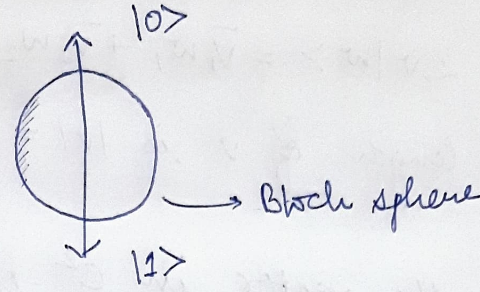
Quantum Computing

→ "Anyone who is not shocked by quantum theory has not understood it."

— Neils Bohr.

→ A quantum bit, or qubit is the fundamental information unit of quantum computing.



$\vec{q} = a\vec{v} + b\vec{w}$, $\vec{q}$ is a linear combination of vectors $\vec{v}$ and $\vec{w}$ and they are orthonormal basis (i.e. $\vec{v}.\vec{w} = 0$). If a and b are non-zero the linear combination is non-trivial.

At any given time, q-bit is in a superposition state represented by linear combination of vectors $|0\rangle$ and $|1\rangle$ in C^2:

$$a|0\rangle + b|1\rangle \text{ where } |a|^2 + |b|^2 = 1$$

probability amplitudes

$\vec{v} = (V_1, V_2, V_3, \ldots V_n)$

$= [V_1 \; V_2 \; V_3 \; \ldots \; V_n]$

$= \langle v|$ ⟶ "bra-v"

transpose complex conjugate of $\langle v|$

$= |w\rangle$ ⟶ "ket-w"

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix}$$

$|v\rangle\langle w|$ is the outer product,

$$|v\rangle\langle w| = \begin{bmatrix} v_1\bar{w}_1 & & v_1\bar{w}_m \\ v_2\bar{w}_2 & & \vdots \\ \vdots & \cdots & \vdots \\ v_n\bar{w}_1 & & v_n\bar{w}_m \end{bmatrix}$$

$\langle v|w\rangle$ is the inner product,

$$\langle v|w\rangle = \bar{v}_1 w_1 + \bar{v}_2 w_2 + \dots \bar{v}_n w_n$$

The length of v is $|v| = \sqrt{\langle v|v\rangle}$

$|0\rangle$ is the vector in C^2 that has coordinates $(1,0)$ and $|1\rangle$ has coordinates $(0,1)$ in the basis e_1 and e_2.

They are frequently called computational basis.

$|+\rangle = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$|-\rangle = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

If L is a linear ~~a~~ transformation on complex vector space, we write,

$$L|w\rangle,$$

If A is a matrise for L

$$A|w\rangle$$

Both A and w must be on the same basis.

property for $\langle v|L$ or $\langle v|A$

$$\langle v|L \times |w\rangle = \langle v| \times L|w\rangle = \langle v|L|w\rangle$$
$$\langle v|A \times |w\rangle = \langle v| \times A|w\rangle = \langle v|A|w\rangle$$

A Hermitian matrix 'A' is a projection matrix if $A^2 = A$. The rank of the projector is the dimension of its image vector space.

$P(i) \rightarrow 1$
$P(i,j) \rightarrow 2$
$P(1, \dots n) = I_n$

Quantum state representation,

$|\psi\rangle = a|0\rangle + b|1\rangle$ —— quantum state.

$|\psi|^2 = |a|^2 + |b|^2 = 1$

$\boxed{\langle\psi|\psi\rangle = 1}$ same

e.g. Let $a = \frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}$. Then $|a|^2 = \frac{1}{4} = 0.25$ and $|b|^2 = \frac{3}{4} = 0.75$. This means that the states of qubit has 25% chance of collapsing to $|0\rangle$ and 75% chance of collapsing to $|1\rangle$.

* If a complex number c in $\mathbb{C}$ is multiplied to $|\psi\rangle$, there's no change is probility if $|c|^2 = 1$.

* The state of single qubit can be represented by -

$$|\psi\rangle = r_1 |0\rangle + r_2 e^{i\phi} |1\rangle$$

$$r_1^2 + r_2^2 = 1 \qquad \boxed{r_1, r_2 \in \mathbb{R}}$$

$$\boxed{0 \leq \phi < 2\pi}$$

$$r_1 = \cos\left(\frac{\theta}{2}\right) \qquad \boxed{0 \leq \theta \leq \pi}$$

$$r_2 = \sin\left(\frac{\theta}{2}\right)$$

$$\boxed{|\psi\rangle = \cos\left(\frac{\theta}{2}\right) |0\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\phi} |1\rangle}$$

