3.3 Markov Process

Markov Process is a random process in which future behaviour of the process depends only on the current state, and not on the states in the past.

Mathematically, A random process $\{X(t), t \in t\}$ is called a Markov process. If

$$P[X(t_{n+1}) = X_{n+1} \,|X(t_n) = X_n\,, X(t_{n-1}) = X_{n-1}\,, \dots, X(t_0) = X_0 \quad \dots (1)$$

$$= P[X(t_{n+1})] = X_{n+1}/X(t_n) = X_n$$

Whenever $t_0 < t_1 < \dots < t_n < t_{n+1}. X_0, X_1, X_2, \dots, X_n, \dots$ are called the states of the process.

Give some examples of Markov process:

Some examples of Markov processes are described below.

(i) Any random process with independent increments.

(ii) Board games played with dice like snakes and ladders etc.

(iii) Weather prediction models.

Markov Chain:

A discrete state Markov process is called a Markov chain.

Markov chain is defined as a set of random variables $\{X_n, n \geq 0\}$ with the Markov property, that, given the present state, the future and past states are independent, i.e., If all values of n

$P[X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \dots, X_0 = a_0]$

$$= P[X_n = a_n / X_{n-1} = a_{n-1}] \quad ...(2)$$

Then the process X_n is called Markov chain $a_0, a_1, a_2, \dots, a_n$ are called states of Markov chain.

One step Transition Probability:

The conditional $P(X_n = a_j / X_{n-1} = a_i)$ is called the one step transition probability from state a_i to state a_j at the nth step and is denoted by

$p_{ij}(n-1, n)$

N – step Transition probability:

The conditional $P(X_n = a_j / X_0 = a_i)$ is called the n step transition probability and is denoted by $p_{ij}{}^{(n)}$

Homogeneous Markov Chain:

If the one – step transition probability is independent of n,

i.e., $P_{ij}(n, n+1) = P_{ij}(m, m+1)$

then the Markov chain is said to have stationary transition probabilities and the process is called as a homogeneous Markov chain. Otherwise, the process is known as non – homogeneous Markov chain.

Transition Probability Matrix (TPM):

Let $\{X_n, n \geq 0\}$ be a homogeneous Markov chain. Then the one – step transition probability from state i to state j is denoted by

$$p_{ij} = P[X_{n+1} = j/X_n = i] \; 1 \leq i \leq m, 1 \leq j \leq m$$

is called the transition probability matrix (TPM) of the process if

(i) $p_{ij} \geq 0$

(ii) $\sum_{j=1}^{m} p_{ij} = 1 \; for \; i = 1,2, \dots, m$ (i.e., row total = 1)

Regular Matrix:

A Stochastic matrix P is said to be a regular matrix if all the entries of $p^m > 0$ for some positive integer m.

A homogeneous Markov chain is said to be regular if its TPM is regular.

Steady State Distribution:

If a homogeneous Markov chain is regular, then every sequence of state probability distributions approaches a unique fixed distribution is called the steady state distribution of the Markov chain.

$$\lim_{n\to\infty} [P^n] = \pi$$

Where $P^{(n)} = \left[p_1{}^{(n)}, p_2{}^{(n)}, \dots, p_k{}^{(n)}\right]$ and $\pi = (\pi_1, \pi_2, \dots, \pi_k)$

Condition for Steady State Distribution:

If P is the TPM of the regular Markov chain and $\pi = (\pi_1, \pi_2, \dots, \pi_k)$ is the steady state distribution, then $\pi P = \pi$ and $\pi_1 + \pi_2 + \cdots + \pi_k = 1$

Chapman – Kolmogrov Theorem:

Let $\{X_n, n \geq 0\}$ be a homogeneous Markov chain with transition probability matrix $P = [p_{ij}]$ and n – step transition probability matrix $P^{(n)} = [p_{ij}{}^{(n)}]$ where

$p_{ij}{}^{(n)} = P\{X_n = j / X_0 = i\}$ and $p_{ij}{}^{(1)} = p_{ij}$

Then the following properties hold:

a) $P^{(n+m)} = P^{(n)}\, P^{(m)}$

b) $P^{(n)} = P^n$

Type: 1

To find the steady state distribution of the chain

Steady state or invariant or stationary or limiting state or long run or 1000th trial or $\lim_{n \to \infty} P^{(n)}$ these are represents the same π.

