Engineering Calculus-EMAT101L

(Lecture-1)

Real Number System

School of Engineering and Applied Sciences
Department of Mathematics
Bennett University
2021

To get maximum benefit from this course

To get maximum benefit from this course here are my suggestions:

- Be interactive.
- Understanding of mathematical concept is the best achieved by discussions.
- Revised the topics discussed.
- Doing the exercises yourself, is very useful thing in strengthening concepts.

Books

Texts

1. Maurice D. Weir and Joel Hass, *Thomas' Calculus*, 12th Edition, Pearson Education India, 2016.
2. K. A. Ross, *Elementary Analysis: The Theory of Calculus*, 2nd Edition, Springer, 2013.

References

1. S. R. Ghorpade and B. V. Limaye, *An Introduction to Calculus and Real Analysis*, Springer India, 2006.
2. James Stewart, *Calculus*, 7th Edition, Brooks Cole Cengage Learning, 2012.
3. Bartle and Shebert, *Introduction to Real Analysis*, 4th Edition, Wiley, 2014.
4. Erwin Kreyszig, *Advanced Engineering Mathematics*, 10th edition, Wiley, 2010.

Aim of the course

Aim of the course

- To understand all the basic fundamental definitions of Calculus.
- To identify the convergence or divergence of a wide class of sequence/series.
- To develop the fundamental ideas of the differential and integral calculus to functions of one variable.
- To understand the concepts of the differential and integral calculus to functions of multivariable.
- To develop the problem-solving skills related to limit, continuity, differentiation, integration etc. using some computational software packages

Learning Outcome of the Lecture

We learn

- What Is Calculus?
- Practical Applications of calculus,
- Real Number System
  - Real Number $\mathbb{R}$,
  - Intervals,
  - Neighbourhood of a point.

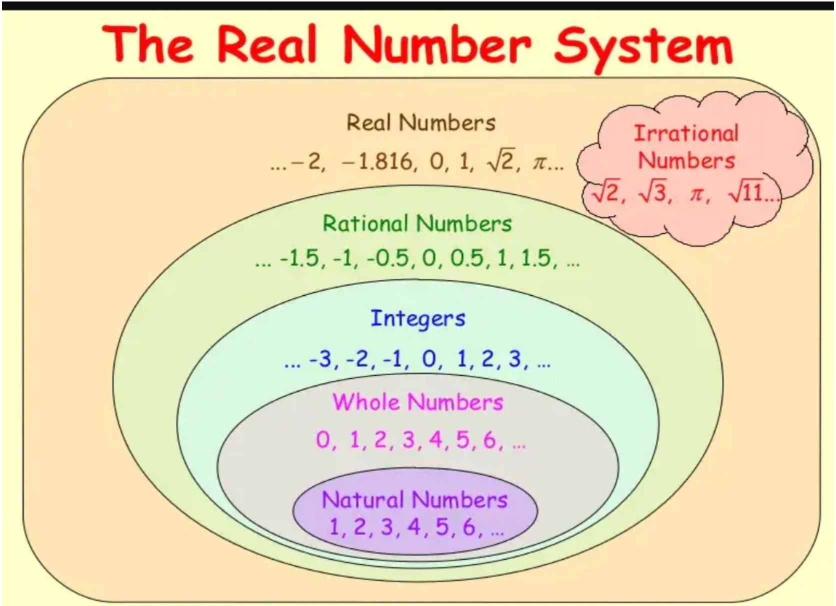
What Is Calculus? Definition and Practical Applications

What Is Calculus?

- **Calculus is a branch of mathematics that involves the study of rates of change.**
- **Gottfried Leibniz and Isaac Newton, 17th-century mathematicians, both invented calculus independently. Newton invented it first, but Leibniz created the notations that mathematicians use today.**
- **There are two types of calculus: Differential calculus determines the rate of change of a quantity, while integral calculus finds the quantity where the rate of change is known.**

Practical Applications

- **Calculus has many practical applications in real life.**
- **calculus is used to help define, explain, and calculate motion, electricity, heat, light.**
- **calculus is used to check answers for different mathematical disciplines such as statistics, analytical geometry, and algebra.**


The Real Number System
Real Numbers
...−2, −1.816, 0, 1, √2, π...
Irrational Numbers
√2, √3, π, √11...
Rational Numbers
... -1.5, -1, -0.5, 0, 0.5, 1, 1.5, ...
Integers
... -3, -2, -1, 0, 1, 2, 3, ...
Whole Numbers
0, 1, 2, 3, 4, 5, 6, ...
Natural Numbers
1, 2, 3, 4, 5, 6, ...

1.jpg

Real Number System

$\mathbb{N}$ = Natural Numbers = {1, 2, 3, ...}

$\mathbb{W}$ or $\mathbb{N}_0$ = Whole Numbers = {0, 1, 2, 3, ...}

$\mathbb{Z}$ = Integers = {..., -1, -2, -3, 0, 1, 2, 3, ...}

$\mathbb{Q}$ = Rational Numbers = {p/q ; p and q are integers}

$\mathbb{I}$ = Irrational Numbers = {non-rational number}

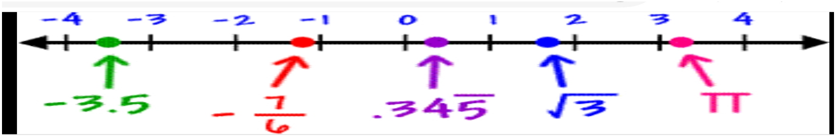
$\mathbb{R}$ = Real Numbers = {All of the above number sets}

3.jpg

Real Line ($\mathbb{R}$)

Definition

A real number line, allows us to visually display real numbers by associating them with unique points on a line. To construct a real number line, draw a horizontal line with arrows on both ends to indicate that it continues without bound.



2.jpg

Absolute Value of a real number

The absolute value of a real number a, denoted $|a|$, is defined as the distance between zero (the origin) and the graph of that real number on the number line. Since it is a distance, it is always positive. For example, $|-4| = 4$ and $|4| = 4$

Note: In general distance between two real numbers a and b is $|a - b|$

Intervals

Definition

A (real) interval is a set of real numbers that contains all real numbers lying between any two numbers of the set.

Open Interval

If a and b are two real numbers such that $a < b$, then the set

$$\{x \in \mathbb{R} : a < x < b\}$$

consisting of all real numbers between a and b (excluding a and b) is called an open interval and is denoted by (a, b).

Closed Interval

The set

$$\{x \in \mathbb{R} : a \leq x \leq b\}$$

consisting of a, b and all real numbers between a and b is called an closed interval and is denoted by $[a, b]$.

Intervals

Half open Interval (Half closed Interval)

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}, \quad (a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

Infinite open interval

$$(a, \infty) = \{x \in \mathbb{R} : x > a\} \quad (-\infty, a) = \{x \in \mathbb{R} : x < a\}$$

Infinite closed interval

$$[a, \infty) = \{x \in \mathbb{R} : x \geq a\} \quad (-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$$

The set R

$$\mathbb{R} = (-\infty, \infty)$$

Neighbourhood of a point

δ neighbourhood of a point

Let $a \in \mathbb{R}$ and $\delta > 0$. Then the δ neighbourhood of a defined as

$$(a-\delta, a+\delta) = \{x \in \mathbb{R} : a-\delta < x < a+\delta\}$$

$\implies N_\delta(a) = (a-\delta, a+\delta) = \{x \in \mathbb{R} : |x-a| < \delta\}$

deleted neighbourhood of a point

If from the δ neighbourhood of a point, the point itself excluded, then we get the deleted neighbourhood of a point.

$$N_\delta(a) - \{a\} = (a-\delta, a+\delta) - \{a\} = \{x \in \mathbb{R} : 0 < |x-a| < \delta\}$$

Limit Point

A real number $b \in \mathbb{R}$ is a limit point of a set $S(\subseteq \mathbb{R})$ if every neighbourhood of b contains at-least one member of S other than b.

Note:

- The set of all limit point of a set S denoted by S'
- A limit point of a set may or may not be member of the set.

Example:1

Let $S = \mathbb{R}$. Then $S' = \mathbb{R}$.

Example:2

Let $S = \{1, 2, 3, 4\}$. Then S has no limit point.

Example:3

Let $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Then $S' = \{0\}$.
Here limit point is not a member of the set.

Example:1

Let $S = (a, b), a, b \in \mathbb{R}$. Then $S' = [a, b]$.


Thank You

Engineering Calculus-EMAT101L

(Lecture-2)

Real Number System

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Learning Outcome of the Lecture

We learn

- Real Number System
  - Neighbourhood of a point.
  - Bounded and Unbounded sets,
  - Supremum and Infimum,
  - Completeness property of $\mathbb{R}$,
  - Archimedean Property of $\mathbb{R}$.

Neighbourhood of a point

δ neighbourhood of a point

Let $a \in \mathbb{R}$ and $\delta > 0$. Then the δ neighbourhood of a defined as

$$(a-\delta, a+\delta) = \{x \in \mathbb{R} : a-\delta < x < a+\delta\}$$

$\implies N_\delta(a) = (a-\delta, a+\delta) = \{x \in \mathbb{R} : |x-a| < \delta\}$

deleted neighbourhood of a point

If from the δ neighbourhood of a point, the point itself excluded, then we get the deleted neighbourhood of a point.

$$N_\delta(a) - \{a\} = (a-\delta, a+\delta) - \{a\} = \{x \in \mathbb{R} : 0 < |x-a| < \delta\}$$

Bounded and Unbounded sets

Bounded above

A subset S of real numbers is said to be bounded above if $\exists$ a real number K such that every member of S is less than or equal to K, i.e,

$$x \leq K \qquad x \in S.$$

The number K is called an upper bound of S.

Note: If no such K exists, the set S is said to be unbounded above or not bounded above.

Bounded below

A subset S of real numbers is said to be bounded below if $\exists$ a real number k such that every member of S is greater than or equal to K, i.e,

$$x \geq K \qquad x \in S.$$

The number k is called an lower bound of S.

Note: If no such k exists, the set S is said to be unbounded below or not bounded below.

Bounded

A subset S of real numbers is said to be bounded if it is bounded above as well as bounded below.

Example of bounded and Unbounded sets

Example of bounded Set

- Let $S = (5, 6)$.
- Lower bounds of $S = (-\infty, 5]$. So S is bounded below.
- Upper bounds of $S = [6, \infty)$. So S is bounded above.
- Thus S is bounded.

Note: Every finite set of numbers is bounded

Example of unbounded Set

- Let $S = \{x \in \mathbb{R} : x > 2\}$.
- Lower bounds of $S = (-\infty, 2]$. So S is bounded below.
- But S has no upper bounds. So S is not bounded above.
- Thus S is unbounded.