* Density Matrix

for $|\psi\rangle = a|0\rangle + b|1\rangle$, the density matrix of $|\psi\rangle$ to be,

$$\rho = |\psi\rangle\langle\psi|$$

$$= \begin{bmatrix} a \\ b \end{bmatrix} \otimes [\bar{a} \ \bar{b}]$$

$$= \begin{bmatrix} a\bar{a} & a\bar{b} \\ b\bar{a} & b\bar{b} \end{bmatrix}$$

$$= \begin{bmatrix} |a|^2 & a\bar{b} \\ b\bar{a} & |b|^2 \end{bmatrix}$$

$$\det|\rho| = |a|^2|b|^2 - a\bar{b}b\bar{a}$$

$$= (ab)^2 - (ab)^2$$

$$= 0$$

* Observables and expectation.

$M_0 = |0\rangle\langle 0|$
$M_1 = |1\rangle\langle 1|$ } projectors

$|\psi\rangle = a|0\rangle + b|1\rangle$

$\langle\psi|M_0|\psi\rangle = \langle\psi|(|0\rangle\langle 0|)|\psi\rangle$

$\cancel{= \langle\psi|0\rangle\langle 0|\psi\rangle}$

$= \langle\psi|0\rangle\langle 0|\psi\rangle$

$= [\bar{a} \ \bar{b}]\begin{bmatrix}1\\0\end{bmatrix}[1 \ 0]\begin{bmatrix}a\\b\end{bmatrix}$

$= [\bar{a} \ \bar{b}]\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}$

$= [\bar{a} \ \bar{b}]\begin{bmatrix}a\\0\end{bmatrix}$

$= |a|^2$

Similarly,

$\langle\psi|M_1|\psi\rangle = |b|^2$

The eigenvalues of M_0 are 0 and 1
eigenvectors of M_0 are $|0\rangle$ and $|1\rangle$
The eigenvalues and eigenvectors of M_1 are same as M_0.

$\therefore$ M_0 and M_1 are examples of observables, i.e. Hermitian matrices whose eigenvectors form a basis for the quantum state space.

Q. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$f_A(x) \Rightarrow |xI - A| = |A - xI| = 0$$

$$\Rightarrow (x-2)(x-3) - 2 = 0$$

$$\Rightarrow x^2 - 5x + 6 - 2 = 0$$

$$\Rightarrow x^2 - 5x + 4 = 0$$

$$x = 4, 1$$

$$\left.\begin{matrix} \lambda_1 = 4 \\ \lambda_2 = 1 \end{matrix}\right\} \text{ eigenvalues}$$

$$[A - 4I][X_1] = 0 \qquad\qquad [A - I][X_2] = 0$$

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2x + 2y = 0$$

$$x - y = 0$$

$$x = y$$

$$-2x + 2x = 0$$

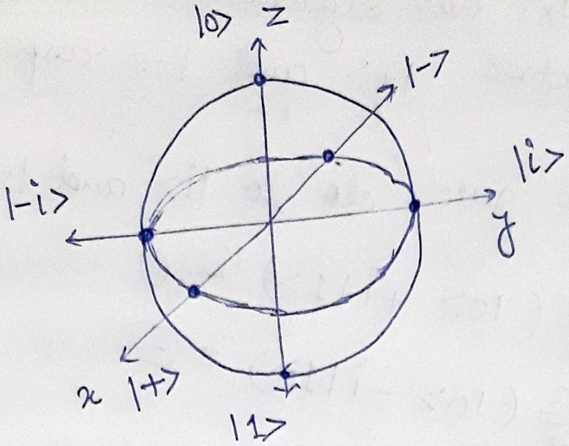
Using elementary transformation

$$\underbrace{X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\text{eigen vectors.}}$$

Measurable :-

$$\underbrace{\begin{matrix} |\langle e_1 | \psi \rangle|^2 \longrightarrow \text{probability of getting } e_1 \\ |\langle e_2 | \psi \rangle|^2 \longrightarrow \text{probability of getting } e_2 \end{matrix}}_{\text{measurable.}}$$

* Bloch sphere



Basis	Common Name	Bloch sphere axis name
$\{\|0\rangle, \|1\rangle\}$	Computational	Z
$\{\|+\rangle, \|-\rangle\}$	Hadamard	X
$\{\|i\rangle, \|-i\rangle\}$	Circular	Y

$\hookrightarrow \{|\circlearrowright\rangle, |\circlearrowleft\rangle\}$.

The Quantum X gate

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$—\boxed{X}—$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\underset{\text{input}}{X} —\boxed{\text{gate}}— \underset{\text{output}}{Y}$$

$$\sigma_x |0\rangle = |1\rangle$$

$$\sigma_x |1\rangle = |0\rangle$$

for, $|\psi\rangle = a|0\rangle + b|1\rangle$

$$X|\psi\rangle = b|0\rangle + a|1\rangle$$

→ The X gate σ_x has eigenvalues +1 and −1 and eigenvectors $|+\rangle$ and $|-\rangle$ respectively.

