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- Young's double slit experiment

In quantum theory state could be represented as column vectors.

Stern Gerlach experiment

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Vector space $\quad V = \{V_1, V_2, V_3 \dots\}$

Rule of Addition (+)

Rule of multiplication (·)

Properties

i) $V_l + V_m = V_m + V_l \in V$

ii) $V_l + (V_m + V_k) = (V_l + V_m) + V_k$

iii) $0 + V_m = V_m + 0 = V_m$

Null Vector

$\exists\, -V_m \in V$ for every V_m

iv) $V_m + (-V_m) = (-V_m) + V_m = 0$

v) $c V_m \in V$ $\qquad c \in$ Complex numbers system.

Ex: $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in ℂ \right\}$

claim: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the null vector.

$+ : \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a+p \\ b+q \end{pmatrix}$ $\quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0+a \\ 0+b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\cdot : \alpha \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix} \in V$ $\quad \alpha \in ℂ$

claim: $\begin{pmatrix} -a \\ -b \end{pmatrix}$ is the additive inverse of $\begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} -a \\ -b \end{pmatrix} = \begin{pmatrix} a+(-a) \\ b+(-b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

A set endowed with these two properties is the vector space.

Ex: $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in ℂ \right\}$

$+ : \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a+q \\ b+p \end{pmatrix}$

$\cdot : \alpha \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix}$

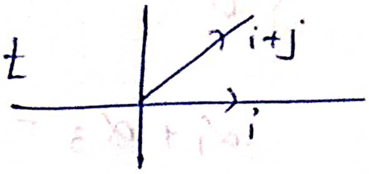
Is this a vector space?

Linear Independence

We cannot write $\hat{i}$ in terms of $\hat{j}$

When we cannot express one vector as constant multiplied with another vector, then we say that those two vectors are linearly independent.

$\hat{i}, \hat{i}+\hat{j}$ are linearly independent



Given n non zero vectors

$v_1, v_2, \ldots v_n$

if $\alpha v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$

$\Rightarrow$ if $\alpha_1 = 0, \alpha_2 = 0, \ldots \alpha_n = 0$

is the ONLY solution then

$\{v_1, v_2, \ldots v_n\}$ is a linearly independent set.

Ex: Are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ LI ?

$$\alpha_1\begin{pmatrix}1\\0\end{pmatrix} + \alpha_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}\alpha_1\\0\end{pmatrix} + \begin{pmatrix}0\\\alpha_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$\alpha_1 = 0, \alpha_2 = 0$

$\therefore$ set is linearly independent.

Ex $\begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix} \begin{pmatrix}1\\1\end{pmatrix}$

$$\alpha_1\begin{pmatrix}1\\0\end{pmatrix} + \alpha_2\begin{pmatrix}0\\1\end{pmatrix} + \alpha_3\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}\alpha_1\\0\end{pmatrix} + \begin{pmatrix}0\\\alpha_2\end{pmatrix} + \begin{pmatrix}\alpha_3\\\alpha_3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$\alpha_1 + \alpha_3 = 0$

$\alpha_2 + \alpha_3 = 0$

$(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$ satisfies but

$(1, 1, -1)$, $(2, 2, -2)$ . . . . . also satisfies

∴ These is not linearly independent

Ex $\begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix} \begin{pmatrix}1\\1\end{pmatrix} \begin{pmatrix}1\\-1\end{pmatrix}$

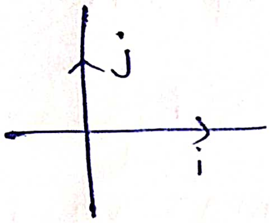
$\alpha_1 + \alpha_3 + \alpha_4 = 0$

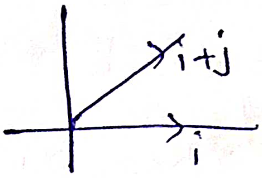
$\alpha_2 + \alpha_3 - \alpha_4 = 0$

If we have a subset which is not linearly independent then we can say that set is not linearly independent.

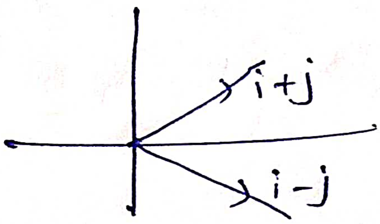
Dimension:

The maximal number of LI vectors in a vector space is the dimension of the space.





orthogonal vectors are always linearly independent



Ex of LI sets

$\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ $\quad$ $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}$

$\left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}$ $\quad$ $\left\{ \begin{pmatrix}\pi\\e\end{pmatrix}, \begin{pmatrix}\sqrt{2}\\-8\end{pmatrix} \right\}$

If we have a maximal LI set then every vector can be uniquely expressed as a linear sum of the vectors from the LI set.

$$\begin{pmatrix}x\\y\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}$$

$a = x,\ b = y$ only possibility.

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$$\vec{A} = a_1\hat{i} + a_2\hat{j}$$

$$|\vec{A}| = \sqrt{a_1^2 + a_2^2}$$

$$|\vec{A}|^2 = a_1^2 + a_2^2$$

Complex numbers

$$c = a + ib$$

$$c^* = a - ib$$

$$cc^* = a^2 + b^2 = |c|^2$$

$$\mathbb{C}^2 \quad \vec{A} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Column vector (n) dimensions

$$\mathbb{C}^n \quad |A\rangle = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Ket vector

$$|v\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Dual $\langle v| = [c_1^* \quad c_2^*]$

Inner product

$$\langle v|v\rangle = [c_1^* \; c_2^*] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1^* c_1 + c_2^* c_2$$

For diff vectors Inner product

$$|v\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$|w\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\langle w|v\rangle = \begin{bmatrix} a_1^* & a_2^* \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\mathbb{C}^n$ n dimensional complex vector space.

$$|v\rangle = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Addition :

$$|v_1\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$|v_2\rangle = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$|v\rangle = |v_1\rangle + |v_2\rangle = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \end{bmatrix}$$

Multiplication by a scalar

$$|v_1\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Multiply with $\frac{1}{\sqrt{2}}$

$$|v_1\rangle = \begin{bmatrix} a_1/\sqrt{2} \\ a_2/\sqrt{2} \end{bmatrix}$$

state of the system ψ and ϕ.

$|\psi\rangle$ or $|\phi\rangle$

Zero vector $|0\rangle$ 0

$$|\psi\rangle + 0 = |\psi\rangle$$

Tensor product

$$|\psi\rangle = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}_{2\times 1} \qquad |\phi\rangle = \begin{bmatrix} 3 \\ 2+i \end{bmatrix}_{2\times 1}$$

$$\langle\psi\rangle \otimes |\phi\rangle = \begin{bmatrix} (1+i)3 \\ (1+i)(2+i) \\ 2\times 3 \\ 2(2+i \end{bmatrix}_{4\times 1}$$

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{2\times 2}$$

$$A\otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$$

Complex conjugate of a matrix

Replace i with $-i$

$$A = \begin{bmatrix} 1 & 6i \\ 3i & 2+4i \end{bmatrix}$$

$$A^* = \begin{bmatrix} 1 & -6i \\ -3i & 2-4i \end{bmatrix}$$

Adjoint/ Hermition conjugate (A digar)

$$A^{+} = (A^*)^T = (A^T)^*$$

Norm of a vector $\| |\psi\rangle \|$

$$\| |\psi\rangle \| = [\langle\psi|\psi\rangle]^{1/2}$$

Making a unit vector

$$|\psi\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Unit vector in the direction $|\psi\rangle$ is (divide $|\psi\rangle$ with $\| |\psi\rangle \|$)

$$= \frac{|\psi\rangle}{\| |\psi\rangle \|}$$

$\underset{\text{operator}}{A}$

Two vectors $|\phi\rangle$

Two vectors $A^{\dagger}|\phi\rangle$ and $|\psi\rangle$

Inner product $\langle\phi| A |\psi\rangle$

A spanning set

$$\vec{A} = a\hat{i} + b\hat{j}$$

Any vector $|v\rangle$ in the vector space can be written as linear combination of two vectors.

$$|v\rangle = \sum_i a_i |v_i\rangle$$

$\mathbb{C}^2$ space

i vector $|v_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ j vector $|v_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\hat{i} = 1\hat{i} + 0\hat{j}$ $\hat{j} = 0\hat{i} + 1\hat{j}$

$$|v\rangle = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 2|v_1\rangle + 3|v_2\rangle$$

$$|v\rangle = \begin{bmatrix} i \\ 2+3i \end{bmatrix} = i\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (2+3i)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$|v_1\rangle$ and $|v_2\rangle$ span $\mathbb{C}^2$ space.

