* $|0\rangle, |1\rangle, |2\rangle$ each with — These will form orthonormal if they have both orthogonal and normalized.

* Eigen fn of e^{kx}

$\rightarrow \frac{d}{dx} e^{kx} = k e^{kx}$

* $A = [\ ]_{2\times 2}$, $A|v\rangle_{2\times 1} = |w\rangle_{m\times 1}$

$V \quad W$

* $A = |w\rangle\langle v|$ → Outer-product

* $A|v'\rangle = (|w\rangle\langle v|)|v'\rangle = |w\rangle \underbrace{\langle v|v'\rangle}_{\text{Complex no.}}$

$\Rightarrow A|v'\rangle = c|w\rangle$

$\Rightarrow A|v'\rangle = |v'\rangle$

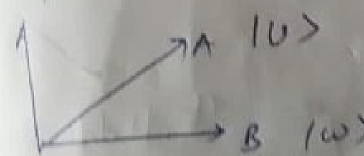
$\Rightarrow I = \sum_i |i\rangle\langle i|$

---

* The Cauchy - Schwarz Inequality

$|v\rangle$ and $|w\rangle$

$|\langle w|v\rangle| \leq \|w\| \, \|v\|$



$\vec{A} = A\cos\theta\,\hat{i} + A\sin\theta\,\hat{j}$

$= \vec{A} - \vec{A}\cos\theta$

$[v_1^* \ v_2^*]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} > 0$

* Sum of square v_1 & v_2

* Hilbert space $\equiv$ inner product

Vector space & inner product

* $\sum_{i=1} \lambda_i |w_i\rangle$ and $|v_i\rangle$

$\sum_i \lambda_i^{\dagger} \langle w_i | v \rangle$

* Orthogonal

$\langle v | w \rangle = 0$

$\| |v\rangle \| = \sqrt{\langle v | v \rangle}$

If norm = 1 → vector is normalized

→ Unit vector in v direction

$\frac{|v\rangle}{\| |v\rangle \|}$

* δ_{ij} ⟶ Kroncker Set

→ A set $|i\rangle$ of vectors is orthonormal

$\langle i | i \rangle = 1$ $\quad \langle j | j \rangle = 1$

$\langle i | j \rangle = 0$ $\quad \langle i | k \rangle = 0$

So, A is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $a+c=1$, $b+d=0$

Identity vector

$I|v\rangle = |v\rangle$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

— X — Pauli matrix X

$|v_0\rangle$ and $\sum_{i=1}^{3} \lambda_i |w_i\rangle$

Inner product of $|0\rangle$ with $\underset{\lambda_1}{2}|0\rangle + \underset{\lambda_2}{3}|1\rangle + \underset{\lambda_3}{4}|2\rangle$

$2\langle 0|0\rangle + 3\langle 0|1\rangle + 4\langle 0|2\rangle$

$\left.\begin{matrix} |0\rangle \\ |1\rangle \\ |2\rangle \end{matrix}\right\}$ Orthogonal $\underbrace{\text{normalized}}$ Orthonormal

$i.\ \bar{A} = i.\ 2\langle 0|0\rangle + i.\ 3\langle 0|1\rangle + i.4\langle 0|2\rangle$

$|\varphi\rangle \quad \sum_{i=1}^{k} \lambda_i |v_i\rangle$

$\langle v|w\rangle = (\langle w|v\rangle)^*$

* $\langle i | j \rangle = \delta_{ij}$

* $|w_1\rangle, |w_2\rangle, \ldots |w_n\rangle$ basis vectors V — Span

$|v_1\rangle, |v_2\rangle, \ldots |v_n\rangle$: Orthonormal set

Gram-Schmidt Procedure converts any set of vectors into orthonormal set.

* $|w_1\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $|w_2\rangle = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ — These are not orthonormal & orthogonal

$|v_1\rangle = \dfrac{|w_1\rangle}{\| |w_1\rangle \|} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ → normalized vector

$|v_2\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$|v_{k+1}\rangle = \frac{|w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle |v_i\rangle}{\|x\|}$$

* $|v_2\rangle = |w_2\rangle - \langle v_1 | w_2\rangle |v_1\rangle$

$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \left(5/\sqrt{2} \right) \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 5/2 \\ 5/2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ +1/\sqrt{2} \end{bmatrix}$

$\sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{1/2}$ $\qquad$ $\overline{\sqrt{2}\ \sqrt{1/2}}$

* $\in \mathbb{C}^2$

$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}_{2\times 1}$

$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$

Then $A|\psi\rangle =$

$\underline{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} [illegible] \\ [illegible] \\ [illegible] \end{bmatrix}$

$\mathbb{C}^3$ space

* **Linear Operator**

$A : V \longrightarrow W$

Linear

$A\left(\sum_i a_i |v_i\rangle\right) = \sum_i a_i A|v_i\rangle$

* $|v\rangle = a_1|v_1\rangle + a_2|v_2\rangle$

$|v_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$|v_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A|v\rangle = 2A|v_1\rangle + 3A|v_2\rangle$

$= 2A\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3A\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

* $\underline{0}|v\rangle = 0$

Operator

(6) Composition of the operator $A\hat{B}|\psi\rangle$ $\begin{bmatrix} 2 \\ 3 \end{bmatrix}_{2\times 1}$

$\underbrace{A_{1\times 3}\{B|\psi\rangle\}_{3\times 1}}$

$[\ ]_{3\times 1}$ $\mathbb{C}^3$

$A_{3\times 3}\{B_{2\times 2}|\psi\rangle\}$ $A : V \to W$, $W \to X$

(7) Show that

$(1,-1)$ $(1,2)$ and $(2,1)$ are linearly dependent

Sol $|v_1\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $|v_2\rangle = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $|v_3\rangle = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$|v_3\rangle = -2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(8) Suppose V is an vector space