Q. What does X gate do to $|i\rangle$ and $|-i\rangle$?

$$|i\rangle = \frac{\sqrt{2}}{2}(|0\rangle + i|1\rangle) \quad — ①$$

$$|-i\rangle = \frac{\sqrt{2}}{2}(|0\rangle - i|1\rangle) \quad — ②$$

$$A = X|i\rangle = \frac{\sqrt{2}}{2}(i|0\rangle + |1\rangle) \quad — ③$$

$$B = X|-i\rangle = \frac{\sqrt{2}}{2}(-i|0\rangle + |1\rangle) \quad — ④$$

Multiplying $-i$ in ③ and ④

$$A = \frac{\sqrt{2}}{2}(|0\rangle - i|1\rangle)$$

$$B = \frac{\sqrt{2}}{2}(-|0\rangle - i|1\rangle)$$

$$\therefore X|i\rangle = -\frac{|-i\rangle}{i}$$

$$X|-i\rangle = \frac{|i\rangle}{i}$$

The quantum Z gate (phase flip gate)

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad -\boxed{Z}-$$

→ It rotates qubit states by π around the z-axis on Bloch sphere.

$$\sigma_z |+\rangle = |-\rangle$$
$$\sigma_z |-\rangle = |+\rangle$$
$$\sigma_z |i\rangle = |-i\rangle$$
$$\sigma_z |-i\rangle = |i\rangle$$

swaps $|+\rangle$, $|-\rangle$, $|i\rangle$, $|-i\rangle$ but leaves $|1\rangle$ and $|0\rangle$.

for,

$$|\psi\rangle = a|0\rangle + b|1\rangle$$
$$Z|\psi\rangle = a|0\rangle - b|1\rangle$$

→ σ_z has eigenvalues $+1$ and -1 and eigenvectors $|0\rangle$ and $|1\rangle$, respectively.

The quantum Y gate

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

→ It rotates ________ y-axis ____.

→ swaps $|0\rangle$, $|1\rangle$, $|+\rangle$ and $|-\rangle$ but leaves $|i\rangle$ and $|-i\rangle$ alone.

$$-\boxed{Y}-$$

for,

$$|\psi\rangle = a|0\rangle + b|1\rangle$$
$$Y|\psi\rangle = -bi|0\rangle + ai|1\rangle$$

→ σ_y has eigenvalues $+1$ and -1 for eigen-vectors $|i\rangle$ and $|-i\rangle$, respectively.

The quantum ID gate

"does nothing"

$$-\boxed{ID}-$$

The quantum H gate

$$-\boxed{H}-$$

$$H = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H|0\rangle = \frac{\sqrt{2}}{2}(|0\rangle + |1\rangle) = |+\rangle$$

$$H|1\rangle = \frac{\sqrt{2}}{2}(|0\rangle - |1\rangle) = |-\rangle$$

$$H|+\rangle = H\left(\frac{\sqrt{2}}{2}(|0\rangle + |1\rangle)\right)$$

$$= \frac{\sqrt{2}}{2}(H|0\rangle + H|1\rangle)$$

$$= |0\rangle$$

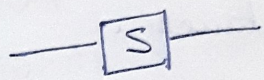
$$H|-\rangle = |1\rangle$$

The quantum R_ϕ^z gates

$$R_\phi^z = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

$$\left.\begin{aligned} R_0^z &= ID \\ R_\pi^z &= Z \end{aligned}\right\}$$

The quantum S gate



$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

The quantum $\sqrt{NOT}$ gate.

$$= \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

Quantum Entanglement..

Quantum entanglement is the phenomenon that occurs when a group of particles are generated, interact or share spatial proximity in a way such that quantum
each particle of group cannot be described independ
state of ~~the~~ others, including when
of the state of .
the particles are separated by a large distance.