But these are not unique.

$$|v_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad |v_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$|v_1\rangle$ and $|v_2\rangle$ span $\mathbb{C}^2$ space.

$$|v\rangle = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{c_1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{c_2}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{c_1+c_2}{\sqrt{2}} \\ \frac{c_1-c_2}{\sqrt{2}} \end{bmatrix}$$

$$c_1 + c_2 = 2\sqrt{2}$$
$$c_1 - c_2 = 3\sqrt{2}$$

$$2c_1 = 5\sqrt{2}$$

$$c_1 = \frac{5}{\sqrt{2}}$$

$$c_1 + c_2 = 2\sqrt{2}$$

$$c_2 = 2\sqrt{2} - \frac{5}{\sqrt{2}}$$

$$c_2 = -\frac{1}{\sqrt{2}}$$

$|v_1\rangle = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $|v_2\rangle = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$|v_1\rangle$ and $|v_2\rangle$ are not linearly independent.

~~Linea~~

Basis set of V

A set of linearly independent vector which span a vector space V

$\mathbb{C}^2$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. They are linearly independent and span $\mathbb{C}^2$ space.

$\mathbb{C}^3$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$|v\rangle = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_{2\times 1}$$

Operator

$$A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}_{3\times 2}$$

$$A|v\rangle = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}_{3\times 2} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_{2\times 1} = \begin{bmatrix} a_1 v_1 + a_2 v_2 \\ b_1 v_1 + b_2 v_2 \\ c_1 v_1 + c_2 v_2 \end{bmatrix}_{3\times 1}$$

Original vector $|v\rangle$ is in F^2 vector space

Resultant of $A|v\rangle$ is in F^3 vector space.

Linear Operator

$$A : V \longrightarrow W$$

$\downarrow$
Operator

Linear $A\left(\sum a_i |v_i\rangle\right) = \sum_i a_i A|v_i\rangle$

Let $|v_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $|v_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$A|v\rangle = 2A|v_1\rangle + 3A|v_2\rangle$$

$$= 2A\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3A\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Zero operator

$$O|v\rangle = 0$$

* Operators are matrices

Composition of two operators on a vector $|v\rangle$

$$AB|v\rangle = A\{B|v\rangle\}$$

$A \, B_{5\times 2} \, |v\rangle$ $\qquad \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{2\times 1}$

$$A_{10\times 5}\{\underbrace{\overbrace{B_{5\times 2}|v\rangle_{2\times 1}}^{\to K^5}\}_{5\times 1}}_{\to K^{10}}$$

$\to$ $A : V \to W$ $\qquad B : W \to X$

Problem 2.1 Show that

$(1,-1)$, $(1,2)$, and $(2,1)$ are linearly dependent

Problem 2.2 $\quad V$ is vector space

Basis vectors $\quad |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A : V \to V$ such that $A|0\rangle = |1\rangle$

linear operator $\qquad A|1\rangle = |0\rangle$

Find A.

$$A|0\rangle = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times 1} = \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$a = 0, \; c = 1$

$$A|1\rangle = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$b = 1 \quad d = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ Bit flip operator.

Identity operator

$$I|v\rangle = |v\rangle$$

For 2 dimensions $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Try
2.3, 2.4

Pg. NO: 65 → Pauli matrices

Inner product

Two vectors $|v\rangle$ and $\sum_i \lambda_i |w_i\rangle$

$|0\rangle$ $\qquad \overset{\lambda_1}{2}|0\rangle + \overset{\lambda_2}{3}|1\rangle + \overset{\lambda_3}{4}|2\rangle$

Inner product $\sum \lambda_i \langle v | w_i\rangle$

$2\langle 0|0\rangle + 3\langle 0|1\rangle + 4\langle 0|2\rangle$

Suppose $|0\rangle$, $|1\rangle$, $|2\rangle$ are orthonormal

orthogonal & normal.
↓
Inner product = 0

Show that
$$\langle v|w\rangle = (\langle w|v\rangle)^*$$

Inner product of two vectors → Complex.

Let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

LHS
$$\langle v|w\rangle = [v_1^* \ \ v_2^*] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= v_1^* w_1 + v_2^* w_2$$

$$\langle w|v\rangle = [w_1^* \ \ w_2^*] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= w_1^* v_1 + w_2^* v_2$$

RHS $(\langle w|v\rangle)^* = (w_1^* v_1 + w_2^* v_2)^*$

$$= v_1^* w_1 + v_2^* w_2$$

$$(a_1^* b_1 + a_2 b_2^*)^* = a_1 b_1^* + a_2^* b_2$$

$|v\rangle \neq 0$ $\langle v|v\rangle > 0$

Hilbert Space $\equiv$ Inner product space

Vector space with Inner product

Inner product of $\sum_i \lambda_i |w_i\rangle$ and $|v_i\rangle$

$$= \sum_i \lambda_i^* \langle w_i|v\rangle$$

Orthogonal

Two vectors are orthogonal $\rightarrow$ Inner product $= 0$

$$\langle v|w\rangle = 0$$

$$|v\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad |w\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\langle v|w\rangle = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 1 - 1$$

$$= 0$$

$\therefore$ $|v\rangle$ and $|w\rangle$ are othogonal.

Norm

$$\| |v\rangle \| = \sqrt{\langle v|v\rangle}$$

if norm $= 1$ $\rightarrow$ vector is normalised.

Kronicker delta δ_{ij}

A set $\langle i \rangle$ of vectors is orthonormal if each vector is a unit vector and distinct vectors are orthogonal.

$$\langle i | i \rangle = 1 \qquad \langle j | j \rangle = 1$$

$$\langle i | j \rangle = 0 \qquad \langle i | k \rangle = 0$$

$$\langle i | j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$|w_1\rangle, |w_2\rangle \ldots |w_n\rangle$ are basis vectors of V

They are not orthonormal

But with these it is difficult to work.

$|v_1\rangle, |v_2\rangle \ldots |v_n\rangle$ are orthonormal set.

How to convert to orthonormal ?

Gram-Schmidt Procedure

Converts any set of vectors to orthonormal.

$$|w_1\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad |w_2\rangle = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

These are neither orthogonal nor a unit vectors

$\langle \cancel{v}T|$

$$|v_1\rangle = \frac{|w_1\rangle}{\| |w_1\rangle \|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

$$\boxed{|v_{k+1}\rangle = \left(|w_{k+1}\rangle\right) - \sum_{i=1}^{k} \langle v_i|w_{k+1}\rangle\, |v_i\rangle}$$

$$|v_2\rangle = |w_2\rangle - \langle v_1|w_2\rangle\, |v_1\rangle$$

$$= \begin{bmatrix}2\\3\end{bmatrix} - \left(\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}\right)|v_1\rangle$$

$$= \begin{bmatrix}2\\3\end{bmatrix} - \frac{5}{\sqrt{2}}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

$$= \begin{bmatrix}2\\3\end{bmatrix} - \begin{bmatrix}5/2\\5/2\end{bmatrix}$$

$$|v_2\rangle = \begin{bmatrix}-1/2\\1/2\end{bmatrix}$$

$\langle \cancel{v}_1$ $$|v_1\rangle = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} \qquad |v_2\rangle = \begin{bmatrix}-1/2\\1/2\end{bmatrix}$$

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Represent quantum states as vectors:

Ex:

$$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}, \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}$$

Spin half particles: $e^{\ominus}$, proton, neutron etc.

Quantum bit

Any quantum system with two levels/states is a qubit.

Cs have closely spaced levels

SQUID

Super ~~x~~ conducting Quantum Interference Device.

