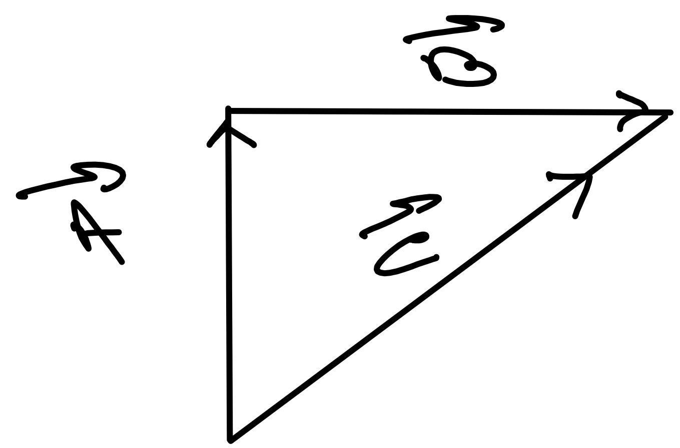
EPHY105L-1 27.9.21

Vector Operations

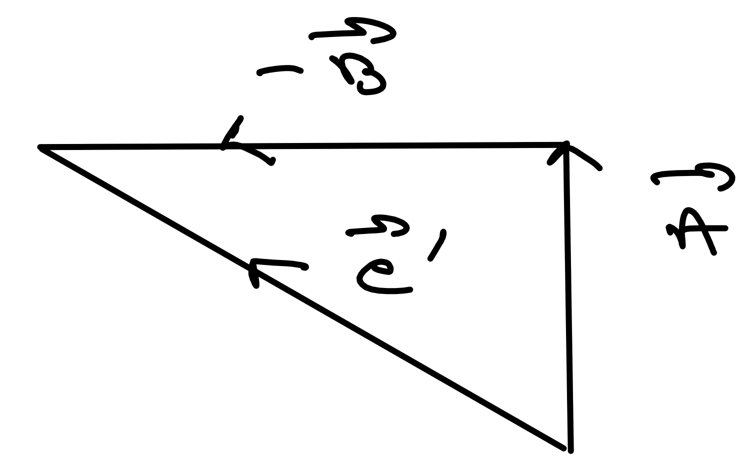
(i) Addition

Two vectors : $\vec{A}$ & $\vec{B}$

$\vec{C} = \vec{A} + \vec{B}$



Subtraction

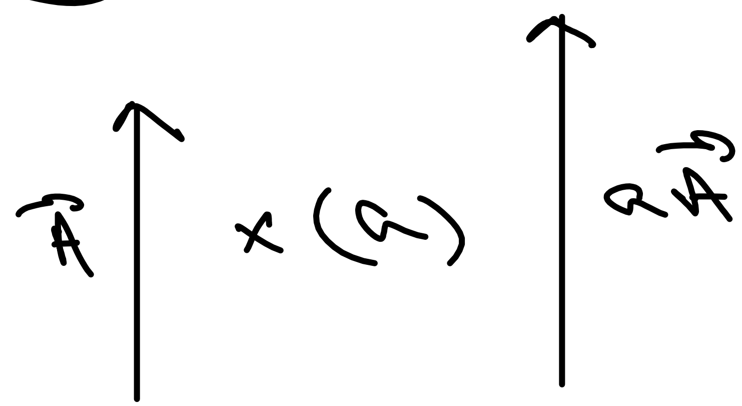


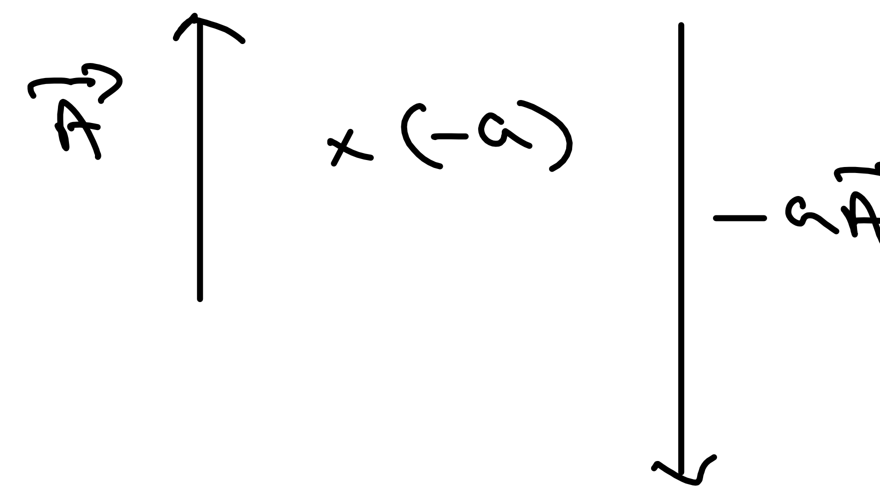
$\vec{C}' = \vec{A} - \vec{B}'$

(+) Addition is associative

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$

② Multiplication by a scalar

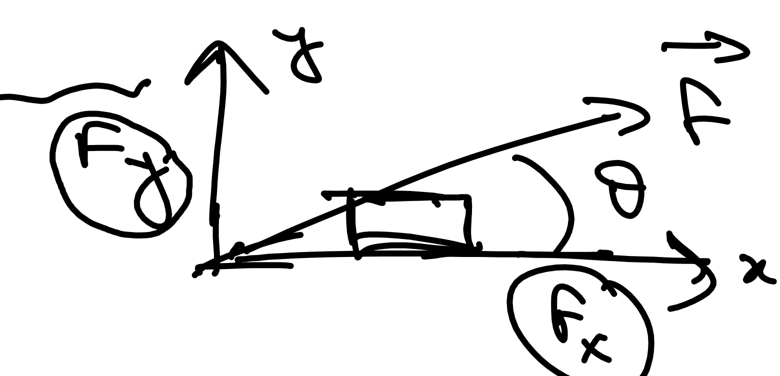




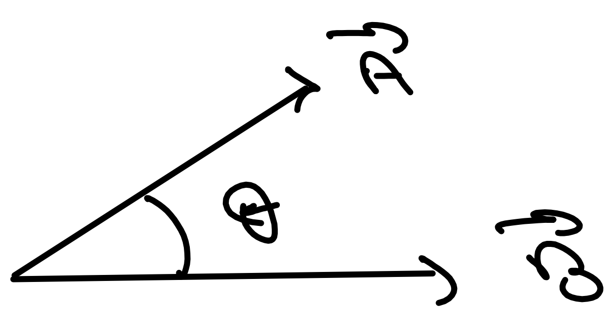
⊛ Multiplication is distributive

$$a(\vec{A} + \vec{B}) = a\vec{A} + a\vec{B}$$