→ Let q_1 and q_2 be two qubits and let $\{|0\rangle_1, |1\rangle_1\}, \{|0\rangle_2, |1\rangle_2\}$ be standard orthonormal basis kets.

$$|\psi\rangle_1 = a_1|0\rangle_1 + b_1|1\rangle_1 \qquad |a_1|^2 + |b_1|^2 = 1$$
$$|\psi\rangle_2 = a_2|0\rangle_2 + b_2|1\rangle_2 \qquad |a_2|^2 + |b_2|^2 = 1$$

The four kets,

$$|0\rangle_1 \otimes |0\rangle_2 \ , \ |0\rangle_1 \otimes |1\rangle_2$$
$$|1\rangle_1 \otimes |0\rangle_2 \ , \ |1\rangle_1 \otimes |1\rangle_2$$

are a basis for the combined set space.

$$|\psi\rangle_1 \otimes |\psi\rangle_2 = a_1 a_2 |0\rangle_1 \otimes |0\rangle_2 + a_1 b_2 |0\rangle_1 \otimes |1\rangle_2 + b_1 a_2 |1\rangle_1 \otimes |0\rangle_2 + b_1 b_2 |1\rangle_1 \otimes |1\rangle_2 .$$

Simplyfying -

$$|\psi\rangle_1 \otimes |\psi\rangle_2 = a_1 a_2 |0\rangle_1 |0\rangle_2 + a_1 b_2 |0\rangle_1 |1\rangle_2 + b_1 a_2 |1\rangle_1 |0\rangle_2 + b_1 b_2 |1\rangle_1 |1\rangle_2 .$$

$$|\psi\rangle_1 \otimes |\psi\rangle_2 = a_1 a_2 |0\rangle |0\rangle + a_1 b_2 |0\rangle |1\rangle + b_1 a_2 |1\rangle |0\rangle + b_1 b_2 |1\rangle |1\rangle$$

$$|\psi\rangle_1 \otimes |\psi\rangle_2 = a_1 a_2 |00\rangle + a_1 b_2 |01\rangle + a_2 b_1 |10\rangle + b_1 b_2 |11\rangle$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|01\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad |01\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \qquad |10\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

$$|11\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

→ $\langle 01|01\rangle = 1$

$\langle 01|11\rangle = 0$

same → 1
diff → 0

$$|\psi\rangle_1 \otimes |\psi\rangle_2 = a_1a_2|00\rangle + a_1b_2|01\rangle + b_1a_2|10\rangle + b_1b_2|11\rangle$$

$$|a_1a_2|^2 + |a_1b_2|^2 + |b_1a_2|^2 + |b_1b_2|^2 = 1$$

we can prove this.

→ Before measurement the qubits are in a state of entanglement.

→ The entangled state is known as Bell state.

$$|\phi^+\rangle = \frac{\sqrt{2}}{2}|00\rangle + \frac{\sqrt{2}}{2}|11\rangle$$

$$|\phi^-\rangle = \frac{\sqrt{2}}{2}|00\rangle - \frac{\sqrt{2}}{2}|11\rangle$$

$$|\psi^+\rangle = \frac{\sqrt{2}}{2}|01\rangle + \frac{\sqrt{2}}{2}|10\rangle$$

$$|\psi^-\rangle = \frac{\sqrt{2}}{2}|01\rangle - \frac{\sqrt{2}}{2}|10\rangle$$

Q How to determine if $|\psi\rangle$ is entangled?

Ans: Let $|\psi\rangle$ be a 2-qubit quantum statet in $C^2 \otimes C^2$.

$|\psi\rangle$ is entangled if it cannot be written as the tensor products of two 1-qubit kets.

$$|\psi^+\rangle = 0|00\rangle + \frac{\sqrt{2}}{2}|01\rangle + \frac{\sqrt{2}}{2}|10\rangle + 0|11\rangle$$

$$= a_1a_2|00\rangle + a_1b_2|01\rangle + a_2b_1|10\rangle + b_1b_2|11\rangle$$

$a_1a_2 = 0$ $\quad a_1b_2 = \frac{\sqrt{2}}{2}$ $\quad a_2b_1 = \frac{\sqrt{2}}{2}$ $\quad b_1b_2 = 0$

either $a_1 = 0$, $a_2 = 0$

which contradicts both a_1b_2 and a_2b_1

This means $|\psi^+\rangle$ cannot be written as the tensor product of two 1-qubit kets and it is an entangled state.

→ If a quantum state is not entangled then it is seperable.

Q. Is $\frac{\sqrt{2}}{2}|00\rangle + \frac{\sqrt{2}}{2}|01\rangle$ an entangled state?

Ans. $\underbrace{a_1a_2 = \frac{\sqrt{2}}{2} \quad a_1b_2 = \frac{\sqrt{2}}{2}}_{}$ $\quad a_2b_1 = 0$, $b_1 = 0$, $a_2 \neq 0$ $\quad b_1b_2 = 0$, $b_1 = 0$, $b_2 \neq 0$

Satisfies 1-qubit state

∴ not-entangled

i.e seperable

* **The general case**

n - bit quantum space

$2^n \longrightarrow$ state space of dimension.

