Uppsala University

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Homework assignment 1

Time Series Analysis

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Introduction :

In this project, we are presented with five stochastic processes: MA (1), MA (2), AR (1), AR (2), and ARMA (1,1). Each model has a theoretical structure that we are asked to get via calculating various theoretical properties: mean, variance, autocovariances, and autocorrelations. We are then able to plot realizations of the times series, autocorrelation functions and partial autocorrelation functions. In this project we will these plots by trying different parameters and interpret them. We will then get various results and have many special cases such as white noise or random walks.

1) MA(1) :

$$Y_t = e_t - \theta e_{t-1}$$

a. Statistical properties:

μ_t	0
γ_0	$1+\theta^2$
γ_1	$-\theta$
γ_2	0
ρ_1	$\frac{-\theta}{1+\theta^2}$
ρ_2	0
ρ_k	$\begin{cases} \frac{-\theta}{1+\theta^2}, k=1 \\ 0, k>1 \end{cases}$

a. Interpretation of simulation results :

- **<u>Interpreting the parameter values :</u>**

The values taken by the parameter θ in the simulations are: -1 , -0.45, 0,0.45, 1 and 2. Given these values, the process remains stationary , as it is stationary by definition as proven in the theoritcal part (constant mean , constant variance and covariance that only depends on the lag and not time).

However, for $\theta = -1, 1 \; or \; 2$ the process is not invertible as it is invertible if and only if $|\theta| < 1.$

For $\theta = 0$, the process is no longer a weighted average of error terms or MA (1) but rather a white noise process (sequence of error terms) which is also stationary:

$$Y_t = e_t - 0 \times e_{t-1} = e_t$$

- **<u>Interpreting the time series plots:</u>**

For all values taken by the parameter θ, the process is not smooth, it oscillates in both directions (up and down) but it remains centered around the mean of 0, in fact the mean is stable throughout the process for values of θ ,and it is crossed at every lag for all θ 's.

We also notice that the variance remains stable (as expected from a theoretically covariance stationary process), for values of θ, hence according to the plots, the process remains stationary (the variance and mean are stable).

The process does not exhibit any drift nor a trend for all the different values taken by θ.

- **Studying the correlograms (ACF and PACF):**

θ	ACF	PACF
-1	The ACF has one spike downward for the first lag, then it completely dies off. This is consistent with the theoretical autocorrelation function: $$\rho_1 = \frac{-\theta}{1+\theta^2} = -\frac{1}{1+1^2} = 0.5$$ The theoretical ACF is equal to 0 when k>1 as well , as no error terms subscripts would overlap in y_t and y_{t-k} after the first lag.	The partial autocorrelation function declines exponentially as the number of lags increases. The decline is slow, and all the spikes are downward (negative)
-0.45	The ACF has one spike downward for the first lag, then it completely dies off. This is consistent with the theoretical autocorrelation function: $$\rho_1 = \frac{-\theta}{1+\theta^2} = -\frac{-0.45}{1-0.45^2} = -0.374 < 0$$ The theoretical ACF is equal to 0 when k>1 as well , as no error terms subscripts would overlap in y_t and y_{t-k} after the first lag.	The partial autocorrelation function declines rapidly, it is barely observable after the 4^{th} lag
0	The spikes in the white noise process are barely noticeable in the correlogram, we might as well say that they do not exist, this is due to the fact that the error terms are identically and independently distributed and that the number of observations simulated is large enough to accurately approximate the theoretical ACF	The partial autocorrelation function oscillates between positive and negative values and its decline is slow, in fact it does not die off until the last lag.
0.45	The ACF has one spike upward for the first lag, then it completely dies off. This is consistent with the theoretical autocorrelation function: $$\rho_1 = \frac{-\theta}{1+\theta^2} = -\frac{0.45}{1+0.45^2} = 0.374 > 0$$ The theoretical ACF is equal to 0 when k>1 as well , as no error terms subscripts would overlap in y_t and y_{t-k} after the first lag.	The partial autocorrelation function declines rapidly, it is barely observable after the 4^{th} lag

1	The ACF has one spike upward for the first lag, then it completely cuts off. This is consistent with the theoretical autocorrelation function: $$\rho_1 = \frac{-\theta}{1+\theta^2} = -\frac{1}{1+1^2} = 0.5 > 0$$ The theoretical ACF is equal to 0 when k>1 as well , as no error terms subscripts would overlap in y_t and y_{t-k} after the first lag.	The partial autocorrelation function declines geometrically as the number of lags increases. The decline is slow, and all the spikes are upward (positive)
2	The ACF has one spike upward for the first lag, then it completely cuts off. This is consistent with the theoretical autocorrelation function: $$\rho_1 = \frac{-\theta}{1+\theta^2} = -\frac{2}{1+4} = 0.4 > 0$$ The theoretical ACF is equal to 0 when k>1 as well , as no error terms subscripts would overlap in y_t and y_{t-k} after the first lag.	The PACF has alternating upward and downward spikes up until the 4th lag, then it cuts off completely.

2) MA(2):

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

a. Statistical properties:

μ_t	0
γ_0	$1+\theta_1^2+\theta_2^2$
γ_1	$\theta_1^2+\theta_2^2$
γ_2	$-\theta_2$
ρ_1	$\frac{\theta_1^2+\theta_2^2}{1+\theta_1^2+\theta_2^2}$
ρ_2	$\frac{-\theta_2}{1+\theta_1^2+\theta_2^2}$
ρ_k	$\begin{cases} \frac{\theta_1^2+\theta_2^2}{1+\theta_1^2+\theta_2^2}, k=1 \\ \frac{-\theta_2}{1+\theta_1^2+\theta_2^2}, k=2 \\ 0, \quad k>2 \end{cases}$

b. Interpretation of simulation results:

- *For* $\theta_1 = -0.8$

For $\theta_2 = 0$

$$Y_t = e_t + 0.8e_{t-1}$$

The time series plot looks choppy. The process looks stationary which is expected from an MA process that is always stationary. The process is also invertible since $|\theta_1| < 1$ and $|\theta_2| < 1$. The time series plot seems to have 0 mean and a stable variance which is expected since the process is stationary. The mean is often crossed. The process does not seem to drift nor have a trend. In this case, we are back to MA (1).

The ACF has one negative spike then it dies off immediately. The process is supposed to be stationary which reflects on the ACF by it going down to zero eventually.

The PACF starts with a small positive spike at lag 0 than a small negative spike at lag 6. Then it has another small positive spike at lag 10. After that it dies off.

For $\theta_2 = 0.7$

$$Y_t = e_t - 0.7e_{t-1}$$

The time series plot looks choppy. The process looks stationary which is expected from an MA process that is always stationary. The process is also invertible since $|\theta_1| < 1$ and $|\theta_2| < 1$. The time series plot seems to have 0 mean and a stable variance which is expected since the process is stationary. The mean is often crossed. The process does not seem to drift nor have a trend. The time series plot overall looks a lot similar to the time series plot of the previous simulation.

The ACF has one negative spike, one positive spike, that is smaller than the previous spike, then it dies off immediately. The process is supposed to be stationary which reflects on the ACF by it going down to zero eventually.

The PACF starts with two negative spikes, then it keeps oscillating between two positive spikes and two negative spikes until it dies off completely.

For $\theta_2 = 1$:

$$Y_t = e_t + e_{t-1}$$

The time series plot looks choppy. The process looks stationary which is expected from an MA process that is always stationary. The process is now not invertible since the condition $|\theta_2| < 1$ is no longer valid. The time series plot seems to have 0 mean and a stable variance which is expected since the process is stationary. The mean is often crossed. The process does not seem to drift nor have a trend. The time series plot overall looks a lot similar to the time series plot of the previous simulation.

The ACF has one negative spike, one positive spike, that is smaller than the previous spike, then it dies off immediately. The process is supposed to be stationary which reflects on the ACF by it going down to zero eventually.

The PACF starts with two negative spikes, then it keeps oscillating between two positive spikes and two negative spikes until it dies off completely.

Both the ACF and the PACF look a lot like the ACF and the PACF from the previous simulation.

- <u>For $\theta_1 = 0$:</u>

  For $\theta_2 = 0$:

$$Y_t = e_t$$

The time series plot looks choppy. The process looks stationary which is expected from an MA process that is always stationary. The process is also invertible since $|\theta_1| < 1$ and $|\theta_2| < 1$. The time series plot seems to have 0 mean and a stable variance which is expected since the process is stationary. The mean is often crossed. The process does not seem to drift nor have a trend. Since both parameters of the model are null, we are back to white noise.

The ACF does not have any relevant spikes. All spikes are practically zero. The process is supposed to be stationary which reflects on the ACF by it being zero.

The PACF has all negative spikes that are gradually decreasing geometrically until they die off.

For $\theta_2 = 0.7$:

$$Y_t = e_t - 0.7e_{t-2}$$

The time series plot looks choppy. The process looks stationary which is expected from an MA process that is always stationary. The process is also invertible since $|\theta_1| < 1$ and $|\theta_2| < 1$. The time series plot seems to have 0 mean and a stable variance which is expected since the process is stationary. The mean is often crossed. The process does not seem to drift nor have a trend. The time series plot overall looks a lot similar to the time series plot of the previous simulation.

The ACF has one very small positive spike, a larger positive spike, then it dies off immediately. The process is supposed to be stationary which reflects on the ACF by it going down to zero eventually.

The PACF starts with very small positive spike, a larger positive spike, then it keeps oscillating between positive and negative spikes until it dies off completely.

For $\theta_2 = 1$:

$$Y_t = e_t - e_{t-2}$$

The time series plot looks choppy. The process looks stationary which is expected from an MA process that is always stationary. The process is now not invertible since the condition $|\theta_2| < 1$ is no longer valid. The time series plot seems to have 0 mean and a stable variance which is expected since the process is stationary. The mean is often crossed. The process does not seem to drift nor have a trend. The time series plot overall looks a lot similar to the time series plot of the previous simulation.

The ACF has one very small positive spike, a larger positive spike, then it dies off immediately. The process is supposed to be stationary which reflects on the ACF by it going down to zero eventually.

The PACF starts with very small positive spike, a larger positive spike, then it keeps oscillating between positive and negative spikes until it dies off completely. Both the ACF and the PACF look a lot like the ACF and the PACF from the previous simulation.

