Binomial Destribution :-

$$P(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

Mean $= E(X) = \sum_{x=0}^{n} x\,p(x)$

$$= \sum_{x=0}^{n} x\, {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!\,(n-x)!} p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{x\, n!}{x(x-1)!\,(n-x)!} p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{n(n-1)!\; p\, p^{x-1}\, q^{(n-1)-(x-1)}}{(x-1)!\,\left((n-1)-(x-1)\right)!}$$

$$= np \sum_{x=0}^{n} {}^{n-1}C_{x-1}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np\,(p+q)^{1} \Rightarrow \boxed{\text{mean} = np}$$

$$\operatorname{Var}(x) = E(x^2) - (E(x))^2$$

Now $$E(x^2) = \sum_{x=0}^{n} x^2 P(x)$$

$$= \sum_{x=0}^{n} [x + (x-1)x] P(x)$$

$$= \sum_{x=0}^{n} x P(x) + \sum_{x=0}^{n} x(x-1) {}^{n}C_{x} p^{x} q^{n-x}$$

$$= np + \sum_{x=0}^{n} \frac{x(x-1)\, n!}{x!\,(n-x)!} p^{x} q^{n-x}$$

$$= np + \sum_{x=0}^{n} \frac{x(x-1)\, n!}{x(x-1)(x-2)!\,(n-x)!} p^{x} q^{n-x}$$

$$= np + \sum_{x=0}^{n} \frac{n(n-1)(n-2)!\, p^{2} p^{x-2} q^{(n-2)-(x-2)}}{(x-2)!\,((n-2)-(x-2))!}$$

$$= np + n(n-1)p^{2} \sum {}^{n-2}C_{x-2}\, p^{x-2} q^{(n-2)-(x-2)}$$

$$= np + n(n-1)p^{2} (p+q)^{n-2}$$

$$\downarrow 1$$

$$E(x^2) = np + n(n-1)p^2$$

Now

$$Var(x) = np + (n-1)np^2 - (np)^2$$

$$= np + n^2p^2 - np^2 - n^2p^2$$

$$= np(1-p)$$

$$\boxed{Var(x) = npq}$$

Poisson Distribution

$$P(x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

mean

$$E(X) = \sum_{x=0}^{\infty} x P(x)$$

$$= \sum_{x=0}^{\infty} \frac{x e^{-\lambda} \lambda^{x}}{x!}$$

$$= 0 + e^{-\lambda}\lambda + \frac{2e^{-\lambda}\lambda^{2}}{2!} + \cdots$$

$$= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right]$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$\boxed{E(X) = \lambda}$$

variance

$$Var(x) = E(x^2) - (E(x))^2$$

$$E(x^2) = \sum_{x=0}^{\infty} x^2 P(x)$$

$$= \sum_{x=0}^{\infty} (x + x(x-1))\, px$$

$$= \sum_{x=0}^{\infty} x\,p(x) + \sum_{x=0}^{\infty} \frac{x(x-1)\, e^{-\lambda} \lambda^{x}}{x!}$$

$$= \lambda + \left[0 + 0 + \frac{2\times1\times e^{-\lambda}\lambda^{2}}{2!} + \frac{3\times2\times e^{-\lambda}\lambda^{3}}{3!} + \frac{4\times3\times e^{-\lambda}\lambda^{4}}{4!} + \cdots \right]$$

$$= \lambda + \left[\frac{e^{-\lambda}\lambda^{2}}{1!} + \frac{e^{-\lambda}\lambda^{3}}{1!} + \frac{e^{-\lambda}\lambda^{4}}{2!} + \cdots \right]$$

$$= \lambda + e^{-\lambda}\lambda^{2} \left[1 + \frac{\lambda}{1!} + \frac{\lambda^{2}}{2!} + \cdots \right]$$

$$= \lambda + e^{-\lambda}\lambda^{2} e^{\lambda}$$

$$E(x^2) = \lambda + \lambda^2$$

Now $\quad var(x) = E(x^2) - E(x)^2$

$$var(x) = \lambda + \lambda^2 - \lambda^2$$

$$\boxed{var(x) = \lambda}$$

Negative Binomial Distribution

$$\boxed{\begin{array}{l} \text{mean} = rq/p \\[2em] \text{Variance} = \dfrac{rq}{p^2} \end{array}}$$