$$|\psi\rangle = a_0|0\rangle_n + a_1|1\rangle_n + a_2|2\rangle_n + \dots + a_{2^n-1}|2^n-1\rangle_n$$

$$|a_0|^2 + |a_1|^2 + \dots + |a_{2^n-1}|^2 = 1$$

$$|\underbrace{00000\dots000}_{n}\rangle = |0\rangle^{\otimes n}$$

* **Density Matrix**

$$\rho = |\psi\rangle\langle\psi|$$

$$= \begin{bmatrix} |a_0|^2 & a_1\bar{a}_0 & a_2\bar{a}_0 & \dots & a_{2^n-1}\bar{a}_0 \\ a_0\bar{a}_1 & |a_1|^2 & \dots & & \vdots \\ \vdots & & \ddots & & \vdots \\ a_0\bar{a}_{2^n-1} & \dots & \dots & \dots & |a_{2^n-1}|^2 \end{bmatrix}$$

$$tr(\rho) = 1$$

The quantum $H^{\otimes n}$ gate

$$H = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$|\psi\rangle_1 = a_1|0\rangle_1 + b_1|1\rangle_1$ $\quad$ $|\psi\rangle_2 = a_2|0\rangle_2 + b_2|1\rangle_2$

$$(H|\psi\rangle_1) \otimes (H|\psi\rangle_2)$$

$$= (H \otimes H)(|\psi\rangle_1 \otimes |\psi\rangle_2)$$

$$= H^{\otimes 2}(|\psi\rangle_1 \otimes |\psi\rangle_2)$$

$$H^{\otimes 2} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{\sqrt{2}}{2}\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

$$H^{\otimes 3} = \frac{\sqrt{2}}{2}\begin{bmatrix} H^{\otimes 2} & H^{\otimes 2} \\ H^{\otimes 2} & -H^{\otimes 2} \end{bmatrix}$$

Recursion? Yes.

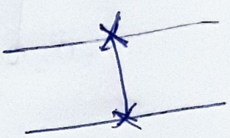
$$H^{\otimes n} = \frac{\sqrt{2}}{2}\begin{bmatrix} H^{\otimes n-1} & H^{\otimes n-1} \\ H^{\otimes n-1} & -H^{\otimes n-1} \end{bmatrix}$$

$$H^{\otimes n}|0\rangle_n = \frac{1}{\sqrt{2^n}}\sum_{j=0}^{n-1}|j\rangle_n$$

Dropping subscripts.

$$\boxed{H^{\otimes n}|0\rangle = \underbrace{\frac{1}{\sqrt{2^n}}}_{\text{Normalization constant.}}\sum_{j=0}^{n-1}|j\rangle}$$

The quantum SWAP gate



$$M(|\psi_1\rangle \otimes |\psi\rangle_2) = |\psi\rangle_2 \otimes |\psi\rangle_1$$

$$M\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_3 \\ v_2 \\ v_4 \end{bmatrix}$$

$v_2, v_3 \rightarrow$ interchange.

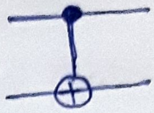
$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The quantum CNOT gate | CX gate.

"commonly used to create entangled qubits".

C → controlled

CNOT is conditional bit flip.



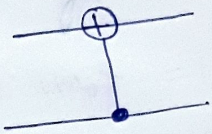
ID $|\psi_1\rangle$ then $X|\psi_2\rangle$ OR
ID $|\psi_2\rangle$ then $X|\psi_1\rangle$.

$$\text{CNOT}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_4 \\ v_3 \end{bmatrix}$$

v_3, v_4 → interchange.

$$\text{CNOT} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \to X$$

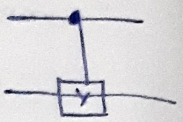
Reverse CNOT



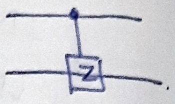
v_1, v_2 → interchange?

The quantum CY and CZ gate

$$CY = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{array}\right] \rightarrow Y$$



$$CZ = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right] \rightarrow Z$$

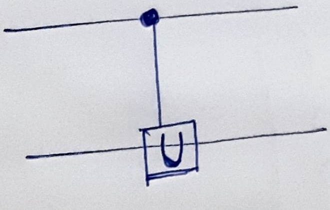


→ The CZ is conditional sign flip.