- <u>For $\theta_1 = 0.8$</u>

  For $\theta_2 = 0$

$$Y_t = e_t - 0.8e_{t-1}$$

The time series plot looks choppy. The process looks stationary which is expected from an MA process that is always stationary. The process is also invertible since $|\theta_1| < 1$ and $|\theta_2| < 1$. The time series plot seems to have 0 mean and a stable variance which is expected since the process is stationary. The mean is often crossed. The process does not seem to drift nor have a trend. In this case, we are back to MA(1).

The ACF has one positive spike then it dies off immediately. The process is supposed to be stationary which reflects on the ACF by it going down to zero eventually.

The PACF starts with a positive spike then it keeps oscillating between negative and positive spikes until it dies off completely.

For $\theta_2 = 0.7$

$$Y_t = e_t - 0.8e_{t-1} - 0.7e_{t-2}$$

The time series plot looks choppy. The process looks stationary which is expected from an MA process that is always stationary. The process is also invertible since $|\theta_1| < 1$ and $|\theta_2| < 1$. The time series plot seems to have 0 mean and a stable variance which is expected since the process is stationary. The mean is often crossed. The process does not seem to drift nor have a trend. The time series plot overall looks a lot similar to the time series plot of the previous simulation.

The ACF has one positive spike, another positive spike that is smaller than the previous spike, then it dies off immediately. The process is supposed to be stationary which reflects on the ACF by it going down to zero eventually.

The PACF starts one positive spikes, then it keeps oscillating between one positive spike and two negative spikes until it dies off completely.

For $\theta_2 = 1$

$$Y_t = e_t - 0.8e_{t-1} - e_{t-2}$$

The time series plot looks choppy. The process looks stationary which is expected from an MA process that is always stationary. The process is now not invertible since the condition $|\theta_2| < 1$ is no longer valid. The time series plot seems to have 0 mean and a stable variance which is expected since the process is stationary. The mean is often crossed. The process does not seem to drift nor have a trend. The time series plot overall looks a lot similar to the time series plot of the previous simulation.

The ACF has one positive spike, another positive spike that is smaller than the previous spike, then it dies off immediately. The process is supposed to be stationary which reflects on the ACF by it going down to zero eventually.

The PACF starts one positive spikes, then it keeps oscillating between one positive spike and two negative spikes until it dies off completely.

Both the ACF and the PACF look a lot like the ACF and the PACF from the previous simulation.

1) AR(1):

$$Y_t = \emptyset Y_{t-1} + e_t$$

a. Statistical properties :

μ_t	0
γ_0	$\frac{1}{1-\emptyset^2}$
γ_1	$\frac{\emptyset}{1-\emptyset^2}$
γ_2	$\frac{\emptyset^2}{1-\emptyset^2}$
ρ_1	$\emptyset$
ρ_2	$\emptyset^2$
ρ_k	$\emptyset^k$

b. Interpretation of simulation results :

- **<u>Interpreting the time series plots :</u>**

The values taken by the parameter ϕ are -1, -0.45, 0, 0.45, 1, 2.

In all simulations except the last one, we notice that the process is never smooth. It is always choppy.

- For $\phi = -1$:

$$Y_t = -Y_{t-1} + e_t$$

the process is not stationary. In the time series plot we can see that even though the process is centered around 0 mean, its variance is not constant throughout the whole process. In fact, we do not notice a drift nor a trend. The process crosses the mean quite often. The process has a small variance from t=0 until t=150. The variance then increases until t is close to 300. This behavior repeats itself starting from t=300 so it seems like the variance is stable.

- For $\phi = -0.45$:

$$Y_t = -0.45Y_{t-1} + e_t$$

the process is stationary as we have $|\phi| < 1$. The time series plot is centered around 0 and seems to have a constant variance which is in line with the stationarity of the process. We do not notice any drift or trend. The process crosses the mean quite often.

- For $\phi = 0$:

$$Y_t = e_t$$

the process is still stationary. In fact, it is white noise. The time series plot is centered around 0 and seems to have a constant variance which is in line with the stationarity of the process. We do not notice any drift or trend. The process crosses the mean quite often.

- For $\phi = 0.45$:

$$Y_t = 0.45Y_{t-1} + e_t$$

the process is stationary as we have $|\phi| < 1$. The time series plot is centered around 0 and seems to have a constant variance which is in line with the stationarity of the process. We do not notice any drift or trend. The process crosses the mean quite often.

- For $\phi = 1$:

$$Y_t = Y_{t-1} + e_t$$

the process is no longer stationary. We obtain a random walk. In the time series plot we can say that even though the process is centered around 0 mean, its variance is not constant throughout the whole process. The process contains many negative and positive drifts which makes it follow some positive and negative trends. The process does not cross the mean often, it only crosses it with a big enough change of a trend. The variance does not look stable. In fact, with huge trends we can have huge variances.

- For $\phi = 2$:

$$Y_t = 2Y_{t-1} + e_t$$

the process is not stationary. The time series plot remains constant then it decreases exponentially and infinitely when it gets close to t=500 (We consider that $-1e150 = -\infty$). Hence the process does not have a finite mean nor a finite variance. The process has one negative drift which constitutes one negative trend.

- **Interpreting the ACF :**

For $\phi = -1$ the ACF has alternated downward and upward spikes. It does not seem to die off. It keeps oscillating between positive and negative values. This can be explained by looking at the theoretical autocorrelation function:

$$\rho_k = \emptyset^k$$

When k is odd the autocorrelation is negative , when k is pair the autocorrelation is positive.

This may indicate the non-stationarity of the process.

For $\phi = -0.45$ the ACF has one downward spike for the first lag then an upward spike for the second then a downward spike for the third. It dies off after those three spikes. The ACF decreases geometrically. This is consistent with the theoretical results as the ACF gets smaller and smaller with each lag as it is $(-0.45)^k$. The process is stationary as we can see the rapid death of the ACF.

For $\phi = 0$ the ACF has infinitesimally small spikes that are practically nonexistent. This is consistent with the theoretical results as the ACF is $0^k = 0$. The process is stationary as we can see the death of the ACF from the first lag.

For $\phi = 0.45$ the ACF has three upward spikes. It dies off after those three spikes. The ACF decreases geometrically. This is consistent with the theoretical results as the ACF gets smaller and smaller with each lag as it is $(0.45)^k$. The process is stationary as we can see the rapid death of the ACF.

For $\phi = 1$ the ACF has positive spikes that do not die off. We no longer have stationarity since we do not have that $|\phi| < 1$, this is apparent in the ACF that does not decay to 0 with lags.

For $\phi = 2$ we can not obtain an ACF with 5000 observations. We had to reduce the number of observations to 500 to be able to have a plot. Hence, this plot may not reflect the real nature of the theoretical ACF. This plot contains three significant upward spikes then it dies off. It decreases exponentially to zero.

- **Interpreting the PACF :**

For $\phi = -1$ the PACF has one negative spike and decreases geometrically to become practically 0.

For $\phi = -0.45$ the PACF has one negative spike that is less pronounced than the spike of the PACF OF $\phi = -1$ .It then decreases geometrically to become practically 0.

For $\phi = 0$ the PACF has infinitesimally small spikes that are practically nonexistent.

For $\phi = 0.45$ the PACF has one positive spike. It then decreases geometrically to become practically 0.

For $\phi = 1$ the PACF has one positive spike and decreases geometrically to become practically 0.

For $\phi = 2$ we can not obtain an ACF with 5000 observations. We had to reduce the number of observations to 500 to be able to have a plot. This plot contains only one downward spike. The PACF of the other lags is zero

2) AR(2):

$$Y_t = \emptyset_1 Y_{t-1} + \emptyset_2 Y_{t-2} + e_t$$

a. Statistical properties :

μ_t	0
γ_0	$\frac{1-\emptyset_2}{1-\emptyset_2(1+\emptyset_1-\emptyset_2^3)-\emptyset_1{}^2(1+\emptyset_2)}$
γ_1	$\gamma_0 \frac{\emptyset_1}{1-\emptyset_2}$
γ_2	$\emptyset_1\gamma_1 + \gamma_0\emptyset_2$
ρ_1	$\frac{\emptyset_1}{1-\emptyset_2}$
ρ_2	$\emptyset_1\rho_1 + \emptyset_2$
ρ_k	$\rho_k = \emptyset_1\rho_{k-1} + \emptyset_2\rho_{k-2}$

b. Interpretation of simulation results :

- **<u>Interpreting the parameter values :</u>**

An AR (2) process is stationary if and only if the roots of its characteristic equation are outside the unit circle, this is equivalent to the following three conditions:

- $\emptyset_1 + \emptyset_2 < 1$
- $\emptyset_2 - \emptyset_1 < 1$
- $|\emptyset_2| < 1$

For the combinations of values taken by the parameters $\emptyset_1$ and $\emptyset_2$ the process is stationary if and only if the three conditions above are satisfied.

The table below indicates if the process is stationary/non-stationary for every combination of parameters, it is based on the conditions detailed above:

	-0.9	0	0.7
0.1	Non-stationary	Stationary	Stationary
0.2	Non-stationary	Stationary	Stationary
0.8	Non-stationary	Stationary	Non-stationary

For the processes with $\emptyset_1 = 0$ the process has a random walk component (when $\emptyset_1 = 0$ one of the roots of the characteristic polynomial is on the unit circle).

- **Interpreting the time series and the correlogram**

The process is choppy for all the combinations of parameters.

- For the process:

$$Y_t = -0.9Y_{t-1} + 0.1Y_{t-2} + e_t$$

We have already established that this process is non-stationary judging by the values of its parameters. This can be observed by looking at the time series plot, the variance is not constant throughout the process. However, the process does not exhibit any drift as it oscillates up and down while being centered around the mean. No trend can be observed in the plot and the mean value is crossed often (on every lag).

By looking at the ACF, the process seems to “hang together” , we observe alternating negative and positive spikes and the function does not die off for the 20 lags observed, the spikes are of comparable lengths.

This is consistent with the theoretical ACF

The PACF has one downward spike for the first lag , after that it begins to die off , we can barely observe the spikes for the 2nd and 3rd lags.

- For the process:

$$Y_t = -0.9Y_{t-1} + 0.2Y_{t-2} + e_t$$

The process is not stationary, and this can be observed from the time series plot. In fact, the variance has a practically null value and we can only observe the process oscillate around the mean after 400th lag. After that, the variance seems to increase exponentially.

This is expected from a non-stationary process, in fact for a simulation with a large number of observations (5000), the variance grows infinitely, and we are not able to observe the autocorrelation function. In order to be able to observe the plot and the ACF, we have shortened the time series to 500 observations.