Geometric Distribution

$$\boxed{\begin{array}{l} \text{mean} = \dfrac{1-p}{p} = \dfrac{q}{p} \\[2em] \text{Var} = \dfrac{1-p}{p^2} = \dfrac{q}{p^2} \end{array}}$$

Gamma function

$$\Rightarrow \int_0^\infty e^{-x} x^{n-1}\, dx = \Gamma n$$

$$\Rightarrow \int_0^\infty e^{-ax} x^{n-1}\, dx = \frac{\Gamma n}{a^n}$$

$$ax = t$$

$$a\,dx = dt$$

$$\int_0^\infty e^{-t} \left(\frac{t}{a}\right)^{n-1} \frac{dt}{a}$$

$$\Rightarrow \frac{1}{a^n} \int_0^\infty e^{-t}\, t^{n-1}\, dt = \frac{\Gamma n}{a^n}$$

$$\Rightarrow \Gamma n = (n-1)! = 1 \times 2 \cdots (n-1)$$

$$\Rightarrow \Gamma 1/2 = \sqrt{\pi}$$

$$\Rightarrow \Gamma n/2 = \sqrt{\pi}\, \frac{(n-2)!!}{2^{(n-1)/2}}$$

→ double factorial

Normal Distribution :-

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $\frac{x-\mu}{\sigma} = z$

$\Rightarrow x = \mu + \sigma z$

$\Rightarrow dx = \sigma dz$

Limit $x \to -\infty \Rightarrow z \to -\infty$

$x \to \infty \Rightarrow x \to \infty$

$$E(x) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} (\mu + \sigma z) e^{-z^2/2} \sigma dz$$

$$= \frac{1}{\sqrt{2\pi}}\left[\underbrace{\int_{-\infty}^{\infty} \mu e^{-z^2/2} dz}_{\text{even fun}} + \underbrace{\int_{-\infty}^{\infty} \sigma z e^{-z^2/2} dz}_{\text{odd func}}\right.$$

odd function

$$f(-x) = -f(x)$$

even function

$$f(-x) = f(x)$$

So

~~E(X) =~~ $\displaystyle\int_{-\infty}^{\infty} \text{odd fun} = 0$

$$\int_{-\infty}^{\infty} \text{even fun} = 2\int_0^{\infty} \text{even fun}$$

hence

$$E(X) = \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} e^{-z^2/2}\, dz$$

$$\text{Let } z^2/2 = K$$

$$\frac{2z\,dz}{2} = dK$$

$$dz = \frac{dK}{z} = \frac{dK}{\sqrt{2K}}$$

Now

$$E(X) = \frac{2\mu}{\sqrt{2}\sqrt{2\pi}} \boxed{\int_0^{\infty} \frac{K^{-1/2} e^{-K}\, dK}{\cancel{\sqrt{K}}}} \Rightarrow \text{gamma fun}$$

$$= \frac{\mu}{\sqrt{\pi}} \Gamma(1/2) \Rightarrow \boxed{E(X) = \frac{\mu}{\sqrt{\pi}} \sqrt{\pi} = \mu}$$