Any physical system having 2 states - qubit

Quantum Gate - transforms qubits

$$NOT \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$NOT \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

Superposition of 2 states of NOT gate

$$NOT\left[\alpha\begin{pmatrix}1\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$= \alpha\ NOT\begin{pmatrix}1\\0\end{pmatrix} + \beta\ NOT\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \alpha\begin{pmatrix}0\\1\end{pmatrix}+\beta\begin{pmatrix}1\\0\end{pmatrix}$$

NOT can be represented as a matrix

Matrix changes vectors to some other vectors

$$NOT \rightarrow \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

Is there a M such that $MM = NOT$?

classically this is not possible to give a NOT gate after two equivalent circuits. But it is possible in quantum.

Linear partial differential equation

Wave eqn

Diffusion eqn

Laplace eqn

$\nabla^2 \phi = \rho$ (Poisson eqn)

$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial t^2}$ (Wave eqn)

$\frac{\partial c}{\partial t} = D \cdot \nabla^2 c$ (Diffusion eqn).

Solving differential eqns

$\psi(x, y, z, t)$ Seperation of variables

$= X(x)\, Y(y)\, Z(z)\, T(t)$

$\psi(x, t) = X(x)\, T(t)$

State of ~~combined~~ systems = State of 1st spin × state of 2nd spin

Two qubits

1st qubit $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

2nd qubit $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

Possible states

corresponds to 1st qubit

corresponds to 2nd qubit

State of system $= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$

$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$

$\alpha \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + \beta \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$

$\gamma \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + \delta \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$ We can assign states here
We do not which belongs to which qubit.

$\begin{pmatrix}1\\0\end{pmatrix} \otimes \left[\alpha \begin{pmatrix}1\\0\end{pmatrix} + \beta \begin{pmatrix}0\\1\end{pmatrix}\right]$

↓ 1st qubit 1st state

↓ 2nd qubit superposition of its possible states.

Not a product of states of the systems (Entangled state)

Tele

Anything that happens at a distance.

Teleportation

Transfer something from one place to another.

State can be teleported in case of superposition.

(We can make 2nd qubit to have the same state as 1st qubit).

$$\text{NOT} \otimes \text{NOT} \; \begin{pmatrix}1\\0\end{pmatrix} \times \begin{pmatrix}0\\1\end{pmatrix} = \text{NOT} \begin{pmatrix}1\\0\end{pmatrix} \otimes \text{NOT} \begin{pmatrix}0\\1\end{pmatrix}$$

Acts on the corresponding states.

$$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\times 1\\1\times 0\\0\times 1\\0\times 0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}a\\b\end{pmatrix} \otimes \begin{pmatrix}c\\d\end{pmatrix} = \begin{pmatrix}ac\\ad\\bc\\bd\end{pmatrix}$$

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Tensor product:

$$\underset{2\times 1}{\begin{pmatrix}a\\b\end{pmatrix}} \otimes \underset{2\times 1}{\begin{pmatrix}c\\d\end{pmatrix}} = \begin{pmatrix}ac\\ad\\bc\\bd\end{pmatrix} = \begin{pmatrix}a\begin{pmatrix}c\\d\end{pmatrix}\\ b\begin{pmatrix}c\\d\end{pmatrix}\end{pmatrix}$$

Tensor product of matrices

$$\begin{pmatrix}A & B\\C & D\end{pmatrix} \otimes \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}A\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} & B\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\\ C\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} & D\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\end{pmatrix}$$

$$= \begin{pmatrix}A\alpha & A\beta & B\alpha & B\beta\\ A\gamma & A\delta & B\gamma & B\delta\\ C\alpha & C\beta & D\alpha & D\beta\\ C\gamma & C\delta & D\gamma & D\delta\end{pmatrix}$$

A matrix is a function on a vector space.

Quantum gates transform qubits

$$\hat{NOT}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\hat{NOT}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

Two NOT gates acting on corresponding two qubits

$$\hat{NOT} \otimes \hat{NOT} \left[\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}\right] = \hat{NOT}\begin{pmatrix}1\\0\end{pmatrix} \otimes \hat{NOT}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \underset{2\times1}{\begin{pmatrix}0\\1\end{pmatrix}} \otimes \underset{2\times2}{\begin{pmatrix}1\\0\end{pmatrix}}$$

$$= \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

Given a vector space the maximal LI set is a <u>basis</u>.

Given two vectors $|a\rangle$, $|b\rangle$

Inner product between them $= \langle |a\rangle, |b\rangle \rangle = \langle a|b\rangle \in \mathbb{C}$

fun of two vectors which gives a number.

When a rule can be an Inner product ?

$$\langle a|b\rangle = \langle b|a\rangle^*$$

$$\langle |a\rangle, \alpha|b\rangle\rangle = \alpha \langle |a\rangle, |b\rangle\rangle = \alpha \langle a|b\rangle$$

$$\langle a|a\rangle \geq 0$$

Inner product:

$$V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \; x, y \in \mathbb{C} \right\}$$

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle = a^* c + b^* d$$

$$|A\rangle, |B\rangle \qquad = \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$$

If $\langle |A\rangle, |B\rangle\rangle = \langle A|B\rangle = 0$

then $|A\rangle$ is orthogonal to $|B\rangle$

Inner product is generalisation of dot product.

Inner product $\rightarrow$ b/w two vector

Norm $\rightarrow$ of a vector.

$$\| |a\rangle \| = \sqrt{\langle a|a\rangle}$$

$$\| |A\rangle \| = \sqrt{\left[\begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right]^2} = \sqrt{|a|^2 + |b|^2}$$

Hilbert space:

Complete normed, Inner product space.

$|\psi\rangle = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix}$

Consider a measurement which leaves the system in $\begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$ as post measurement states [Stern Gerlach experiment]

Probability of getting $\begin{pmatrix}1\\0\end{pmatrix}$ as the post measurement state $= |\alpha|^2$

Probability of getting $\begin{pmatrix}0\\1\end{pmatrix}$ as the post measurement state $= |\beta|^2$

Normalisation Condition $\quad |\alpha|^2 + |\beta|^2 = 1 \quad$ Sum of all the probabilities of mutually exclusive sets

$\begin{pmatrix}1\\0\end{pmatrix}$: Norm $\sqrt{(1^* \; 0^*)\begin{pmatrix}1\\0\end{pmatrix}} = \sqrt{1^2+0^2} = 1$

$\begin{pmatrix}0\\1\end{pmatrix}$: Norm $\sqrt{(0^* \; 1^*)\begin{pmatrix}0\\1\end{pmatrix}} = \sqrt{0^2+1^2} = 1$

$\left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\rangle = 0 \Rightarrow$ Inner product $= 0$

$\begin{pmatrix}1\\0\end{pmatrix}$ is orthogonal to $\begin{pmatrix}0\\1\end{pmatrix}$

Case ii)

$$\begin{pmatrix}1\\1\end{pmatrix} \quad \begin{pmatrix}1\\-1\end{pmatrix}$$

Normalising

$$\left\langle \alpha\begin{pmatrix}1\\1\end{pmatrix}, \alpha\begin{pmatrix}1\\1\end{pmatrix} \right\rangle = 1^2 = 1$$

$$\begin{pmatrix}\alpha^* & \alpha^*\end{pmatrix}\begin{pmatrix}\alpha\\\alpha\end{pmatrix} = 2|\alpha|^2 = 1$$

$$|\alpha| = \frac{1}{\sqrt{2}}$$

$$\begin{pmatrix}1\\1\end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\-1\end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

These are orthogonal and linearly independent.

