\, t^{\frac{1}{2}-1}\, dt$$

$$\Rightarrow \frac{1}{\sqrt{a}} \Gamma\tfrac{1}{2} = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\frac{\pi}{1}} = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$$

Q.35. $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dx\,dy = \int_0^{\infty} e^{-x^2} dx \cdot \int_0^{\infty} e^{-y^2} dy$

$$= \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} = \pi/4$$

Q. $\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2+z^2)} dx\,dy\,dz = \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} = \frac{\pi\sqrt{\pi}}{8}$

Application of Integrations:

* Length of curves :

① If $y = f(x)$, $a \le x \le b$

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

② If $x = f(y)$; $c \le y \le d$

$$L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

③ If $x = x(t)$, $y = y(t)$, $t_1 < t < t_2$

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

④ If $r = f(\theta)$, $\theta_1 < \theta < \theta_2$

$$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Q.60. Find length of $r = a(1+\cos\theta)$; $\theta : 0$ to π.

$$L = \int_0^{\pi} \sqrt{a^2(1+\cos\theta)^2 + (-a\sin\theta)^2}\, d\theta$$

$$= a\int_0^{\pi} \sqrt{2+2\cos\theta}\, d\theta$$

$$\Rightarrow \sqrt{2}\,a \int_0^{\pi} \sqrt{2\cos^2\theta/2}\, d\theta = 2a\int_0^{\pi} \cos\theta/2\, d\theta$$

$$\Rightarrow 2a\left[\frac{\sin\theta/2}{1/2}\right]_0^{\pi} = 4a$$

Q. Find length of 3D spatial curve $x(t) = \cos t$, $y(t) = \sin t$; $z(t) = \frac{2}{\pi} t$; $0 < t < \pi/2$?

$$L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_0^{\pi/2} \sqrt{(-\sin t)^2 + (\cos t)^2 + \left(\frac{2}{\pi}\right)^2}\, dt$$

$$= \sqrt{1+\frac{4}{\pi^2}}\, [t]_0^{\pi/2} = \frac{\pi}{2}\sqrt{\frac{\pi^2+4}{\pi^2}} = \frac{\sqrt{\pi^2+4}}{2}$$

Volume of solid of revolutions:

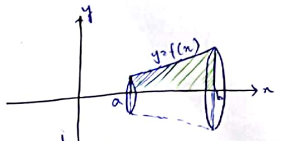
→ Volume of solid of revolutions of area bounded by curve $y=f(x)$ and lines $x=a$ and b about x-axis, given by,

$$\boxed{V = \int_a^b \pi y^2\, dx}$$

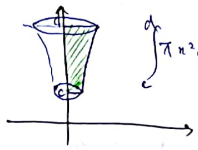
→ Volume of solid of revolutions of area bounded by curve $x=f(y)$ and lines $y=c$ & d about y-axis is given by:

$$\boxed{V = \int_c^d \pi x^2\, dy.}$$

Q. Find the length of the curve $y = \log \cos x$; $0 \le x \le \pi/4$
from Q①.

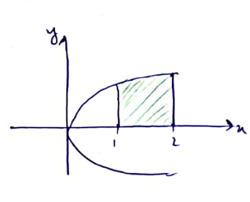


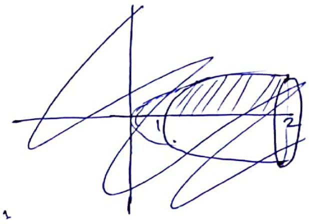
$\int_a^b \pi y^2 dx$



$\int_c^d \pi x^2 dy$

Q. Find the vol of solid of revolution of the arc $y=\sqrt{x}$, $1 \le x \le 2$ about x-axis.





$$\int_1^2 \pi y^2\, dx = \pi \int_1^2 x\, dx = \pi\left[\frac{x^2}{2}\right]_1^2$$

$\frac{3\pi}{2}$

③ If $x = x(t)$, $y = y(t)$, $t_1 < t < t_2$

about x-axis

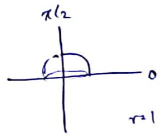
$$V = \int_{t_1}^{t_2} \pi y^2 \frac{dx}{dt}\, dt$$

about y-axis

$$V = \int_{t_1}^{t_2} \pi x^2 \frac{dy}{dt}\, dt$$

Q. Find the volume of the solid of revolution. $x^2+y^2=1$
$x = \cos t$ $y = \sin t$, $0 \le t \le \pi/2$ about y-axis

$\int_0^{\pi/2} \pi.$



$r=1$

$= \frac{2\pi}{3} r^3 = \frac{2\pi}{3}$