The process does not exhibit any drift nor trend.

The correlogram shows that the ACF has several alternating negative and positive (upward and downward) spikes) that decrease in length quickly.

The PACF, has one downward spike for the first lag, then it completely dies off

- For the process :

$$Y_t = -0.9Y_{t-1} + 0.8Y_{t-2} + e_t$$

This process is similar to the previous process. It is non-stationary and we had to shorten the time series to 500 observations in order to be able to observe the ACF.

The variance is practically null until shortly before the 500th lag, after that it starts growing which is expected of a non-stationary process.

No drift nor trend can be observed.

The ACF has few alternating upward and downward spikes and it dies off quickly.

The PACF has one downward spike for the first lag then nothing.

- For the process:

$$Y_t = 0.1Y_{t-2} + e_t$$

The process ,judging by the parameters , has a random walk component. From the time series plot, we can observe that the variance does exhibit instability in very few instances.

The mean can be described as stable as no noticeable drift or rend can be observed in the plot.

The mean value is crossed fairly often.

The ACF has one upward spike then it completely dies off.

The PACF has alternating upward and downward spikes annd it starts dying off quickly right after the 3rd lag.

- For the process:

$$Y_t = 0.2Y_{t-2} + e_t$$

According to the time series plot, the process is choppy. The overall mean does not change, the mean value is crossed on every lag throughout the process. and no visible trend or drift can be identified from the time series plot. The variance also seems fairly stable.

The ACF has few alternating upward and downward spikes and it dies off quickly.

The PACF has one downward spike for the first lag then nothing.

- For the process:

$$Y_t = 0.8Y_{t-2} + e_t$$

The overall mean does not change, but we can observe some brief drifts throughout the process.The mean is not crossed as often as what's observed for processses with $\emptyset_1$ =0 and smaller $\emptyset_2$. The variance exhibits some instability on several occasions throughout the process.

We do not observe a trend nor a drift in the time series plot.

Even though we have concluded that the process is stationary based on the values of its parameters, the simulation makes us question if it is indeed stationary, this is might be due to the fact that the value of the parameter $\emptyset_2$is close to 1.

The ACF has upward spikes for all lags and it decreases slowly.

The PACF has 1 upward spike for lag 1 then it completely dies off.

- For the process:

$$Y_t = 0.7Y_{t-1} + 0.1Y_{t-2} + e_t$$

The process is expected to be stationary jaudging by its parameters, however, the variance does not seem to be stable throught the process as expected.The overall mean does not change, there is no visible trend.

The mean does experience brief instabilities in few instances and the theoritical mean value is not crossed at every lag.

The ACF has upward spikes that decrease relatively quickly.

The PACF has one upward spike for lag 1, a very short upward spike for the 2nd lag then it completely cuts off.

- For the process:

$$Y_t = 0.7Y_{t-1} + 0.2Y_{t-2} + e_t$$

The variance and the mean are not stable as expected of a stationary process. However no visible trend is observed. The process briefly drifts from the mean on several occasions and the mean value is not crossed very often throughout the process.

Even though the parameters satisfy the stationarity conditions, the simulation results suggests otherwise, this might be due to the value of the first parameter which is close to one.

The ACF has upward spikes that slowly decrease in length and the PACF has 2 upward spikes for the 1^{st} and 2^{nd} lag then nothing.

- For the process:

$$Y_t = 0.7Y_{t-1} + 0.8Y_{t-2} + e_t$$

The process is non-stationary, according to the time series plot, the process does not oscilate around the mean, in fact , we observe a constant null mean and variance until right before the 500th lag. On the 500th lag, the process drifts and exhibits a downward trend.

It was not possible to observe the ACF for this process until the simulation was conducted with a shorter time series (500 observations).

The ACF has upward spikes that quicly decrease in length and the PACF has 1 upward spike for the first lag then it cuts off completely.

3) ARMA (1,1) :

$$Y_t = \emptyset Y_{t-1} + e_t - \theta e_{t-1}$$

a. Statistical properties :

$\boldsymbol{\mu_t}$	0
$\boldsymbol{\gamma_0}$	$\dfrac{\theta^2 + 1 - 2\theta\emptyset}{1 - \emptyset^2}$
$\boldsymbol{\gamma_1}$	$\emptyset\gamma_0 - \theta$
$\boldsymbol{\gamma_2}$	$\emptyset\gamma_1$
$\boldsymbol{\rho_1}$	$\emptyset^2 - \dfrac{\theta(1 - \emptyset^2)}{\theta^2 + 1 - 2\theta\emptyset}$
$\boldsymbol{\rho_2}$	$\emptyset\boldsymbol{\rho_1}$
$\boldsymbol{\rho_k}$	$\rho_k = \emptyset^{k-1}\rho_1 , k \geq 1$

b. Interpretation of simulation results :

For an ARMA(1,1) process to be stationary and invertible , the AR part has to be stationary , and the MA part has to be invertible.

For the AR part to be stationary , the parameter $|\emptyset|$ has to be less than 1.

For the MA part to be invertible, the parameter $|\theta|$ has to be less than 1.

The values taken by both parameters satisfy the conditions of invertibility and stationarity, thus judging by the parameter values alone , we expect the process to be stationary for all values of $\emptyset$

- **For $\boldsymbol{\theta = -0.4}$:**

For $\phi = -0.9$:

$$Y_t = -0.9Y_{t-1} + e_t + 0.4e_{t-1}$$

the time series plot is choppy. It is supposed to be stationary since the AR part of the model is stationary, $|\phi| < 1$. It is also supposed to be invertible since the MA part of the model is invertible, $|\theta| < 1$. However, in our simulation the process seems to be not stationary. In fact, we can see in the time series plot that the variance from t=0 to t=90 is lower than the variance from t=90 to t=110. We also notice that this behavior is cyclic. We suspect that we are getting these results because we chose

$\phi = -0.9$ which is very close to -1. The process does not seem to drift nor to have a trend and it often crosses the mean.

The ACF starts with a negative spike and it keeps oscillating between positive and negative spikes. It is decreasing rather slowly. The correlogram eventually dies off even though it is slow which makes us think the process is stationary.

The PACF has three negative spikes then it decreases exponentially until it dies off.

For $\phi = 0.8$:

$$Y_t = 0.8Y_{t-1} + e_t + 0.4e_{t-1}$$

the time series plot is choppy. It is supposed to be stationary since the AR part of the model is stationary, $|\phi| < 1$. It is also supposed to be invertible since the MA part of the model is invertible, $|\theta| < 1$. However, in our simulation the process seems to be not stationary. In fact, it looks more like a random walk. The process does drift and it starts a trend around t=350. We suspect that we are getting these results because we chose $\phi = 0.8$ which is close to 1. The process does cross the mean more often than a random walk would.

The spikes of the ACF are positive and decreasing geometrically with every lag. It is decreasing rather slowly. The correlogram eventually dies off even though it is slow which makes us think the process is stationary.

The PACF has three positive spikes then it decreases exponentially until it dies off.

For $\phi = 0.9$:

$$Y_t = 0.9Y_{t-1} + e_t + 0.4e_{t-1}$$

the time series plot is choppy. It is supposed to be stationary since the AR part of the model is stationary, $|\phi| < 1$. It is also supposed to be invertible since the MA part of the model is invertible, $|\theta| < 1$. However, in our simulation the process seems to be not stationary. In fact, it looks more like a random walk. The process does drift and it starts a trend around t=350. It is very similar to the previous simulation with $\phi = 0.8$ .We suspect that we are getting these results because we chose $\phi = 0.9$ which is very close to 1. The process does cross the mean more often than a random walk would.

The spikes of the ACF are positive and decreasing geometrically with every lag. It is decreasing rather slowly, even slower than the correlogram of the previous simulation of $\phi = 0.8$. The correlogram eventually dies off even though it is slow which makes us think the process is stationary.

The PACF has three positive spikes then it decreases exponentially until it dies off. It looks very similar to the PACF of the previous simulation.

- **For $\boldsymbol{\theta = 0.4}$:**

For $\phi = -0.9$

$$Y_t = -0.9Y_{t-1} + e_t - 0.4e_{t-1}$$

The time series is choppy and it oscillates around the mean throught the process , it crosses the mean value on every lag. No visible drift nor trend can be observed.

The variance does not look to be stable all along the process. Judging by the value of the parameter $\emptyset = -0.9$ this process was expected to be stationary. The variance instability in the time series process leads to suspicions regarding its stationarity, this is most likely due to the large value taken by ∅ , -0.9 is very close to -1.

The ACF has alternating negative and positive spikes that die off slowly.

The PACF dies off quickly , it has a downward spike for lag 1 , a shorter upward spike for lag 2 , then it cuts off completely.

For $\phi = 0.8$

$$Y_t = 0.8Y_{t-1} + e_t - 0.4e_{t-1}$$

The time series seems to have a stable overall mean, no visible trend can be detected from the plot. However, the process (briefly) drifts from its mean on several occasions and it does not seem to cross the mean values often.

The variance does not seem to be stable thoughout the process.

By looking at the value of the parameter ∅ , the process is supposed to be stationary , this does not seem to be the case according to the simulation results. This is perhaps due to the large value taken by the parameter ∅.

The ACF dies off somewhat slowly , it has several upward spikes.

The PACF dies off quickly, it has 1 upward spike for lag 1, a downward spike for lag 2 and no pronounced spikes for the rest of the lags.

For $\phi = 0.9$

$$Y_t = 0.9Y_{t-1} + e_t - 0.4e_{t-1}$$

The process does not look smooth. It has no visible upward or downward trend, however, it does seem to (briefly) drift from the mean on several occasions and it does not cross the mean value often.

The variance is non-stable throughout the process.

This process was expected to be stationary judging by the parameters but this does not seem to be the case according to the simulation ressults, this is due to the large value , close to 1 taken by the parameter Ø.

The ACF is composed of upward spikes that decline geometrically(slowly) in length.

The PACF dies off quickly, it has 3 spikes alternating between negative and postive values for the first 3 lags then it cuts off completely.

Conclusion:

In this project we studied five different stochastic processes, two autoregressive processes AR(1) and AR(2), two moving average processes MA(1) and MA(2), and one autoregressive moving average process ARMA(1). We have calculated for each process many theoretical properties such as the mean, the variance, autocovariances and autocorrelations. Then, we simulated each process with different parameters and interpreted the graphs we obtained. Many processes fall into special cases such as white noise or random walk with specific parameters. We have discussed the stationarity and invertibility of each process with specific parameters. This project helps us get a deeper understanding of the processes so we can get to estimation and forecasting in the next project.