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix} = a\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + b\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

If can only do the experment in x-axis

$\vec{V} = 3\hat{i} + 4\hat{j}$

Dotoproduct with $\hat{i}$ $\quad \vec{i}\cdot\vec{V} = 3$

$\hat{j}$ $\quad \vec{j}\cdot\vec{V} = 4$

$\hat{i}, \hat{j}$ are unit vectors and orthogonal.

if $\vec{V} = a\left(\frac{i+j}{\sqrt{2}}\right) + b\left(\frac{i-j}{\sqrt{2}}\right)$

$\frac{i+j}{\sqrt{2}}, \frac{i-j}{\sqrt{2}}$ are unit vectors and orthognal to each other

So to get 'a'

Do dot product with $\frac{i+j}{\sqrt{2}}$

$\frac{i+j}{\sqrt{2}}, \frac{i-j}{\sqrt{2}}$ form a basis.

$$a = \left\langle \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}\alpha\\\beta\end{pmatrix}\right\rangle$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{*}\begin{pmatrix}1^{*} & 1^{*}\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

i, j form a basis

$$a = \frac{\alpha+\beta}{\sqrt{2}}$$

Both these form basis for the same vector space

$$b = \frac{1}{\sqrt{2}}^{*}\begin{pmatrix}1^{*} & -1^{*}\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

$$b = \frac{\alpha-\beta}{\sqrt{2}}$$

$$a = \left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right)\cdot\vec{V}, \quad b = \left(\frac{\hat{i}-\hat{j}}{\sqrt{2}}\right)\cdot\vec{V}$$

$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1+1}{(\sqrt{2})^2} = 1$

$\boxed{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}}$ $\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix}$

choose another vector such that its norm is 1 and orthogonal to $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

Let $\begin{pmatrix}e\\f\end{pmatrix}$ $\sqrt{|e|^2+|f|^2} = 1$

$\left\langle \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \begin{pmatrix}e\\f\end{pmatrix} \right\rangle = 0$

$e = 1$ $f = -i$

$\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \gamma\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} + \delta\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

γ = Inner product with $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

$\gamma = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}$

$\gamma = \frac{1}{\sqrt{2}}(\alpha - i\beta)$

$\delta = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}$

$\delta = \frac{1}{\sqrt{2}}(\alpha + i\beta)$

NOT gate operation $\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}0\\1\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}1\\0\end{pmatrix}$

Special matrices (operations)

1. Hermition matrix

Adjoint $= (M^T)^*$

matrix = adjoint

Ex 1 $\quad M = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad M^T = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad (M^T)^* = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ Eigenvalu

$$M = (M^T)^*$$

Ex 2 $\quad \sigma_z = M = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \quad M^T = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \quad (M^T)^+ = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \quad 1, -1$

Ex 3 $\quad \sigma_y = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \quad 1, -1$

Ex 4 $\quad \sigma_x = \begin{pmatrix}0 & i\\-i & 0\end{pmatrix} \quad \begin{pmatrix}0 & -i\\+i & 0\end{pmatrix} \quad \begin{pmatrix}0 & i\\-i & 0\end{pmatrix} \quad 1, -1$

$\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix} \quad -1, 3$

To calculate eigen values $|A - \lambda I| = 0$

$$\left| \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \right| = 0$$

$$(1-\lambda)^2 - 4 = 0$$

$$1 - \lambda = \pm 2$$

$1 - \lambda = 2$ $\qquad$ $1 - \lambda = -2$

$\lambda = -1$ $\qquad$ $\lambda = 3$

Eigen values of a Hermition matrix are real.

Eigen vectors corresponding to different eigen values are orthogonal.

Any eigen vector can be scaled by a number, eigen values don't change.

$$M\vec{v} = \lambda \vec{v}$$

$$\vec{u} = 3\vec{v}$$

$$M\vec{v} = M(3\vec{v}) = 3M\vec{v} = 3\lambda\vec{v}$$
$$= \lambda(3\vec{v})$$
$$= \lambda \cdot \vec{u}$$

Eigen vectors are orthogonal to each other.

Eigen vectors form a basis.

Unitary matrix

$$M^{+}M = MM^{+} = I$$

transpse conjugate

$$(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M^{+} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

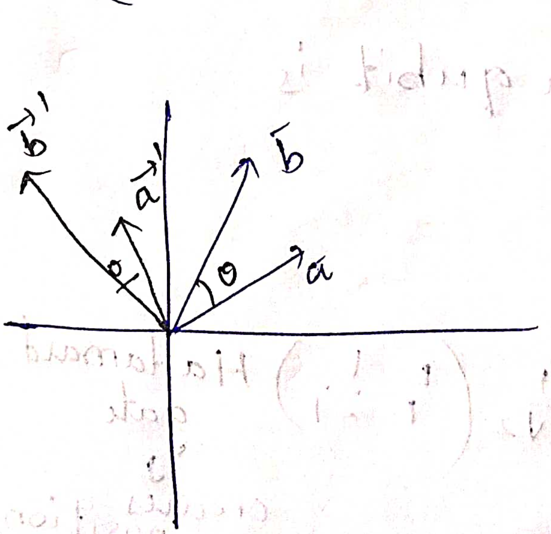
$$MM^{+} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

matric equivalent

M is an unitary matrix.

$$(e^{j\theta})^{*} e^{j\theta} = 1$$



Rotation → Inner product preserving transformation

Even if we rotate Inner product of the vector remains same.

$\langle |b'\rangle, |a'\rangle \rangle$

$= \langle M|b\rangle, M|a\rangle \rangle$

$= \langle b| M^{\dagger} M |a\rangle$

$= \langle b|a\rangle$

Transpose of a matrix, original matrix will have same eigen values.

Eigen values of a unitary matrix is of the form $e^{i\theta}$.

Eigen values of matrix $= \dfrac{1}{\text{Eigen values of Inverse matrix}}$.

i) Qubits are described by vector of the form $\begin{pmatrix} a \\ b \end{pmatrix}$ $a, b \in \mathbb{C}$

ii) Time evolution of a qubit is

$\hat{U} \begin{pmatrix} a \\ b \end{pmatrix}$

Ex: $\hat{U}$: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ Hadamard gate.

creates superposition of particles.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

iii) Observables are represented by Hermition matrices.

Ex: position
momentum,
spin.
excited level of $e^{\ominus}$

Whatever we can measure and get a number in a lab are observables.

You cannot get the same number in quantum mechanics.

Only we can find the expected value of that number bcoz of fluctuations

iv) Expectation value of an observable $\hat{M}$ is

$$\langle \hat{M} \rangle = \langle \psi \rangle, M \psi \rangle \rangle$$

$$= \langle \psi | M | \psi \rangle$$

Hermition matrix is guaranteed to be real.

V_1, V_2 be eigen vectors of M.

ψ can be expanded as combination of V_1, V_2

$$M|V_1\rangle = \lambda_1 |V_1\rangle$$

$$M|V_2\rangle = \lambda_2 |V_2\rangle$$

$$|\psi\rangle = \alpha|V_1\rangle + \beta|V_2\rangle$$

$$M|\psi\rangle = \alpha\lambda_1 |V_1\rangle + \beta\lambda_2 |V_2\rangle$$

$$\langle\psi|M|\psi\rangle$$

$$= \langle \lambda_1\alpha |V_1\rangle + \lambda_2\beta|V_2\rangle, \quad \alpha|V_1\rangle + \beta|V_2\rangle\rangle$$

$$= \lambda_1 \alpha^* \alpha \langle V_1|V_1\rangle + \lambda_2 \beta^* \beta \langle V_2|V_2\rangle$$

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Qubits

Any two level system spin ½ particle

Two chosen energy levels of an atom/molecule.

General quantum state of a

qubit $|\psi\rangle = \underbrace{\alpha|0\rangle + \beta|1\rangle}_{\text{Super position of } |0\rangle \text{ \& } |1\rangle}$ $\alpha, \beta \rightarrow$ complex.

$$|\alpha|^2 + |\beta|^2 = 1$$

$$\alpha = \cos \theta/2 \quad \beta = \sin \theta/2$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$= e^{j\phi}[\alpha|0\rangle + \beta|1\rangle]$$

$\downarrow$ overall phase factor $\rightarrow$ is irrelevant in quantum Mechanics.