If given is 2 × 2 matrix

$\pi = (\pi_1, \pi_2)$

$\pi P = \pi \quad ...(1)$

$\pi_1 + \pi_2 = 1 \quad ... \ (2)$

If given is 3 × 3 matrix

$\pi = (\pi_1, \pi_2, \pi_3)$

$\pi P = \pi \quad ...(1)$

$\pi_1 + \pi_2 + \pi_3 = 1 \quad ... \ (2)$

Problems Under Steady State:

1. **A college student has the following study habits. If he studies one night, then 70% sure not to study in the next night. If he does not study one night, then he is only 60% sure not to study in the next night also. (i) Find the TPM (ii) How often he studies in the long run?**

Solution:

Let the state space = $\{study, not\ study\}$

The transition probability matrix (TPM) = $P = \begin{pmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{pmatrix}$

Let the steady state distribution is $\pi = (\pi_0, \pi_1)$

$\pi_0 + \pi_1 = 1 \quad ... \ (1)$

Condition for steady state $\pi P = \pi$

$$(\pi_0, \pi_1)\begin{pmatrix}0.3 & 0.7\\ 0.4 & 0.6\end{pmatrix} = (\pi_0, \pi_1)$$

$0.3\,\pi_0 + 0.4\,\pi_1 = \pi_0 \quad ...(2)$

$0.7\,\pi_0 + 0.6\,\pi_1 = \pi_1 \quad ...(3)$

$(2) \Rightarrow 0.3\,\pi_0 - \pi_0 = -0.4\,\pi_1$

$\Rightarrow -0.70\,\pi_0 = -0.4\,\pi_1$

$\Rightarrow \pi_0 = \frac{0.4}{0.70}\pi_1 \quad ...(4)$

Sub (4) in equation (1)

$\Rightarrow \frac{0.4}{0.70}\pi_1 + \pi_1 = 1$

$\Rightarrow \pi_1\left(\frac{0.4}{0.70} + 1\right) = 1$

$\Rightarrow \pi_1 = \frac{7}{11}$

$(4) \Rightarrow \pi_0 = \frac{0.4}{0.70} \times \frac{7}{11} = \frac{4}{11}$

The steady state distribution is $(\pi_0, \pi_1) = \left(\frac{4}{11}, \frac{7}{11}\right)$

Probability of he studies in the long run $\pi_0 = \frac{4}{11}$

2. **A salesman territory consists of 3 cities A, B and C. He never sells in the same city on successive days. If he sells in city A, then next day**

sells in city B. However if he sell in either B or C, then the next day he is twice as likely to sell in the city A as in the other city. How often does he sell in each of cities in the steady state.

Solution:

Let the state space = $\{A, B, C\}$

The transition probability matrix (TPM) = $P = \begin{pmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}$

Let the steady state distribution is $\pi = (\pi_1, \pi_2, \pi_3)$

$\pi_1 + \pi_2 + \pi_3 = 1 \quad ... \text{ (1)}$

Condition for steady state $\pi P = \pi$

$$(\pi_1, \pi_2, \pi_3)\begin{pmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{pmatrix} = (\pi_1, \pi_2, \pi_3)$$