Appendix A

We have that $e_t \sim NID(0, 1)$ so,

$$\forall t, E[e_t] = 0 \ \& \ Var(e_t) = 1$$
$$\forall t \neq s, Cov(e_t, e_s) = 0$$

MA(1)

$$Y_t = e_t - \theta e_{t-1}$$

1) $\mu_t = E[Y_t]$
$= E[e_t - \theta e_{t-1}]$
$= E[e_t] - \theta E[e_{t-1}]$
$= 0 - \theta 0$
$= 0$

2) $\gamma_0 = Var(Y_t)$
$= Var(e_t) + \theta^2 Var(et - 1) - 2\theta Cov(e_t, e_{t-1})$
$= 1 + \theta^2 + 0$, e_t and e_{t-1} are independent
$= 1 + \theta^2$

3) $\gamma_1 = Cov(Y_t, Y_{t-1})$
$= Cov(e_t, e_{t-1}) - \theta Cov(e_t, e_{t-2}) - \theta Cov(e_{t-1}, e_{t-1}) + \theta^2 Cov(e_t, e_{t-2})$
$= -\theta$

4) $\gamma_2 = Cov(Y_t, Y_{t-2})$
$= Cov(e_t, e_{t-2}) - \theta Cov(e_t, e_{t-3}) - \theta Cov(e_{t-1}, e_{t-2}) + \theta^2 Cov(e_{t-1}, e_{t-3})$
$= 0$

5) $\rho_1 = \dfrac{\gamma_1}{\gamma_0} = \dfrac{-\theta}{1+\theta^2}$

6) $\rho_2 = \dfrac{\gamma_2}{\gamma_0} = 0$

7) $\rho_k = ?$

For $k > 1$ we have,

$$\begin{aligned}
&\gamma_k = Cov(Y_t, Y_{t-k}) \\
&= Cov(e_t, e_{t-k}) - \theta Cov(e_t, e_{t-k-1}) - \theta Cov(e_{t-1}, e_{t-k}) + \theta^2 Cov(e_{t-1}, e_{t-k-1}) \\
&= 0 \text{ , since } \forall t \neq s, Cov(e_t, e_s) = 0
\end{aligned}$$

Hence,

$$\rho_k = \begin{cases} \frac{-\theta}{1+\theta^2}, & k = 1 \\ 0, & k > 1 \end{cases}$$

MA(2)

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

1) $\mu_t = E[et] - \theta_1 E[et-1] - \theta_2 E[et_2] = 0$

2) $\gamma_0 = Var(e_t) + \theta_1^2 Var(e_{t-1}) + \theta_2^2 Var(e_{t-2}) + \text{ null covariance terms}$
$= 1 + \theta_1^2 + \theta_2^2$

3) $\gamma_1 = Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-1} - \theta_1 e_{t-2} - \theta_2 e_{t-3})$
$= -\theta_1 Cov(e_{t-1}, e_{t-1}) + \theta_1\theta_2 Cov(e_{t-2}, e_{t-2}) + \text{ null covariance terms}$
$= -\theta_1 + \theta_1\theta_2$

4) $\gamma_2 = Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-2} - \theta_1 e_{t-3} - \theta_2 e_{t-4})$
$= -\theta_2 Cov(e_{t-2}, e_{t-2}) +$ null covariance terms
$= -\theta_2$

5) $\rho_1 = \dfrac{\gamma_1}{\gamma_0} = \dfrac{-\theta_1 + \theta_1\theta_2}{1+\theta_1^2+\theta_2^2}$

6) $\rho_2 = \dfrac{\gamma_2}{\gamma_0} = \dfrac{-\theta_2}{1+\theta_1^2+\theta_2^2}$

7) For $k > 2$ we have,

$\gamma_k = Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-k} - \theta_1 e_{t-k-1} - \theta_2 e_{t-k-2})$
$= Cov(e_t, e_{t-k}) - \theta_1 Cov(e_t, e_{t-k-1}) - \theta_2 Cov(e_t, e_{t-k-2}) - \theta_1 Cov(e_{t-1}, e_{t-k})$
$+ \theta_1^2 Cov(e_{t-1}, e_{t-k-1}) + \theta_1\theta_2 Cov(e_{t-1}, e_{t-k-2}) - \theta_2 Cov(e_{t-2}, e_{t-k}) + \theta_1\theta_2 Cov(e_{t-2}, e_{t-k-1})$
$+ \theta_2^2 Cov(e_{t-2}, e_{t-k-2})$
$= 0$, since e_t, e_s are independent if $t \neq s$

so,

$$\rho_k = \begin{cases} \frac{-\theta_1+\theta_1\theta_2}{1+\theta_1^2+\theta_2^2}, & k=1 \\ \frac{-\theta_2}{1+\theta_1^2+\theta_2^2}, & k=2 \\ 0, & k>2 \end{cases}$$

AR(1)

$$Y_t = \phi Y_{t-1} + e_t$$

1) $\mu_t = E[Y_t] = \phi E[Y_{t-1}] + E[e_t]$
$\Rightarrow E[Y_t] - \phi E[Y_{t-1}] = 0$
$\Rightarrow \mu - \phi\mu = 0,$ if stationarity
$\Rightarrow \mu = 0$

2) $\gamma_0 = Var(\phi Y_{t-1} + e_t)$
$= \phi^2 Var(Y_{t-1}) + Var(e_t) + 2\phi Cov(Y_{t-1}, e_t)$
$Cov(Y_{t-1}, e_t)$ is a function of $e_0, e_1, ..., e_{t-1}$
$\Rightarrow Cov(Y_{t-1}, e_t) = 0$
$\Rightarrow \gamma_0 - \phi^2 \gamma_0 = 1$
$\Rightarrow \gamma_0 = \dfrac{1}{1 - \phi^2}$

3) $\gamma_1 = Cov(Y_t, Y_{t-1})$
$= Cov(\phi Y_{t-1} + e_t, Y_{t-1})$
$= \phi Cov(Y_{t-1}, Y_{t-1}) + Cov(e_t, Y_{t-1})$
$= \phi Var(Y_{t-1}) + 0$
$= \phi \gamma_0$

4) $\gamma_2 = Cov(Y_t, Y_{t-2})$
$= Cov(\phi Y_{t-1} + e_t, Y_{t-2})$
$= Cov(\phi(\phi Y_{t-2} + e_{t-1}) + e_t, Y_{t-2})$
$= Cov(\phi^2 Y_{t-2} + \phi e_{t-1} + e_t, Y_{t-2})$
$= \phi^2 Var(Y_{t-2}) + \phi Cov(e_{t-1}, Y_{t-2}) + Cov(e_t, Y_{t-2})$
$= \phi^2 \gamma_0$

5) $\rho_1 = \dfrac{\gamma_1}{\gamma_0} = \phi$

6) $\rho_2 = \dfrac{\gamma_2}{\gamma_0} = \phi^2$

7) Let us prove by induction that $\gamma_k = \phi^k \gamma_0, \forall k$
For $k = 1$: $\gamma_1 = \phi\gamma_0 = \phi^1\gamma_0$
Assuming for a given k that $\gamma_k = \phi^k\gamma_0$:

$$\begin{aligned}
&\gamma_{k+1} = Cov(Y_t, Y_{t-(k+1)}) \\
&= Cov(\phi Y_{t-1} + e_t, Y_{t-k-1}) \\
&= \phi Cov(Y_{t-1}, Y_{t-k-1}) + Cov(e_t, Y_{t-k-1}) \\
&= \phi\gamma_k + 0 \\
&= \phi\phi^k\gamma_0 \\
&= \phi^{k+1}\gamma_0
\end{aligned}$$

Hence, $\forall k, \rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k$

AR(2)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

1) $\mu = \phi_1\mu + \phi_2\mu + 0$ assuming stationarity
$\Rightarrow \mu = 0$

2) $\gamma_0 = Var(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t)$
$= \phi_1^2\gamma_0 + \phi_2^2\gamma_0 + 1 + 2\phi_1\phi_2 Cov(Y_{t-1}, Y_{t-2}) + 2\phi_1 Cov(Y_{t-1}, e_t) + 2\phi_2 Cov(Y_{t-2}, e_t)$

$$\Rightarrow (1 - \phi_1^2 - \phi_2^2)\gamma_0 = 1 + 2\phi_1\phi_2\gamma_1 \tag{1}$$

3) $\gamma_1 = Cov(Y_t, Y_{t-1})$
$= Cov(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t, Y_{t-1})$
$= \phi_1 Var(Y_{t-1}) + \phi_2 Cov(Y_{t-1}, Y_{t-2}) + 0$
$\Rightarrow \gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1$

$$\Rightarrow \gamma_1 = \frac{\phi_1}{1-\phi_2}\gamma_0 \tag{2}$$

Equations 1 and 2 give us:

$$(1-\phi_1^2-\phi_2^2)\gamma_0 = 1 + 2\phi_1\phi_2\frac{\phi_1}{1-\phi_2}\gamma_0$$
$$\gamma_0(1-\phi_1^2-\phi_2^2-\frac{2\phi_1^2\phi_2}{1-\phi_2}) = 1$$
$$\gamma_0 = \frac{1}{1-\phi_1^2-\phi_2^2-\frac{2\phi_1^2\phi_2}{1-\phi_2}}$$

4) $\gamma_2 = Cov(Y_t, Y_{t-2})$
$= Cov(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t, Y_{t-2})$
$= \phi_1 Cov(Y_{t-1}, Y_{t-2}) + \phi_2 Cov(Y_{t-2}, Y_{t-2}) +$ null covariance terms
$= \phi_1\gamma_1 + \phi_2\gamma_0$

5) $\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\phi_1}{1-\phi_2}$

6) $\rho_2 = \frac{\gamma_2}{\gamma_0} = \phi_1\rho_1 + \phi_2$

7) for $k > 1, \gamma_k = Cov(Y_t, Y_{t-k})$
$= Cov(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t, Y_{t-k})$
$= \phi_1 Cov(Y_{t-1}, Y_{t-k}) + \phi_2 Cov(Y_{t-2}, Y_{t-k}) +$ null covariance terms
$= \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2}$

Hence,
$$\rho_k = \begin{cases} \frac{\phi_1}{1-\phi_2}, & k = 1 \\ \phi_1\rho_{k-1} + \phi_2\rho_{k-2}, & k \geq 2 \end{cases}$$