$$|\psi\rangle = |\alpha| e^{j\theta_1}|0\rangle + |\beta| e^{i\theta_2}|1\rangle$$

$$= e^{i\theta_1}\left[|\alpha||0\rangle + |\beta| e^{i(\theta_2-\theta_1)}|1\rangle\right]$$

$$= e^{i\theta_1}\left[|\alpha||0\rangle + |\beta| e^{i\gamma}|1\rangle\right]$$

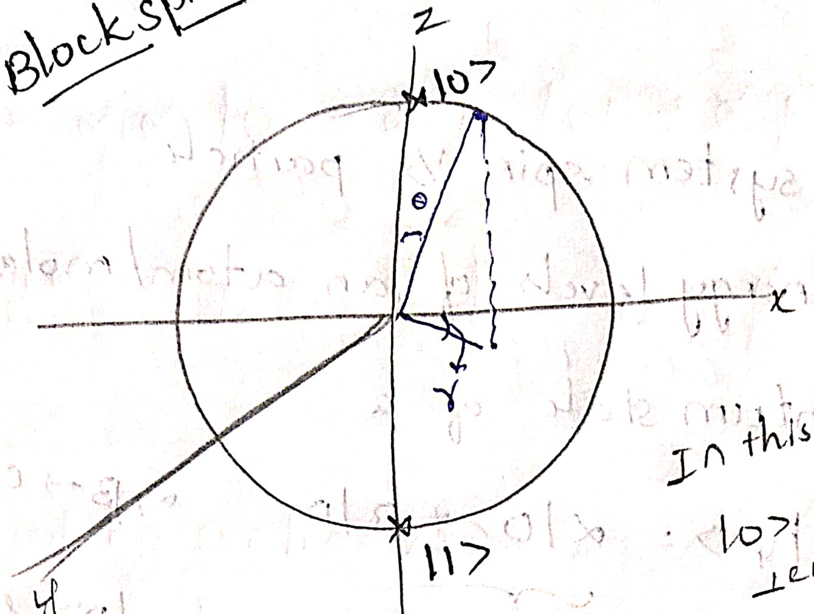
$$\Rightarrow |\alpha||0\rangle + |\beta| e^{i\gamma}|1\rangle$$

We made one of the coefficients to a real numb

$$|\alpha| = \cos \theta/2 \quad |\beta| = \sin \theta/2$$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\gamma}\sin\theta/2\,|1\rangle$$

Blocksphere



In this representation $|0\rangle$ and $|1\rangle$ are not $\perp$er to each other geometrically

H.W Qutrit $|0\rangle, |1\rangle, |2\rangle$

Represent

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$$

$$|0\rangle \Rightarrow \theta = 0, \gamma = 0$$

$$|1\rangle \Rightarrow \theta = \pi, \gamma = 0$$

→ Jacobian

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{2}, \gamma = 0$$

$$\frac{|0\rangle + i|1\rangle}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{2}, \gamma = \frac{\pi}{2}$$

Two qubits:

$|0\rangle \otimes |0\rangle \rightarrow |00\rangle$

$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$|0\rangle \otimes |1\rangle \rightarrow |01\rangle$

Four basis states $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$\frac{|00\rangle + |01\rangle}{2}$ states correspond to 1st state are not orthogonal.
But states corresponding to 2nd state are orthogonal.

$\langle |ab\rangle, |cd\rangle \rangle$

$= \langle |a|, |c| \rangle \langle |b\rangle, |d\rangle \rangle$

$\therefore |00\rangle, |01\rangle$ are orthogonal

$|\psi_1\rangle = \frac{|00\rangle + |01\rangle}{\sqrt{2}}$

$|\psi_2\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

$|\psi_1\rangle = |0\rangle \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$.

$|\psi_2\rangle \rightarrow$ Total qubit is in a state

You cannot write this state as

$|\text{I qubit}\rangle \otimes |\text{II qubit}\rangle$

$\cdot$ This state is an "Entangled state"

How to know that it is an entangled state?

Assume $|\psi_2\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle)$

$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \alpha_2\beta_1|10\rangle + \beta_1\beta_2|11\rangle$

$\alpha_1\alpha_2 = \frac{1}{\sqrt{2}} \quad \beta_1\beta_2 = \frac{1}{\sqrt{2}} \quad \alpha_1\beta_2 = 0 \quad \alpha_2\beta_1 = 0$

Proof by contradiction

Dense Coding:

Using 1 qubit generating 4 states, sending it to other qubit (fixed).

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \Rightarrow 00$$

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
flips sign of 0.

$$\sigma_z \otimes I|\psi\rangle = \frac{\sigma_z \otimes I|00\rangle + \sigma_z \otimes I|11\rangle}{\sqrt{2}} = \frac{\sigma_z \otimes I(|0\rangle \otimes |0\rangle) + \sigma_z \otimes I(|1\rangle \otimes |1\rangle)}{\sqrt{2}}$$

$$= \frac{-|00\rangle + |11\rangle}{\sqrt{2}} \Rightarrow 01$$

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$NOT \otimes I|\psi\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}} \Rightarrow 10$$

$$\sigma_y \otimes I|\psi\rangle = \frac{-|10\rangle + |01\rangle}{\sqrt{2}} \Rightarrow 11$$

$\sigma_z \otimes I |\psi\rangle$

$$= \sigma_z \otimes I |00\rangle + \sigma_z \otimes I |11\rangle$$

$$= \frac{\sigma_z \otimes I (|0\rangle \otimes |0\rangle) + \sigma_z \otimes I (|1\rangle \otimes |1\rangle)}{\sqrt{2}}$$

$$= \frac{\sigma_z |0\rangle \otimes I |0\rangle + \sigma_z |1\rangle \otimes |1\rangle}{\sqrt{2}}$$

$$= \frac{-|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{\sqrt{2}}$$

$$= \frac{-|00\rangle + |11\rangle}{\sqrt{2}}$$

$\sigma_z \otimes I |\psi\rangle$

$= \sigma_z \otimes I \dfrac{|00\rangle + \sigma_z \otimes I |11\rangle}{\sqrt{2}}$

$= \dfrac{\sigma_z \otimes I (|0\rangle \otimes |0\rangle) + \sigma_z \otimes I (|1\rangle \otimes |1\rangle)}{\sqrt{2}}$

$= \dfrac{\sigma_z |0\rangle \otimes I |0\rangle + \sigma_z |1\rangle \otimes |1\rangle}{\sqrt{2}}$

$= \dfrac{-|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{\sqrt{2}}$

$= -\dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

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Measurements:

Galvanometer - shunt resistance

Ammeter - connect in series

Voltmeter - connect in parallel

Make error as small as possible by increasing resistance

- Heisenberg uncertainity

Measurement disturbs a system

$$\vec{V} = 3\hat{i} + 4\hat{j} + 5\hat{k}$$

$$3 = \hat{i} \cdot \vec{V}$$

$$\vec{V} \xrightarrow[\text{post measurement}]{} 3\hat{i}$$

$$(\hat{i} \cdot \vec{V})\hat{i}$$

$$\hat{i} \to |i\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\hat{j} \to |j\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\vec{V} = \begin{pmatrix}3\\4\\5\end{pmatrix}$$

$$\hat{i} \cdot \vec{V} = \langle \hat{i}, \vec{V} \rangle$$

$$= \langle i | \vec{V} \rangle$$

$$|\vec{V}\rangle \xrightarrow[\text{post measurement}]{} 3|\hat{i}\rangle$$

$$= \langle |i\rangle, |\vec{V}\rangle \rangle \, |i\rangle$$

$$= \langle i | v \rangle \, |i\rangle$$

$$|i\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\langle i | = (1 \; 0 \; 0)$$

$$\begin{pmatrix}0&0\\0&1\end{pmatrix}_{2\times2}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$\vec{v} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

$$\langle i | v \rangle | i \rangle = \left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Projection operator $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$|i\rangle\langle i|$ projection along x axis

$|j\rangle\langle j|$ projection along y axis

$|k\rangle\langle k|$ projection along z axis.

AH

* If a matrix is Hermition → then its eigen vectors form basis

Corresponding to every state → there is a projection.

$$|\psi\rangle = \alpha |+\rangle + \beta |-\rangle \qquad |+\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\alpha|^2 + |\beta|^2 = 1 \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Projection operator corresponding to this

$$|+\rangle\langle +| = P_+ \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \qquad |-\rangle\langle -| = P_-$$

$$\begin{pmatrix} 0 & 0 \\ & \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$P_+ |\psi\rangle = \alpha |+\rangle$

$P_- |\psi\rangle = \beta |-\rangle$

Stern-Gerlac
Projecting state $|\psi\rangle$ along the stream

$P_+ P_+ = P_+^2$

$P_+^2 |\psi\rangle = P_+ (P_+ |\psi\rangle)$

$= P_+ (\alpha |+\rangle)$

$= \alpha (P_+ |+\rangle)$

$= \alpha |+\rangle$

$P_+^2 = P_+$

sum of all orthogonal projectors, total sum will be an identity.