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, A - linear operator

$A : V \to V$
$\mathbb{C}^2$ $\mathbb{C}^2$

and that $A|0\rangle = |1\rangle$,
$A|1\rangle = |0\rangle$

$\begin{pmatrix} 0 & \\ 1 & \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$

* ψ and ϕ

→ State of the system

$$|\psi\rangle = \begin{bmatrix} i+1 \\ 2 \end{bmatrix}_{2\times 1} \qquad |\phi\rangle = \begin{bmatrix} 3 \\ 2+i \end{bmatrix}_{2\times 1}$$

* $$A = \begin{bmatrix} 1 & 6i \\ 3i & 2+4i \end{bmatrix}$$

$$A^* = \begin{bmatrix} 1 & -6i \\ -3i & 2-4i \end{bmatrix}$$

dagger

A^+

Adjoint or hermition conjugate $= (A^*)^T$

→ Norm of a vector

$$\| |\psi\rangle \| = (\langle\psi|\psi\rangle)^{1/2}$$

Let a vector is given as $|\psi\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

So, unit vector $= \dfrac{|\psi\rangle}{\| |\psi\rangle \|}$

* Operator $|\phi\rangle, A|\psi\rangle$

Inner product $\langle\phi|A\psi\rangle$

* $\underbrace{A^+|\phi\rangle \text{ and } |\psi\rangle}_{\text{Inner product}} = \langle\phi|A|\psi\rangle$

* $\vec{A} = a\hat{i} + b\hat{j}$.

A spanning set

Any vector $|v\rangle$ in the V.S. can be written

$$= \begin{bmatrix} 6 \\ 8 \\ 18i \\ 24i \end{bmatrix}_{4\times 1}$$

Two matrices tensor producttt

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} a_{11} B & a_{12} B \\ a_{21} B & a_{22} B \end{bmatrix}_{4\times 4}$$

$$= \begin{bmatrix} a_{11} b_{11} & a_{11} b_{12} & \_\_\_ \\ a_{11} b_{21} & a_{11} b_{22} & \\ & & \\ \_\_\_ & & \_\_\_ \end{bmatrix}$$

$$\langle w | v \rangle = \begin{bmatrix} a_1^* & a_2^* \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\mathbb{C}^n$ $$|v\rangle = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

<u>Addition</u> $\mathbb{C}^2$

$$|v_1\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad |v_2\rangle = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\Rightarrow |v\rangle = |v_1\rangle + |v_2\rangle = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

~~Multiplicat~~
Multiplication by scalar

$$|v_1| = \frac{1}{\sqrt{2}} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

as linear combination

- $\mathbb{C}^2$ space

$|v_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; |v_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\hat{i} = 1\hat{i} + 0\hat{j}, \; \hat{j} = 0\hat{i} + 1\hat{j}$

- $\cancel{|v\rangle = \begin{bmatrix} i \\ 2+3i \end{bmatrix}}$

$= i\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (2+3i)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $|v\rangle = \begin{bmatrix} a \\ b \end{bmatrix} = \frac{c_1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{c_2}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{pmatrix} \frac{c_1+c_2}{\sqrt{2}} \\ \frac{c_1-c_2}{\sqrt{2}} \end{pmatrix}$

- **<u>Linearly dependent & Independent vector</u>**

Eg. $|v_1\rangle = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad |v_2\rangle = 2|v_1\rangle$

$|v_2\rangle = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

- Basis set of $\vec{V}$:- A set of linearly independent vector which span a vector space V.

In $\mathbb{C}^2$

→ Basis set is

$|v_1\rangle \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

In $\mathbb{C}^3$

$|v_1\rangle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, |v_2\rangle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, |v_3\rangle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Quantam Mechanics

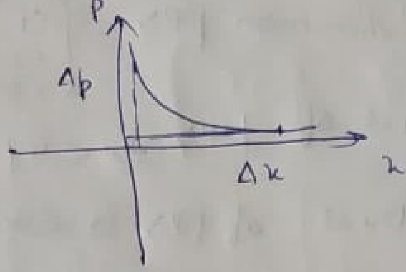
① Heisenberg uncertainity

$$(\Delta x)(\Delta p) \geq \frac{h}{4\pi}$$

(Δx): uncertainity in position → standard deviation in position

(Δp): uncertainity in H.b → standard deviation in momentum

$xy = 1$



In C.M

1 particle, state

x, p, F ✓

In Q.M

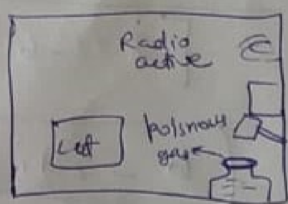
x, p ×

We cann't specify the particle by x & p.
because of uncentenity

② In 1925
Schrödinger find equation $\psi(x,t)$

We find future state of a particle by $\vec{F} = m\vec{a}$ in classical mechanics. But we find future state of a particle by schrodinger wave equation in quantam mechanics.

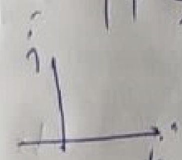
$$\psi_{cat} = c_1 \psi_{cat} + c_2 \psi_{alive}$$ → superposition or linear combination



$$H = \frac{-13.6}{n^2} \text{ e.V}$$

③ $cc^* = a^2 + b^2 = |c|^2$

④ $|\psi\rangle$ ket → state vector



$$\vec{A} = 2\hat{i} + 3\hat{j} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\vec{A} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$|\psi\rangle = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

State vector

n-dimension

→ 2-dimension $|\psi\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

→ 3-dimension $|\psi\rangle = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

→ Dual of $|\psi\rangle \Rightarrow \langle\psi| = [c_1^* \;\; c_2^*]$ → In 2-dimension

→ Dual of $|\psi\rangle \Rightarrow \langle\psi| = [c_1^* \;\; c_2^* \;\; c_3^*]$ → In 3 dimension