ARMA(1,1)

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}$$

1) $\mu_t = \phi E[Y_{t-1}] + E[e_t] - \theta E[e_{t-1}]$
$\mu = \phi\mu,$ if stationarity
$\Rightarrow \mu = 0$

2) $\gamma_0 = Var(Y_t)$
$= \phi^2 Var(Y_{t-1}) + Var(e_t) + \theta^2 Var(e_{t-1}) - 2\phi\theta Cov(Y_{t-1}, e_{t-1}) +$ null covariance terms
$= \phi^2\gamma_0 + 1 + \theta^2 - 2\phi\theta Cov(\phi Y_{t-2} + e_{t-1} - \theta e_{t-2}, e_{t-1})$
$= \phi^2\gamma_0 + 1 + \theta^2 - 2\phi\theta Cov(e_{t-1}, e_{t-1})$
$= \phi^2\gamma_0 + 1 + \theta^2 - 2\phi\theta \Rightarrow \gamma_0 = \dfrac{1 + \theta^2 - 2\phi\theta}{1 - \phi^2}$

3) $\gamma_1 = Cov(Y_t, Y_{t-1})$
$= Cov(\phi Y_{t-1} + e_t - \theta e_{t-1}, Y_{t-1})$
$= \phi Cov(Y_{t-1}, Y_{t-1}) - \theta Cov(e_{t-1}, Y_{t-1})$
$= \phi\gamma_0 - \theta Cov(e_{t-1}, \phi Y_{t-2} + e_{t-1} - \theta e_{t-2}) +$ null covariance terms
$= \phi\gamma_0 - \theta$

4) $\gamma_1 = Cov(Y_{t-1} =, Y_{t-2})$
$= Cov(\phi Y_{t-1} + e_t - \theta e_{t-1}, Y_{t-2})$
$= \phi Cov(Y_{t-1}, Y_{t-2})$
$= \phi\gamma_1$

5) $\rho_1 = \dfrac{\gamma_1}{\gamma_0}$

$= \dfrac{\phi\gamma_0 - \theta}{\gamma_0}$

$= \phi - \dfrac{\theta(1-\phi^2)}{1+\theta^2-2\phi\theta}$

6) $\rho_2 = \dfrac{\gamma_2}{\gamma_0}$

$= \phi\rho_1$

7) $\rho_k = \dfrac{\gamma_k}{\gamma_0}$

We prove by induction that $\forall k > 1, \gamma_k = \phi\gamma_{k-1}$:
For $k = 2, \gamma_2 = \phi^1\gamma_1$
Let us assume that for a given $k, \gamma_k = \phi\gamma_{k-1}$

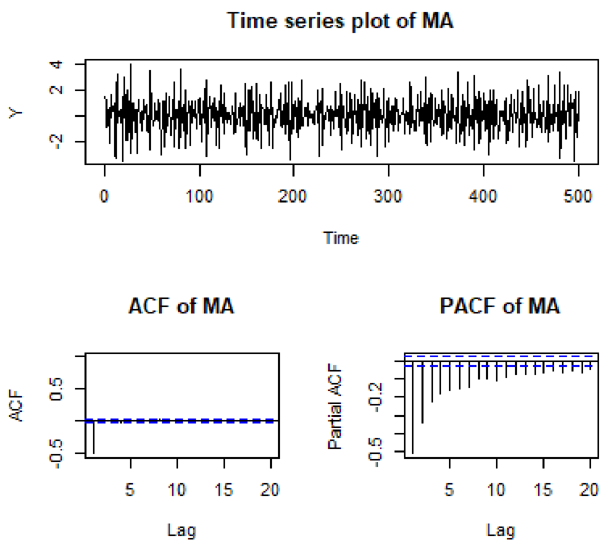
$\gamma_{k+1} = Cov(\phi Y_{t-1} + e_t - \theta e_{t-1}, Y_{t-k-1})$
$= \phi Cov(Y_{t-1}, Y_{t-k-1}) + 0$
$= \phi\gamma_k$

Hence,

$$\rho_k = \phi\rho_{k-1} = \phi^2\rho_{k-1} = ... = \phi^{k-1}\rho_1$$

So we get, $\rho_k = \phi^{k-1}\rho_1, k \geq 1$

APPENDIX B

<table>
<tr><th>θ</th><th>Visualisation of MA(1)</th></tr>
<tr><td>-1</td><td>


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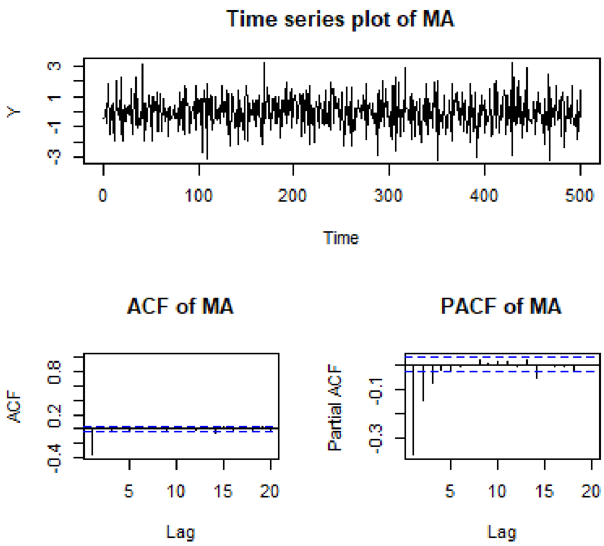

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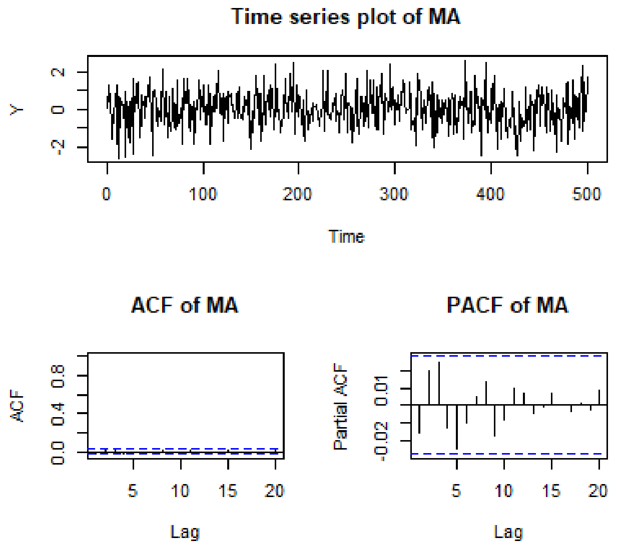
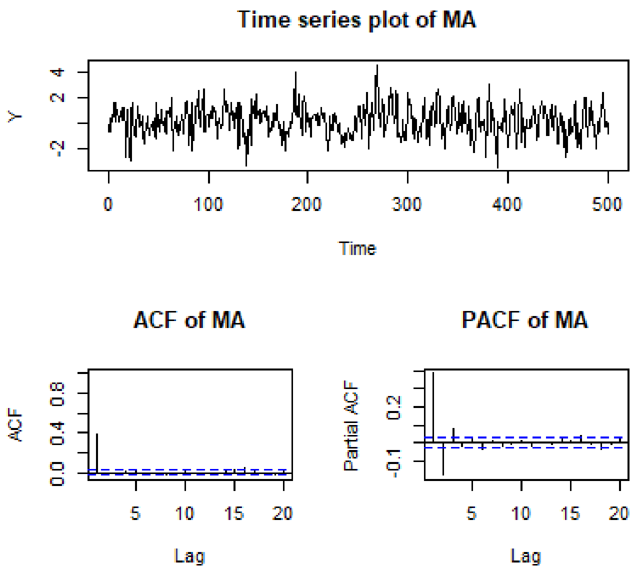
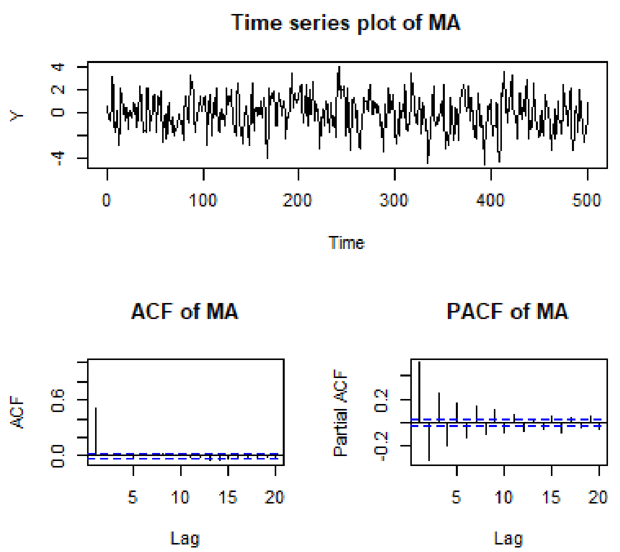
0	Time series plot of MA Y Time ACF of MA ACF Lag PACF of MA Partial ACF Lag
0.45	Time series plot of MA Y Time ACF of MA ACF Lag PACF of MA Partial ACF Lag
1	Time series plot of MA Y Time ACF of MA ACF Lag PACF of MA Partial ACF Lag