$\frac{2}{3}\pi_2 + \frac{2}{3}\pi_3 = \pi_1 \quad ... \text{(2)}$

$\pi_1 + \frac{1}{3}\pi_3 = \pi_2 \quad ... \text{(3)}$

$\frac{1}{3}\pi_2 = \pi_3 \quad ... \text{(4)} \Rightarrow \pi_2 = 3\pi_3$

$(2) \Rightarrow \frac{2}{3}\pi_2 + \frac{2}{3}\pi_3 = \pi_1$

$\Rightarrow \frac{2}{3}(\pi_2 + \pi_3) = \pi_1$

$$\Rightarrow (\pi_2 + \pi_3) = \frac{3}{2}\pi_1$$

$$(1) \Rightarrow \pi_1 + \frac{3}{2}\pi_1 = 1$$

$$\Rightarrow \frac{5}{2}\pi_1 = 1$$

$$\Rightarrow \pi_1 = \frac{2}{5}$$

$$\Rightarrow \frac{2}{5} + \frac{1}{3}\pi_3 = 3\pi_3$$

$$\Rightarrow \frac{2}{5} = 3\pi_3 - \frac{1}{3}\pi_3$$

$$\Rightarrow \frac{2}{5} = \frac{8}{3}\pi_3 \Rightarrow \pi_3 = \frac{2}{5} \times \frac{3}{8}$$

$$\Rightarrow \pi_3 = \frac{3}{20}$$

$$(4) \Rightarrow \pi_2 = 3 \times \frac{3}{20} = \frac{9}{20}$$

3. An Engineer analysing a series of digital signals generated by a testing system observes that only 1 out of 15 highly distorted signals follows a highly distorted signals, with no recognizable signal between, whereas 20 out of 23 recognizable signals, with no highly distorted signal between. Given that any highly distorted signals are not recognizable, find the TPM and fraction of signals that are highly distorted.

Solution:

Let the state space = $\{recognizable\ signal, distorted\ signal\}$

The transition probability matrix (TPM) = $P = \begin{pmatrix} 20/23 & 3/23 \\ 14/15 & 1/15 \end{pmatrix}$

Let the steady state distribution is $\pi = (\pi_1, \pi_2)$

$\pi_1 + \pi_2 = 1 \quad ...\ (1)$

Condition for steady state $\pi P = \pi$

$$(\pi_1, \pi_2)\begin{pmatrix} 20/23 & 3/23 \\ 14/15 & 1/15 \end{pmatrix} = (\pi_1, \pi_2)$$

$\frac{20}{23}\pi_1 + \frac{14}{15}\pi_2 = \pi_1 \quad ...\ (2)$

$\frac{3}{23}\pi_1 + \frac{1}{15}\pi_2 = \pi_2 \quad ...\ (3)$

$(3) \Rightarrow \frac{3}{23}\pi_1 = \pi_2 - \frac{1}{15}\pi_2$

$\Rightarrow \frac{3}{23}\pi_1 = \frac{14}{15}\pi_2$

$\Rightarrow \pi_1 = \frac{14}{15} \times \frac{23}{3}\pi_2$

$\Rightarrow \pi_1 = \frac{322}{45}\pi_2 \quad ...\ (4)$

$(1) \Rightarrow \pi_1 + \pi_2 = 1$

$\Rightarrow \frac{322}{45}\pi_2 + \pi_2 = 1$

$$\Rightarrow \frac{322}{45}\pi_2 + \pi_2 = 1$$

$$\Rightarrow \left(\frac{322}{45} + 1\right)\pi_2 = 1$$

$$\Rightarrow \left(\frac{367}{45}\right)\pi_2 = 1$$

$$\Rightarrow \pi_2 = \frac{45}{367}$$

$$(4) \Rightarrow \pi_1 = \frac{322}{45} \times \frac{45}{367}$$

$$\Rightarrow \pi_1 = \frac{322}{367}$$

The fraction of signals that are highly distorted = $\pi_2 = \frac{45}{367}$

Type: 2

To find the probability distribution based on the initial distribution

1. Suppose that the probability of a dry day following a rainy day is 1/3 and that the probability of a rainy day following a dry day is ½. Given May 1 is a dry day, find the probability that (i) May 3 is also a dry day (ii) May 5 is also a dry day.

