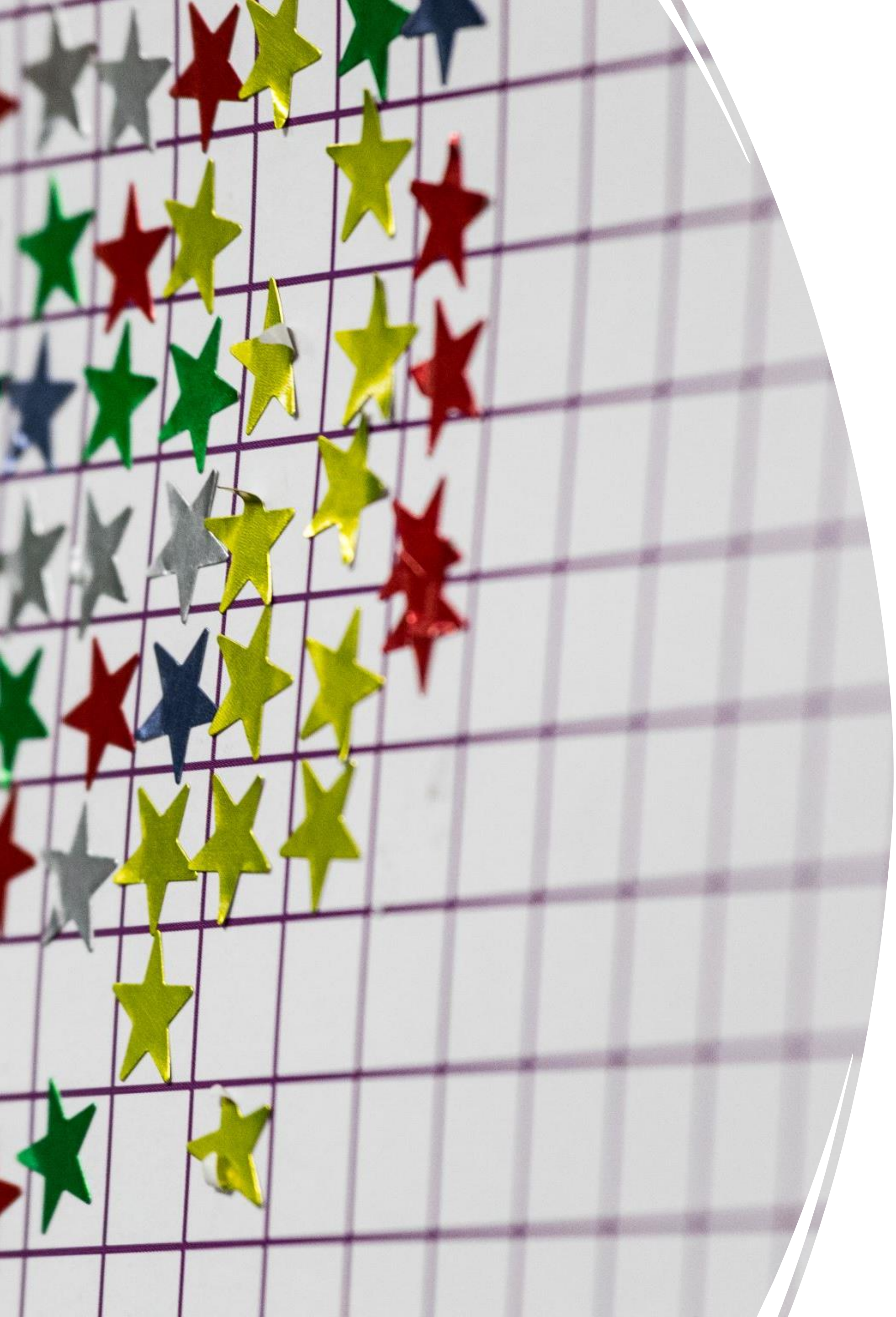


Example-7

- Determine the constant c so that the following p.m.f. of the random variable y is a valid probability mass function:

$$f(y) = c\left(\frac{1}{4}\right)^{y} \quad for\ y = 1, 2, 3, \ldots$$

Example-8

I have an unfair coin for which $P(H) = p$, where $0 < p < 1$. I toss the coin repeatedly until

I observe a heads for the first time. Let Y be the total number of coin tosses (times).

Find the distribution of Y.