Inner product

$\vec{A} = 2\hat{i} + 3\hat{j}$, $\vec{B} = 5\hat{i} + 9\hat{j}$

Let $|\phi\rangle$ is one vector and $|\psi\rangle$ is another vector

So, inner product $= \langle\phi|\psi\rangle = [a_1^* \;\; a_2^*]|\psi\rangle$

Let $|\psi\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ $\cancel{\langle\phi|\psi\rangle =} \begin{bmatrix} \cancel{c_1 a_1 + c_2 a_2} \\ c_2 a_1 + c_2 a_2 \end{bmatrix}$

$|\phi\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$= a_1^* c_1 + a_2^* c_2$

Ex $|\phi\rangle = \begin{bmatrix} 2 \\ 6i \end{bmatrix}$

$|\psi\rangle = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\boxed{\text{Inner product}}$

$\begin{bmatrix} 2 & -6i \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\boxed{6 - 24i}$

Tensor product

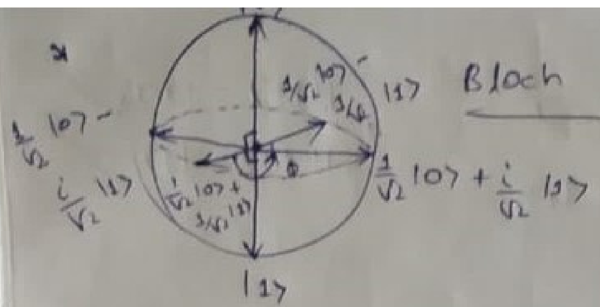
$|\psi\rangle$ and $|\phi\rangle$

$|\psi\rangle \otimes |\phi\rangle$

Let $|\psi\rangle = \begin{bmatrix} 2 \\ 6i \end{bmatrix}$ & $|\phi\rangle = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\underset{2\times1}{\begin{bmatrix} 2 \\ 6i \end{bmatrix}} \otimes \underset{2\times1}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}$

$\cancel{\begin{bmatrix} 2 & 6i \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}}$



Bloch Sphere

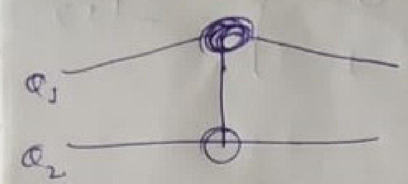
r, θ, ϕ

$$|0\rangle \cos\frac{\theta}{2} + e^{i\phi} \sin\frac{\theta}{2} |1\rangle$$

at P, $\phi = 0$
Q, $\phi = \pi$

⊛ Two Qubit Gate

CNOT → (Controlled NOT)



$|0\rangle$ —[Gate]— $|+\rangle$
↓
H-gate

$\alpha_{00} = |00\rangle \longrightarrow |00\rangle \longrightarrow 1|00\rangle + 0|01\rangle + 0|10\rangle + 0|11\rangle$

$\alpha_{01} = |01\rangle \longrightarrow |01\rangle$

$\alpha_{10} = |10\rangle \longrightarrow |11\rangle \longrightarrow 0|00\rangle + 0|01\rangle + 0|10\rangle + 1|11\rangle$

If first is 1 then it will change second

$\alpha_{11} = |11\rangle \longrightarrow |10\rangle$

CNOT GATE

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$|00\rangle \Rightarrow |00\rangle$
$|01\rangle$
$|10\rangle \Rightarrow |11\rangle$
$|11\rangle$

Problem

Q. $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|0\rangle = 1|0\rangle + 0\cdot|1\rangle$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$|1\rangle = 0\cdot|0\rangle + 1\cdot|1\rangle$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$X|0\rangle = |1\rangle$
$X|1\rangle = |0\rangle$

$\overset{a}{0}|0\rangle + \overset{c}{1}|1\rangle$

$\overset{b}{1}|0\rangle + \overset{d}{0}|1\rangle$

$\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = |-\rangle$$

2 Qubits

$$|\psi\rangle = (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$$

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

$$\begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix}, \quad |00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Leftarrow |00\rangle$$

Ex: $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle + 0\cdot|01\rangle + 0|10\rangle$

$\neq |\psi\rangle \otimes |\phi\rangle$

Bell state or entangled state.

Chapter

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, Is it Hermitian.

Pauli matrix

For Hermitian $X = X^{+}$.

For unitary $X^{+} = X^{-1}$.

$\boxed{\text{GATE}} \longrightarrow$ Bit-Flip Gate or not gate

$X(\alpha|0\rangle + \beta|1\rangle) = \alpha X|0\rangle + \beta X|1\rangle$

not gate

$= \alpha|1\rangle + \beta|0\rangle = \beta|0\rangle + \alpha|1\rangle$

$H|0\rangle = |+\rangle \longrightarrow \boxed{H} \longrightarrow |0\rangle$ $\qquad$ $H|+\rangle = |0\rangle$

$H|1\rangle = |-\rangle \longrightarrow \boxed{H} \longrightarrow |1\rangle$ $\qquad$ $H|-\rangle = |1\rangle$

Z gate $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$Z|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -|1\rangle$$

$\alpha|0\rangle + \beta|1\rangle$

$\alpha|0\rangle - \beta|1\rangle$

$\boxed{\text{Phase - Flip gate}}$

Gates

X $\quad$ Y

H $\quad$ S

Z $\quad$ T ($\pi/4$ rotation)