Note: The set $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are not bounded

Supremum or least upper bound (l.u.b)

Let S be a nonempty subset of $\mathbb{R}$. Then $M \in \mathbb{R}$ is said o be supremum or least upper bound of S if it satisfies the conditions:

- M is an upper bound of S and
- if M' is another upper bound of S, then $M \leq M'$

Note: Supremum of S denoted by $sup\ S$

Infimum or greatest lower bound (g.l.b)

Let S be a nonempty subset of $\mathbb{R}$. Then $m \in \mathbb{R}$ is said o be infimum or greatest lower bound of S if it satisfies the conditions:

- m is an lower bound of S and
- if m' is another lower bound of S, then $m \geq m'$

Note: Infimum of S denoted by $inf\ S$
Note: Supremum and infimum of a set may not belong to the set.

Example: Find the supremum and infimum of the following sets

Example:1

- Let $S = \{2, 4, 6, 8, 10\}$.
- Lower bounds of $S = (-\infty, 2]$ and $inf\ S = 2$.
- Upper bounds of $S = [10, \infty)$ and $sup\ S = 10$.

Note: Here $sup\ S$ and $inf\ S$ exist and belong to the set.

Example: 2

- Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.
- Lower bounds of $S = (-\infty, 0]$ and $inf\ S = 0$.
- Upper bounds of $S = [1, \infty)$ and $sup\ S = 1$.

Note: Here $sup\ S$ exist and belong to the set. $inf\ S$ exist but does not belongs to the set.

Example: 3

- Let $S = \{x \in \mathbb{R} : x > 0\}$.
- Lower bounds of $S = (-\infty, 0]$ and $inf\ S = 0$.
- But it has no upper bounds. So $sup\ S$ does not exist.

Note: Here $sup\ S$ does not exist.

Completeness property and Archimedean property of $\mathbb{R}$

least upper bound property

Every nonempty subset S of $\mathbb{R}$ which is bounded above has a least upper bound i.e $sup\ S$ exit.

Greatest lower bound property

Every nonempty subset S of $\mathbb{R}$ which is bounded below has a greatest lower bound i.e $inf\ S$ exit.

Archimedean property

For any $\epsilon > 0$, there $\exists$ a natural number n, such that $\frac{1}{n} < \epsilon$.

Thank You

Engineering Calculus-EMAT101L

(Lecture-3)

Sequence

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Definition

A mapping $f : \mathbb{N} \to \mathbb{R}$ is said to be a sequence in $\mathbb{R}$ or a real sequence.

Note:

- A sequence f is generally denoted by the symbol $\{f(n)\}_{n=1}^{\infty}$. Simply we can write $\{f(n)\}$.
- Here $f(n)$ is the n^{th} element of the sequence.
- The symbols like $\{u_n\}$, $\{v_n\}$, $\{x_n\}$ etc. shall also be used to denote a sequence.

Examples:

1. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = n$. Then the sequence is $\{n\} = \{1, 2, 3, ...\}$
2. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = \frac{n}{n+1}$. Then the sequence is $\{\frac{n}{n+1}\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\}$
3. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = sin\frac{n\pi}{2}$. Then the sequence is $\{sin\frac{n\pi}{2}\} = \{1, 0, -1, , 0, 1, 0...\}$

Bounded Sequences

Definition

A sequence $\{x_n\}$ is:

- bounded above: if there exists $M \in \mathbb{R}$ such that, for all $n \in \mathbb{N}$, $x_n \leq M$;
- bounded below: if there exists $m \in \mathbb{R}$ such that, for all $n \in \mathbb{N}$, $x_n \geq m$;
- bounded: if it is both bounded above and bounded below.

examples:

1. The sequence $\{\frac{1}{n}\}$ is bounded since $0 < \frac{1}{n} \leq 1$.
2. The sequence $\{n\}$ is bounded below but is not bounded above.
3. The sequence $\{-n\}$ is bounded above but is not bounded below.

Monotonic Sequences

Definition

A sequence $\{x_n\}$ is:

- strictly increasing: if, for all $n \in \mathbb{N}$, $x_n < x_{n+1}$;
- increasing: if, for all $n \in \mathbb{N}$, $x_n \leq x_{n+1}$;
- strictly decreasing if, for all $n \in \mathbb{N}$, $x_n > x_{n+1}$;
- decreasing if, for all $n \in \mathbb{N}$, $x_n \geq x_{n+1}$;
- monotonic if it is increasing or decreasing or both;

Note:

- Each increasing sequence $\{x_n\}$ is bounded below.
- Each decreasing sequence $\{x_n\}$ is bounded above.

examples:

1. The sequence $\{n\}$ is is strictly increasing, since $n < n+1$;
2. The sequence $\{-n\}$ is is strictly decreasing, since $-n > -(n+1)$;

Definition

We say that a sequence $\{x_n\}$ converges to a real number L, if for every $\epsilon > 0$ (given), there exists $N \in \mathbb{N}$ (depending on ϵ) such that
for all $n \geq N \implies |x_n - L| < \epsilon$ i.e. $x_n \in (L - \epsilon, L + \epsilon)$

- The same thing expressed in symbol is

$x_n \to L$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = L$.

Examples;1

1. Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Solution: Let $\epsilon > 0$,

$$|x_n - L| = \left|\frac{1}{n} - 0\right| = \frac{1}{n}$$

From the Archimedean Property, we know that, For each $\epsilon > 0$, there $\exists$ a natural number N, such that

$$\frac{1}{N} < \epsilon$$

Then if
.

$$n \geq N \implies \frac{1}{n} \leq \frac{1}{N} < \epsilon \implies \frac{1}{n} < \epsilon$$

. Consequently, if $n \geq N$, then

$$|x_n - L| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon$$

Example:2

Show that $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1$.

Solution: Let $\epsilon > 0$,

$$|x_n - L| = \left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \frac{1}{n}$$

From the Archimedean Property, we know that, For each $\epsilon > 0$, there $\exists$ a natural number N, such that

$$\frac{1}{N} < \epsilon$$

Then if

.

$$n \geq N \implies \frac{1}{n} \leq \frac{1}{N} < \epsilon \implies \frac{1}{n} < \epsilon$$

. Consequently, if $n \geq N$, then

$$|x_n - L| = \left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \frac{1}{n} < \epsilon$$

Theorem

- A monotonic increasing sequence which is bounded above is convergent and converges to its least upper bound.
- A monotonic decreasing sequence which is bounded below is convergent and converges to its greatest lower bound.

Example:

Show that the sequence $\{x_n\}$ is convergent where $x_n = 1 - \frac{1}{n}, \forall n \in \mathbb{N}$.

Now

$$x_{n+1} - x_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0, \forall n$$

Therefore, the sequence $\{x_n\}$ is monotonic increasing.
Again

$$x_n = 1 - \frac{1}{n} < 1$$

. i.e., the sequence $\{x_n\}$ is bounded above.

Hence, the sequence being monotonic increasing and bounded above, is convergent.


Thank You

Engineering Calculus-EMAT101L

(Lecture-4,5,6)

Sequence

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Learning Outcome of the Lecture

We learn

- Sequence
    - Some theorems of convergent sequence,
    - Null sequence,
    - Divergent sequence,
    - Oscillatory sequence.

Some theorems of convergent sequence

Theorem

Let $\{x_n\}_1^\infty$ and $\{y_n\}_1^\infty$ be two convergent sequences such that $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} y_n = M$. Then

(i) $\lim_{n\to\infty} \{x_n + y_n\} = L + M.$

(ii) $\lim_{n\to\infty} \{x_n - y_n\} = L - M.$

(iii) $\lim_{n\to\infty} \{cx_n\} = cL, \quad c \in \mathbb{R}.$

(iv) $\lim_{n\to\infty} \{x_n y_n\} = LM.$

(v) $\lim_{n\to\infty} \left\{\frac{x_n}{y_n}\right\} = \frac{L}{M}$ if $M \neq 0.$

Examples:

Find the limit of the following sequences:

(i) $\left\{\dfrac{5}{n^2}\right\}_1^\infty$, (ii) $\left\{\dfrac{3n^2-6n}{5n^2+4}\right\}_1^\infty$, (iii) $\lim_{n\to\infty} \left(\frac{n-1}{n}\right)$.

Solution: (i)

$$\lim_{n\to\infty}\frac{5}{n^2}=\lim_{n\to\infty}5\cdot\frac{1}{n}\cdot\frac{1}{n}=5\cdot\lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{1}{n}=5\cdot 0\cdot 0=0.$$

(ii) Notice that

$$\frac{3n^2-6n}{5n^2+4}=\frac{3-6/n}{5+4/n^2}.$$

Now

$$\lim_{n\to\infty}(3-6/n)=3-6\lim_{n\to\infty}1/n=3-6\cdot 0=3$$

and

$$\lim_{n\to\infty}(5+4/n^2)=5+4\lim_{n\to\infty}1/n^2=5+4\cdot 0=5.$$

Therefore,

$$\lim_{n\to\infty}\frac{3n^2-6n}{5n^2+4}=\lim_{n\to\infty}\frac{3-6/n}{5+4/n^2}=\frac{3}{5}.$$

(iii)

$$\lim_{n\to\infty}\left(\frac{n-1}{n}\right)=\lim_{n\to\infty}\frac{1-1/n}{1}=1-\lim_{n\to\infty}\frac{1}{n}=1-0=1.$$

Theorem:1

A convergent sequence has a unique limit.

Proof: Let, if possible, a sequence $\{x_n\}$ converges to two real numbers L and L'
Therefore, $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} x_n = L'$.
Now, $L - L' = \lim_{n\to\infty} x_n - \lim_{n\to\infty} x_n = \lim_{n\to\infty}(x_n - x_n) = 0$
$\implies L = L'$.
Hence, a convergent sequence has a unique limit.

Theorem:2

Every convergent sequence is bounded.

Proof: Let $\{x_n\}$ be a convergent sequence and L be its limit.
Then, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that
$|x_n - L| < \epsilon$ ∀n$\geq N$
i.e. $L - \epsilon < x_n < L + \epsilon \; \forall n \geq N$.
Now, let $G = max\{L + \epsilon, x_1, x_2, ..., x_{N-1}\}$ and $g = min\{L - \epsilon, x_1, x_2, ..., x_{N-1}\}$.
Thus, we have $g \leq x_n \leq G \; \forall n$.
Hence, $\{x_n\}$ is a bounded sequence.

Note: Converse of the theorem: 2, i.e., A bounded sequence may not be convergent sequence.
Example: $\{x_n\} = \{(-1)^n\}$. This sequence is bounded but not convergent.

Theorem

- A monotonic increasing sequence which is bounded above is convergent and converges to its least upper bound.
- A monotonic decreasing sequence which is bounded below is convergent and converges to its greatest lower bound.

Example:

Show that the sequence $\{x_n\}$ is convergent where $x_n = 1 - \frac{1}{n}, \forall n \in \mathbb{N}$.