③ Dot product of two vectors



$$\vec{A} \cdot \vec{B} = \underbrace{AB\cos\theta}_{\text{Scalar quantity}}$$



$$A = |\vec{A}|$$
$$B = |\vec{B}|$$

$$F_x = \vec{F} \cdot \hat{x}$$
$$\vec{F} \cdot \hat{x} = F\cos\theta$$

* Dot product is commutative

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

* Dot product is distributive

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

Geometrically, $\vec{A} \cdot \vec{B}$ is simply product A times the projection of $\vec{B}$ along $\vec{A}$ or vice versa.

* If the two vectors are parallel:

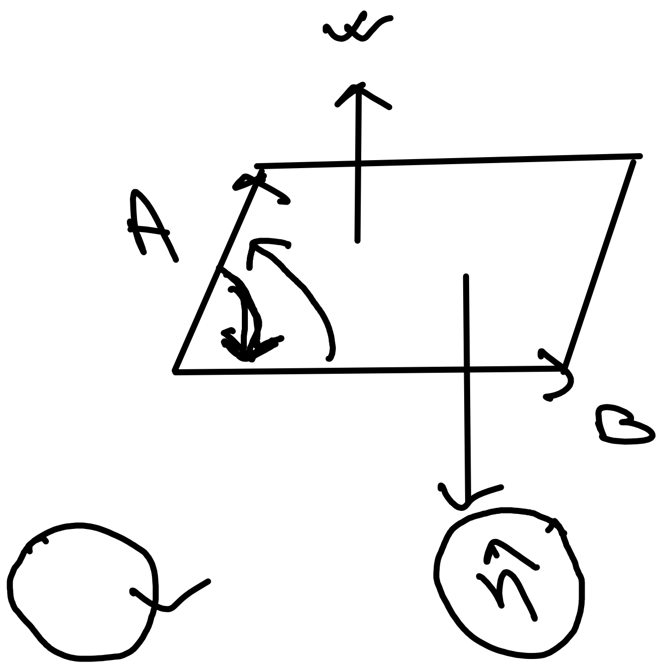
$$\vec{A} \cdot \vec{B} = AB \leftarrow$$