$P_+ |\psi\rangle = \alpha |+\rangle$

$P_- |\psi\rangle = \beta |-\rangle$

$(P_+ + P_-) |\psi\rangle = \alpha |+\rangle + \beta |-\rangle$

$= |\psi\rangle$

Ideal Measurement is not supposed to disturb the system.

Representation of states for stern Gerlac oriented along y.

$|+, z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|-, z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|+; y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$|-, y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Projection operation (measurement along z axis)

$P_{+z}|\psi\rangle = \alpha|+\rangle$

$P_{-z}|\psi\rangle = \beta|-\rangle$

$P_{+y} = |+y\rangle\langle +y|$

$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}(1, 1)$

$= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_{2\times1}(1 \quad 1)_{1\times2} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$P_{+y}|+\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}_{2\times2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{2\times1} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

SG-y

outcome $= +\hbar/2$

What is the state after measurement?

state $\propto P_{+y}|\psi\rangle$

$P_{+y} [\alpha |+, z\rangle + \beta |-, z\rangle]$

$|+, y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}|+, z\rangle\\|-, z\rangle\end{bmatrix}$

$|-, y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\Big[|+, z\rangle$

$\alpha \; P_{+,y} |\psi\rangle$

$= P_{+y} [\alpha |+, z\rangle, \beta |-, z\rangle]$

$= \underbrace{|+, y\rangle \;\; \langle +, y|}_{\text{Projector}} \; \alpha \left[\frac{|+, y\rangle + |-, y\rangle}{\sqrt{2}}\right] +$

$\beta \left[\frac{|+, y\rangle - |-, y\rangle}{\sqrt{2}}\right]$

$= \alpha \dfrac{\overset{|+,y\rangle}{\langle +, y|} \cancel{|}+, y\rangle + \langle +, y| \cancel{|}-y\rangle}{\sqrt{2}}$

$+ \beta$

$$= \frac{\alpha\,|+,y\rangle + \beta\,|+,y\rangle}{\sqrt{2}}$$

$$= \frac{(\alpha+\beta)}{\sqrt{2}}\,|+,y\rangle \xrightarrow[\text{normalization}]{} |+,y\rangle$$

Bell states $|00\rangle = |0\rangle \otimes |0\rangle$

$$|B_1\rangle \ \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|B_2\rangle \ \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|B_3\rangle \ \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|B_4\rangle \ \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

All observables are represented by self adjoint operator.

All outcomes are eigen vectors of these observable

$|B_1\rangle \langle B_1 \psi$

$$|B_1\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\langle B_1| = (a^+ \ b^+ \ c^+ \ d^+)$$

$\cancel{|B_1\rangle \langle B_1}$

$$|B_1\rangle\langle B_1| = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}_{4\times1} (a^+ \ b^+ \ c^+ \ d^+)_{1\times4}$$

$$(|B_1\rangle\langle B_1|)^\dagger = |B_1\rangle\langle B_1|$$

$$(|B_1\rangle\langle B_1|)\,|B_1\rangle = |B_1\rangle$$

projector

$$(|B_1\rangle\langle B_1|)\,|B_2\rangle = 0$$

Eigen values = 1, 0, 0, 0.

$|B_1\rangle, |B_2\rangle$ are states

$$|\psi\rangle = 0.5\,|B_1\rangle - 0.8\,|B_2\rangle$$

Probability of $|B_1\rangle$ as post measurement state?

Not normalised.

$$|\psi\rangle = \frac{1}{\sqrt{0.5^2 + 0.8^2}}\,(0.5\,|B_1\rangle - 0.8\,|B_2\rangle)$$

$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$

Probability that $|+\rangle$ is a post measurement state is $|\alpha|^2$

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$$= \frac{0.5^2}{0.5^2 + 0.8^2}$$

$P((|B_1\rangle + |B_2\rangle)/\sqrt{2}$ as the post measurement state)

$\cancel{|\psi\rangle} =$

$$|S_1\rangle = \frac{|B_1\rangle + |B_2\rangle}{\sqrt{2}}$$

$$|S_2\rangle = \frac{|B_1\rangle - |B_2\rangle}{\sqrt{2}}$$

$$\cancel{|\psi\rangle = a|s_1\rangle + b|s_2\rangle}$$

$$\cancel{= a}$$

$$|\psi_1\rangle \propto 0.5\frac{|s_1\rangle + |s_2\rangle}{\sqrt{2}} - 0.8\frac{|s_1\rangle - |s_2\rangle}{\sqrt{2}}$$

$$|\psi_1\rangle \propto \frac{(0.5-0.8)}{\sqrt{2}}|s_1\rangle + \frac{0.5+0.8}{\sqrt{2}}|s_2\rangle$$

P($|s_1\rangle$ as post measurement state)

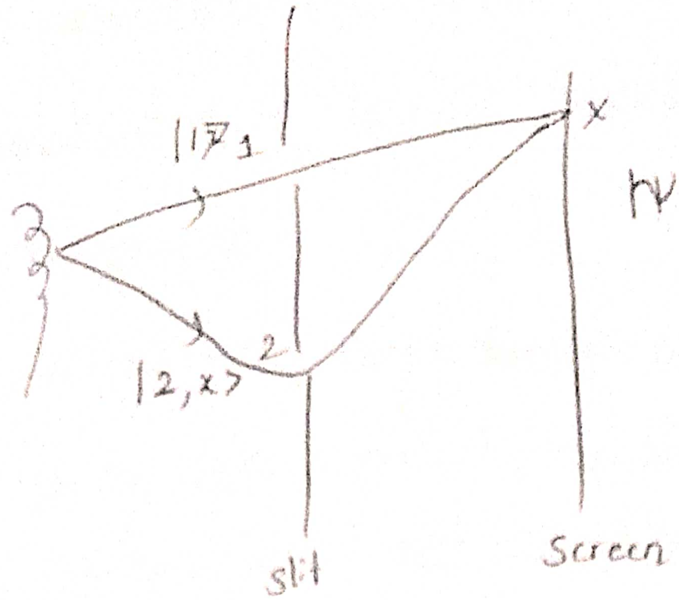
$$= \frac{\left(\frac{0.5-0.8}{\sqrt{2}}\right)^2}{\left(\frac{0.5-0.8}{\sqrt{2}}\right)^2 + \left(\frac{0.5+0.8}{\sqrt{2}}\right)^2}$$

If $|s_1\rangle = |B_1\rangle + 0.5|B_2\rangle$

what would be the orthogonal vector?

Find (P(($|B_1\rangle + 0.5|B_2\rangle$) as post measurement state)

Young's Double Slit Experiment



$|\psi\rangle = \alpha|1,x\rangle + \beta|2,x\rangle$

(Since we donot know the path information)

With one particle measurement, we cannot say its input state

probability of photon to detect at x)

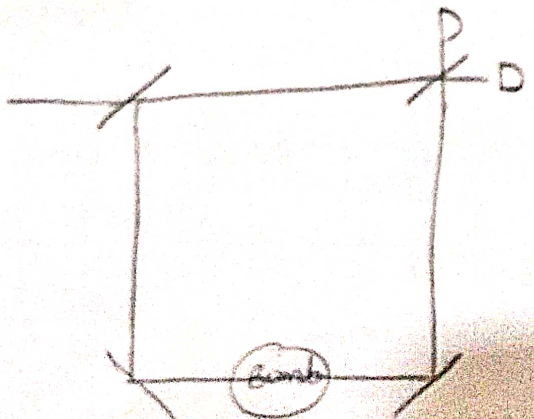
$P(\text{photon } x) = |\alpha+\beta|^2$

$$= |\alpha|^2 + |\beta|^2 + \underbrace{2\,Re(\alpha\beta^*)}_{\max(Re(\alpha\beta^*)) \leq |\alpha||\beta|}$$