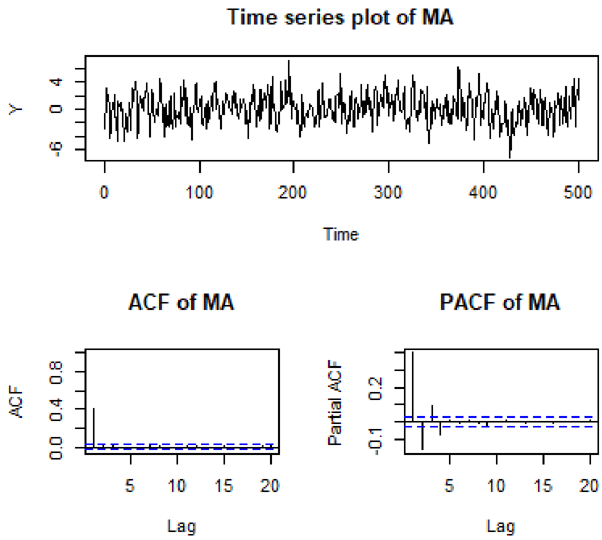
2	

Table 1: Time series plot, ACF and PACF of MA(1) for different parameters

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<tr><td rowspan="2" colspan="2">MA(2) visualization</td><td colspan="3">θ_1</td></tr>
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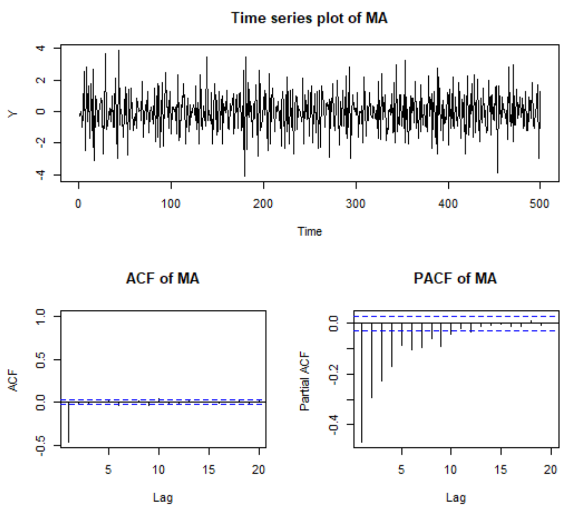
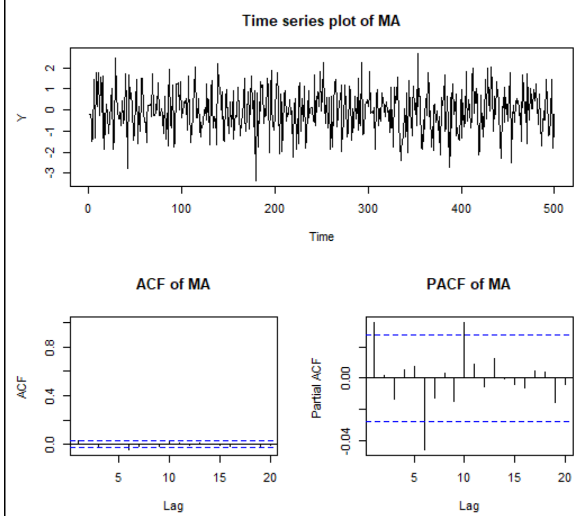
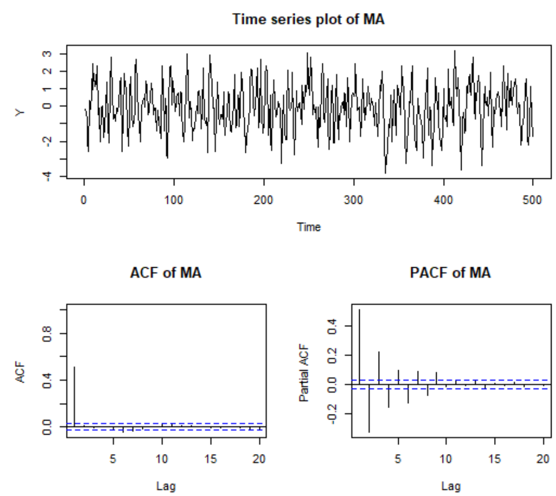
<tr><td>θ_2</td><td>0</td><td>
</td><td>
</td><td>
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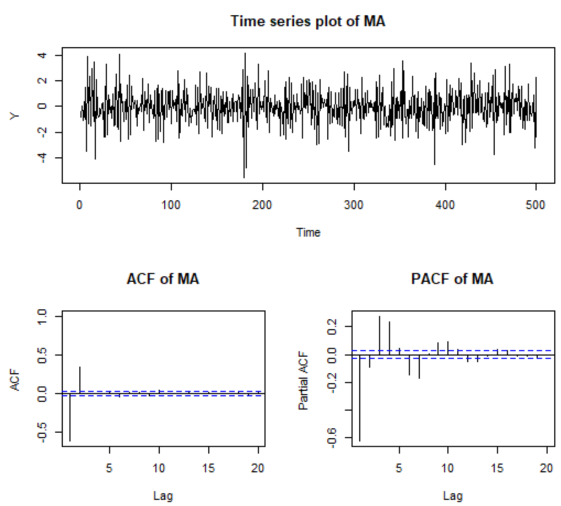
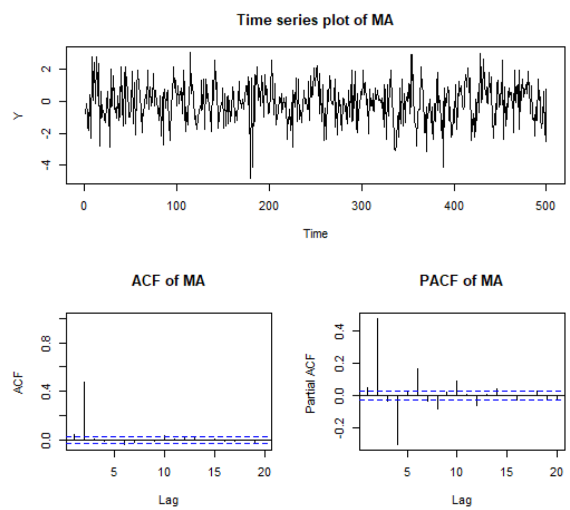
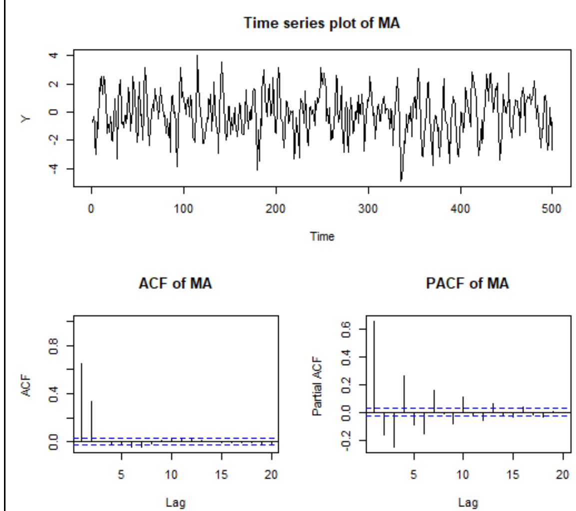
<tr><td></td><td>0.7</td><td>
</td><td>
</td><td>
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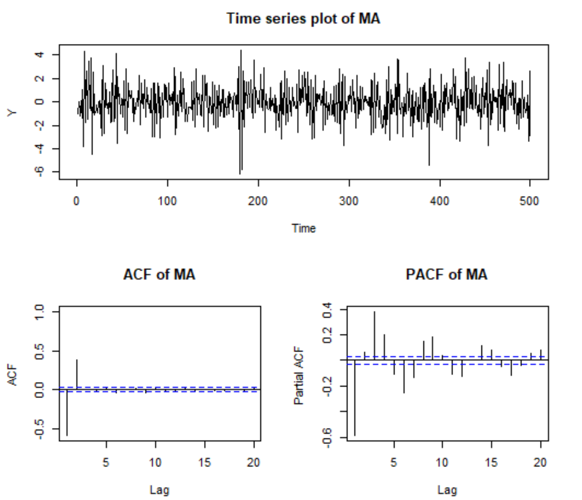
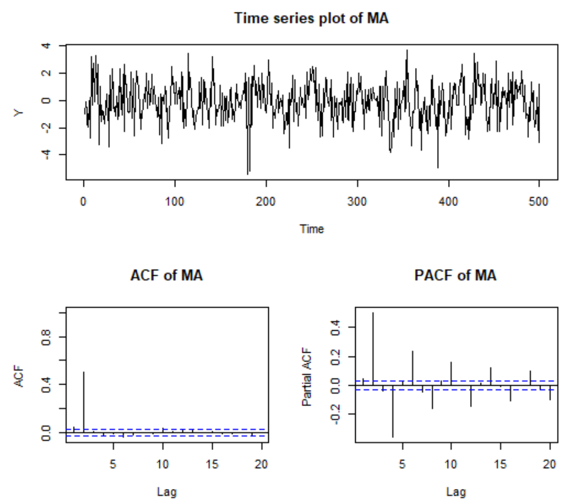
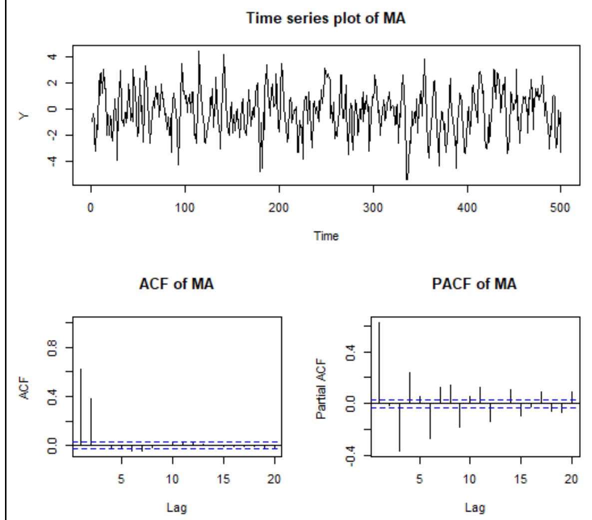
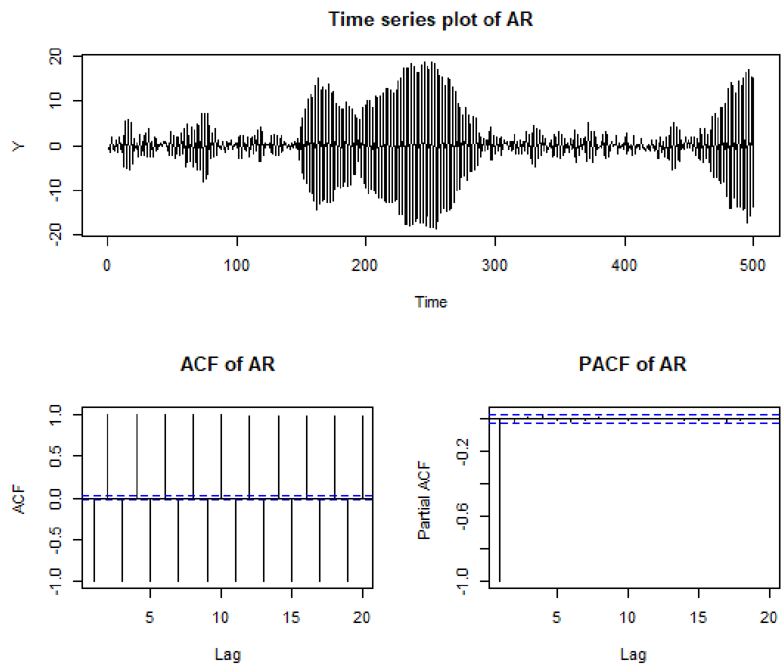
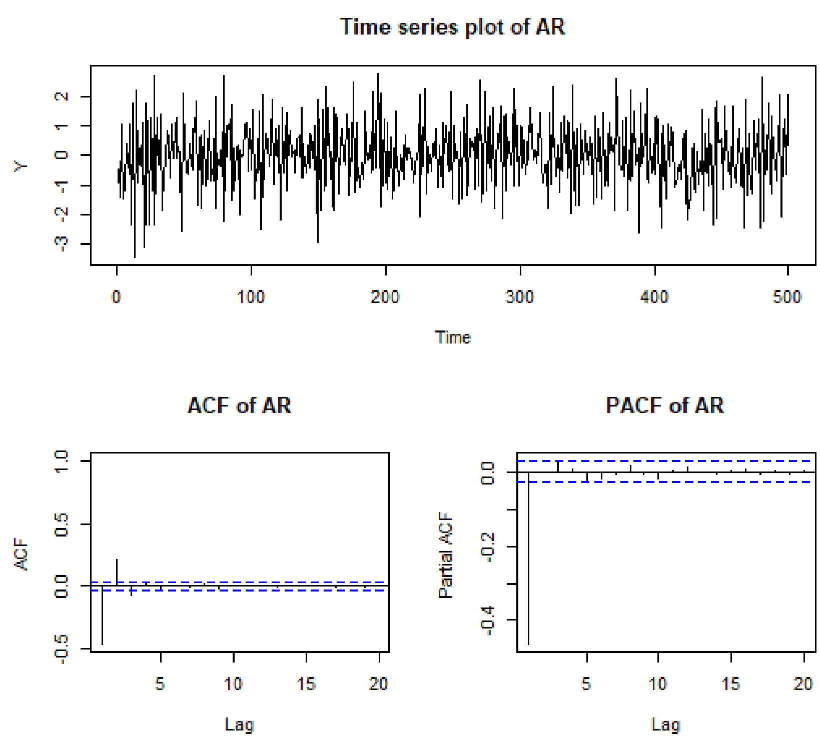
	1			