Example-8-Solution

First, we note that the random variable Y can potentially take any positive integer, so we have $R_Y = \mathbb{N} = \{1, 2, 3, \ldots\}$. To find the distribution of Y, we need to find $P_Y(k) = P(Y = k)$ for $k = 1, 2, 3, \ldots$. We have

$$P_Y(1) = P(Y = 1) = P(H) = p,$$

$$P_Y(2) = P(Y = 2) = P(TH) = (1 - p)p,$$

$$P_Y(3) = P(Y = 3) = P(TTH) = (1 - p)^2 p,$$

$$\begin{array}{cccc} . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{array}$$

$$P_Y(k) = P(Y = k) = P(TT\ldots TH) = (1 - p)^{k-1} p.$$

Thus, we can write the PMF of Y in the following way

$$P_Y(y) = \begin{cases} (1-p)^{y-1} p & \text{for } y = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Example-9

I have an unfair coin for which $P(H) = p$, where $0 < p < 1$. I toss the coin repeatedly until

I observe a heads for the first time. Let Y be the total number of coin tosses.

Find the distribution of Y.

1. Check that $\sum_{y \in R_Y} P_Y(y) = 1$.
2. If $p = \frac{1}{2}$, find $P(2 \leq Y < 5)$.

1. Check that $\sum_{y \in R_Y} P_Y(y) = 1$.
2. If $p = \frac{1}{2}$, find $P(2 \le Y < 5)$.

$$P_Y(k) = P(Y = k) = (1-p)^{k-1}p, \text{ for } k = 1, 2, 3, \ldots$$

Thus,

1. to check that $\sum_{y \in R_Y} P_Y(y) = 1$, we have

$$\begin{aligned} \textstyle\sum_{y \in R_Y} P_Y(y) &= \textstyle\sum_{k=1}^{\infty} (1-p)^{k-1} p \\ &= p \textstyle\sum_{j=0}^{\infty} (1-p)^j \\ &= p \tfrac{1}{1-(1-p)} && \text{Geometric sum} \\ &= 1; \end{aligned}$$

2. if $p = \frac{1}{2}$, to find P$(2 \le Y < 5)$, we can write

$$\begin{aligned} \mathrm{P}(2 \le Y < 5) &= \textstyle\sum_{k=2}^{4} P_Y(k) \\ &= \textstyle\sum_{k=2}^{4} (1-p)^{k-1} p \\ &= \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}\right) \\ &= \tfrac{7}{16}. \end{aligned}$$

Example-9-Solution

Example-10

Let X be a discrete random variable with range $R_X = \{1, 2, 3, \ldots\}$. Suppose the PMF of X is given by

$$P_X(k) = \frac{1}{2^k} \text{ for } k = 1, 2, 3, \ldots$$

1. Find CDF of $X, F_X(x)$
2. Find $P(2 < X \leq 5)$
3. Find $P(X > 4)$

Example-10-Solution

First, note that this is a valid PMF. In particular,

$$\sum_{k=1}^{\infty} P_X(k) = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 \text{ (geometric sum)}$$

a. To find the CDF, note that

$$\text{For } x < 1, \qquad F_X(x) = 0.$$
$$\text{For } 1 \leq x < 2, F_X(x) = P_X(1) = \tfrac{1}{2}.$$
$$\text{For } 2 \leq x < 3, F_X(x) = P_X(1) + P_X(2) = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}.$$

In general we have

$$\text{For } 0 < k \leq x < k+1,$$

$$F_X(x) = P_X(1) + P_X(2) + \ldots + P_X(k)$$

$$= \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}.$$

Example-10-Solution

b. To find $P(2 < X \leq 5)$, we can write

$$P(2 < X \leq 5) = F_X(5) - F_X(2) = \frac{31}{32} - \frac{3}{4} = \frac{7}{32}.$$