$\alpha = 1,\ \beta = re^{i\theta}$

Vaidman experiment

interferometer



How to detect the bomb without exploding

06/02/23

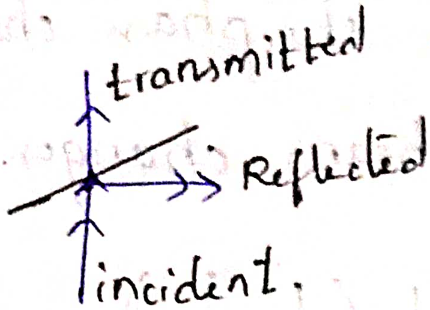
Mach Zehnder Interferometer

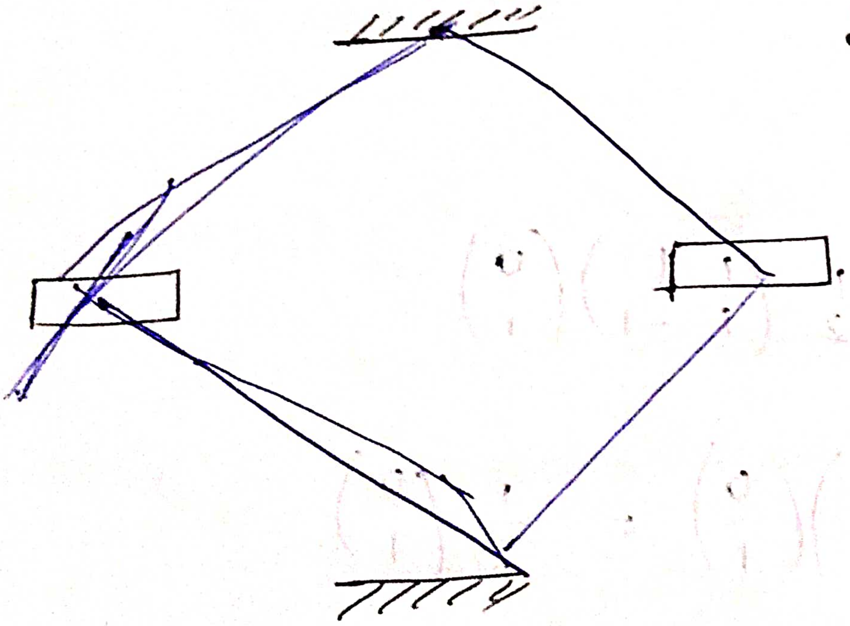
Generalisation of Young's double slit experiment

Basic components

1. Beam splitters (semi transparent plate)
   when light is passed, incident beam is split into reflected and transmitted beams)

2. Mirrors
3. Light Detector

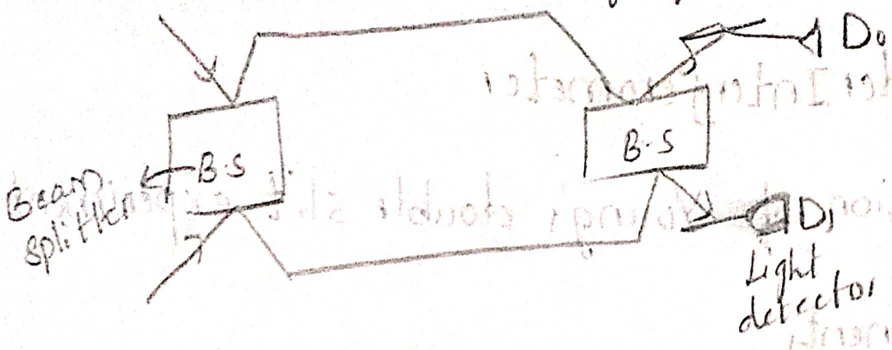




Symmetric beam splitter
50% of the energy flows
50% transmitted
50% reflected

Assuming ideal mirrors - only changes direction.

2 input 2 output gadget



This is a linear device.

light only in lower part $\Rightarrow$ lower beam $= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

light in upper part $\Rightarrow$ upper beam $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

After reflection the phase changes.

When the lower beam changes its phase by reflecting

$$B_l = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

For the ~~higher~~ upper beam $\quad B_u = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

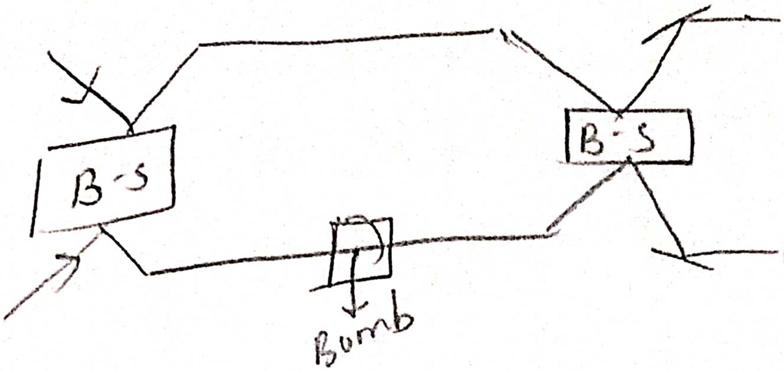
Input light $= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\frac{1}{4}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Beam splitter has the equal probability that it ~~spits~~ splits to transmitted and reflected.

Both the detectors has to work



$$\begin{pmatrix}0\\1\end{pmatrix} \longrightarrow \begin{pmatrix}0\\0\end{pmatrix}$$

$$A = \begin{pmatrix}1&0\\0&0\end{pmatrix} \rightarrow \text{Absorption of photon}$$

$$\frac{1}{2}\begin{pmatrix}-1&1\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$\frac{1}{4}\begin{pmatrix}-1&0\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$\frac{1}{4}\begin{pmatrix}-1&-1\\1&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}-1\\1\end{pmatrix}$$

When lower beam is input

D_1 is detected we cannot say if bomb is active or not

D_0 is detected $\Rightarrow$ bomb is active

01 - qubit gates

NOT gate

$$\hat{NOT}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\hat{NOT}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\hat{NOT}\left[\alpha\begin{pmatrix}0\\1\end{pmatrix} + \beta\begin{pmatrix}1\\0\end{pmatrix}\right] = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix}$$

How

- Linear operator acting on basis sets

$$|0\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

$$|0\rangle\langle 0| = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$|0\rangle\langle 0|\,|0\rangle = |0\rangle$$

$|0\rangle\langle 0|\,|1\rangle$ = Null vector

$|1\rangle\langle 1|\,|0\rangle$ = Null vector

$$|1\rangle\langle 1|\,|1\rangle = |1\rangle$$

$$\hat{M} = |0\rangle\langle 0|$$

$\hat{M}$	$\|0\rangle$	$\|1\rangle$
$\langle 0\|$	$\langle 0\|\hat{M}\|0\rangle$	$\langle 0\|\hat{M}\|1\rangle$
$\langle 1\|$	$\langle 1\|\hat{M}\|0\rangle$	$\langle 1\|\hat{M}\|1\rangle$

$\langle 0|M|0\rangle = \langle |0\rangle, M|0\rangle\rangle$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$NOT\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

$\hookrightarrow (|0\rangle\langle 1| + |1\rangle\langle 0|)\,|0\rangle = 0 + |1\rangle = |1\rangle$

$NOT\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

$\hookrightarrow (|0\rangle\langle 1| + |1\rangle\langle 0|)\,(|1\rangle) = |0\rangle$

$\hat{O}\,|0\rangle = -|0\rangle$ $\qquad |0\rangle = \begin{pmatrix}0\\1\end{pmatrix} \quad |1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$\hat{O}\,|1\rangle = |1\rangle$

outer product → matrix

Inner product → number/complex

$$\hat{O} = |1\rangle\langle 1| - |0\rangle\langle 0|$$

$$= \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hadamard gate $H = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$H|0\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \frac{1}{2}\left[\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$H|1\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

2 Qubit gates

Transforms states of two qubits

How to represent 2 qubits? Using tensorproduct

Ex of 2 qubit state $|0\rangle|0\rangle = |0\rangle \otimes |0\rangle$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{2\times1} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{2\times1}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_{4\times1}$$

Ex of 2 qubit gate $\hat{NOT} \otimes \hat{NOT} \; |0\rangle|0\rangle$

NOT Composition

$$\underset{N_1}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \otimes \underset{N_2}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} 0\times N_2 & 1\times N_2 \\ 1\times N_2 & 0\times N_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Ex

$$\underbrace{(\hat{NOT} \otimes \hat{NOT})}_{\hat{N}} \left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)/\sqrt{2}$$

$$\frac{\hat{N}\,|0\rangle|0\rangle + \hat{N}\,|1\rangle|1\rangle}{\sqrt{2}}$$

$$\frac{|1\rangle|1\rangle + |0\rangle|0\rangle}{\sqrt{2}} \rightarrow \text{same as input}$$

$\hat{NOT} \otimes \hat{NOT}$ is eigen vector of $\frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}}$

with corresponding eigen value 1.