$|\varphi\rangle = (\alpha|0\rangle + \beta|1\rangle)$

If we multiply by $e^{i\theta}$, then no changes will occur,

$|\varphi\rangle'_{\theta} = (\alpha|0\rangle + \beta|1\rangle)e^{i\theta} \longrightarrow$ global Phase

$$\boxed{|\varphi\rangle = |\varphi\rangle'}$$

$|\langle 1|\psi\rangle|^2 = |\beta|^2$

* Similarly → In 1 Qubit

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$

* Two Qubits

→ $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$

→ $|\phi\rangle = \beta_0|0\rangle + \beta_1|1\rangle$

→ $(\alpha_0|0\rangle + \alpha_1|1\rangle)(\beta_0|0\rangle + \beta_1|1\rangle)$

→ $\alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$

→ It is basically tensor product b/w

$$\begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix}_{2\times 1} \otimes \begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix}_{2\times 1} = \begin{bmatrix}\alpha_0\beta_0\\ \alpha_0\beta_1\\ \alpha_1\beta_0\\ \alpha_1\beta_1\end{bmatrix}_{4\times 1}$$

* If $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$|\phi\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$

$$\begin{bmatrix}1\\ -\frac{\sqrt{3}}{2\sqrt{2}}\\ \frac{1}{2\sqrt{2}}\\ -\frac{\sqrt{3}}{4}\end{bmatrix}_{4\times 1}$$

* Bell state - When we cannot factorize qubits or we cannot write tensor product of two states.
or Entangled state

* In $\mathbb{C}^4$ $\quad |\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

* $|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$

* $|\psi\rangle = \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix}$

* <u>Ex:-</u> $\frac{1}{\sqrt{2}}|00\rangle + \frac{i}{2}|01\rangle + \frac{1}{2}|11\rangle$ → It is not normalized

* You measure only first qubit

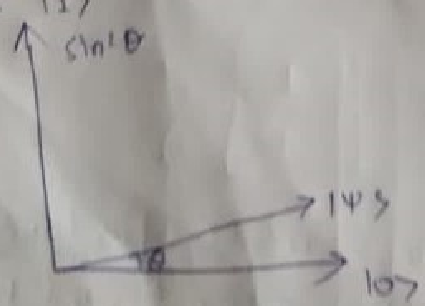
$P[0,\ ] = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

So, $\dfrac{\frac{1}{\sqrt{2}}|00\rangle + \frac{i}{2}|01\rangle}{\sqrt{\frac{1}{2}+\frac{1}{4}}}$ $\quad \sqrt{\frac{1}{2}+\frac{1}{4}} = \sqrt{\frac{2+1}{4}} = \frac{\sqrt{3}}{2}$

* $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \neq \underbrace{|\psi\rangle}_{\alpha_0|0\rangle + \alpha_1|1\rangle} \otimes |\phi\rangle$

$\beta_0|0\rangle + \beta_1|1\rangle$

* <u>Recap</u> ~~Revision~~



if $\theta \approx 0$, Then probability of $\cos^2\theta = 1$ be 100%.

$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$

$= 1 \cdot 1 \cos\theta$

$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1 i}{\sqrt{2}} |1\rangle$

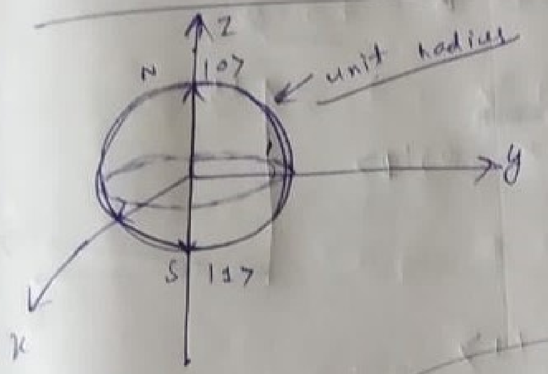
$\cos 45 |0\rangle + \sin 45 |1\rangle$

$\boxed{\text{Probability} = \cos\theta}$

Now, $\langle 0 | \psi \rangle = \frac{1}{\sqrt{2}}$

$\hat{i} \cdot \bar{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

Visualize Bloch sphere

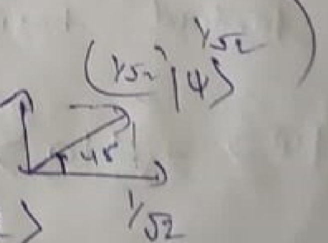


state of $x = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$

$\mathbb{C}^2$ $|v\rangle$ Lin. $|0\rangle$ and $|1\rangle$ Standard-basis

$\mathbb{C}^2$ $|v\rangle$ Lin. $\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$ and $\frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle$ → sign basis

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 (\ ) + c_2 (\ )$



$|\psi\rangle = \cos\theta |0\rangle + \sin\theta |1\rangle$

$\cos^2\theta$

$\sin^2\theta$

$\cos^2\theta + \sin^2\theta = 1$

$\langle 0 | \psi \rangle = \cos\theta$

$|0\rangle$

Measure in $|1\rangle$

spin $\overset{\uparrow}{|0\rangle}$ $\overset{\downarrow}{|1\rangle}$. $\frac{1}{4}+\frac{9}{4}=\frac{10}{4}$

State $\psi = \alpha|0\rangle + \beta|1\rangle$

$|\alpha|^2$ = Probability to get state $|0\rangle$

$|\beta|^2$ = " " " $|1\rangle$

$|\alpha|^2 + |\beta|^2 = 1$

(a) $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ = $\frac{1}{2}+\frac{1}{2}=1$

$\cancel{0.64 + 0.3}$

(b) $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$

(c) $|\psi\rangle = 0.8|0\rangle + 0.6|1\rangle$ $0.64 + 0.36$

X (d) $\psi = \frac{1}{2}|0\rangle + \frac{3}{2}|1\rangle$ $\boxed{\frac{1}{4}+\frac{3}{4}=\frac{10}{4}\neq 1}$

3-Level function

$\to |\psi\rangle = \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \left(\frac{1}{2}+\frac{i}{2}\right)|2\rangle$

$\frac{1}{4}+\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{4}\right)$ $\frac{4}{4}=1$

Outcome	P	new state
if 0	1/4	$\|0\rangle$
if 1	1/4	$\|1\rangle$
if 2	1/2	$\|2\rangle$