In general,

if $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

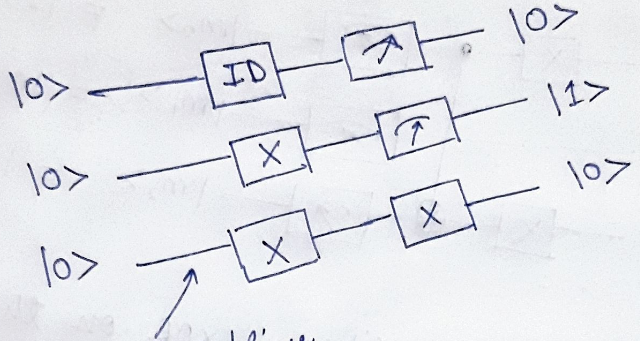
$$\text{controlled-}U\ (CU) = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & a & b \\ 0 & 0 & c & d \end{array}\right]$$



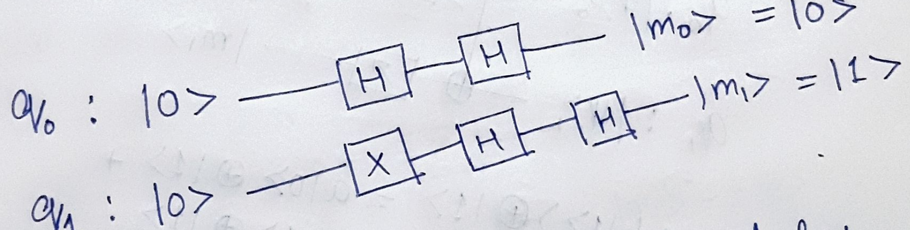
WIRING UP THE CIRCUITS

→ quantum register :- A quantum register is a collection of qubits we use for computation.

→ A quantum circuit is a sequence of gates applied to one or more qubits in a quantum register.



→ If nothing is happening just remove the intermediate gates.

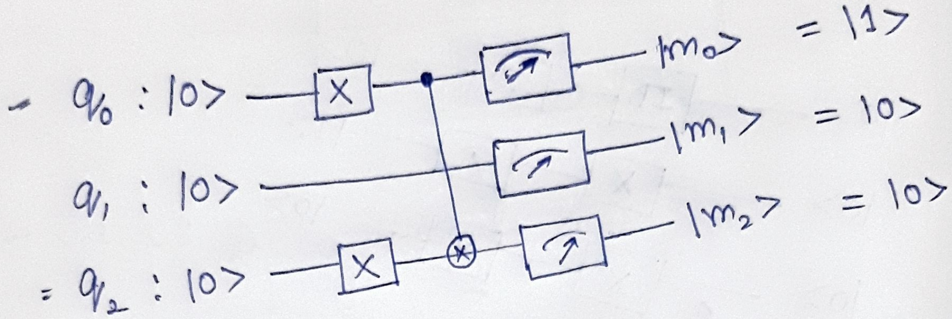


The first wire has depth 2 and second has depth 3.

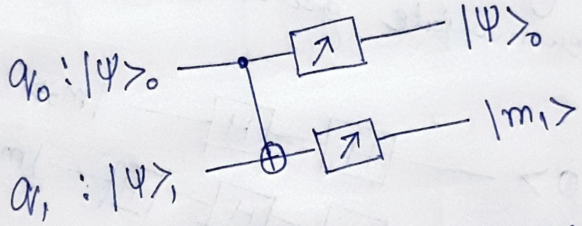
*

$$CNOT\,|00\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle$$

$$CNOT\,|11\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |1$$



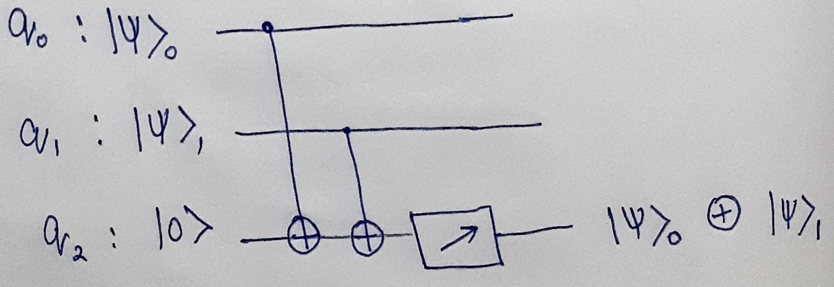
* The basic CNOT acts like a XOR on the standard kets.



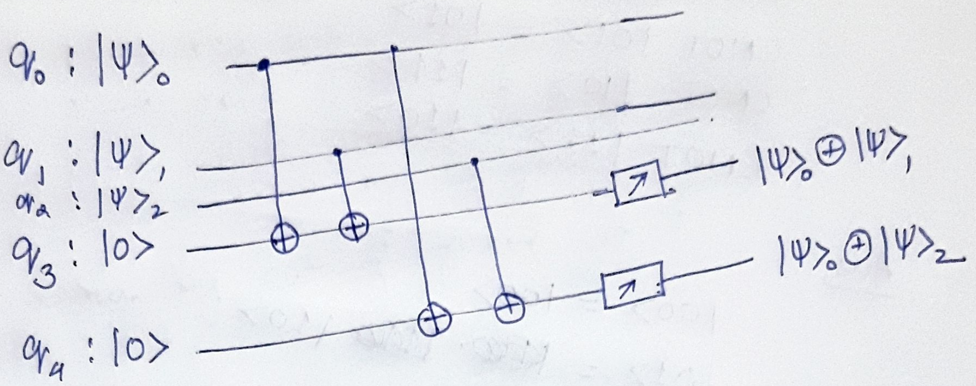
$$(a|0\rangle + b|1\rangle) \oplus |1\rangle = a|0\rangle \oplus |1\rangle + b|1\rangle \oplus |1\rangle$$