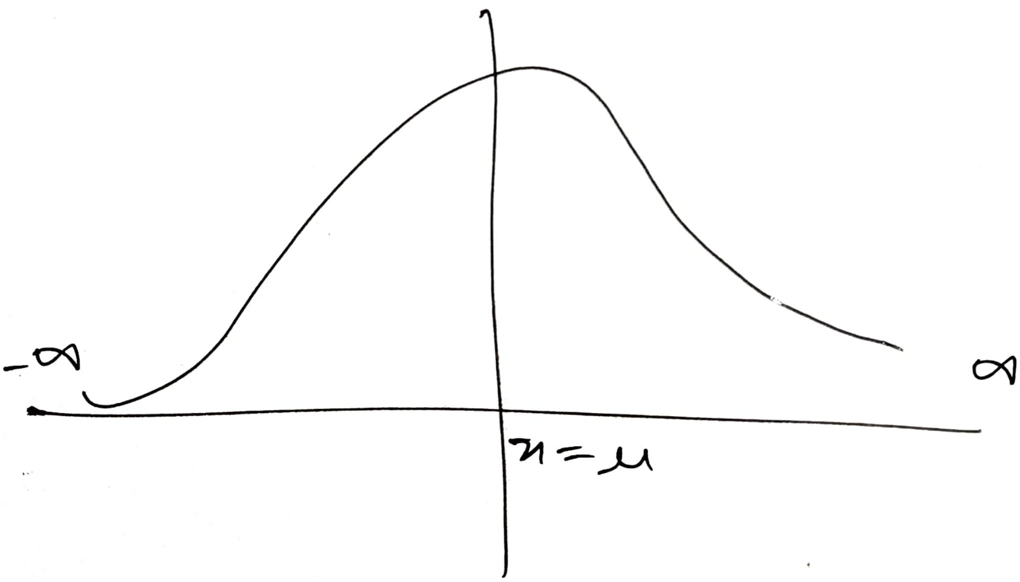
variance

$$E(x^2) = \mu^2 + \sigma^2$$

$$var(x) = E(x^2) - (E(x))^2$$

$$= \mu^2 + \sigma^2 - \mu^2$$

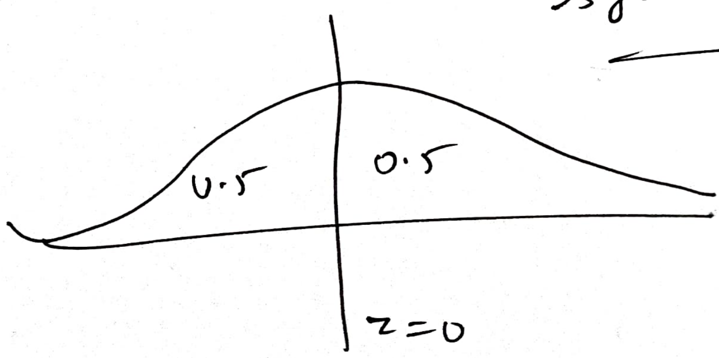
$$\boxed{var(x) = \sigma^2}$$



$$-\infty < x < \infty$$

$$z = \frac{x - \mu}{\sigma}$$

symmetric



Mean of Exponential Distribution :-

$$E(X) = \int_0^\infty x \, f_X(x) \, dx$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \int_0^\infty \lambda x e^{-\lambda x} \, dx$$

Let $\lambda x = t$

$dx = dt/\lambda$

$$E(X) = \frac{1}{\lambda} \int_0^\infty t e^{-t} \, dt$$

$\Rightarrow$ using gamma function

$$E(X) = \frac{1}{\lambda} \sqrt{2} = \frac{1!}{\lambda}$$

$$\boxed{E(X) = 1/\lambda}$$

Variance of Exponential Distribution

—— x —— x ——

$$Var(X) = E(X^2) - (E(X))^2$$

$$\downarrow$$

$$1/\lambda$$

Now $E(X^2) = \int_0^\infty x^2 \times e^{-\lambda x} \, dx$

$\hookrightarrow \int_0^\infty x^2 f(x) \, dx$

$\Rightarrow$ Let $\lambda x = t$

$$dx = dt/\lambda$$

$$E(X^2) = \frac{1}{\lambda^2} \int_0^\infty t^2 \times e^{-t} \, \frac{dt}{\lambda}$$

$$= \frac{1}{\lambda^2} \int_0^\infty t^2 e^{-t} \, dt$$

$$= \frac{1}{\lambda^2} \Gamma 3 \quad \text{(using gamma func)}$$

$$= 2/\lambda^2$$

Now $Var(X) = E(X^2) - (E(X))^2$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \Rightarrow \boxed{Var(X) = \frac{1}{\lambda^2}}$$