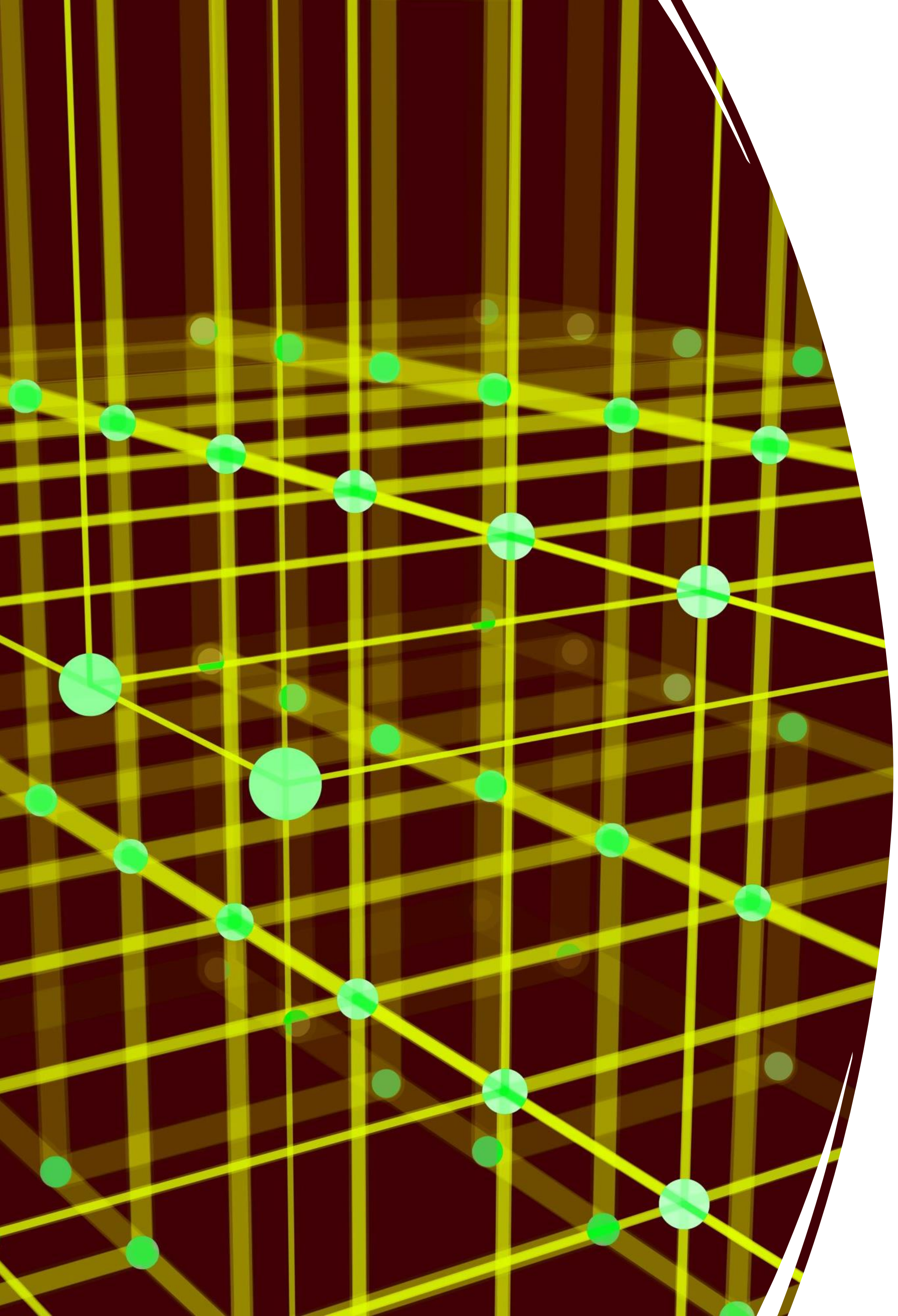
Or equivalently, we can write

$$P(2 < X \leq 5) = P_X(3) + P_X(4) + P_X(5) = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{7}{32},$$

which gives the same answer.