Now

$$x_{n+1} - x_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0, \forall n$$

Therefore, the sequence $\{x_n\}$ is monotonic increasing.
Again

$$x_n = 1 - \frac{1}{n} < 1$$

. i.e., the sequence $\{x_n\}$ is bounded above.

Hence, the sequence being monotonic increasing and bounded above, is convergent.

Sandwich Theorem

Sandwich theorem for sequences

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be three sequences such that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} z_n = L$, then $\lim_{n\to\infty} y_n = L$.

Proof: Let $\epsilon > 0$ be given. As $\lim_{n\to\infty} x_n = L$, there exists $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |x_n - L| < \epsilon. \tag{1}$$

Similarly as $\lim_{n\to\infty} z_n = L$, there exists $N_2 \in \mathbb{N}$

$$n \geq N_2 \implies |z_n - L| < \epsilon. \tag{2}$$

Let $N = \max\{N_1, N_2\}$. Then, $L - \epsilon < x_n$ (from (1)) and $z_n < L + \epsilon$ (from (2)). Thus

$$L - \epsilon < x_n \leq y_n \leq z_n < L + \epsilon.$$

Thus $|y_n - L| < \epsilon$ for all $n \geq N$. Hence the proof.

Examples

Using Sandwich theorem, prove the following:

(i) $\lim_{n\to\infty} \frac{\cos n}{n} = 0.$

(ii) $\lim_{n\to\infty} \frac{1}{2^n} = 0.$

(iii) $\lim_{n\to\infty} (-1)^n \frac{1}{n} = 0.$

(iv) If $0 < b < 1$, then $\lim_{n\to\infty} b^n = 0.$

(v) $\lim_{n\to\infty} \sqrt[n]{n} = 1.$

Solution: (i) Consider the sequence $\left\{\frac{\cos n}{n}\right\}_{n=1}^{\infty}$. Then $\frac{-1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$. Hence by Sandwich theorem, $\lim_{n\to\infty} \frac{\cos n}{n} = 0.$

(ii) As $0 < \frac{1}{2^n} < \frac{1}{n}$ and $\frac{1}{n} \to 0$ as $n \to \infty$, $\frac{1}{2^n}$ also converges to 0 by Sandwich theorem.

(iii) As $\frac{-1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$ for all $n \geq 1$ and $\frac{1}{n} \to 0$ as $n \to \infty$, $(-1)^n \frac{1}{n}$ also converges to 0 by Sandwich theorem.

(iv) Since $0 < b < 1$, So, $\frac{1}{b} > 1$.
Let $\frac{1}{b} = 1 + a$, where $a > 0$. Also we have $(1+a)^n \geq 1 + na$.
Hence

$$0 < b^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} < \frac{1}{na}.$$

So, by sandwich Theorem, we conclude that $\lim_{n\to\infty} b^n = 0$.

(v) Let $a_n = n^{\frac{1}{n}} - 1$. Then $0 \leq a_n$ for all $n \in \mathbb{N}$. Further,

$$n = (1 + a_n)^n \geq \frac{n(n-1)}{2} a_n^2.$$

Thus $0 \leq a_n \leq \sqrt{\frac{2}{(n-1)}}$ $(n \geq 2)$. As $\sqrt{\frac{2}{(n-1)}} \to 0$ as $n \to \infty$, by Sandwich theorem, $a_n \to 0$, i.e., $n^{\frac{1}{n}} \to 1$ as $n \to \infty$.

Null Sequence, Divergent sequence and oscillatory sequence

Null Sequence

A sequence $\{x_n\}$ is said to be null sequence if it converges to 0, i.e. $\lim_{n\to\infty} x_n = 0$

Divergent Sequence

A sequence $\{x_n\}$ is said to be divergent sequence if $\lim_{n\to\infty} x_n = -\infty$ or $\lim_{n\to\infty} x_n = +\infty$

Example: $\{n^2\}$ diverges to $+\infty$ and $\{-n^2\}$ diverges to $-\infty$

Oscillatory sequence

A sequence which is neither convergent nor divergent is called oscillatory sequence.

- A bounded sequence which does not convergent, and has at least two limit points, is said to finitely oscillating sequence.
  Example:$\{x_n\} = \{(-1)^n\} = \{-1, 1, -1, 1, ...\}$. It is bounded and has two limits -1 and 1. So it is not convergent. But this sequence oscillate finitely between -1 and 1.
- An unbounded sequence which is diverges neither to ∞ and to $-\infty$ is said to oscillate infinitely.
  Example:$\{x_n\} = \{(-1)^n n\}$. It is unbounded sequence and it is not properly divergent. It is an oscillatory sequence of infinite oscillation.

Thank You

Engineering Calculus-EMAT101L

(Lecture-7 and 8)

Sequence

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Learning Outcome of the Lecture

We learn

- Sequence
  - Some result of positive real number sequence.
  - Cauchy sequence,
  - Cauchy's criterion for convergence,
  - Bolzano-Weierstrass Theorem

Result:1

Let $\{x_n\}$ be a sequence of positive real number such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$.
(i) If $0 \leq L < 1$, then $\lim_{n\to\infty} x_n = 0$.
(ii) If $L > 1$, then $\lim_{n\to\infty} x_n = \infty$.

Examples

(i) Let $\{x_n\} = \{\frac{n}{2^n}\}$.
Now,
$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+1}{n} \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)\frac{1}{2} = \frac{1}{2}$$
. Since, $\frac{1}{2} < 1$, so $\{x_n\} = \{\frac{n}{2^n}\} \to 0$.

(ii) Let $\{x_n\} = \{ny^{n-1}\}$, where $y \in (0, 1)$
Now,
$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+1}{n} \frac{y^n}{y^{n-1}} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)y = y$$
. Since, $0 < y < 1$, so $\{x_n\} = \{ny^{n-1}\} \to 0$.

Remark:1

If $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L = 1$, we cannot make any conclusion. For example, consider the sequence $\{n\}$, $\{\frac{1}{n}\}$ and $\{\frac{2+n}{n}\}$.

Result:2

Let $\{x_n\}$ be a sequence of positive real number such that $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = L$.
(i) If $0 \leq L < 1$, then $\lim\limits_{n\to\infty} x_n = 0$.
(ii) If $L > 1$, then $\lim\limits_{n\to\infty} x_n = \infty$..

Examples:

(i) Let $\{x_n\} = \{\frac{1}{n^n}\}$.
Now,
$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{1}{n} = 0$$
. Since, $L = 0$, so $\{x_n\} = \{\frac{1}{n^n}\} \to 0$.

(ii) Let $\{x_n\} = \{\frac{4^{3n}}{3^{4n}}\}$
Now,
$$\lim_{n\to\infty} \sqrt[n]{x_n} = \frac{64}{81}$$
. Since, $\frac{64}{81} < 1$, so $\{x_n\} = \{\frac{4^{3n}}{3^{4n}}\} \to 0$.

Remark:2

If $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = L = 1$, we cannot make any conclusion.

Some result of positive real number sequence

Result:3

Let $\{x_n\}$ be a sequence of positive real number such that

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$ (finite or infinite). Then,

$\lim_{n\to\infty} \sqrt[n]{x_n} = L.$

Example:

Prove that

$$\lim_{n\to\infty} \sqrt[n]{n+1} = 1$$

Solution: Let $\{x_n\} = \{n+1\}$.

Now,
$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+2}{n+1} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{1}{n}} = 1$$

. Since, $L = 1$ is a finite number, so $\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{n+1} = 1.$

Cauchy sequence

A sequence $\{x_n\}$ is called a **Cauchy sequence** if for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \epsilon \text{ for all } n, m \geq N.$$

Example

Show that the sequence $\{\frac{1}{n}\}$ is a Cauchy sequence.

Solution: Let $\epsilon > 0$ be given,
Now,

$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m}$$

If, we choose a natural number N such that $N > 2/\epsilon$.
Then for $m, n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} < \frac{\epsilon}{2}$, $\frac{1}{m} \leq \frac{1}{N} < \frac{\epsilon}{2}$.
Hence,

$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, we conclude that $\{\frac{1}{n}\}$ is a Cauchy sequence.

Cauchy's criterion for convergence

A sequence of real number $\{x_n\}$ converges if and only if for every $\epsilon > 0$, there exists N such that

$$|x_n - x_m| < \epsilon, \;\; \forall \;\; m, n \geq N.$$

i.e. Every convergent sequence is a Cauchy sequence and every Cauchy sequence of real number is convergent sequence.

Subsequence

Let $\{x_n\}$ be a sequence and $\{n_k\}$ be a strictly monotonic increasing sequence of natural number. Then the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ is called a subsequence of $\{x_n\}_{n=1}^{\infty}$.

Example:

Let $\{x_n\}= \{\frac{1}{n}\}$.

Then for $\{n_k\} = \{2k\}, k \in \mathbb{N}$,

$\{x_{n_k}\}_{k=1}^{\infty}$=$\left\{\frac{1}{n_k}\right\}_{k=1}^{\infty}$=$\{\frac{1}{2k}\}_{k=1}^{\infty}$ is a subsequence of $\{\frac{1}{n}\}$.

Similarly, for $\{n_k\} = \{2k-1\}, k \in \mathbb{N}$,

$\{x_{n_k}\}_{k=1}^{\infty}$=$\left\{\frac{1}{n_k}\right\}_{k=1}^{\infty}$=$\left\{\frac{1}{2k-1}\right\}_{k=1}^{\infty}$ is a subsequence of $\{\frac{1}{n}\}$.

Theorem:

Let $\{x_n\}$ be a sequence.

- if $\{x_n\}$ converges to L, then every subsequence of $\{x_n\}$ converges to L.
- if $\{x_n\}$ has two subsequence that converges two different limits, then $\{x_n\}$ does not converges.

Example: Let $\{x_n\}$=$\{1, 0, 1, 0, \cdots\}$. This sequence is not convergent.

- Now, $\{x_{2n-1}\}$=$\{1, 1, 1, \cdots\}$ is a subsequence, converges to 1.
- Again, $\{x_{2n}\}$=$\{0, 0, 0, \cdots\}$ is a subsequence, converges to 0.

Therefore, $\{x_n\}$ has two subsequence that converges two different limits 1 and 0 but $\{x_n\}$ is not convergent.

Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.


Thank You

Engineering Calculus-EMAT101L

(Lecture-9 and 10)

Series

School of Engineering and Applied Sciences
Department of Mathematics
Bennett University
2021

Learning Outcome of the Lecture

We learn

- Series
  - Definition of Series,
  - Necessary and sufficient condition for convergence,
  - Necessary condition for convergence of an series.
  - Geometric Series.