$\cancel{|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle}$

$\cancel{\langle 0|\psi\rangle} =$

* $\mathbb{C}^2$

prob. at stage $|0\rangle$ $\leftarrow |\alpha|^2$

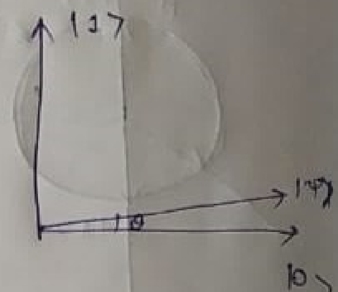
$|\beta|^2 \rightarrow$ probability at this stage $|1\rangle$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

* $\langle 0|\psi\rangle = \alpha \underbrace{\langle 0|0\rangle}_{\| 0 \|^2 = 1} + \beta \underbrace{\langle 0|1\rangle}_{0}$

→ Now, $|\langle 0|\psi\rangle|^2 = |\alpha|^2 = \cos^2\theta$.

* $|\psi\rangle = \cos\theta\,|0\rangle + \cos(\pi/2 - \theta)|1\rangle$



* $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Sign basis or
Hadamard Basis

* standard basis $|0\rangle$ & $|1\rangle$

* $|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ Std. basis

Measure in sign basis $|+\rangle$ and $|-\rangle$

$$|\psi\rangle = \beta_0|+\rangle + \beta_1|-\rangle$$

$$= \frac{\beta_0}{\sqrt{2}}$$

* $|\langle 0|\psi\rangle|^2 = 1/2 = 1/4 \rightarrow$ standard basis

* $|\langle +|\psi\rangle|^2 = |\beta_0|^2$ Sign basis

Now, Put the value of λ_1 & λ_2

$(A - \lambda_1 I)\,|v\rangle = 0$ & where, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$(A - \lambda_2 I)\,|v\rangle = 0$

Hermitian Matrices

A is Hermitian if $A = A^{\dagger}$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} = A^{\dagger}$.

→ a and d are real.

Let $b = x + iy \rightarrow b^* = x - iy$

$c = x - iy \rightarrow c^* = x + iy$

So, Example of A ~~should~~ be like

$A = \begin{bmatrix} 2 & 2-3i \\ 2+3i & 4 \end{bmatrix}$ = Hermitian Matrices

Spectral Theorem

A is Hermitian then A will have Orthonormal set of eigen vectors and real eigen values.

Orthonormal :- Inner product with itself =1 but with other = 0.

$A|v\rangle = \underbrace{v}_{\text{complex no.}}\,|v\rangle$ → Eign value

Eigen vector for operator A and value $= v$.

2.11 Find eign value
eign vector
diagonal rep.
Y, Z

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$X - \lambda I = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$

$\det(X - \lambda I) = 0 \quad \lambda = \pm 1$

$\Rightarrow \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$c_1 = c_2$

$|c_1|^2 + |c_2|^2 = 1$

So, $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

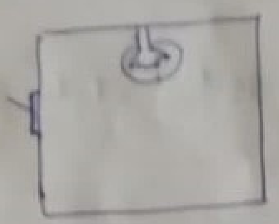
$P = \cancel{|\phi\rangle\langle\phi|}$

$\det |A - \lambda I| \Rightarrow A|v\rangle = \lambda|v\rangle$

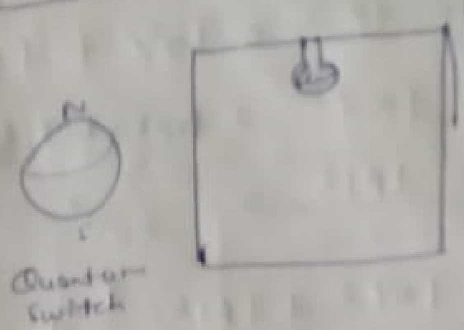
$\Rightarrow \underline{(A - \lambda I)}\,|v\rangle = 0$

$F^2 = (\lambda_1$ → Eign values

Classical Switch



Quantum Switch



State of room (in Quantum switch) $= \frac{1}{\sqrt{2}} |light\rangle + \frac{1}{\sqrt{2}} |dark\rangle$ → When we touch at middle

State of room $= \underset{90\%}{\sqrt{0.9}} |l,\rangle + \underset{10\%}{\sqrt{0.1}} |dark\rangle$ → When we touch near the north pole or south pole

© Qubit → (Quantum Bit)

↳ basic unit of information in QC (quantum computing)

→ It is 2-level system

→ Spin $\underset{up}{\uparrow}$ $\underset{down}{\downarrow}$

→ Photon → (polarization) → (horizontal or vertical)

Triangle Inequality

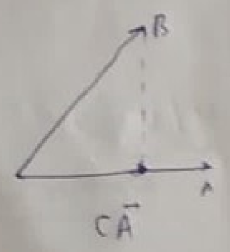
$\|v+w\| \le \|v\| + \|w\|$

- $U^{+} = U^{-1}$. Unitary Matrix $\Rightarrow$ Preserves inner product

- $U|\phi\rangle = |\phi'\rangle$
- $U|\psi\rangle = |\psi'\rangle$
- $\langle\phi'| = \langle\phi|U^{+}$

- $\langle\phi'|\psi'\rangle \Rightarrow \langle\phi|U^{+}U|\psi\rangle \equiv \langle\phi|\psi\rangle$

Projection Matrix



$P = |\phi\rangle\langle\phi|$

$$\phi^2 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{2\times1} \begin{bmatrix} c_1^* & c_2^* \end{bmatrix}_{1\times2} \Rightarrow \begin{bmatrix} c_1c_1^* & c_1c_2^* \\ c_2c_1^* & c_2c_2^* \end{bmatrix}_{2\times2}$$