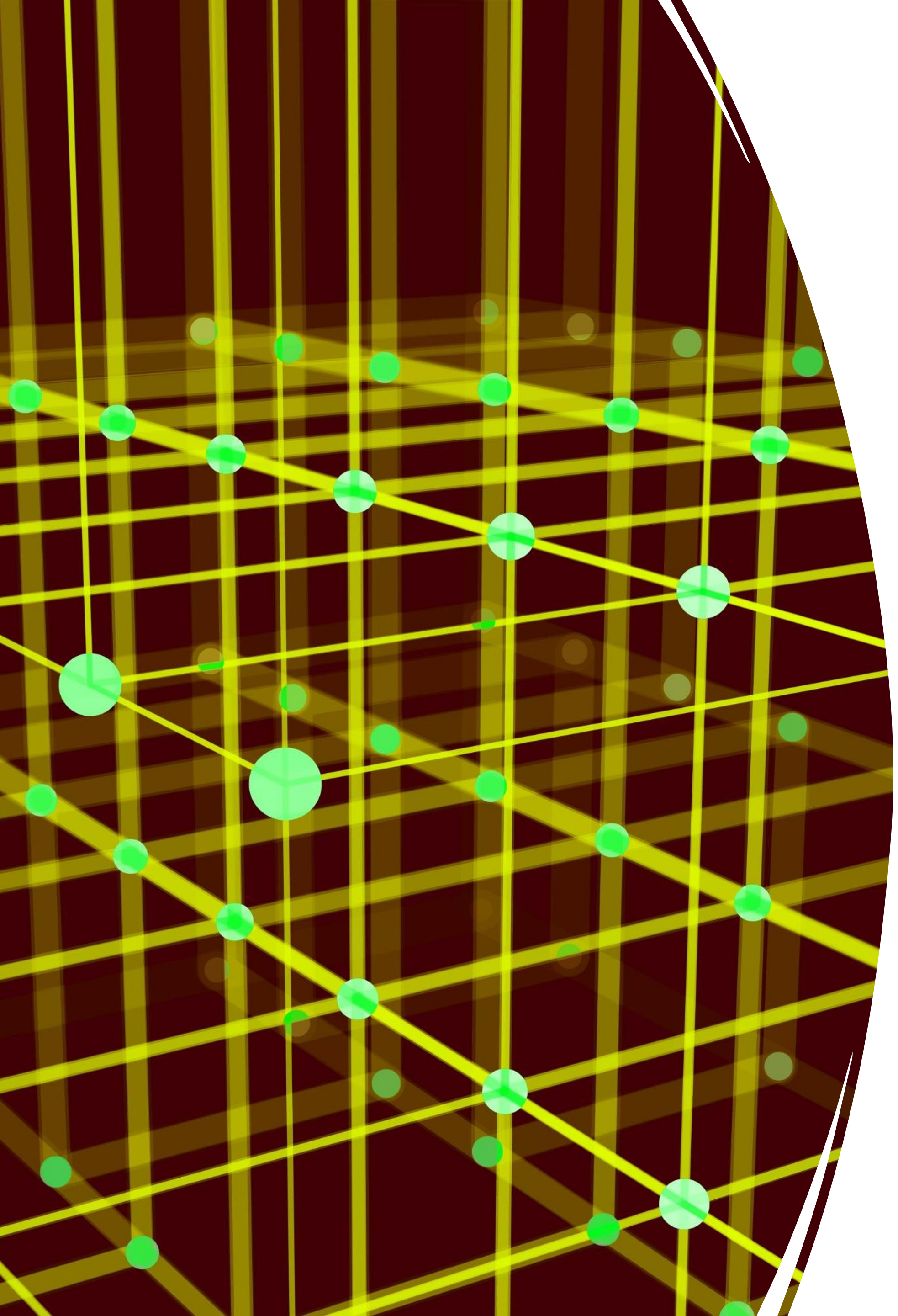
c. To find $P(X > 4)$, we can write

$$P(X > 4) = 1 - P(X \leq 4) = 1 - F_X(4) = 1 - \frac{15}{16} = \frac{1}{16}.$$



Bivariate Random Variable

Let S be a sample space associated with random experiment. Let $X(t), Y(t)$ be two random function each assigning a real number to each outcome $t \epsilon S$. Then (X, Y) is called a Bivariate or Two Dimensional Random variable



Probability (Join) Mass Function

Let (X, Y) is a Two dimensional discrete RV, where $X = x$; $Y = y$; i and $j = 1,2,\ldots$

Then $P(X = xi, Y = y_j) = P_{ij}$ is called PMF of (X, Y) provided:

1. $P_{ij} => 0$
2. $\sum_j^m \sum_i^n P_{ij} = 1$

The set of triple (xi, y_i, P_{ij}) is called Probability (Join) Mass Function

Example-11

Two balls are selected at random from a box containing 3-red, 2-green, 4-white. If X and Y are the number of red balls and green balls respectively, included among the two balls drawn from the box, find

1) Join probability of X and Y
2) Marginal Probability of X and Y
3) Conditional distribution of X given Y = 1

3 - r
2 - g
4 - w

x => # red
y => # green

P(x, y)

x \ y	0	1	2		Total
0	1/6	2/9	1/36		15/36
1	1/3	1/6	0		18/36
2	1/12	0	0		3/36
3	0	0	0		
Total	21/36	14/36	1/36		1