Series

Definition

A series is the sum of the terms of a sequence. Thus if $\{u_n\}$ be a sequence of real numbers, then the sum

$$u_1 + u_2 + \ldots + u_n + \ldots$$

of all the terms is called an infinite series and is denoted by

$$\sum_{n=1}^{\infty} u_n \quad \text{or simply by} \quad \sum u_n$$

Sequence of partial sums of the series

Let, $\{u_n\}$ be a sequence of real numbers. Then the sequence $\{s_n\}$ defined by

$$s_n = \sum_{k=1}^{n} u_k,$$

i.e.

$$s_1 = u_1, \quad s_2 = u_1 + u_2, \quad s_3 = u_1 + u_2 + u_3, \ldots \quad \ldots$$

is called the sequence of partial sums of the series $\sum u_n$.

Necessary and sufficient condition for the convergence of an infinite series

Theorem (Necessary and sufficient condition for convergence)

A necessary and sufficient condition for the convergence of an infinite series $\sum_{n=1}^{\infty} u_n$ is that the sequence of its partial sums $\{s_n\}$ is convergent.

Example:1

Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.

Solution: Let $\{s_n\}$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Then

$$s_n = \frac{1}{1.2} + \frac{1}{2.3} + ... + \frac{1}{n(n+1)}$$

$$\implies s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + ... + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

Now, since $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1,$ so the sequence $\{s_n\}$ converges to 1.

Hence, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and converges to 1.

Lemma

(a) If $\sum_{n=1}^{\infty} u_n$ converges to L and $\sum_{n=1}^{\infty} v_n$ converges to M, then $\sum_{n=1}^{\infty} (u_n + v_n)$ converges to $L + M$.

(b) If $\sum_{n=1}^{\infty} u_n$ converges to L and if $c \in \mathbb{R}$, then the series $\sum_{n=1}^{\infty} cu_n$ converges to cL.

Theorem (Necessary condition for convergence)

If $\sum_{n=1}^{\infty} u_n$ converges, then $\lim_{n\to\infty} u_n = 0$.

Proof: Suppose $\sum_{n=1}^{\infty} u_n = L$. Then the sequence of partial sums $\{s_n\}$ also converges to L. Now

$$u_n = s_n - s_{n-1} \to L - L = 0 \text{ as } n \to \infty.$$

Hence for a convergent series, $\lim_{n\to\infty} u_n = 0$.
In other words, a series cannot converge if its nth term does not tend to zero i.e

if $\lim_{n\to\infty} u_n \neq 0$, then $\sum_{n=1}^{\infty} u_n$ diverges.

Note: Converse of the above theorem i.e. $\lim_{n\to\infty} u_n = 0$ does not prove that a series $\sum_{n=1}^{\infty} u_n$ is convergent.

Example: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, however $\lim_{n\to\infty} \frac{1}{n} = 0$.

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + ...ar^{n-1} + ... = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real number and $a \neq 0$.
If $|r| < 1$, the geometric series convergent and converges to $\frac{a}{1-r}$:

If $|r| \geq 1$, then the geometric series diverges.


Thank You

Engineering Calculus-EMAT101L

(Lecture-11 and 12)

Series

School of Engineering and Applied Sciences
Department of Mathematics
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2021

Learning Outcome of the Lecture

We learn

- Series
  - Series of positive terms
  - Convergence Tests for positive terms series
    - Comparison test
    - Limit Comparison test
    - D'Alembert's ratio test
    - Cauchy's root test

Series of positive terms

A series $\sum_{n=0}^{\infty} u_n$ is said to be a series of positive terms if u_n is a positive real number for all n.
This type of series are the simplest and the most important type of series one comes across.
The simplicity arises mainly from the fact that the sequence of its partial sums is monotonic increasing.

Theorem

A series of positive terms converges if and only if the sequence of its partial sums is bounded above.

Theorem

A positive term series

$$\sum \frac{1}{n^p} \text{ converges for } p > 1$$

$$\sum \frac{1}{n^p} \text{ diverges for } p \leq 1$$

.

Theorem (Comparison Test)

Let $\sum u_n$, $\sum v_n$ and $\sum w_n$ be series with non-negative terms. Suppose that for some positive integer N we have

$$w_n \leq u_n \leq v_n \text{ for all } n \geq N$$

,

1. If $\sum v_n$ is convergent, then $\sum u_n$ also convergent.
2. If $\sum w_n$ is divergent, then $\sum u_n$ also divergent.

Examples

1. The series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges, because $\frac{1}{(n+1)^2} \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.
2. The series $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ diverges, because $\frac{1}{2n} \leq \frac{1}{n+\sqrt{n}} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Theorem (Limit comparison test)

Let $\sum u_n$ and $\sum v_n$ be two series of positive real numbers. Then

(a) If $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = l > 0$, then $\sum\limits_{n=1}^{\infty} u_n$ and $\sum\limits_{n=1}^{\infty} v_n$ both converge or diverge together.

(b) if $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = 0$ and $\sum\limits_{n=1}^{\infty} v_n$ converges, then $\sum\limits_{n=1}^{\infty} u_n$ converges.

(c) if $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \infty$ and $\sum\limits_{n=1}^{\infty} v_n$ diverges, then $\sum\limits_{n=1}^{\infty} u_n$ diverges.

Example

(1) Consider the series $\sum\limits_{n=1}^{\infty} \dfrac{2n+1}{(n+1)^2}$.

Here $u_n = \dfrac{2n+1}{(n+1)^2}$.

Let $v_n = \dfrac{1}{n}$.

Then $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{\left(\dfrac{2n+1}{(n+1)^2}\right)}{\dfrac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{2n^2+n}{n^2+2n+1} = 2 > 0.$

Now, $\sum\limits_{n=1}^{\infty} v_n = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges.

Thus by limit comparison test, the given series i.e $\sum\limits_{n=1}^{\infty} u_n = \sum\limits_{n=1}^{\infty} \dfrac{2n+1}{(n+1)^2}$ diverges.

D'Alembert's ratio test

Let $\sum u_n$ be a series of positive real numbers such that $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = l.$
(i) If $l < 1$, then $\sum u_n$ is convergent.
(ii) If $l > 1$, then $\sum u_n$ is divergent.

Example

(1) Consider the series $\sum\limits_{n=1}^{\infty} \dfrac{n^n}{n!}$.

Here $u_n = \dfrac{n^n}{n!}$.

Then $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1.$

Then by D'Alembert's ratio test, the given series i.e $\sum\limits_{n=1}^{\infty} u_n = \sum\limits_{n=1}^{\infty} \dfrac{n^n}{n!}$ diverges.

Cauchy's root test

Let $\sum u_n$ be a series of positive real numbers such that $\lim_{n\to\infty} \sqrt[n]{u_n} = l$.
(i) If $l < 1$, then $\sum u_n$ is convergent.
(ii) If $l > 1$, then $\sum u_n$ is divergent.

Example

(1) Consider the series $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$.

Here $u_n = \frac{n^n}{3^{1+2n}}$.

Then $\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \frac{n}{3^{\frac{1}{n}+2}} = \frac{\infty}{3^2} > 1$.

Then by Cauchy's root test, the given series i.e $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$ diverges.


Thank You

Engineering Calculus-EMAT101L

(Lecture-13)

Series

School of Engineering and Applied Sciences
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2021

Learning Outcome of the Lecture

We learn

- Series
  - Absolute convergent Series
  - Tests for absolute convergence
    - D'Alembert's ratio test
    - Cauchy's root test
  - Alternating series
    - Leibniz's test
  - Conditionally Convergent Series

Definition (Absolute convergence)

(a) Let $\sum_{n=1}^{\infty} u_n$ be a series of real numbers.

If $\sum_{n=1}^{\infty} |u_n|$ converges, we say that $\sum_{n=1}^{\infty} u_n$ converges absolutely.

Result

If $\sum_{n=1}^{\infty} u_n$ converges absolutely, then $\sum_{n=1}^{\infty} u_n$ converges.

Tests for absolute convergence

Theorem (Ratio test)

Let $\sum_{n=1}^{\infty} u_n$ be a series of real numbers and suppose that

$$L = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|.$$

Then

(a) $\sum_{n=1}^{\infty} u_n$ converges absolutely if $L < 1$.

(b) $\sum_{n=1}^{\infty} u_n$ diverges if $L > 1$.

(c) the test fails if $L = 1$.

Examples

(a) The series $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$ diverges.

Here

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)^{n+1}}{(n+1)!}\,\frac{n!}{n^n}\right| = \left|\left(\frac{n+1}{n}\right)^n\right| = \left|\left(1+\frac{1}{n}\right)^n\right| \to e,$$

which is greater than 1. So $L > 1$. Thus the given series diverges.

(a) The series $\sum_{n=1}^{\infty} (-1)^n \frac{n^2(n+2)!}{3^{2n} n!}$ converges.

Here

$$\left|\frac{u_{n+1}}{u_n}\right| \to \frac{1}{9},$$

which is less than 1. So $L < 1$. Thus the given series converges absolutely

Theorem (Root test)

Let $\sum_{n=1}^{\infty} u_n$ be a series of real numbers and suppose that

$$L = \lim_{n\to\infty} \sqrt[n]{|u_n|}.$$

Then

(a) the series converges absolutely if $L < 1$;

(b) the series diverges if $L > 1$;

(c) the test fails if $L = 1$.

Examples

(1) Find the value of $x \in \mathbb{R}$ for which the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges or diverges.

Here $u_n = \frac{x^n}{n}$. Therefore, $\sqrt[n]{\left|\frac{x^n}{n}\right|} = \left|\frac{x}{\sqrt[n]{n}}\right| \to |x|$. Thus the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$.

(2) Find the value of $x \in \mathbb{R}$ for which the series $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ converges.

Here $u_n = \frac{x^n}{n^n}$. Then, $\sqrt[n]{|a_n|} = \left|\frac{x}{n}\right| \to 0$. Thus the series converges absolutely for $x \in \mathbb{R}$.

Alternating series

Definition

An alternating series is an infinite series whose terms alternate in sign. i.e.

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n$$

is an alternating series.

Theorem (Leibniz's test)

The alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges if all three conditions are satisfied:

(a) The u_n's are all positive.

(b) $u_n \geq u_{n+1}$ for all $n \in \mathbb{N}$

(c) $\lim_{n \to \infty} u_n = 0$,

Examples

(a) Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Then a_n's of this series satisfies the hypothesis of the above theorem and hence the series converges.

(b) Consider the series $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\log n}$.

Then $a_n = \frac{1}{\log n}$ satisfy the hypothesis of the above theorem and hence the series converges.

Conditionally Convergent Series

A convergent series that is not absolutely convergent is Conditionally Convergent Series

Examples

(3) The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

(4) The series $\sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{2n-1}$ converges conditionally.


Thank You

Engineering Calculus-EMAT101L

(Lecture-14, 15 and 16)

limit of a function of one variable

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2021

Function

A function f from a set A to a set B is a rule that assigns each element of A to a unique element of B.

$$f : A \to B$$

Domain of the function : set A

Range of the function : set $\{f(x) \in B : x \in A\}$

Functions of One Variable

Functions which has only one input variable.