- $P|\psi\rangle = |\phi\rangle\langle\phi|\psi\rangle$

$\Rightarrow P|\psi\rangle = c|\phi\rangle$

$\Rightarrow P = c.$

$P^2 = c|\phi\rangle\langle\phi|$

$P^2 - P = 0$

$P(P-1) = 0$

$\boxed{P=0}$ $\boxed{P=1}$

$\Rightarrow$

$P^2 = |\phi\rangle\langle\phi|\phi\rangle\langle\phi|$

$= |\phi\rangle\langle\phi| = P \leftarrow$ If $\|\phi\| = 1$

⊛ If f is balanced

$f(0)=0,\ f(1)=1$ or $f(0)=1,\ f(1)=0$

$U_f \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$

$= \left(\frac{(-1)^{f(x)}|0\rangle + (-1)^{f(x)}|1\rangle}{\sqrt{2}}\right) \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$= \frac{(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle}{\sqrt{2}}$ $\quad K$

$= \frac{|0\rangle - |1\rangle}{\sqrt{2}}\ K$ if $f(0)=0,\ f(1)=1$

$|\psi_3\rangle = \frac{-(|0\rangle + |1\rangle)}{\sqrt{2}} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$ if $f(0)=1, f(1)=0$

$\Rightarrow |\psi_3\rangle = \pm \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$, if f is balanced.

⊛ $|\psi_4\rangle = H \otimes I\ |\psi_3\rangle$

$= I\ |1\rangle \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$ if f is balanced

$|\psi_4\rangle = H \otimes I\ |\psi_3\rangle$

$= \pm |0\rangle \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$ if f is constant.

$\boxed{\text{Deutsch-Jossa Algorithm}}$

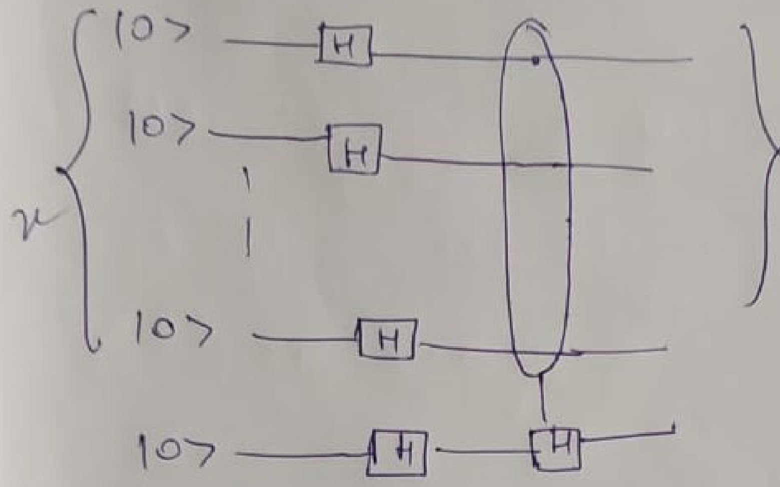
⊛ $f : Z_2^n \longrightarrow Z_2 \quad \rightarrow |0\rangle\rangle \text{---} |2^n - 1\rangle\rangle$

$\underbrace{f(|x\rangle\rangle) = f(|y\rangle\rangle)}_{\text{Constant function}}$

$f(x) = 0$ for 2^{n-1} input

$f(x) = 1$ for the remaining 2^{n-1} input

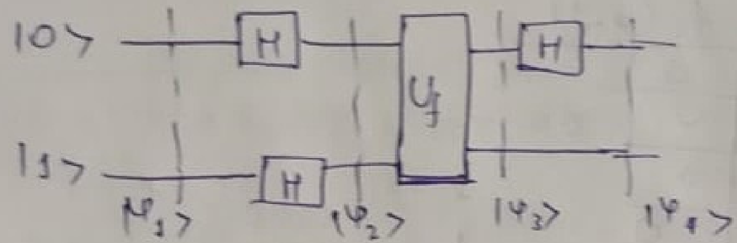
Q) Is f is balanced or constant ?



Hint $Z_2^2 \to Z_2$

Compare

Q - version :-



$U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$

* $|\psi_1\rangle = |0\rangle|1\rangle$

$$|\psi_2\rangle = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

$$= \frac{|0\rangle|0\rangle+|1\rangle|0\rangle-|0\rangle|1\rangle-|1\rangle|1\rangle}{2}$$

$$= \frac{|0\rangle|0\oplus f(0)\rangle-|0\rangle|1\oplus f(0)\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle|0\rangle-|0\rangle|1\rangle}{\sqrt{2}} \quad \text{if } f(0)=0$$

$$= \frac{|0\rangle(|0\rangle-|1\rangle)}{\sqrt{2}} \quad \text{if } f(0)=0.$$

* $$U_f \frac{|0\rangle(|0\rangle-|1\rangle)}{\sqrt{2}} = \frac{|0\rangle|f(0)\rangle-|0\rangle|1\oplus f(0)\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle|1\rangle-|0\rangle|0\rangle}{\sqrt{2}} \quad \text{if } f(0)=1$$

$$= \frac{|0\rangle(|1\rangle-|0\rangle)}{\sqrt{2}} = -|0\rangle\frac{(|1\rangle-|0\rangle)}{\sqrt{2}}.$$

$$U_f\,|1\rangle \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} = \frac{|1\rangle|0 \oplus f(1)\rangle - |1\rangle|1 \oplus f(1)\rangle}{\sqrt{2}}$$

$$= \frac{|1\rangle|0\rangle - |1\rangle|1\rangle}{\sqrt{2}} \quad \text{if } f(1) = 0$$

$$= |1\rangle \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$$

$$U_f\,|1\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{|1\rangle|1\rangle - |1\rangle|0\rangle}{} \text{ if } f(1)=1$$

$$= \cancel{|1\rangle|0\rangle} = \frac{|1\rangle\ (|1\rangle - |0\rangle)}{\sqrt{2}}$$

$$= -|1\rangle \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$$

$$U_f\,|x\rangle \frac{(|0\rangle - |1\rangle)}{2} = (-1)^{f(x)}\,|x\rangle \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$$