Table 2: Time series plot, ACF and PACF of MA(2) for different parameters

ϕ	Visualization of AR(1)
-1	
-0.45	

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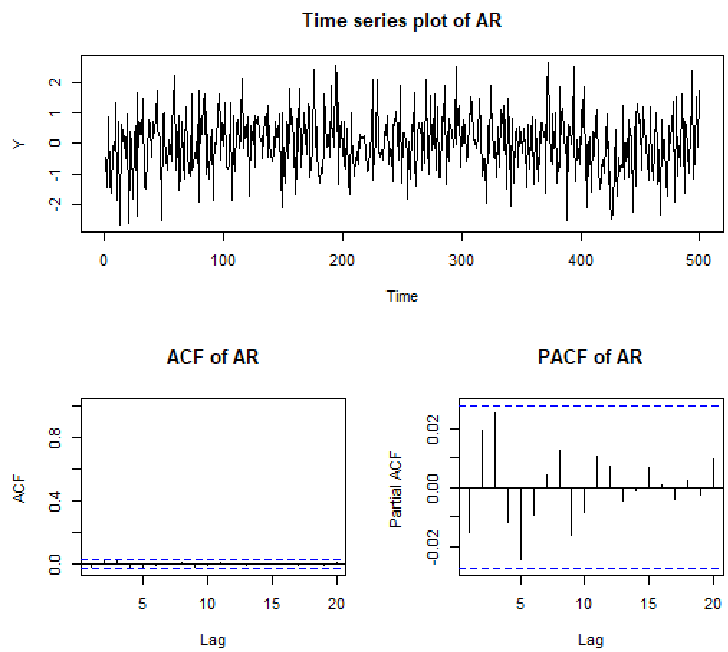

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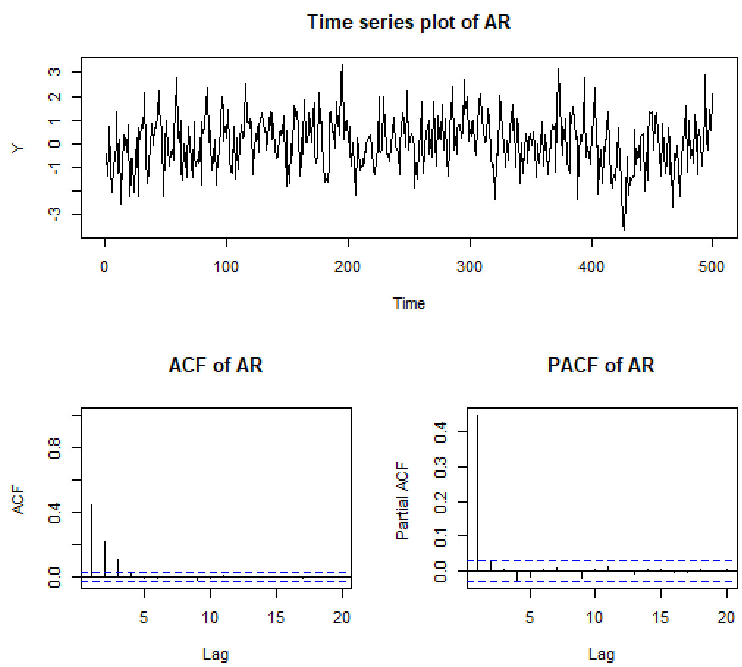

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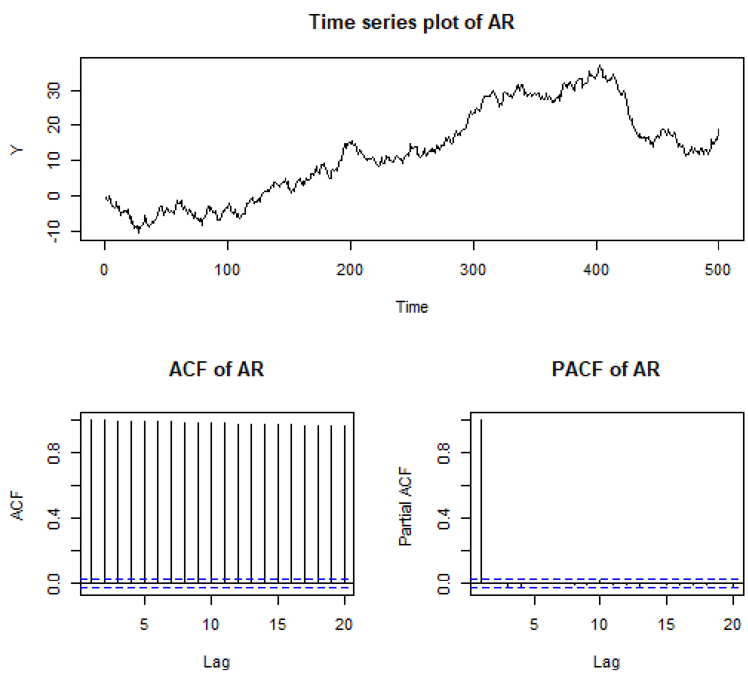
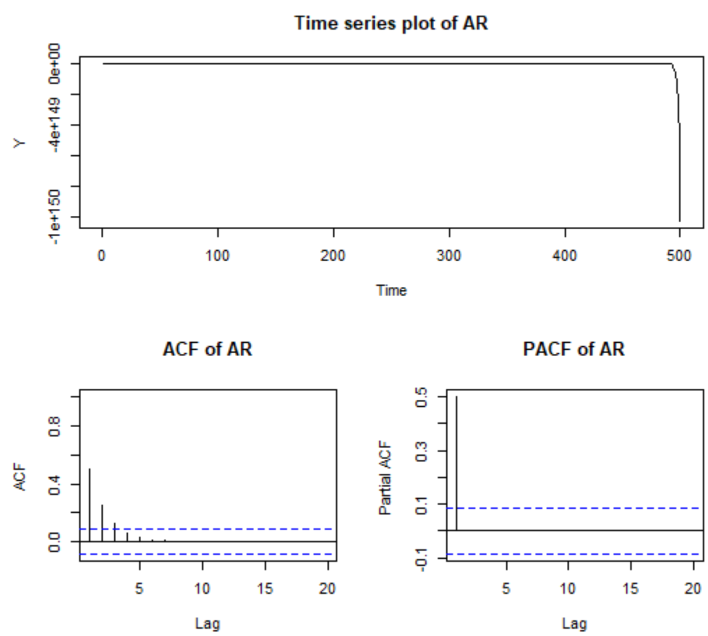
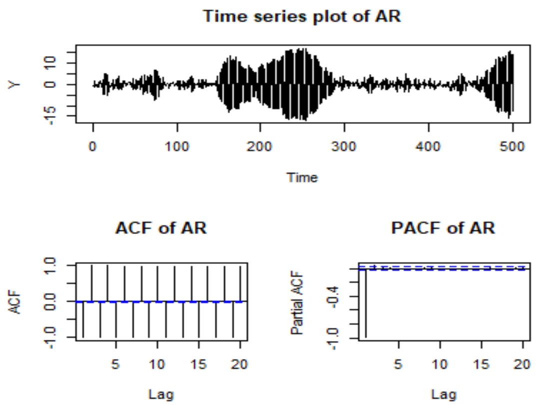
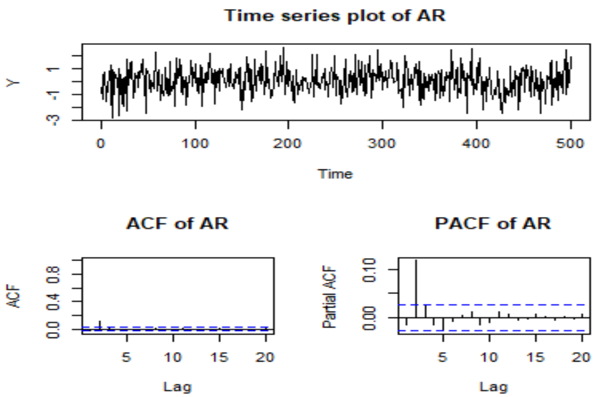
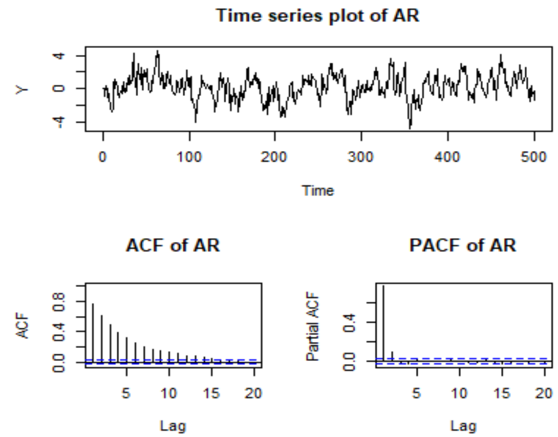
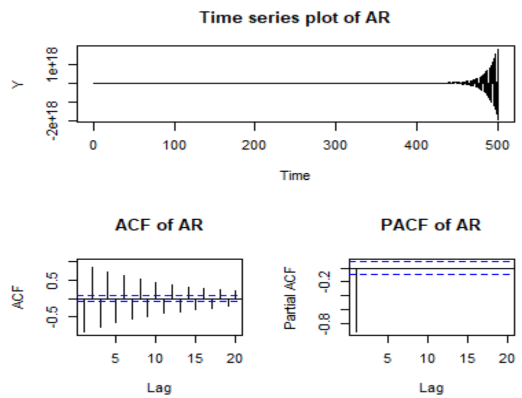
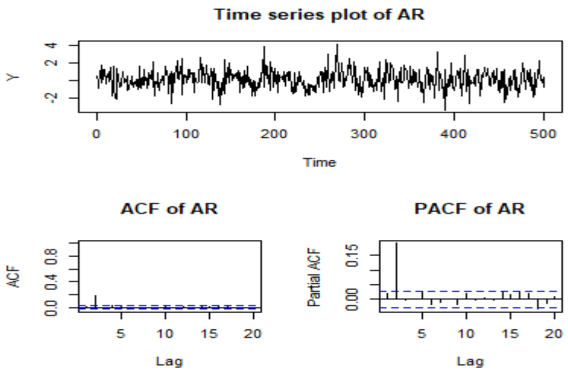
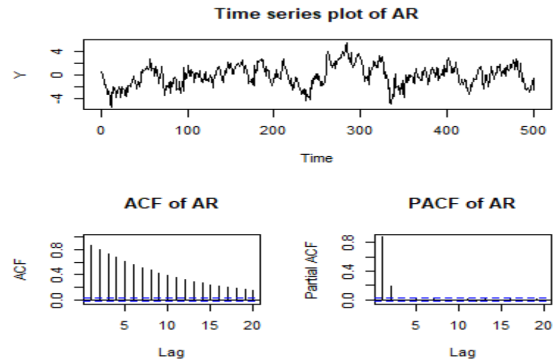
1	
2	

