Probability and Statistics

Lecture: Conditional probability

Example

Alice and Bob toss two fair coins separately. Denote the event that Alice gets a H by E_A and the event that Bob gets a H by E_B.
We know that $P(E_A) = P(E_B) = 1/2$.

Example

Alice and Bob toss two fair coins separately. Denote the event that Alice gets a H by E_A and the event that Bob gets a H by E_B.
We know that $P(E_A) = P(E_B) = 1/2$.

- $P(E_A \cup E_B) = P(E_A) + P(E_B) = 1.$
- So its certain that at least one of them will get a head.

Example

Alice and Bob toss two fair coins separately. Denote the event that Alice gets a H by E_A and the event that Bob gets a H by E_B.

We know that $P(E_A) = P(E_B) = 1/2$.

- $P(E_A \cup E_B) = P(E_A) + P(E_B) = 1$.
- So its certain that at least one of them will get a head.
- What's wrong with this argument?

Example

Alice and Bob toss two fair coins separately. Denote the event that Alice gets a H by E_A and the event that Bob gets a H by E_B.

We know that $P(E_A) = P(E_B) = 1/2$.

- $P(E_A \cup E_B) = P(E_A) + P(E_B) = 1$. So its certain that at least one of them will get a head.
- What's wrong with this argument?
- But both could get tails!

Example

Alice and Bob toss two fair coins separately. Denote the event that Alice gets a H by E_A and the event that Bob gets a H by E_B.
We know that $P(E_A) = P(E_B) = 1/2$.

- $P(E_A \cup E_B) = P(E_A) + P(E_B) = 1$. So its certain that at least one of them will get a head.
- What's wrong with this argument?
- But both could get tails!
- Are E_A and E_B disjoint?

Example

Alice and Bob toss two fair coins separately. Denote the event that Alice gets a H by E_A and the event that Bob gets a H by E_B.

We know that $P(E_A) = P(E_B) = 1/2$.

- $P(E_A \cup E_B) = P(E_A) + P(E_B) = 1$. So its certain that at least one of them will get a head.
- Whats wrong with this argument?
- But both could get tails!
- Are E_A and E_B disjoint?
- Not quite.
- The sample space you are considering is: HT, TH, TT, HH.

  So $E_A = \{HT, HH\}$ and $E_B = \{TH, HH\}$.

Example

Alice and Bob toss two fair coins separately. Denote the event that Alice gets a H by E_A and the event that Bob gets a H by E_B.

We know that $P(E_A) = P(E_B) = 1/2$.

- $P(E_A \cup E_B) = P(E_A) + P(E_B) = 1$. So its certain that at least one of them will get a head.
- What's wrong with this argument?
- But both could get tails!
- Are E_A and E_B disjoint?
- Not quite.
- The sample space you are considering is: HT, TH, TT, HH.

  So $E_A = \{HT, HH\}$ and $E_B = \{TH, HH\}$.
- So the union is really

  $P(E_A \cup E_B) = P(E_A) + P(E_B) - P(E_A \cap E_B) = 3/4.$

Example

Alice and Bob toss two fair coins separately. Denote the event that Alice gets a H by E_A and the event that Bob gets a H by E_B.

We know that $P(E_A) = P(E_B) = 1/2$.

- $P(E_A \cup E_B) = P(E_A) + P(E_B) = 1$. So its certain that at least one of them will get a head.
- What's wrong with this argument? But both could get tails!
- Are E_A and E_B disjoint? Not quite. The sample space you are considering is: HT, TH, TT, HH.

  So $E_A = \{HT, HH\}$ and $E_B = \{TH, HH\}$.
- So the union is really
  $P(E_A \cup E_B) = P(E_A) + P(E_B) - P(E_A \cap E_B) = 3/4$.
- May be what you are really thinking about is independent and not disjoint events.

Partial information

- So far, we have assumed we know nothing about the outcome of our experiment, except for the information encoded in the probability law.
- Sometimes, however, we have **partial information** that may affect the likelihood of a given event.
  - The experiment involves rolling a die. You are told that the number is odd.
  - The experiment involves the weather tomorrow. You know that the weather today is rainy.
  - The experiment involves the presence or absence of a disease. A blood test comes back positive.
- In each case, knowing about some event B (e.g., "it is raining today") changes our beliefs about event A ("Will it rain tomorrow?").
- We want to **update** our probability law to incorporate this new knowledge.

Conditional Probability

Original problem:

- What is the probability of some event A.

  e.g. What is the probability that we roll a number less than 4?

- In other words, what is $P(A)$?
- This is given by our probability law.

Conditional Probability

Original problem:

- What is the probability of some event A.

  e.g., What is the probability that we roll a number less than 4?

- In other words, what is $P(A)$?
- This is given by our probability law.

New problem:

- Assuming event B (equivalently given event B), what is the probability of event A?

  e.g., Given that the number rolled is an odd number, what is the probability that it is less than 4?