So, $U_f \frac{(|0\rangle + |1\rangle)}{\sqrt{2}} \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$

$$= U_f \frac{|0\rangle}{\sqrt{2}} \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} + U_f \frac{|1\rangle}{\sqrt{2}} \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$$

$$= (-1)^{f(x)} \frac{|0\rangle}{\sqrt{2}} \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} + (-1)^{f(x)} \frac{|1\rangle}{\sqrt{2}} (|0\rangle - |1\rangle)$$

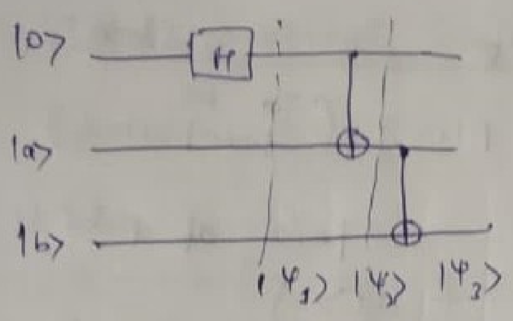
$$= \left[\frac{(-1)^{f(x)}\,|0\rangle + (-1)^{f(x)}\,|1\rangle)}{\sqrt{2}}\right] \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$$

$$|\psi_3\rangle = \frac{-(|0\rangle - |1\rangle)}{\sqrt{2}} \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} \quad f(0) = f(1) = 1.$$

$$|\psi_3\rangle = \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} \quad f(0) = f(1) = 0$$

$$|\psi_3\rangle = \pm \frac{(|0\rangle + |1\rangle)}{\sqrt{2}} \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} \quad \text{if } f = \text{constant}.$$

(2)



$$|\psi_1\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|a\rangle|b\rangle$$

$$= \frac{|0\rangle|a\rangle|b\rangle + |1\rangle|a\rangle|b\rangle}{\sqrt{2}}$$

$$|\psi_2\rangle = \frac{|0\rangle|0\oplus a\rangle|b\rangle + |1\rangle|1\oplus a\rangle|b\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle|a\rangle|b\rangle + |1\rangle|1\oplus a\rangle|b\rangle}{\sqrt{2}}$$

$$|\psi_3\rangle = \frac{|0\rangle|a\rangle|a\oplus b\rangle + |1\rangle|1\oplus a\rangle|1\oplus a\oplus b\rangle}{\sqrt{2}}$$

a	b	$\vert\psi_3\rangle$
0	0	$\frac{\vert 000\rangle + \vert 111\rangle}{\sqrt{2}}$
0	1	
1	0	
1	1	

A $\{0,1\}$ B $\{0,1\}$

$f_1 : A \to B$ (Boolean Function)

$\{0,1\}$ $\{0,1\}$

$\therefore$ $f_2(x,y) = g(z)$ $x \in A_1,\ y \in A_2,\ z \in B_1$

All possible combinations are

$f(0) = 1$ $f_1(0) = 0$ $f_1(0) = 1$ $f_1(0) = 0$

$f(1) = 0$ $f_1(1) = 0$ $f_1(1) = 1$ $f_1(1) = 1$

$\underbrace{f_1(0)=0,\ f_1(1)=0 \quad f_1(0)=1,\ f_1(1)=1}_{\text{Constant}}$

Teleportation of "Quantum State"

$$|\psi\rangle = (\alpha|0\rangle + \beta|1\rangle) \quad (\text{To be teleported})$$

Resource available: a pair of qubits in $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$|\text{3-qubit system}\rangle = |\psi\rangle \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$$

$$= (\alpha|0\rangle + \beta|1\rangle)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$$

$$= \frac{\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle}{\sqrt{2}}$$

$$\cancel{\frac{\alpha^2}{2} + \frac{\beta^2}{2}} \quad = \frac{1}{2}|00\rangle\left(\frac{\alpha|0\rangle + \beta|1\rangle}{\sqrt{2}}\right) + \frac{1}{2}|00\rangle\left(\frac{\alpha|0\rangle - \beta|1\rangle}{\sqrt{2}}\right)$$

$$+ \left[\frac{1}{2}|01\rangle\left(\frac{\alpha|1\rangle + \beta|0\rangle}{\sqrt{2}}\right) + \frac{1}{2}|01\rangle\left(\frac{\alpha|1\rangle - \beta|0\rangle}{\sqrt{2}}\right)\right]$$

$$+ \_\_\_$$

$$= \frac{1}{2}\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)(\alpha|0\rangle + \beta|1\rangle) +$$

$$\frac{1}{2}\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right)(\alpha|1\rangle + \beta|0\rangle) +$$

$$\frac{1}{2}\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right)(\alpha|0\rangle - \beta|1\rangle) + \frac{1}{2}\left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)$$

$$(\alpha|1\rangle - \beta|0\rangle)$$

So, apply X & Z at appropriate place.

Ex 2, Ex 3 :- Constant function

Ex 1, Ex 4 :- Balanced function.

x	y	$f_2(x,y)$		
0	0	0		1
0	1	0	or	1
1	1	0		1
1	0	0		1

$f_3 : A_1 \times A_2 \longrightarrow A_1 \times A_2$

Ex. $f_3(x,y) = (x \oplus 1, y \oplus 1)\ x \in A_1, y \in A_2$

Ex: $f_3(x,y) = (x, x \cdot y)$

$\underbrace{Z_2 \times Z_2 \times Z_2 \times \dots Z_2}_{m\text{-times}} = Z_2^m$

$|x\rangle|y\rangle \quad x \in Z_2, y \in Z_2$

$|0\rangle|0\rangle = |00\rangle \equiv |0\rangle\rangle_2$

$|0\rangle|1\rangle = |01\rangle = |1\rangle\rangle_2$

$|1\rangle|0\rangle = |10\rangle \equiv |2\rangle\rangle_2$

$|1\rangle|1\rangle = |11\rangle \equiv |3\rangle\rangle_2$

$(X \otimes Z)|0\rangle|0\rangle = X|0\rangle \otimes Z|0\rangle$

$(X \otimes X)|0\rangle|0\rangle = X|0\rangle \otimes |0\rangle = X^{\otimes 2}|0\rangle_2$

Ex $X \otimes X \otimes Z \otimes Z \otimes Z|3\rangle_5 = X^{\otimes 2} \otimes Z^{\otimes 3}|3\rangle_5$

$f : \{0,1\} \longrightarrow \{0,1\}$

Given :- f is either constant or balanced

Qn :- Is f constant or balanced?