Solution:

Let the state space = $\{dry\ day, rainy\ day\}$

The transition probability matrix (TPM) = $P = \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$

Since initial probability distribution of May 1 is given by

$$\therefore\ P^{(1)} = [1 \quad 0]$$

The probability distribution of the May 2 is given by

$$P^{(2)} = P^{(1)}P = (1 \quad 0)\begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

$$= (1/2 \quad 1/2)$$

The probability distribution of the May 3 is given by

$$P^{(3)} = P^{(2)}P = (1/2 \quad 1/2)\begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

$$= (5/12 \quad 7/12)$$

P(May 3 dry day) = $\frac{5}{12}$

The probability distribution of the May 4 is given by

$$P^{(4)} = P^{(3)}P = (5/12 \quad 7/12)\begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

$$= (29/72 \quad 43/72)$$

The probability distribution of the May 5 is given by

$$P^{(5)} = P^{(4)}P = (29/72 \quad 43/72)\begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

$$= (173/432 \quad 259/432)$$

P(May 5 dry day) = $\frac{173}{432}$

2. A welding process is considered to be a two state Markov chain, where the state 0 denotes the process is running in the manufacturing firm and the state 1 denotes the process is not running in the firm. Suppose that the transition probability matrix for this Markov chain is given by $P = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$ (i) Find the probability that the welding process will run on the third day from today given that the welding process is running today.

(ii) Find the probability that the welding process will run on the third day from today if the initial probability of state 0 and 1 are equally likely.

Solution:

Let the state space = $\{run, not\ run\}$

The transition probability matrix (TPM) = $P = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$

The initial probability distribution is given by

$$\therefore\ P^{(0)} = [1 \quad 0]$$

The probability distribution of the first day is given by

$$P^{(1)} = P^{(0)}P = (1 \quad 0)\begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

$$= (0.8 \quad 0.2)$$

The probability distribution on the second day is given by

$$P^{(2)} = P^{(1)}P = (0.8 \quad 0.2)\begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

$$= (0.7 \quad 0.3)$$

The probability distribution on the third day is given by

$$P^{(3)} = P^{(2)}P = (0.7 \quad 0.3)\begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

$$= (0.65 \quad 0.35)$$

P(welding process is running on the third day) = 0.65

(ii)Given initial probabilities of 0 and 1 are equally likely

P(running today) = P(not running today) = ½

The initial probability distribution is given by

$$\therefore\ P^{(0)} = [1/2 \quad 1/2]$$

The probability distribution of the first day is given by

$$P^{(1)} = P^{(0)}P = (1/2 \quad 1/2)\begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

$$= (0.55 \quad 0.45)$$

The probability distribution on the second day is given by

$$P^{(2)} = P^{(1)}P = \begin{pmatrix} 0.55 & 0.45 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

$$= \begin{pmatrix} 0.575 & 0.425 \end{pmatrix}$$

The probability distribution on the third day is given by

$$P^{(3)} = P^{(2)}P = \begin{pmatrix} 0.575 & 0.425 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

$$= \begin{pmatrix} 0.5875 & 0.4125 \end{pmatrix}$$

P(welding process is running on the third day) = 0.5875

Problems based on Type 1 and Type 2:

1. A man either drives a car or catches a train to go office each day. He never goes two days in a row by train. But he drives one day, then the next day he is just as likely to drive again as he used to travel like train. Suppose that on the first day of the week the man tosses a fair die and drove to work if and only if a “6” appears. Find the probability that he takes a train on the third day. The probability that he drives to work in the long run.

Solution:

Let the state space = $\{train, car\}$

The transition probability matrix (TPM) = $P = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$

(i) P(first day he drives the car) = P (getting 6 on a die) = $\frac{1}{6}$ (Given)

P(first day he catches the train) = $1 - \frac{1}{6} = \frac{5}{6}$

The Probability distribution of the first day is given by

$$P^{(1)} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix}$$

The probability distribution of the second day is given by

$$P^{(2)} = P^{(1)}P = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{12} & \frac{11}{12} \end{pmatrix}$$

The probability distribution of the third day is given by

$$P^{(3)} = P^{(2)}P = \begin{pmatrix} \frac{1}{12} & \frac{11}{12} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{11}{24} & \frac{13}{24} \end{pmatrix}$$

P(third day he catches the train) = $\frac{11}{24}$

(ii)Let the steady state distribution is $\pi = (\pi_1, \pi_2)$

$\pi_1 + \pi_2 = 1 \quad ... \ (1)$

Condition for steady state $\pi P = \pi$

$$(\pi_1, \pi_2) \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (\pi_1, \pi_2)$$