If they are perpendicular;

$$\vec{A} \cdot \vec{B} = 0$$

- Cross product betn. two vectors

$$\vec{A} \times \vec{B} = \underbrace{AB \sin\theta \, \hat{n}}_{\text{vector quantity}}$$



$\hat{n}$ ≡ unit vector perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$

$\vec{A} \times \vec{B}$ ≡ Points into the page

$\vec{B} \times \vec{A}$ ≡ Points out of the page.

- Cross-product is distributive

$$\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C})$$

- Cross product is not commutative

$$\vec{B} \times \vec{A} = -(\vec{A} \times \vec{B})$$

Geometrically, $|\vec{A} \times \vec{B}|$ is the area of the parallelogram generated by $\vec{A}$ and $\vec{B}$

(x) If the vectors are parallel,

$$\vec{A} \times \vec{B} = 0$$

If they are perpendicular,

$$\vec{A} \times \vec{B} = AB\,\hat{n}$$

In component form

In Cartesian co-ordinate system

Basis vectors $\equiv \hat{x}, \hat{y}, \hat{z}$

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

components

A_x, A_y, A_z are simply just projections of $\vec{A}$ along three axes.

(i) Addition

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} + (A_z + B_z)\hat{z}$$

(ii) Multiplication by scalar

$$a\vec{A} = (aA_x)\hat{x} + (aA_y)\hat{y} + (aA_z)\hat{z}$$

(iii) Dot product

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$$

$$\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0$$

$$\vec{A} \cdot \vec{A} = |\vec{A}|^2 = A_x^2 + A_y^2 + A_z^2$$

⊗ Cross Product

$$\vec{A} \times \vec{B} = (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \times (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

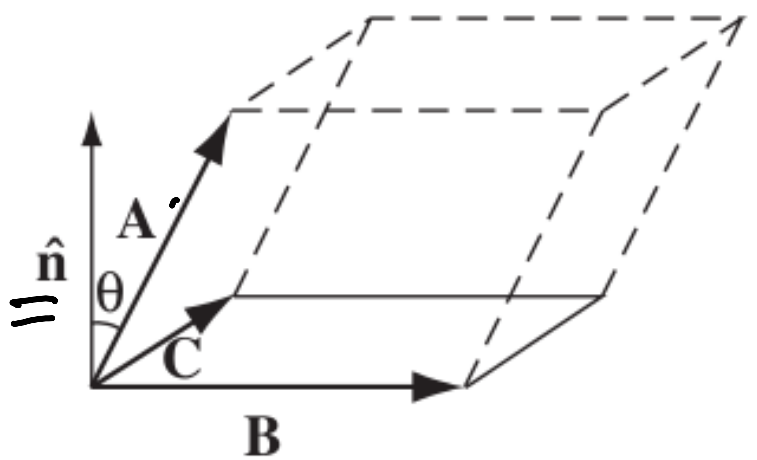
$$\hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0$$
$$\hat{x} \times \hat{y} = -\hat{y} \times \hat{x} = \hat{z}$$
$$\hat{y} \times \hat{z} = -\hat{z} \times \hat{y} = \hat{x}$$
$$\hat{z} \times \hat{x} = -\hat{x} \times \hat{z} = \hat{y}$$

$$= \hat{x}(A_y B_z - A_z B_y) + \hat{y}(A_z B_x - A_x B_z) + \hat{z}(A_x B_y - A_y B_x)$$

Triple Product

Scalar triple product $\equiv \vec{A} \cdot (\underline{\vec{B} \times \vec{C}})$

→ Geometrically, $|\vec{A}\cdot(\vec{B}\times\vec{C})|$ is the volume of a parallelopiped.