- We call this the **conditional distribution** of A given B.
- We write this as $P(A|B)$
- Read | as "given" or "conditioned on the fact that".
- Our conditional probability is still describing "the probability of something", so we expect it to behave like a probability distribution.

Conditional Probability

- Consider rolling a fair 6-sided die (uniform, discrete probability distribution).
- Let A be the event "outcome is equal to 1'.
  - What is $P(A)$?
- Let's now assume that the number rolled is an odd number.
  - What is the set, B, that we are conditioning on?

- What do you think $P(A|B)$ should be?

Conditional Probability

- You are conditioning on event B. So your new sample space is $\{1,3,5\}$ instead of $\{1,2,3,4,5,6\}$.

Conditional Probability

- You are conditioning on event B. So your new sample space is $\{1,3,5\}$ instead of $\{1,2,3,4,5,6\}$.
- Each of these are equally likely. So $P(A|B) = 1/3$.

Conditional Probability

- You are conditioning on event B. So your new sample space is $\{1,3,5\}$ instead of $\{1,2,3,4,5,6\}$.
- Each of these are equally likely. So $P(A|B) = 1/3$.

Formally, if all outcomes are equally likely, we have

$$P(A|B) = \frac{\#\text{ elements in } A \cap B}{\#\text{ elements in } B}$$

Conditional Probability

- You are conditioning on event B. So your new sample space is $\{1,3,5\}$ instead of $\{1,2,3,4,5,6\}$.
- Each of these are equally likely. So $P(A|B) = 1/3$.

  Formally, if all outcomes are equally likely, we have

$$P(A|B) = \frac{\#\text{ elements in } A \cap B}{\#\text{ elements in } B}$$

More generally, $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

Conditional Probability

- You are conditioning on event B. So your new sample space is $\{1,3,5\}$ instead of $\{1,2,3,4,5,6\}$.
- Each of these are equally likely. So $P(A|B) = 1/3$.

Formally, if all outcomes are equally likely, we have

$$P(A|B) = \frac{\#\text{ elements in } A \cap B}{\#\text{ elements in } B}$$

More generally, $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

A conditional probability is only defined if $P(B) > 0$.

Conditional Probability Axioms

- **Nonnegativity** – Check.
- **Normalization** –
- **Additivity** –

Conditional Probability Axioms

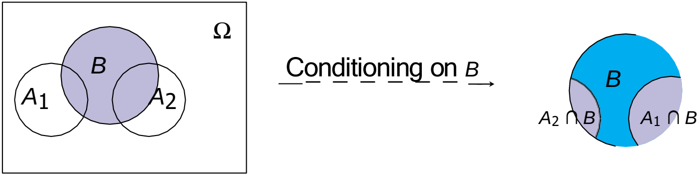
- **Nonnegativity** – Check.
- **Normalization** – Your new universe is now B, and we know that $P(B|B) = 1$.
- **Additivity** –

Conditional Probability Axioms

- **Nonnegativity** – Check.
- **Normalization** – Your new universe is now B, and we know that $P(B|B) = 1$.
- **Additivity** – $P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B)$ for two disjoint sets, A_1 and A_2.

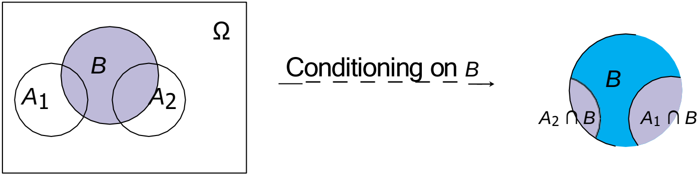
Conditional Probability Axioms

- **Nonnegativity** – Check.
- **Normalization** – Your new universe is now B, and we know that $P(B|B) = 1$.
- **Additivity** – $P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B)$ for two disjoint sets, A_1 and A_2.



Conditional Probability Axioms

- **Nonnegativity** – Check.
- **Normalization** – Your new universe is now B, and we know that $P(B|B) = 1$.
- **Additivity** – $P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B)$ for two disjoint sets, A_1 and A_2.



Using additivity, $P((A_1 \cup A_2) \cap B) = P(A_1 \cap B) + P(A_2 \cap B)$, so

$$P(A_1 \cup A_2|B) = \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)} = P(A_1|B) + P(A_2|B)$$

Properties of conditional probability

If $P(B) > 0$,

If A_i for $i \in \{1,...,n\}$ are all pairwise disjoint, then

$$P \bigcup_{i=1}^{n} A_i | B = \sum_{i=1}^{n} P(A_i|B)$$

Properties of conditional probability

If $P(B) > 0$,

If A_i for $i \in \{1,\dots,n\}$ are all pairwise disjoint, then

$$P \bigcup_{i=1}^{n} A_i | B = \sum_{i=1}^{n} P(A_i|B)$$

If $A_1 \subseteq A_2$, then $P(A_1|B) \le P(A_2|B)$.