- For example, the following are Real valued functions of single variable x;
  - $f(x) = x^3, \ x \in \mathbb{R}$
  - $f(x) = \frac{1}{x}, \ x \in \mathbb{R}$

Limits for Functions of One Variable at a point

Suppose $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is defined on the real line and $c, L \in D$.

Question

What do we mean when we say that

$$\lim_{x \to c} f(x) = L$$

Informally, we might say that as x gets 'closer and closer' to c, $f(x)$ should get 'closer and closer' to L.

Existence of limit at a point

Left Hand Limit (LHL)

Let $f(x)$ be a given function. Then left hand limit of $f(x)$ at c is

$$\lim_{h \to 0} f(c-h) = \lim_{x \to c^-} f(x)$$

Right Hand Limit(RHL)

Let $f(x)$ be a given function. Then right hand limit of $f(x)$ at c is

$$\lim_{h \to 0} f(c+h) = \lim_{x \to c^+} f(x)$$

Existence of limit at a point

Let $f(x)$ be a given function. Then we say that the function $f(x)$ has a limit L at $x = c$, if

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

Some examples of functions where limit do NOT exist

Example:1

Let

$$f(x) = \frac{|x|}{x}, \ x \in \mathbb{R}$$

Evaluate, $\lim_{x \to 0} f(x)$, does it exist?

When $x > 0, |x| = x$,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{x} = 1$$

When $x < 0, |x| = -x$,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

Since,

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

Conclusion: the limit does not exist

Some examples of functions where limit do NOT exist

Example:2

Let

$$f(x) = \begin{cases} x-1 & x \leq 0 \\ x+2 & x > 0 \end{cases}$$

Evaluate, $\lim_{x \to 0} f(x)$, does it exist?

$$\lim_{x \to 0^+} f(x) = 2$$

$$\lim_{x \to 0^-} f(x) = -1$$

Since,

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

Conclusion: the limit does not exist

Home work

- $f(x) = \begin{cases} x^2 & x < 1 \\ 2 & x \geq 1 \end{cases}$ Evaluate, $\lim_{x \to 1} f(x)$, does it exist?

- $f(x) = \begin{cases} x^2 - x^3 & x < 2 \\ 5x - 14 & x \geq 2 \end{cases}$ Evaluate, $\lim_{x \to 2} f(x)$, does it exist?

- $f(x) = \begin{cases} 2x^2 & x \leq 6 \\ x - 8 & x > 6 \end{cases}$ Evaluate, $\lim_{x \to 6} f(x)$, does it exist?

Limits of real valued functions

Theorem

Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

(a) $\lim_{x \to c} (f(x) \pm g(x)) = L \pm M$.

(b) $\lim_{x \to c} (f(x).g(x)) = LM$

(c) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, when $M \neq 0$,

(d) $\lim_{x \to c} [f(x)]^n = L^n$ where n is a positive integer.

(d) Sandwich Therem
Suppose that $h(x)$ satisfies $f(x) \leq h(x) \leq g(x)$ in an interval containing c, and $L = M$. Then $\lim_{x \to c} h(x) = L$.

How to Determine the Limits of Functions?

- If $P(x)$ be polynomial function, then

$$\lim_{x \to c} P(x) = P(c)$$

or If $P(x)$ and $Q(x)$ be two polynomial functions and $Q(c) \neq 0$then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

- if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$, then

$$\lim_{x \to c} f(x) = L$$

Example: $f(x) = \begin{cases} x^2 - x^3 & x < 2 \\ 5x - 14 & x \geq 2 \end{cases}$ Evaluate, $\lim_{x \to 2} f(x)$.

Solution: Here, $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = -4$.

Therefore,

$$\lim_{x \to 2} f(x) = -4$$

How to Determine the Limits of Functions?

- Find the limit by factorization

Example: Evaluate, $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}$

Solution: $\lim\limits_{x\to 2}\frac{x^2-4}{x-2} = \lim\limits_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim\limits_{x\to 2} x+2 = 4$

- Find the limit by rationalization

Example: Evaluate, $\lim\limits_{x\to 0}\frac{x}{1-\sqrt{1-x}}$

Solution: $\lim\limits_{x\to 0}\frac{x}{1-\sqrt{1-x}} = \lim\limits_{x\to 0}\frac{x(1+\sqrt{1-x})}{1-1+x} = \lim\limits_{x\to 0}(1+\sqrt{1-x}) = 2$

- Find the limit by substitution

Example: Evaluate, $\lim\limits_{x\to a}\frac{\sin x-\sin a}{x-a}$

Solution: $\lim\limits_{x\to a}\frac{\sin x-\sin a}{x-a} = \lim\limits_{x\to a}\frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a} = \lim\limits_{y\to 0}\frac{2\cos\frac{y+2a}{2}\sin\frac{y}{2}}{y} = \cos a$

(Here put $x-a=y$ and use $\lim\limits_{y\to 0}\frac{\sin y}{y} = 1$)

ϵ, δ definition of Limit of function at a point

Definition

$$\lim_{x \to c} f(x) = L$$

Means that

- given any $\epsilon > 0$ for L
- we can find a $\delta > 0$ for c such that
- if x is between $c - \delta$ and $c + \delta$, but x is not c,
- $f(x)$ will be between $L - \epsilon$ and $L + \epsilon$.

i.e. We say that a function $f(x)$ converges to L as x approaches c, if for every $\epsilon > 0$ (given), there exists $\delta > 0$ (depending on ϵ) such that for all x,

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$

Note: If limit exists, then it is unique.

Examples: ϵ, δ definition of Limit

Using $\epsilon - \delta$ definition, show following limits:

$(a)\ \lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0 \qquad (b)\ \lim_{x\to a} x^2 = a^2$

(a)

$$\begin{aligned} \left| x^2 \cos\frac{1}{x} - 0 \right| &\leq |x|^2 \\ &< \epsilon \text{ iff } |x| < \sqrt{\epsilon} \end{aligned}$$

Choose $\delta = \sqrt{\epsilon}$, then for $|x - 0| < \delta$, $\left| x^2 \cos\frac{1}{x} - 0 \right| < \epsilon$.

$$\Rightarrow \lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0.$$

(b)

$$\begin{aligned} \left| x^2 - a^2 \right| &= |x-a||x+a| = |x-a||x-a+2a| \leq |x-a|(|x-a| + |2a|) \\ &< \delta(\delta + 2a) \text{ whenever } |x-a| < \delta \end{aligned}$$

Choose $\delta > 0$ such that $\delta(2a + \delta) = \epsilon$, then for $|x - a| < \delta$, $\left| x^2 - a^2 \right| < \epsilon$.

$$\Rightarrow \lim x^2 = a^2.$$

Limits at infinity

Limits at infinity

Evaluate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$, where $f(x) = \frac{1}{x}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x} = 0$$

Horizontal Asymptote

A line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either $\lim\limits_{x\to\infty} f(x) = b$ or $\lim\limits_{x\to-\infty} f(x) = b$.

For, $f(x) = \frac{1}{x}$, x-axis is a horizontal asymptote.

Infinite Limits

Infinite Limits

Evaluate $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ where $f(x) = \frac{1}{x}$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

Vertical Asymptote

A line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.

For, $f(x) = \frac{1}{x}$, y-axis is a vertical asymptote.


Thank You

Engineering Calculus-EMAT101L

(Lecture-17 and 18)

Continuous functions of one variable

BENNETT UNIVERSITY
TIMES OF INDIA GROUP

School of Engineering and Applied Sciences
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2021

Let $f : D \to \mathbb{R}$ be a function defined on a subset D of the set $\mathbb{R}$ of real numbers. This subset D is the domain of f. Some possible choices include

- $D = \mathbb{R}$ (D is the whole set of real numbers).
- $D = [a, b]$, for a and b real numbers.
- $D = (a, b)$, for a and b real numbers.

Definition

A real valued function $f(x)$ is said to be continuous at $x = c$ of its domain if

(i) $f(c)$ is defined,

(ii) $\lim_{x \to c} f(x)$ exists,

(iii) $\lim_{x \to c} f(x) = f(c)$.

Definition (Another)

A real valued function $f(x)$ is said to be continuous at $x = c$ of its domain if

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

.

Examples: Continuous functions

Example

Show that $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ is continuous at 0.

Solution:

- $f(0) = 1$
- $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim\limits_{x \to 0} \frac{\sin x}{x} = f(0)$

Example

Show that $f(x) = \begin{cases} \frac{3}{2}x & x \geq 2 \\ 2x - 1 & x < 2 \end{cases}$ is continuous at 2.

Solution:

- $f(2) = \frac{3}{2}.2 = 3$
- $\lim\limits_{x \to 2+} \frac{3}{2}x = \lim\limits_{x \to 2+} \frac{3}{2}2 = 3$
- $\lim\limits_{x \to 2-} 2x - 1 = \lim\limits_{x \to 2-} 2.2 - 1 = 3$
- $\lim\limits_{x \to 2+} f(x) = \lim\limits_{x \to 2-} f(x) = f(2) = 3.$

Examples: Continuous functions

Example

Find k, so that the function f defined by

$$f(x) = \begin{cases} kx^2 & x \leq 2 \\ x-3 & x > 2 \end{cases}$$

continuous at $x = 2$.

Solution:

- $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} kx^2 = 4k$
- $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} x - 3 = -1$
- f continuous at $x = 2$, if

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

$$\implies 4k = -1$$

$$\implies k = -\frac{1}{4}$$

Discontinuous Function

- A function which is not continuous is called discontinuous function.

Removable discontinuity

- The right-hand limit and the left-hand limit both exist and equal to each other

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$$

- But

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) \neq f(c)$$

Example: $\lim_{x\to 1} f(x)$ where f(x)=$\begin{cases} 4-2x & x<1 \\ 6x-4 & x>1 \\ 3 & x=1 \end{cases}$

Solution: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) \neq f(1)$

Jump discontinuity

The right-hand limit and the left-hand limit both exist but not equal:

$$\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$$

Example: $\lim_{x \to 2} f(x)$ where f(x)= $\begin{cases} x+3 & x \leq 2 \\ -2x+5 & x > 2 \end{cases}$

Solution: $\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$

Second kind discontinuity

A function $f(x)$ is said to have discontinuity of second kind at $x = c$, if at least one of right-hand limit $\lim_{x \to c^+} f(x)$ or the left-hand limit $\lim_{x \to c^-} f(x)$ doesn't exist.