$$= a|1\rangle + b|0\rangle$$

→ If we do not want to modify one of the input qubits, we can put the value of the XOR in a third output, or ancilla qbit.

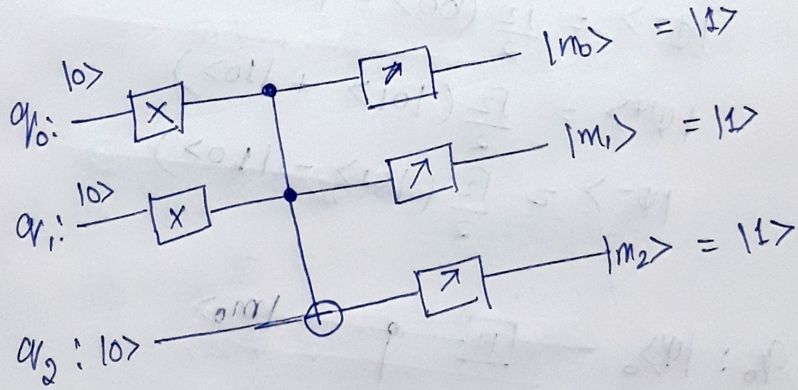


Q. Suppose we have three input qubits, q_0, q_1 and q_2 in states $|\psi\rangle_0$, $|\psi\rangle_1$ and $|\psi\rangle_2$ respectively. We want to put $|\psi\rangle_0 \oplus |\psi\rangle_1$ in ancilla qubit q_3 and put $|\psi\rangle_0 \oplus |\psi\rangle_2$ in ancilla qubit q_4. Draw the circuit.



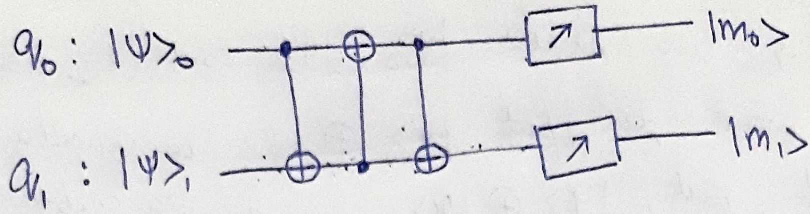
The quantum Toffoli gate

The quantum Toffoli gate operates on three qubits. If the first two qubits are $|1\rangle$ then flips the third, otherwise does nothing.



→ The tofolli gate can be used to create the quantum equivalent of a NAND gate.

Q.



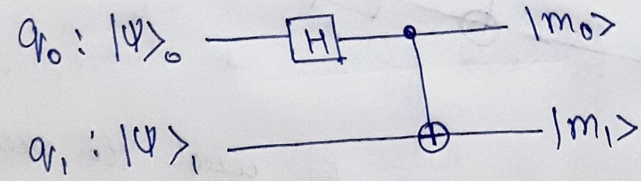
$$CNOT|00\rangle = |00\rangle$$
$$CNOT|01\rangle = |01\rangle$$
$$CNOT|10\rangle = |11\rangle$$
$$CNOT|11\rangle = |10\rangle$$

Ans.

$$|00\rangle = |00\rangle$$
$$|01\rangle = |10\rangle$$
$$|10\rangle = |01\rangle$$
$$|11\rangle = |11\rangle$$

Q. Show that circuit creates bell states.

$$|\phi^+\rangle = \frac{\sqrt{2}}{2}(|00\rangle + |11\rangle)$$
$$|\phi^-\rangle = \frac{\sqrt{2}}{2}(|00\rangle - |11\rangle)$$
$$|\psi^+\rangle = \frac{\sqrt{2}}{2}(|01\rangle + |10\rangle)$$
$$|\psi^-\rangle = \frac{\sqrt{2}}{2}(|01\rangle - |10\rangle)$$



$$|00\rangle = CNOT(H|0\rangle \otimes |0\rangle)$$
$$= \frac{\sqrt{2}}{2}(|0\rangle + |1\rangle)$$
$$|P\rangle = \frac{\sqrt{2}}{2}(|00\rangle + |10\rangle)$$
$$CNOT|P\rangle = \frac{\sqrt{2}}{2}(|00\rangle + |11\rangle)$$

Quantum Teleportation

→ My qubit Q is in some arbitary quantum state $|\psi\rangle_Q = a|0\rangle + b|1\rangle$.