Ex: $(\hat{NOT} \otimes \hat{NOT})(|0\rangle|0\rangle - |1\rangle|1\rangle)/\sqrt{2}$

$$= \frac{|1\rangle|1\rangle - |0\rangle|0\rangle}{\sqrt{2}} \quad \text{eigen value} = -1$$

4 eigen vectors, self adjoint

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{self adjoint}$$

This is a bell state

$$\frac{|0\rangle|1\rangle + |1\rangle|0\rangle}{\sqrt{2}} \rightarrow \text{another bell state}$$

Is this an eigen state of $\hat{NOT} \otimes \hat{NOT}$?

$\hat{NOT} \otimes \hat{NOT}$ have bell states as eigen states.

Bell states are degenerate eigen states with eigen value -1.

Ex: CNOT gate - 2 qubit gate

$$\hat{CNOT}\,|0\rangle|0\rangle = |0\rangle|0\rangle$$
$$\hat{CNOT}\,|0\rangle|1\rangle = |0\rangle|1\rangle$$
$$\hat{CNOT}\,|1\rangle|0\rangle = |1\rangle|1\rangle$$
$$\hat{CNOT}\,|1\rangle|1\rangle = |1\rangle|0\rangle$$

o/p tells uniquely what the i/p is
So this is ~~an~~ a reversible gate

Dimension of hilbert space in a 2 qubit system = 4

$(|00\rangle\langle 00|)|00\rangle = |00\rangle$

$(|00\rangle\langle 00|)|01\rangle = \text{Null}$

$(|00\rangle\langle 00|)|10\rangle = \text{Null}$

$(|00\rangle\langle 00|)|11\rangle = \text{Null}$

$\hat{CNOT}$	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
$\langle 00\|$	$\langle 00\|M\|00\rangle$			
$\langle 01\|$				
$\langle 10\|$				
$\langle 11\|$				

M = CNOT

$|00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$

As a whole it is self adjoint

$(\hat{NOT})^{\dagger} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$

It is unitary

entangled $\qquad\qquad\qquad\qquad$ Not an entangled state

$$\hat{CNOT}\left[\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right] = \frac{|00\rangle+|10\rangle}{\sqrt{2}}$$

$$= \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|0\rangle$$

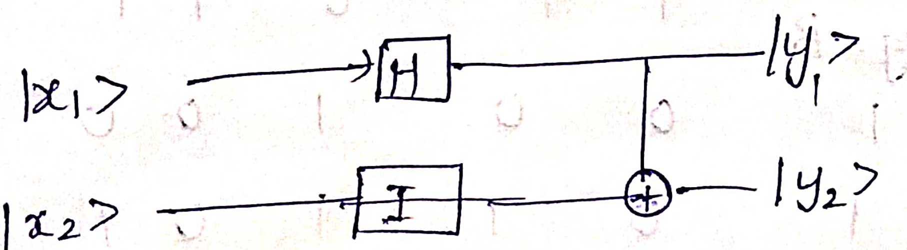
$\hat{CNOT}$ can change entangled to non-entangled states

$$\hat{CNOT}\left[\frac{|00\rangle-|11\rangle}{\sqrt{2}}\right] = \frac{|00\rangle-|10\rangle}{\sqrt{2}} \rightarrow \text{not entangled}$$

$$\hat{CNOT}\left[\frac{|10\rangle+|01\rangle}{\sqrt{2}}\right] = \frac{|11\rangle-|01\rangle}{\sqrt{2}} \rightarrow \text{not entangled}$$

$$\hat{CNOT}\,(\hat{H}\otimes I)\,|x_1\rangle|x_2\rangle$$

$$= \hat{CNOT}\;\hat{H}|x_1\rangle \otimes I\,|x_2\rangle$$



Controlled NOT gate

A	B	A $\hat{CNOT}$ B	
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0
		A	Addition %2

Is this reversible?

Yes.

Taffoli gate - reversible gate

Control control not

2 controls one gate

All the conventional gates can be generated from taffoli gate.

No. of i/ps = 3

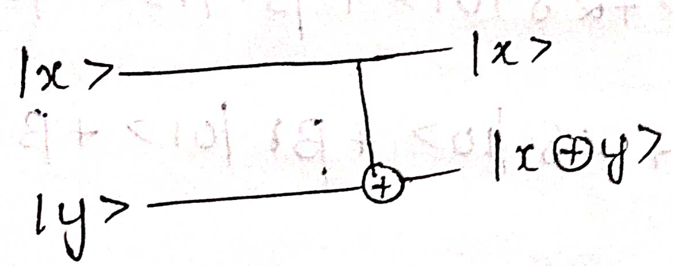
A	B	C	x	y	z
			A	B	C XOR (A AND B)
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0 ~~y=B~~	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

(A=1, B=1, C) = NOT(C)

08/2/23

Two qubit gates

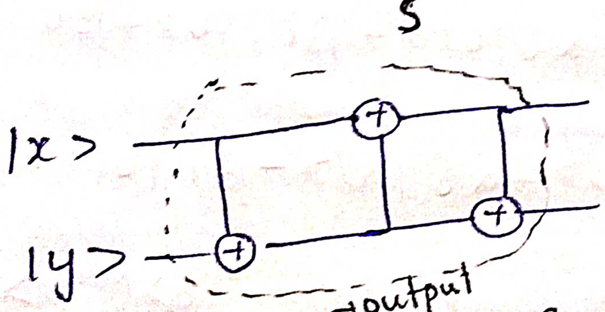
2 input 2 output gate



$\|x\rangle$	$\|y\rangle$	$\|x \oplus y\rangle$
0	0	0
0	1	1
1	0	1
1	1	0

In Quantum ⇒ no of inputs = no. of outputs
else it wont be a unitary matrix and reversible.

SWAP



Input	S_1	S_2	output
$\|00\rangle$ →	$\|00\rangle$ →	$\|00\rangle$ →	$\|00\rangle$
$\|11\rangle$ →	$\|10\rangle$ →	$\|10\rangle$ →	$\|11\rangle$
$\|10\rangle$ →	$\|11\rangle$ →	$\|01\rangle$ →	$\|01\rangle$
$\|01\rangle$ →	$\|01\rangle$ →	$\|11\rangle$ →	$\|10\rangle$

Exchanging x and y.

If $|x\rangle$ is in superposition $\alpha|0\rangle + \beta|1\rangle$

$|y\rangle$ is in superposition $\gamma|0\rangle + \delta|1\rangle$

$S\,[\alpha|0\rangle + \beta|1\rangle][\gamma|0\rangle + \delta|1\rangle]$

$= S\,[\alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle]$

$= \alpha\gamma|00\rangle + \alpha\delta|10\rangle + \beta\gamma|01\rangle + \beta\delta|11\rangle$

$= \gamma|0\rangle[\alpha|0\rangle + \beta|1\rangle] + \delta|1\rangle[\alpha|0\rangle + \beta|1\rangle]$

$= (\alpha|0\rangle + \beta|1\rangle)(\gamma|0\rangle + \delta|1\rangle)$

$S'\,[\alpha|0\rangle + \beta|1\rangle][\gamma|0\rangle + \delta|1\rangle]$

$= [\gamma|0\rangle + \delta|1\rangle][\alpha|0\rangle - \beta|1\rangle]$

$S'[\alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle]$

$= [\alpha\gamma|00\rangle + \alpha\delta|10\rangle - \beta\gamma|01\rangle - \delta\beta|11\rangle]$

$-(|0\rangle \otimes |1\rangle)$

0	0	0	0	
0	1	1	0	
1	0	0	1	(-)
1	1	1	1	(-)