Example: $\lim_{x \to 0} f(x)$ where f(x)= $\begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

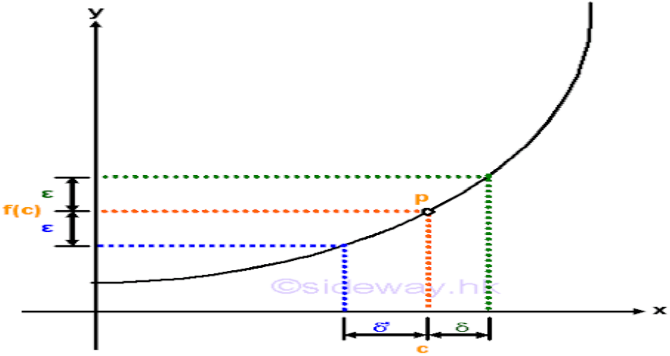
Solution: $\lim_{x \to 0^+} f(x)$ dose not exist,

Also, $\lim_{x \to 0^-} f(x)$ dose not exist.

The ϵ, δ - definition of continuity

We say that a function $f(x)$ continuous at c of its domain [i.e. $\lim_{x \to c} f(x) = f(c)$],
if for every $\epsilon > 0$ (given), there exists $\delta > 0$ (depending on ϵ) such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$$



Continuous functions

Theorem

Suppose f and g are continuous at c of its domain. Then

1. $f \pm g$ is also continuous at c.
2. fg is continuous at c.
3. $\dfrac{f}{g}$ is continuous at c if $g(c) \neq 0$.
4. $|f|$ is also continuous at c and $\lim\limits_{x \to c} |f(x)| = |f(c)|$.

Theorem

1. Every polynomial function is continuous everywhere on $(-\infty, \infty)$.
2. Every rational function is continuous everywhere it is defined, i.e., at every point in its domain. Its only discontinuities occur at the zeros of its denominator.

Composite of Continuous Functions

If f is continuous at c of its domain and g is continuous at $f(c)$, then the composite function $g \circ f$ given by $(g \circ f)(x) = g(f(x))$ is continuous at c.

Example: Since both $f(x) = x^2 + 1$ and $g(x) = \cos x$ are continuous on $(-\infty, \infty)$.
Therefore, $(g \circ f)(x) = cos(x^2 + 1)$ are continuous on $(-\infty, \infty)$.

Properties of continuous functions

Continuity in an interval

A function f is said to be continuous in an interval if it is continuous at every point of the interval.

Theorem

Let $f(x)$ be a continuous function on a closed interval $[a, b]$ and let

$$f(a)f(b) < 0 \text{ for some } a, b \in \mathbb{R}$$

. Then there exits $c \in (a, b)$ such that $f(c) = 0$.

Example

Show that $f(x) = x^2 - 2$ has at least one root in $(1, 2)$.

Solution: $f(1) = -1$, and $f(2) = 2$

Now, $f(1)f(2) = -2 < 0$

Therefore, from above theorem, there exits $c \in (1, 2)$

such that $f(c) = 0 \implies c^2 - 2 = 0 \implies c = \pm\sqrt{2}$.
Therefore $c = \sqrt{2}$ is a root of $f(x) = x^2 - 2$ in $(1, 2)$.

Intermediate Value Theorem

Let $f(x)$ be a continuous function on a closed interval $[a, b]$ and let

$$f(a) < y < f(b)$$

. Then there exists

$$c \in (a, b) \text{ such that } f(c) = y$$

.
i.e. it assumes every value between $f(a)$ and $f(b)$.

Remark

From the IVT, we can conclude that A continuous function assumes all values between its maximum and minimum.

Thank You

Engineering Calculus-EMAT101L

(Lecture-19, 20 and 21)

Differentiability

School of Engineering and Applied Sciences
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2021

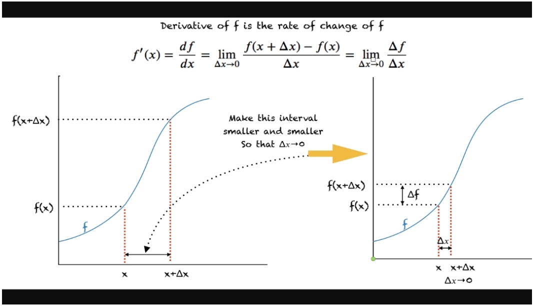
What is the Derivative of a Function

The derivative of a function $f(x)$ represents it's rate of change and

is denoted by either

$$f'(x) \text{ or } \frac{df}{dx}$$

. Let's first look at its definition and a pictorial illustration of the derivative.



In the figure, Δx represents a change in the value of x. We keep making the interval between x and $(x + \Delta x)$ smaller and smaller until it is infinitesimal. Hence, we have the limit $(\Delta x \to 0)$. The numerator $f(x + \Delta x) - f(x)$ represents the corresponding change in the value of the function f over the interval Δx. This makes the derivative of a function f at a point x, the rate of change of f at that point.

Differentiability

Let $I = [a, b]$ be an interval and a function $f : I \to \mathbb{R}$ and let $c \in (a, b)$.

Derivative at a point

$$f'(c) = \left.\frac{df}{dx}\right|_{x=c}$$
$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Left Hand Derivative (LHD)

Then left hand derivative of $f(x)$ at c is

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c - h) - f(c)}{-h}$$

and is denoted by $Lf'(c)$

Right Hand Derivative (RHD)

Let $f(x)$ be a given function. Then right hand derivative of $f(x)$ at c is

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

and is denoted by $Rf'(c)$

Differentiability

Definition

Let $I = [a, b]$ be an interval and a function $f : I \to \mathbb{R}$.

(a) If c is an interior point of I $(a < c < b)$, then f is said to be differentiable at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{exists},$$

i.e. when both the limits

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

exist and be equal i.e.

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \quad \text{i.e} \quad Lf'(c) = Rf'(c)$$

The derivative of f at c is denoted by $f'(c)$.

(b) f is said to be differentiable at the end point a if $\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$ exists. derivative of f at a is denoted by $f'(a)$.

(c) f is said to be differentiable at the end point b if $\lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$ exists. The derivative of f at b is denoted by $f'(b)$.

Example:1

Show that the function $f(x) = x^2 \quad \forall \quad x \in \mathbb{R}$, is differentiable at $x = 3$.

Solution:

$$\lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{x^2 - 3^2}{x - 3} = \lim_{x \to 3} x + 3 = 6$$

Therefore,

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = 6.$$

Example:2

Discuss the derivability of the following function at $x = 1$

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ 1 & x \geq 1. \end{cases}$$

$$Lf'(1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x - 1}{x - 1} = 1$$
$$Rf'(1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{1 - 1}{x - 1} = 0$$

Since, $Lf'(1) \neq Rf'(1)$, therefore $f'(1)$ does not exist.

Examples:

Example:3

Discuss the derivability of the following function at $x = 2$

$$f(x) = \begin{cases} x - 1 & x < 2 \\ 2x - 3 & x \geq 2. \end{cases}$$

Example:4

Show that the function $F(x) = |x+1| + |x-1| \quad \forall \quad x \in \mathbb{R}$
is not differentiable at $x = 1$.

Example:5

Show that the function $f(x) = |x| \quad \forall \quad x \in \mathbb{R}$
is not differentiable at $x = 0$.

Example:6

Show that the function $f(x) = x|x| \quad \forall \quad x \in \mathbb{R}$
is differentiable at $x = 0$.

Theorem (Differentiability implies continuity)

If $f(x)$ is differentiable at c, then it is continuous at c.

Proof: For $x \neq c$, we may write,

$$f(x) = (x - c)\frac{f(x) - f(c)}{(x - c)} + f(c).$$

Now taking the limit $x \to c$ and noting that $\lim_{x \to c}(x - c) = 0$ and $\lim_{x \to c} \frac{f(x)-f(c)}{(x-c)} = f'(c)$, we get the result.

Remark (Not all continuous functions are differentiable)

The continuity of $f : I \to \mathbb{R}$ at a point does not assure the existence of the derivative at that point.

Example if $f(x) = |x|$ for $x \in \mathbb{R}$, then for $x \neq 0$

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} 1 & x > 0 \\ -1 & x < 0. \end{cases}$$

Since, $Lf'(0) \neq Rf'(0)$, therefore $f'(0)$ does not exist.
But, $f(x) = |x|$ is continuous at $x = 0$, (since LHL=RHL=f(0))

Theorem

Let f, g be differentiable at c. Then $f \pm g,\ fg,\ \dfrac{f}{g}$ $(g(c) \neq 0)$ are also differentiable at c and

- $(f \pm g)'(c) = f'(c) \pm g'(c)$
- $(f.g)'(c) = f'(c)g(c) + f(c)g'(c)$
- $(\frac{f}{g})'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g^2(c)}$, if $(g(c) \neq 0)$

Theorem (Chain Rule)

Suppose $f(x)$ is differentiable at c and g is differentiable at $f(c)$, then $h(x) := g \circ f(x) = g(f(x))$ is differentiable at c and

$$h'(c) = g'(f(c))\ .f'(c).$$

Example: Let $f(x) = x^2$ and $g(x) = e^x$, then

$$h(x) := g \circ f(x) = g(f(x)) = g(x^2) = e^{x^2}$$

. Now,

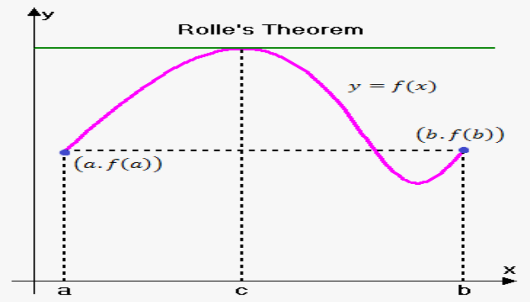
$$h'(x) = (e^{x^2})' = (e^{x^2}).(2x)$$

Rolle's Theorem

If a function $f(x)$ defined on $[a, b]$ is

- continuous on $[a, b]$,
- differentiable on (a, b), and
- $f(a) = f(b)$

Then there exists at least one real number $c \in (a, b)$ such that $f'(c) = 0$.
i.e. there is a point $c \in (a, b)$ where the tangent to curve $f(x)$ is horizontal or we can say it is parallel to the X-axis.



Algebraic Interpretation of Rolle's Theorem

Algebraically Rolle's Theorem can be interpreted as follows:
Between any two roots of polynomial $f(x)$, there is always a root of its derivative $f'(x)$.

Example:

Verify Rolle's theorem for the function $f(x) = x^2 + 2$, $a = -2$ and $b = 2$.

Solution: $f(x) = x^2 + 2$ = polynomial function.
Hence the function $f(x) = x^2 + 2$ is continuous in $[-2, 2]$ and differentiable in $(-2, 2)$.
Now, $f(-2) = (-2)^2 + 2 = 4 + 2 = 6$ and $f(2) = (2)^2 + 2 = 4 + 2 = 6$.
Thus, $f(-2) = f(2) = 6$
Now, $f'(x) = 2x$
Rolle's theorem states that there is a point $c \in (-2, 2)$ such that

$$f'(c) = 0 \implies 2c = 0 \implies c = 0.$$

Here, $0 \in (-2, 2)$.
Hence verified.