$\frac{1}{2}\pi_2 = \pi_1 \quad ...(2) \quad \Rightarrow \pi_2 = 2\pi_1$

$\pi_1 + \frac{1}{2}\pi_2 = \pi_2 \quad ...(3)$

$(1) \Rightarrow \pi_1 + 2\pi_1 = 1$

$\Rightarrow 3\pi_1 = 1$

$\Rightarrow \pi_1 = \frac{1}{3}$

$(2) \quad \Rightarrow \pi_2 = 2 \times \frac{1}{3} = \frac{2}{3}$

P(he drives to work in the long run) = $\frac{2}{3}$

2. A fair die is tossed repeatedly. If X_n denotes the maximum numbers occurring in the first n tosses, find TPM of the Markov chain. Find also P^2 and $P(X_2 = 6)$.

Solution:

Let the State space = {1, 2, 3, 4, 5, 6}

X_n denotes maximum of the numbers occurring in the first n tosses

The TPM is $P = X_n \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 6/6 \end{bmatrix}$

$$= \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$P^2 = P.P$

$$P^2 = \frac{1}{36}\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{bmatrix}$$

Since initial probability is not given assume all are equally likely

$$\therefore\ P^{(0)} = [1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6]$$

$P[X_2 = 6] = \sum_{i=1}^{6} P(X_2 = 6)\ /\ X_0 = i)P(X = i)$

$$= P(X_2 = 6/X_0 = 1)P(X_0 = 1) + P(X_2 = 6/X_0 = 2)P(X_0 = 2)$$

$$+ P(X_2 = 6/X_0 = 3)P(X_0 = 3) + P(X_2 = 6/X_0 = 4)P(X_0 = 4)$$

$$+ P(X_2 = 6/X_0 = 5)P(X_0 = 5) + P(X_2 = 6/X_0 = 6)P(X_0 = 6)$$

$$= p_{16}^2\, P(X_0 = 1) + p_{26}^2\, P(X_0 = 2) + p_{36}^2\, P(X_0 = 3)$$

$$+ p_{46}^2\, P(X_0 = 4) + p_{56}^2\, P(X_0 = 5) + p_{66}^2\, P(X_0 = 6)$$

$$= \frac{11}{36} \times \frac{1}{6} \times \frac{11}{36} \times \frac{1}{6} \times \frac{11}{36} \times \frac{1}{6} \times \frac{11}{36} \times \frac{1}{6} \times \frac{11}{36} \times \frac{1}{6} \times \frac{36}{36} \times \frac{1}{6}$$

$P[X_2 = 6] = \frac{91}{216}$

3. A gambler has Rs. 2/ -. He bets Rs. 1/- at a time and wins Rs.1/-with probability $\frac{1}{2}$. He stops playing, if he loses Rs. 2/- or he wins Rs.4/- Write down the tpm of the associated Markov chain. What is the probability that he lost his money at the end of his 5th play. What is the probability that the game lasts more than 7 plays?

Solution:

Let X_n represents the amount with the player at the end of the n^{th} round of the play.

Initially he has Rs. 2. He bets Rs. 1 at a time. The game ends when he loses Rs. 2, [i.e., he has $2 - 2 = 0$] or wins Rs. 4 [i.e., $2 + 4 = 6$]

State space of $\{X_n\} = (0,1,2,3,4,5,6)$.

If he has Rs, 0, then he can't play $\therefore p_{00} = 1$

If he has Rs. 6 , then the play ends $\therefore p_{66} = 1$

If he has Rs. 1 , then he get rupees 2 with probability $\frac{1}{2}$ is $p_{12} = \frac{1}{2}$

If he loses Rs. 1 , then he has Rs. 0 i.e $p_{10} = \frac{1}{2}$ and so on.