When we are done, you will know this state but I will no longer have access to it.

M → my qubit
Y → your qubit
Q → whose state I want to transfer from me to you.

We begin by entangling M and Y for that,

$$|\phi^+\rangle_{MY} = \frac{\sqrt{2}}{2}\left(|00\rangle_{MY} + |11\rangle_{MY}\right)$$

$$|\psi\rangle_Q \otimes |\phi^+\rangle_{MY} = (a|0\rangle_Q + b|1\rangle_Q) \otimes \left(\frac{\sqrt{2}}{2}(|00\rangle_{MY} + |11\rangle_{MY})\right)$$

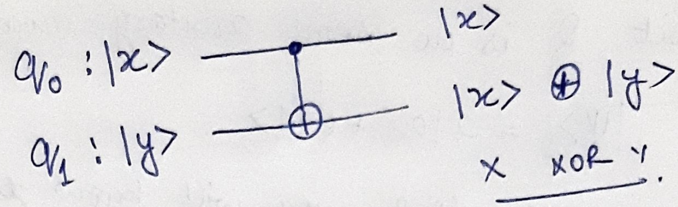
$$= \frac{\sqrt{2}}{2}\left(a|000\rangle_{QMY} + a|011\rangle_{QMY} + b|100\rangle_{QMY} + b|111\rangle_{QMY}\right)$$

$$= \frac{\sqrt{2}}{2}\left(a|00\rangle_{QM} \otimes |0\rangle_Y + a|01\rangle_{QM} \otimes |1\rangle_Y + b|10\rangle_{QM} \otimes |0\rangle_Y + b|11\rangle_{QM} \otimes |1\rangle_Y\right)$$

$$= \frac{1}{2}\Big(|\phi^+\rangle_{QM} \otimes (a|0\rangle_Y + b|1\rangle_Y) + |\phi^-\rangle_{QM} \otimes (a|0\rangle_Y - b|1\rangle_Y) + |\psi^+\rangle_{QM} \otimes (b|0\rangle_Y + a|1\rangle_Y) + |\psi^-\rangle_{QM} \otimes (-b|0\rangle_Y + a|1\rangle_Y)\Big)$$

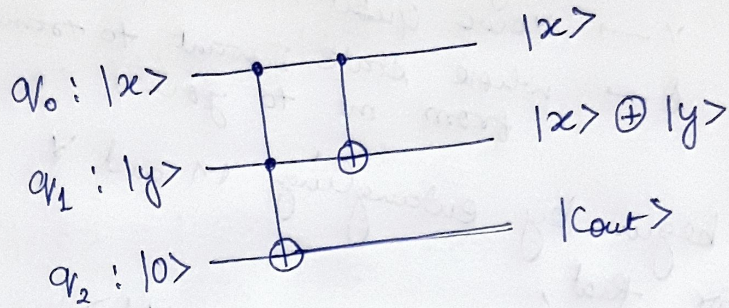
Arithmetic

* XOR

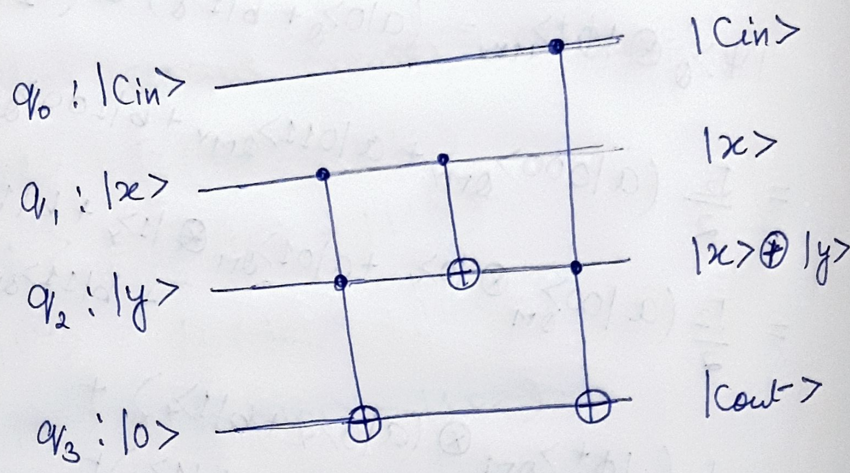


x XOR y.

"without carry out"



"with carry out".



"with carry in"

* SUM