Statement of Lagrange's mean value theorem (LMVT)

Lagrange's mean value theorem (LMVT) states that if a function $f(x)$

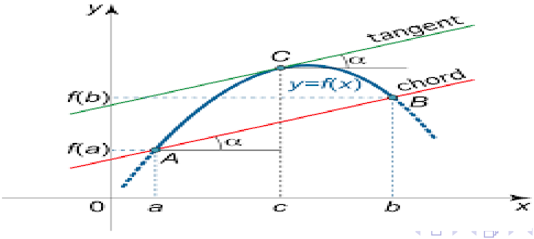
- continuous on a closed interval $[a, b]$
- differentiable on the open interval (a, b),

Then there exists at least one real number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Geometric interpretation

Geometrically, the LMVT describes a relationship between the slope of the tangent line and the slope of a secant line.



Relationship to the Rolle's Theorem

Rolle's theorem is a special case of the LMVT:
it has the same requirements about continuity on $[a, b]$ and differentiability on (a, b), and the additional requirement that $f(a) = f(b)$. In that case, the LMVT says that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = 0$$

What are the consequence of the LMVT?

Relationship to Monotonically increasing function

In LMVT, if $f'(c) \geq 0, \forall c \in (a, b)$, then f is increasing.

Relationship to Monotonically decreasing function

In LMVT, if $f'(c) \leq 0, \forall c \in (a, b)$, then f is decreasing function.

Relationship to constant function

In LMVT, if $f'(c) = 0, \forall c \in (a, b)$ then f is constant function.

Example:

Show that $|\cos x - \cos y| \leq |x - y|$ for all x, $y \in \mathbb{R}$.

Solution: Let x, $y \in \mathbb{R}$ and $f(x) = \cos x$.
By the Lagrange's mean value theorem (LMVT),

$$\frac{\cos x - \cos y}{x - y} = -\sin c \quad \text{for some} \quad c \in (x, y)$$

.

$$\implies \cos x - \cos y = -(x - y) \times \sin c$$

Using the fact that $|\sin x| \leq 1$,
we obtain that $|\cos x - \cos y| \leq |x - y|$.

Example:

- Show that $|\sin x - \sin y| \leq |x - y|$ for all x, $y \in \mathbb{R}$.

Indeterminate Forms

• Let $\lim_{x \to c} f(x) = A$ and $\lim_{x \to c} g(x) = B$. If $B \neq 0$ then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

• If $B = 0$ and $A \neq 0$, then the limit is infinite.

• If $B = 0$ and $A = 0$, then the limit is said to be **indeterminate**.

The symbolism $\frac{0}{0}$ is used to refer this situation.

• If $B = \infty$ and $A = \infty$, then the limit is said to be **indeterminate**.

The symbolism $\frac{\infty}{\infty}$ is used to refer this situation.

L'Hospital's Rules

L'Hospital's Rule

L'Hospital's Rule states that for functions f and g which are differentiable on an open interval I except possibly at a point c contained in I, if $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = 0$ or $\pm\infty$ and $g'(x) \neq 0$ for all x in I with $x \neq c$, and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Examples

Evaluate (i) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$, (ii) $\lim_{x\to 0} \frac{e^x-1}{x}$, (iii) $\lim_{x\to 1} \frac{\ln x}{x-1}$.

Solution: (i)

$$\begin{aligned}
& \lim_{x\to 0} \frac{1-\cos x}{x^2} \quad \left(\frac{0}{0} \text{ form}\right) \\
= & \lim_{x\to 0} \frac{\sin x}{2x} \qquad \left(\frac{0}{0} \text{ form}\right) \\
= & \lim_{x\to 0} \frac{\cos x}{2} \\
= & \frac{1}{2}.
\end{aligned}$$

(ii) $\lim_{x\to 0} \frac{e^x-1}{x} \quad \left(\frac{0}{0} \text{ form}\right) = \lim_{x\to 0} \frac{e^x}{1} = 1.$

(iii) $\lim_{x\to 1} \frac{\ln x}{x-1} \quad \left(\frac{0}{0} \text{ form}\right) = \lim_{x\to 1} \frac{(1/x)}{1} = 1.$

Examples

Evaluate (i) $\lim\limits_{x\to\infty} \dfrac{\ln x}{x}$, (ii) $\lim\limits_{x\to\infty} e^{-x}x^2$, (iii) $\lim\limits_{x\to 0} \dfrac{\ln \sin x}{\ln x}$.

Solution: (i) $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} \quad \left(\dfrac{\infty}{\infty} \text{ form}\right) = \lim\limits_{x\to\infty} \dfrac{(1/x)}{1} = 0.$

(ii)

$$\begin{aligned}
\lim_{x\to\infty} e^{-x}x^2 &= \lim_{x\to\infty} \frac{x^2}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right) \\
&= \lim_{x\to\infty} \frac{2x}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right) \\
&= \lim_{x\to\infty} \frac{2}{e^x} = 0.
\end{aligned}$$

(iii) $\lim\limits_{x\to 0} \dfrac{\ln \sin x}{\ln x} \quad \left(\dfrac{\infty}{\infty} \text{ form}\right) = \lim\limits_{x\to 0} \dfrac{(\cos x/\sin x)}{(1/x)} = \lim\limits_{x\to 0} \left[\dfrac{x}{\sin x}\right] \cdot \lim\limits_{x\to 0} \cos x = 1.$

Definition

A function $f : I \to \mathbb{R}$ is said to be **strictly increasing** on I, if for $x, y \in I$ with $x < y$ we have $f(x) < f(y)$. Also, we say f is **strictly decreasing** if $x < y$ in I implies $f(x) > f(y)$.

Theorem

Let $f : I \to \mathbb{R}$ be differentiable function on I. Then

(a) f is strictly increasing on I iff $f'(x) > 0$ for all $x \in I$.

(b) f is strictly decreasing on I iff $f'(x) < 0$ for all $x \in I$.

- **Stationary points**
  Stationary point is a value of x where f is defined, and where

$$f'(x) = 0$$

Example

For $f(x) = x^4 - 8x^2$ determine all intervals where f is strictly increasing or strictly decreasing.

Solution : The domain of $f(x)$ is all real numbers, and its Stationary points occur at $x = -2, 0,$ and 2.
Testing all intervals to the left and right of these values for

$$f'(x) = 4x^3 - 16x$$

, you find that

$$f'(x) < 0, \text{ on } (-\infty, -2)$$
$$f'(x) > 0, \text{ on } (-2, 0)$$
$$f'(x) < 0, \text{ on } (0, 2)$$
$$f'(x) > 0, \text{ on } (2, \infty)$$

hence, f is strictly increasing on $(-2, 0)$ and $(2, \infty)$ and strictly decreasing on $(-\infty, -2)$ and $(0, 2)$.

Local extremum

A point $x = c$ is called **local maximum** of $f(x)$, if there exists $\delta > 0$ such that

$$c - \delta < x < c + \delta \implies f(c) \geq f(x).$$

Similarly, one can define **local minimum**: $x = b$ is a local minimum of $f(x)$ if there exists $\delta > 0$ such that

$$b - \delta < x < b + \delta \implies f(b) \leq f(x).$$

Theorem

If $f(x)$ has a local maximum or a local minimum value at an interior point c of its domain and f' is defined at c, Then

$$f'(c) = 0$$

.

Second Derivative Test for local maximum and local minimum

Let us consider a function f defined in the interval I and let $c \in I$. Let the function be twice differentiable at c.

- If $f'(c) = 0$ and $f''(c) < 0$, then f has local maximum at c.
- If $f'(c) = 0$ and $f''(c) > 0$, then f has local minimum at c.
- If $f'(c) = 0$ and $f''(c) = 0$, then the test fails.

Examples

Example

Find all the local maxima and minima of the given function

$$f(x) = \frac{3}{4}x^4 + 8x^3 + \frac{45}{2}x^2 + 250$$

$$f'(x) = 3x^3 + 24x^2 + 45x = 0 \implies x = 0, -3, -5$$

Now,

$$f''(x) = 3 \times (3x^2 + 16x + 15)$$

- $f'(0) = 0$ and $f''(0) = 45 > 0$, then f has local minimum at $x = 0$.
- $f'(-5) = 0$ and $f''(-5) = 30 > 0$, then f has local minimum at $x = -5$.
- $f'(-3) = 0$ and $f''(-3) = -18 < 0$, then f has local maximum at $x = -3$.

Example

Find all the local maxima and minima of the given function

$$f(x) = x^4 - 8x^2$$

Thank You

Engineering Calculus-EMAT101L

(Lecture-22 and 23)

Power series and Taylor series

School of Engineering and Applied Sciences
Department of Mathematics
Bennett University
2021

Power series

Definition

power series centered at c : If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + ...$$

is called a power series centered at c, where $a_n \in \mathbb{R}$ represents the coefficient of the nth term and $c \in \mathbb{R}$ is a constant.

power series centered at 0 : An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ...$$

is called a power series centered at 0,

Remark

- A power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + ...$$

  always converges for $x = c$ and the sum of the series is a_0.

- The following power series is centered at 0.

$$\sum_{n=0}^{\infty} x^n$$

This is the geometric series. It converges for $|x| < 1$ and diverges for $|x| \geq 1$.

- The following power series is centered at 0.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

This series converges absolutely for all x. (using Ratio test)

- The following power series is centered at 1.

$$\sum_{n=0}^{\infty} (x-1)^n$$

It converges for $|x-1| < 1$ and diverges for $|x-1| \geq 1$.

Power series

Theorem

For a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ exactly one of the following three cases is true:

- **Case 1:** The series converges only for $x = c$.
- **Case 2:** The series converges for all x.
- **Case 3:** There exists a positive real number R such that the series converges absolutely for all real x satisfying $|x-c| < R$ and diverges for all x satisfying $|x-c| > R$.

Radius of convergence (R)

The radius of convergence (R) of a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ is defined to be number

- $R = 0$ if the series is divergent for all $x \neq c$.
- $R = \infty$ if the series is absolutely convergent for all x.
- R, the positive member such that the series converges absolutely for all real x satisfying $|x-c| < R$ and diverges for all x satisfying $|x-c| > R$.

Computation of Radius of Convergence and finding Interval of Convergence

Theorem: Ratio Test for power series

Consider the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$. Then the radius of convergence is given as follows:

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

Theorem: Root Test for power series

Consider the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$. Then the radius of convergence is given as follows:

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

Finding Interval of Convergence

- If, $\frac{1}{R} = 0 \implies R = \infty \implies$ the series is absolutely convergent for all x.
- If, $\frac{1}{R} = \infty \implies R = 0 \implies$ the series is divergent for all $x \neq c$.
- If, $\frac{1}{R} = A$ (finite number) $\implies R = \frac{1}{A} \implies$ the series converges absolutely for all real x satisfying $|x-c| < R = \frac{1}{A}$ and diverges for all x satisfying $|x-c| > R = \frac{1}{A}$.