The TPM of the Markov chain is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Since initially he has 2 Rs., $p^{(0)}(2) = 1$

lnitial probability distribution of $\{X_n\}$ is $P^{(0)} = (0,0,1,0,0,0,0)$

The probability distribution at the first round is given by

$$p^{(1)} = P^{(0)}P = P^{(1)} = \left(0,\frac{1}{2},0,\frac{1}{2},0,0,0\right)$$

The probability distribution at the second round is given by

$$P^{(2)} = P^{(1)}P = \left(\frac{1}{4},0,\frac{1}{2},0,\frac{1}{4},0,0\right)$$

The probability distribution at the third round is given by

$$P^{(3)} = P^{(2)}P = \left(\frac{1}{4},\frac{1}{4},0,\frac{3}{8},0,\frac{1}{8},0\right)$$

The probability distribution at the fourth round is given by

$$P^{(4)} = P^{(3)}P = \left(\frac{3}{8},0,\frac{5}{16},0,\frac{1}{4},0,\frac{1}{16}\right)$$

The probability distribution at the fifth round is given by

P(5)=$P^{(4)}P$

$$=\left(\frac{3}{8},\ \frac{5}{32},0,\frac{9}{32},0,\frac{1}{8},\frac{1}{16}\right)$$

P [the man has lost his money at the end of his 5^{th} play] $= P^5(0) = \frac{3}{8}$.

The probability distribution at the sixth round is given by

$$P^{(6)} = P^{(5)}P = \left(\frac{29}{64},\ 0,\ \frac{7}{32},0,\frac{13}{64},0,\frac{1}{8}\right)$$

The probability distribution at the seventh round is given by

$$P^{(7)} = P^{(6)}p$$

$$=\left(\frac{29}{64},\frac{7}{64},0,\frac{27}{128},0,\frac{13}{128},\frac{1}{8}\right)$$

P (the game lasts more than 7 rounds)

$= P$ (the system is neither in state 0 nor in 6 at the end of the 7^{th} round)

$$= P^7(1) + P^7(2) + P^7(3) + P^7(4) + P^7(5)$$

$$= \frac{7}{64} + 0 + \frac{27}{128} + 0 + \frac{13}{128}$$

$$= \frac{27}{64}$$

Problem under 'n' Step Probability

(1) $P[X_n = j/X_m = i] = {p_{ij}}^{n-m}$

(2) $P_j^{(n)} = P[X_n = j] = \sum_{i=0}^{k} P[X_n = j/X_0 = i]P[X_0 = i]$

1. The transition probability matrix (TPM) of a Markov chain X_n has three states 1,2 and 3 is $P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$ and initial distribution is $P^{(0)} = (0.7 \quad 0.2 \quad 0.1)$.

Find (i) $P[X_3 = 2,\ X_2 = 3,\ X_1 = 3,\ X_0 = 2]$ (ii) $P[X_3 = 1\ /X_2 = 3\ X_0 = 2]$ (iii) $P[X_2 = 3]$

Solution:

State space = {1,2,3}

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \end{array}$$

Given initial probability distribution is

$$P^{(0)} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ [0.7 \quad 0.2 \quad 0.1] \end{array}$$

$\therefore P(X_0 = 1) = 0.7, P(X_0 = 2) = 0.2, P(X_0 = 3) = 0.1$

(i) $P[X_3 = 2,\ X_2 = 3,\ X_1 = 3,\ X_0 = 2]$

$$= P[X_3 = 2/X_2 = 3]P[X_2 = 3/X_1 = 3]P[X_1 = 3/X_0 = 2]P[X_0 = 2]$$

$$= p_{32}^{3-2} p_{33}^{2-1} p_{23}^{1-0} P[X_0 = 2]$$

$$= p_{32}^{1} p_{33}^{1} p_{23}^{1} P[X_0 = 2] = (0.4)(0.3)(0.2)(0.2)$$

$$= 0.0048$$

(ii)$P\ [X_3 = 1/X_2 = 3\ X_0 = 2]\ = P[X_3 = 1/X_2 = 3]$[By property]

$$= P_{31}^{3-2} = P_{31}^{1}$$

$$= 0.3$$

(iii) $P(X_2 = 3) = \sum_{i=1}^{3} P(X_2 = 3)/X_0 = i\big)P(X_0 = i)$

$$= P(X_2 = 3/X_0 = 1)P(X_0 = 1) + P(X_2 = 3/X_0 = 2)P(X_0 = 2)$$

$$+P(X_2 = 3/X_0 = 3)P(X_0 = 3)$$

$P(X_2 = 3) = p_{13}^{2} P(X_0 = 1) + p_{23}^{2} P(X_0 = 2) + p_{33}^{2} P(X_0 = 3)$ ... (1)