Classical :- (i) Calculate $f(0)$ (ii) Calculate $f(1)$

$$\alpha = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |output\rangle \rangle$$

$$= (1 \quad 0) * \frac{1}{2}\begin{pmatrix} 1+\omega \\ 1-\omega \end{pmatrix}$$

$$= \frac{1+\omega}{2}$$

$$\cancel{|\alpha|^2 = \frac{1+\omega^2+2\omega}{4} = 1+1+1}, \quad \omega = \frac{1+i}{\sqrt{2}}$$

$$|\alpha|^2 = \left|\frac{1+\omega}{2}\right|^2 \rightarrow \text{Irrational}$$

$$\cancel{= 1+\omega^2+2\omega}$$

$$= \left|\frac{1+\left(\frac{1+i}{\sqrt{2}}\right)}{2}\right|^2 = \left|\frac{(\sqrt{2}+1)+i}{2\sqrt{2}}\right|^2$$

$$= \frac{(\sqrt{2}+1)^2+1}{8} = \frac{2+1+2\sqrt{2}+1}{8} = \frac{4+2\sqrt{2}}{8}$$

$$= \frac{2\sqrt{2}\,(\sqrt{2}+1)}{8} = \frac{\sqrt{2}+1}{2\sqrt{2}}$$

$$|\beta|^2 = 1-|\alpha|^2 = 1-\frac{(\sqrt{2}+1)}{2\sqrt{2}} = \frac{2\sqrt{2}-\sqrt{2}-1}{2\sqrt{2}}$$

$$= \frac{\sqrt{2}-1}{2\sqrt{2}}$$

$\boxed{\text{Bell states}}$

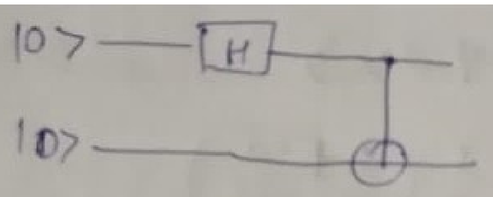
$$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |0\rangle$$

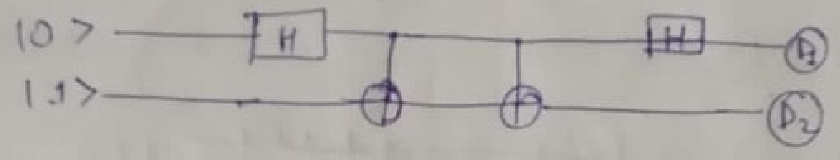
0 $\quad 1+\rangle$

$$|01\rangle + |10\rangle$$

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

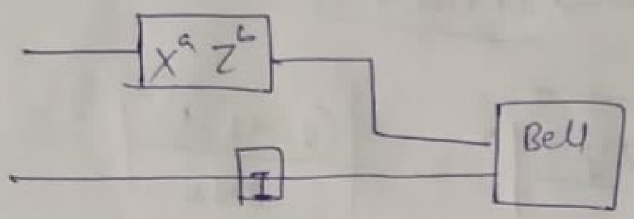
$\boxed{\text{Using CNOT or addition modulo 2}}$





D1 – detects $|0\rangle$ or $|1\rangle$

D2 – " " " "



$a, b \in \{0,1\}$

$$X^a Z^b \otimes I$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

a	b	output
0	0	$\|B_{00}\rangle$
1	0	$\|B_{01}\rangle$
0	1	$\|B_{10}\rangle$
1	1	$\|B_{11}\rangle$

$$\begin{bmatrix} 0I & 1I \\ 1I & 0I \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now, $\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{4\times 1} \underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{4\times 4}$ $\underbrace{(X^a Z^b \otimes I)}_{4\times 4} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

Alice and Bob

$$B|00\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |B_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|B_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \quad |B_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

$\boxed{QIC}$

(*) $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$-\boxed{H_1}-\boxed{T}-\boxed{H_2}-$

$T(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$, $T(\theta) = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{|0\rangle + e^{i\theta}|1\rangle}{\sqrt{2}}$

$\theta = \pi/4$

$H_2 T H_1 |input\rangle$

$\Downarrow$

$\frac{1}{2}\begin{pmatrix} 1+\omega & 1-\omega \\ 1-\omega & 1+\omega \end{pmatrix}$, $e^{i\theta} = \frac{1+i}{\sqrt{2}} = \omega$

(*) If inputs $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|output\rangle = \frac{1}{2}\begin{pmatrix} 1+\omega \\ 1-\omega \end{pmatrix}$

$T^{+} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = T(-\theta)$

(*) $(AB)^{+} = B^{+}A^{+}$

(*) $(H_1 T H_2)^{+} = (H_2^{+} T^{+} H_1^{+}) = H_2 T^{+} H_1$ ← because they are unitary

$\Rightarrow (H_1 T H_2)^{+} (H_1 T H_2)$

$= (H_2^{+} T^{+} H_1^{+} H_1 T H_2)$

$= (H_2^{+} T^{+} T H_2) = (H_2^{+} H_2) = I.$

→ Make measurement such that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the post - measure states

(*) $|output\rangle$

Prob $\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = |\alpha|^2$

where $|output\rangle = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix}$