Examples

Find the radius of convergence and interval of Convergence of

$(i)\ \sum \frac{x^n}{n},\ (ii)\ \sum \frac{x^n}{n!},\ (iii)\ \sum 2^{-n}x^n.$

(i) $\frac{1}{R} = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1,$. So $R = 1$ and the series converges absolutely for all real x satisfying $|x| < R = 1$ and diverges for all x satisfying $|x| > R = 1$.

(ii) $\frac{1}{R} = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 0$. So $R = \infty$, and series converges everywhere.

(iii) $\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = 2^{-1} = \frac{1}{2}$. Therefore, $R = 2$ and the series converges absolutely for all real x satisfying $|x| < R = 2$ and diverges for all x satisfying $|x| > R = 2$.

Taylor's series

The Taylor series of a real-valued function $f(x)$ that is infinitely differentiable at a real number c is the power series

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots$$

$$\implies f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

where $f^{(n)}(c)$ denotes the nth derivative of f evaluated at the point c.

Maclaurin's series

- If $c = 0$, the formula obtained in Taylor's theorem is known as *Maclaurin's series*

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\implies f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

Examples

(i) Find Taylor series of $f(x) = e^x$ about $c = 0$.
We have $f^{(n)}(x) = e^x$. So $f^{(n)}(0) = e^0 = 1$.

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(ii) Find Taylor series of $f(x) = e^x$ about $c = -1.5$.
We have $f^{(n)}(x) = e^x$. So $f^{(n)}(-1.5) = e^{-1.5}$.

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1.5)}{n!}(x+1.5)^n = \sum_{n=0}^{\infty} \frac{e^{-1.5}}{n!}(x+1.5)^n.$$


Thank You

Engineering Calculus-EMAT101L

Lecture 24 and 25

Riemann Integration

Bernhard Riemann (1826–1866)

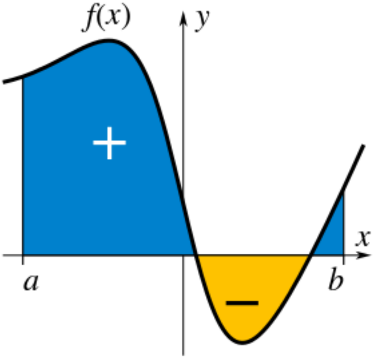
BENNETT UNIVERSITY
TIMES OF INDIA GROUP

Integration

- An **integral** assigns numbers to functions in a way that describes displacement, area, volume, and other concepts that arise by combining infinitesimal data.
- The process of finding integrals is called **integration**.
- Integrals can be categorized into two types.
  1. Definite integrals
  2. Indefinite integrals

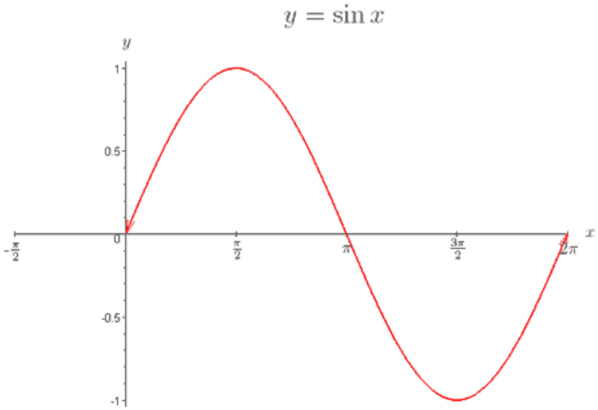
Definite Integrals

- The definite integrals can be interpreted as the signed area of the region in the plane that is bounded by the graph of a given function between two points in the real line.
- Conventionally, areas above the horizontal axis of the plane are positive while areas below are negative.



- **Example :**

$$\int_0^{2\pi} \sin x \; dx = \Big[-\cos x \Big]_0^{2\pi} = 0$$



Benefits/Use of Integrals

- Integrals appear in many practical situations. For instance, from the length, width and depth of a swimming pool which is rectangular with a flat bottom, one can determine the volume of water it can contain, the area of its surface, and the length of its edge.
- But if it is oval with a rounded bottom, integrals are required to find exact and rigorous values for these quantities. In each case, one may divide the sought quantity into infinitely many infinitesimal pieces, then sum the pieces to achieve an accurate approximation.

Interpretations of Integrals

- **Example :** To find the area of the region bounded by the graph of the function $f(x) = \sqrt{x}$ between $x = 0$ and $x = 1$, one can cross the interval in five steps $(0, 1/5, 2/5, ..., 1)$, then fill a rectangle using the right end height of each piece (thus $\sqrt{0}, \sqrt{1/5}, \sqrt{2/5}, ..., \sqrt{1}$) and sum their areas to get an approximation of

$$\sqrt{\tfrac{1}{5}}\left(\tfrac{1}{5} - 0\right) + \sqrt{\tfrac{2}{5}}\left(\tfrac{2}{5} - \tfrac{1}{5}\right) + \cdots + \sqrt{\tfrac{5}{5}}\left(\tfrac{5}{5} - \tfrac{4}{5}\right) \approx 0.7497,$$

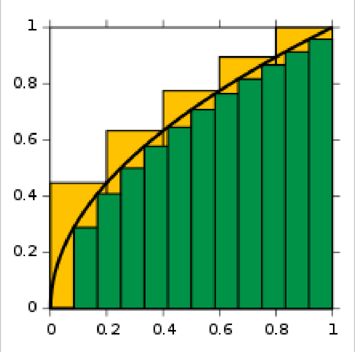
which is larger than the exact value.

- Alternatively, when replacing these subintervals by ones with the left end height of each piece, the approximation one gets is too low: with twelve such subintervals the approximated area is only 0.6203.

- However, when the number of pieces increase to infinity, it will reach a limit which is the exact value of the area sought (in this case, $2/3$). One writes

$$\int_0^1 \sqrt{x}\, dx = \frac{2}{3},$$

which means $2/3$ is the result of a weighted sum of function values, $\sqrt{x}$, multiplied by the infinitesimal step widths, denoted by dx, on the interval $[0, 1]$.



Approximations to integral of $\sqrt{x}$ from 0 to 1, with 5 yellow right endpoint partitions and 12 green left endpoint partitions

Some links for further Visualization

For upper sum of the function $y = x^2$, follow the below link.

https://en.wikipedia.org/wiki/File:Riemann_Integration_and_Darboux_Upper_Sums.gif

For lower sum

https://en.wikipedia.org/wiki/File:Riemann_Integration_and_Darboux_Lower_Sums.gif

Riemann Integrals

- The Riemann integral is defined in terms of Riemann sums of functions with respect to **tagged partitions** of an interval.
- A tagged partition of a closed interval $[a, b]$ on the real line is a finite sequence

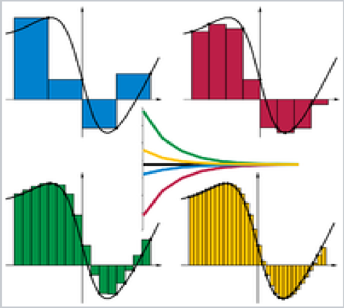
$$a = x_0 \leq t_1 \leq x_1 \leq t_2 \leq x_2 \leq \cdots \leq x_{n-1} \leq t_n \leq x_n = b.$$

- This partitions the interval $[a, b]$ into n sub-intervals $[x_{i-1}, x_i]$ indexed by i, each of which is "tagged" with a distinguished point $t_i \in [x_{i-1}, x_i]$.
- A Riemann sum of a function f with respect to such a tagged partition is defined as

$$\sum_{i=1}^{n} f(t_i)\, \Delta_i.$$

- Each term of the sum is the area of a rectangle with height equal to the function value at the distinguished point of the given sub-interval, and width the same as the width of sub-interval, $\Delta_i = x_i - x_{i-1}$.

Sum Covering.PNG



Riemann sums converging

- The Riemann integral of a function f over the interval $[a, b]$ is equal to S if

For all $\varepsilon > 0$, there exists $\delta > 0$ such that, for any tagged partition $[a, b]$ with mesh less than δ,

$$\left| S - \sum_{i=1}^{n} f(t_i)\, \Delta_i \right| < \varepsilon.$$

Riemann Integrals (simplified)

To visualize a sequence of Riemann sums over a regular partition of an interval, follow the below link.

https://upload.wikimedia.org/wikipedia/commons/2/28/Riemann_integral_regular.gif

Simplified formulation of Riemann integral:

$$\boxed{\int_a^b f(x)dx = \lim_{\|\Delta x\| \to 0} \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}).}$$

Some other integrals (not in our course, but for more curious students)

1. Cauchy integral
2. Riemann-Stieltjes integral
3. Lebesgue integral
4. Lebesgue-Stieltjes integral
5. Daniell integral
6. Haar integral
7. Henstock-Kurzweil (HK) integral
8. Wiener integral
9. Feynman integral

Question

How can we calculate the value of a definite integral?

Is it always by using the definition, i.e. by the limit-of-sum approach?

Or, is there any easy approach?

Question

How can we calculate the value of a definite integral?

Is it always by using the definition, i.e. by the limit-of-sum approach?

Or, is there any easy approach?

Rescue: Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

- The fundamental theorem of calculus is a theorem that links the concept of differentiating a function (calculating the gradient) with the concept of integrating a function (calculating the area under the curve).
- The two operations are inverses of each other apart from a constant value which depends where one starts to compute area
- So, we call integrals as anti-derivatives.

Fundamental Theorem of Calculus

Let f be a continuous real-valued function defined on a closed interval $[a, b]$.

Let F be the function defined, for all x in $[a, b]$, by

$$F(x) = \int_a^x f(t)\, dt.$$

Then F is uniformly continuous on $[a, b]$ and differentiable on the open interval (a, b), and

$$F'(x) = f(x) \text{ for all } x \in (a, b).$$

Application of Fundamental Theorem of Calculus

- The fundamental theorem is often employed to compute the definite integral of a function f for which an anti-derivative F is known.
- Let f be a real-valued function on a closed interval $[a, b]$ and F an anti-derivative of f in (a, b) such that

$$F'(x) = f(x).$$

If f is Riemann integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

- **Example :** Suppose the following is to be calculated:

$$\int_2^5 x^2\,dx.$$

Here, $f(x) = x^2$ and we can use $F(x) = \dfrac{x^3}{3}$ as the anti-derivative.

$$\therefore \quad \int_2^5 x^2\,dx = F(5) - F(2) = \frac{5^3}{3} - \frac{2^3}{3} = \frac{125}{3} - \frac{8}{3} = \frac{117}{3} = 39.$$

Properties of Definite Integrals

- Linearity: $\int_a^b (\alpha f + \beta g)(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$
- An integrable function f on $[a, b]$, is necessarily bounded on that interval. Thus there are real numbers m and M so that $m \leq f(x) \leq M$ for all $x \in [a, b]$. Hence we have

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Thank You.