Properties of conditional probability

If $P(B) > 0$,

If A_i for $i \in \{1,...,n\}$ are all pairwise disjoint, then

$$P \bigcup_{i=1}^{n} A_i | B = \sum_{i=1}^{n} P(A_i|B)$$

If $A_1 \subseteq A_2$, then $P(A_1|B) \leq P(A_2|B)$.

$$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2|B)$$

Properties of conditional probability

If $P(B) > 0$,

If A_i for $i \in \{1,...,n\}$ are all pairwise disjoint, then

$$P \bigcup_{i=1}^{n} A_i | B = \sum_{i=1}^{n} P(A_i | B)$$

If $A_1 \subseteq A_2$, then $P(A_1|B) \leq P(A_2|B)$.

$$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2|B)$$

Union bound: $P(A_1 \cup A_2|B) \leq P(A_1|B) + P(A_2|B)$

Properties of conditional probability

If $P(B) > 0$,

If A_i for $i \in \{1,\ldots,n\}$ are all pairwise disjoint, then

$$P\bigcup_{i=1}^{n} A_i|B = \sum_{i=1}^{n} P(A_i|B)$$

If $A_1 \subseteq A_2$, then $P(A_1|B) \leq P(A_2|B)$.

$$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2|B)$$

Union bound: $P(A_1 \cup A_2|B) \leq P(A_1|B) + P(A_2|B)$

$$P\bigcup_{i=1}^{n} A_i|B \leq \sum_{i=1}^{n} P(A_i|B)$$

Example: Coin toss

Consider the experiment of tossing a fair coin three times. What is the probability of getting alternating heads and tails conditioned on the event that your first toss gives a head?

- Notation: Let $A := \{\text{Tosses yield alternating tails and heads}\}$ and $B := \{\text{The first toss is a head}\}$.
- We want $P(A|B)$.
- What if you wanted to follow the formula?

Example: Coin toss

Consider the experiment of tossing a fair coin three times. What is the probability of getting alternating heads and tails conditioned on the event that your first toss gives a head?

- Notation: Let A := {Tosses yield alternating tails and heads} and B := {The first toss is a head}.
- We want $P(A|B)$.
- What if you wanted to follow the formula?
  - Sample space for three coin tosses $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

Example: Coin toss

Consider the experiment of tossing a fair coin three times. What is the probability of getting alternating heads and tails conditioned on the event that your first toss gives a head?

- Notation: Let $A :=$ {Tosses yield alternating tails and heads} and $B :=$ {The first toss is a head}.
- We want $P(A|B)$.
- What if you wanted to follow the formula?
  - Sample space for three coin tosses $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.
  - $A = \{HTH, THT\}$ and $A \cap B = \{HTH\}$.

Example: Coin toss

Consider the experiment of tossing a fair coin three times. What is the probability of getting alternating heads and tails conditioned on the event that your first toss gives a head?

- Notation: Let A := {Tosses yield alternating tails and heads} and B := {The first toss is a head}.
- We want $P(A|B)$.
- What if you wanted to follow the formula?
  - Sample space for three coin tosses $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.
  - $A = \{HTH, THT\}$ and $A \cap B = \{HTH\}$.
  - We have $P(B) = 4/8$ and $P(A \cap B) = 1/8$ and so $P(A|B) = 1/4$.

Example: Coin toss

Consider the experiment of tossing a fair coin three times. What is the probability of getting alternating heads and tails conditioned on the event that your first toss gives a head?

- Notation: Let A := {Tosses yield alternating tails and heads} and B := {The first toss is a head}.
- We want $P(A|B)$.
- What if you wanted to follow the formula?
  - Sample space for three coin tosses $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.
  - $A = \{HTH, THT\}$ and $A \cap B = \{HTH\}$.

  We have $P(B) = 4/8$ and $P(A \cap B) = 1/8$ and so $P(A|B) = 1/4$.

  - Your new world/sample space when you condition is $B = \{HHT, HTH, HTT, HHH\}$. Each of these are equally likely. Out of these 1 event satisfies alternating heads and tails.

Example: Coin toss

Consider the experiment of tossing a fair coin three times. What is the probability of getting alternating heads and tails conditioned on the event that your first toss gives a head?

- Notation: Let A := {Tosses yield alternating tails and heads} and B := {The first toss is a head}.
- We want $P(A|B)$.
- What if you wanted to follow the formula?
  - Sample space for three coin tosses $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.
    $A = \{HTH, THT\}$ and $A \cap B = \{HTH\}$.
  - We have $P(B) = 4/8$ and $P(A \cap B) = 1/8$ and so $P(A|B) = 1/4$.

  - Your new world/sample space when you condition is $B = \{HHT, HTH, HTT, HHH\}$. Each of these are equally likely. Out of these 1 event satisfies alternating heads and tails.
  - So, $P(A|B) = 1/4$.