Table 3: Time series plot, ACF and PACF of AR(1) for different parameters

AR(2) Process

		$\emptyset_1$		
		-0.9	0	0.7
	0.1			
	0.2			

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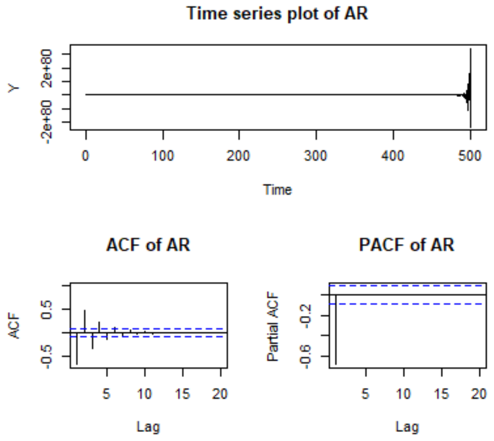
<td>
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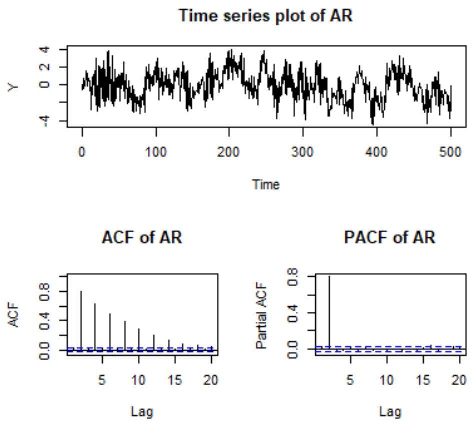
<td>
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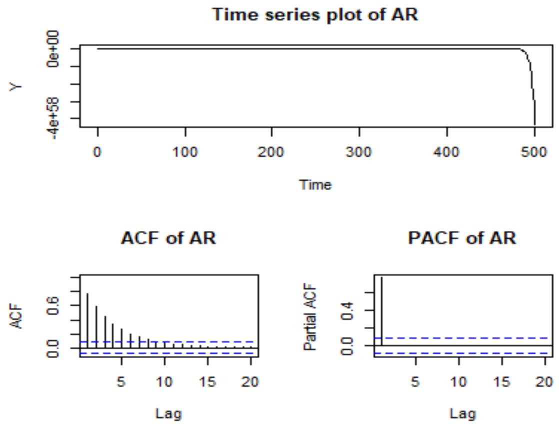
<td>
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Table 4: Time series plot, ACF and PACF of AR(2) for different parameters

ARMA(1,1)

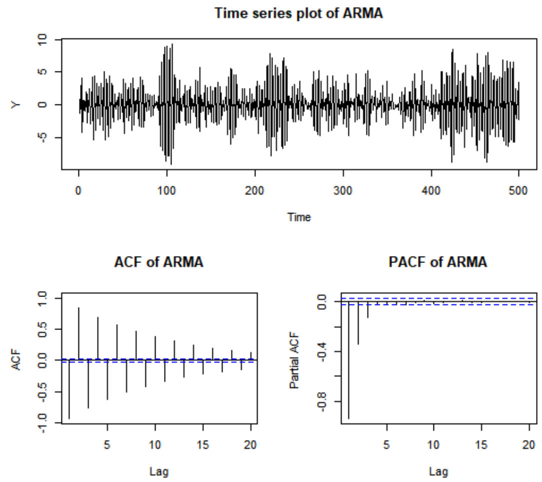
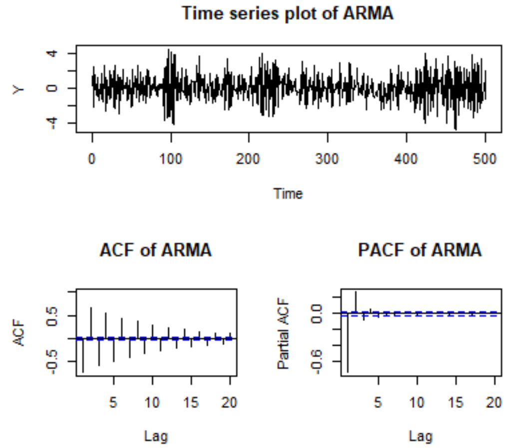
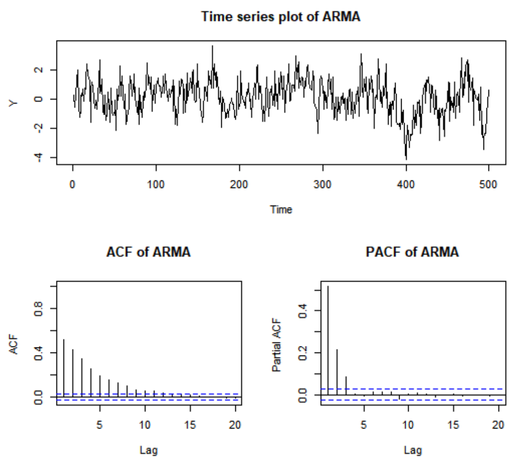
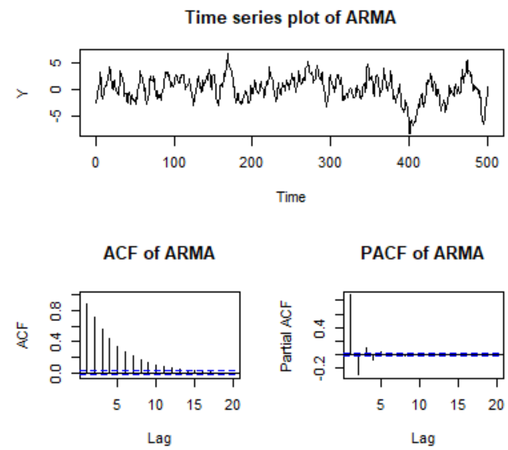
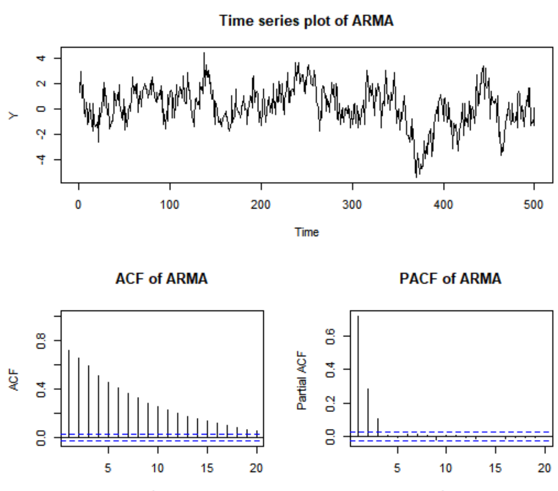
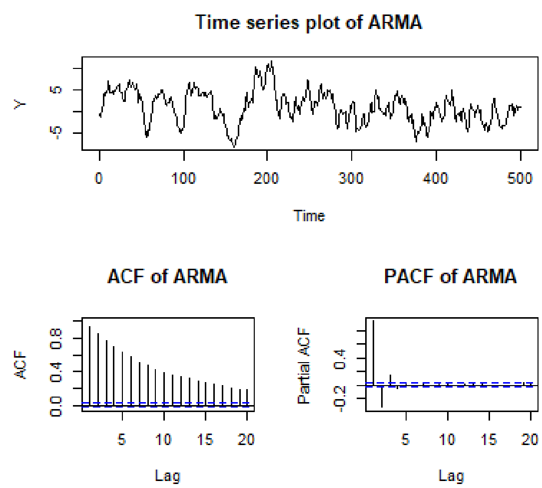
	$\theta = -0.4$	$\theta = 0.4$
$\phi =$ -0.9		
$\phi =$ 0.8		
$\phi =$ 0.9		

Table 5: Time series plot, ACF and PACF of ARMA(1,1) for different parameters