Example-5

In a game , a man wins 5 rupees for getting a number greater than 4 and loss rupees 1 otherwise, when a fair die is thrown.

The man decide to throw a die thrice but to quit as and when he wins. Find the expected value of the amount he win/loss.

Expected Value

Expected value

Expected value: measure of location, central tendency

X continuous with range $[a, b]$ and pdf $f(x)$:

$$E(X) = \int_a^b x f(x)\, dx.$$

X discrete with values $x_1, \ldots, x_n$ and pmf $p(x_i)$:

$$E(X) = \sum_{i=1}^{n} x_i p(x_i).$$

View these as essentially the same formulas.

Variance and Standard Deviation

Variance and standard deviation

Standard deviation: measure of spread, scale

For *any* random variable X with mean μ

$$\mathrm{Var}(X) = E((X-\mu)^2), \qquad \sigma = \sqrt{\mathrm{Var}(X)}$$

X continuous with range $[a, b]$ and pdf $f(x)$:

$$\mathrm{Var}(X) = \int_a^b (x-\mu)^2 f(x)\,dx.$$

X discrete with values $x_1, \ldots, x_n$ and pmf $p(x_i)$:

$$\mathrm{Var}(X) = \sum_{i=1}^{n} (x_i-\mu)^2 p(x_i).$$

View these as essentially the same formulas.

Properties

Properties: (the same for discrete and continuous)

1. $E(X+Y) = E(X) + E(Y)$.
2. $E(aX+b) = aE(X) + b$.
3. If X and Y are independent then
   $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$.
4. $\text{Var}(aX+b) = a^2\text{Var}(X)$.
5. $\text{Var}(X) = E(X^2) - E(X)^2$.

Covariance

Measures the degree to which two random variables vary together, e.g. height and weight of people.

X, Y random variables with means μ_X and μ_Y

$$\mathrm{Cov}(X, Y) = E((X - \mu_X)(Y - \mu_Y)).$$

X	Y
10	40
12	48
14	56
8	32

$\mu_x = 11 \qquad \mu_y = 44$

$$Cov(X,Y) = E(X-\mu_x)(Y-\mu$$

$$= \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{N}$$

$$\Rightarrow \frac{(-1)(-4) + (1)\times(4) + (3)\times(12}{4}$$

$$=$$

Properties of Covariance

Properties

1. $\mathrm{Cov}(aX + b, cY + d) = ac\mathrm{Cov}(X, Y)$ for constants a, b, c, d.
2. $\mathrm{Cov}(X_1 + X_2, Y) = \mathrm{Cov}(X_1, Y) + \mathrm{Cov}(X_2, Y)$.
3. $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$
4. $\mathrm{Cov}(X, Y) = E(XY) - \mu_X\mu_Y$.
5. If X and Y are independent then $\mathrm{Cov}(X, Y) = 0$.
6. **Warning:** The converse is not true, if covariance is 0 the variables might not be independent.

Concept Question

Suppose we have the following joint probability table.

$Y \backslash X$	-1	0	1	$p(y_j)$
0	0	1/2	0	1/2
1	1/4	0	1/4	1/2
$p(x_i)$	1/4	1/2	1/4	1

→ 1/2

At your table work out the covariance $\mathrm{Cov}(X, Y)$.

Because the covariance is 0 we know that X and Y are independent

1. True
2. False

Key point: covariance measures the linear relationship between X and Y. It can completely miss a quadratic or higher order relationship.

$$\mu_x = E(x) = \sum x p_i = -1 \times \frac{1}{4} + 0 \times \frac{1}{2} + 1 \times \frac{1}{4}$$

$$= 0$$

$$\mu_y\, E(y) = 0 \times \frac{1}{2} + 1 \times 1/2 = 1/2$$

$$\mathrm{Cov}(x, y) = E\left((x - \mu_x)(y - \mu_y) \right)$$

$$= E\left(\underline{x \cdot} \; y - \cancel{\mu_y} \mu_y \right)$$

Correlation

Like covariance, but removes scale.
The *correlation coefficient* between X and Y is defined by

$$\mathrm{Cor}(X, Y) = \rho = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}.$$

Properties:
1. ρ is the covariance of the standardized versions of X and Y.
2. ρ is dimensionless (it's a ratio).
3. $-1 \leq \rho \leq 1$. $\rho = 1$ if and only if $Y = aX + b$ with $a > 0$ and $\rho = -1$ if and only if $Y = aX + b$ with $a < 0$.

X	Y
65	67
66	68
67	68
67	68
68	72
70	69
72	71

X: height of father
Y: height of son

$$r = \frac{Cov(X,Y)}{\sigma_x \sigma_y}$$

$$Cov(X,Y) = E(XY) - E(X)\,E(Y)$$

$$\sigma_x = \sqrt{E(X^2) - (E(X))^2}$$

$$\sigma_y = \sqrt{E(Y^2) - (E(Y))^2}$$

68 69 $\Sigma u = 0$ $\Sigma v = 0$

X	Y	$u = X - \bar{X}$	$v = Y - \bar{Y}$	u^2	v^2	uv
65	67	−3	−2			
66	68	−2	−1			
67	68	−1	−4			
67	68	−1	−1			
68	72	0	3			
69	72	1	3			
70	69	2	0			
72	71	4	2			