To compute $p_{13}^{2}, p_{23}^{2} p_{33}^{2}$, find P^2 :

$$P^2 = P \times P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$$P^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix} \end{array}$$

$(1) \Rightarrow P[X_2 = 3] = 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1 = 0.279$

$P[X_2 = 3] = 0.279$

2. The TPM 'P' of the Markov chain with 3 states $(\mathbf{0}, \mathbf{1}, \mathbf{2})$ is

$\boldsymbol{P} \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & \mathbf{0} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \mathbf{0} & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$ **and initially state distribution of the chain is**

$\boldsymbol{P(X_0 = i) = \frac{1}{3}; i = 0, 1, 2.}$ **Find (i)** $\boldsymbol{P[X_2 = 2] (ii) P[X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2]}$

Solution:

State space is {0,1,2}

$$P = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$$

The initial probability distribution is

$$P(X_0 = i) = \frac{1}{3}; i = 0,1,2$$

$$\therefore P(X_0 = 0) = \frac{1}{3}, P(X_0 = 1) = \frac{1}{3}, P(X_0 = 2) = \frac{1}{3}$$

(ii) $P[X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2]$

$P(X_1 = 1/X_2 = 2)P(X_2 = 2/X_1 = 1)P(X_1 = 1/X_0 = 2)P(X_0 = 2)$

$$= p_{21}^{3-2} p_{12}^{2-1} p_{21}^{1-0} P(X_0 = 2)$$

$$= p_{21}^{1} p_{2}^{1} p_{21}^{1} P(X_0 = 2)$$

$$= \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{3}\right)$$

$P(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2) = 3/64$

(i) $P(X_2 = 2) = \sum_{i=0}^{2} P(X_2 = 2)/X_0 = i\big) P(X_0 = i)$

$$= P(X_2 = 2/X_0 = 0)P(X_0 = 0) + P(X_2 = 2/X_0 = 1)P(X_0 = 1)$$

$$+ P(X_2 = 2/X_0 = 2)P(X_0 = 2)$$

$$= p_{02}^{2} P(X_0 = 0) + p_{12}^{2} P(X_0 = 1) + p_{22}^{2} P(X_0 = 2) \dots\dots\dots (1)$$

To compute $p_{02}^2, p_{12}^2, p_{22}^2$

Find P^2:

$$P^2 = \begin{pmatrix} \frac{10}{16} & \frac{5}{16} & \frac{1}{16} \\ \frac{5}{16} & \frac{8}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{9}{16} & \frac{4}{16} \end{pmatrix}$$

(1) => $P(X_2 = 2) = \frac{1}{16}\frac{1}{3} + \frac{3}{16}\frac{1}{3} + \frac{4}{16}\frac{1}{3}$

$$= \frac{1}{3}\left(\frac{8}{16}\right) = \frac{1}{6}$$

Classification of Markov Chain

State diagram:

The pictorial representation of a Markov chain is called state diagram.

Accessibility:

Suppose that the state j has the property that can be reached from any state I, then j is said to be accessible from i.

i.e., $p_{ij}{}^{(n)} > 0, \forall n > 0$

Communication:

If two states i and j are accessible from each other, then they are said to be communicative with each other.

Irreducible Markov chain:

If it is possible to reach one state to another, then the Markov chain is irreducible.

i.e., All the states communicate with each other.

Transient state:

A state is transient if and only if there is a positive probability that the process will not return to this state.

Absorbing state:

A state i is called an absorbing state if $p_{ii} = 1$

Note:

All the absorbing states of a Markov chain is recurrent.

Ergodic State:

A non – null persistent and aperiodic state is called an ergodic state.

i.e., all its states are positive recurrent and aperiodic. A Markov chain is said to be ergodic, if all the states are ergodic.

An irreducible non – null persistent Markov chain is ergodic.