$|\vec{B}\times\vec{C}| \equiv$ Area of the base

$|\vec{A}\cos\theta| =$ Altitude

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

Link to the Recording:

https://bennettu.sharepoint.com/sites/EPHY105L-Odd2021/Shared%20Documents/General/Recordings/EPHY105L%20Theory%20Class-20210927_154620-Meeting%20Recording.mp4

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$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

$$\vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

$(\hat{x}, \hat{y}, \hat{z})$
$\;\;\;\;\parallel\;\;\;\; \parallel$
$(\hat{i}, \hat{j}, \hat{k})$

$$\vec{A}\cdot\vec{B} = (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \cdot (\underline{B_x} \hat{x} + \underline{B_y} \hat{y} + \underline{B_z} \hat{z})$$

$$= A_x B_x + A_y B_y + A_z B_z$$

$$\vec{A}\times\vec{B} = (\underline{A_x} \hat{x} + A_y \hat{y} + A_z \hat{z}) \times (B_x \hat{x} + \underline{B_y} \hat{y} + \underline{B_z} \hat{z})$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}(A_y B_z - A_z B_y) + \hat{y}(A_z B_x - A_x B_z) + \hat{z}(A_x B_y - A_y B_x)$$

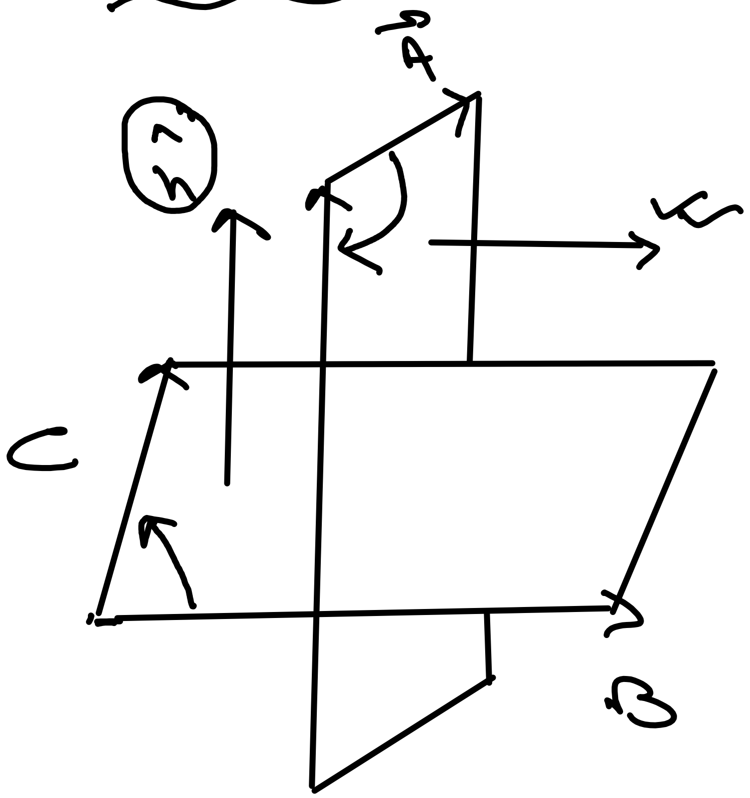
Triple product

Scalar product

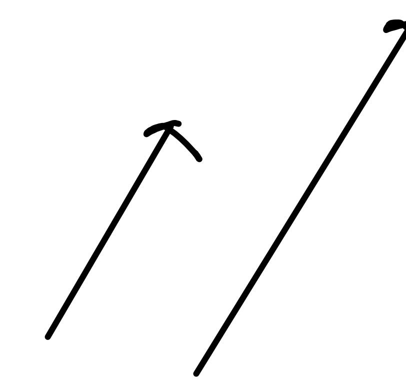
$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

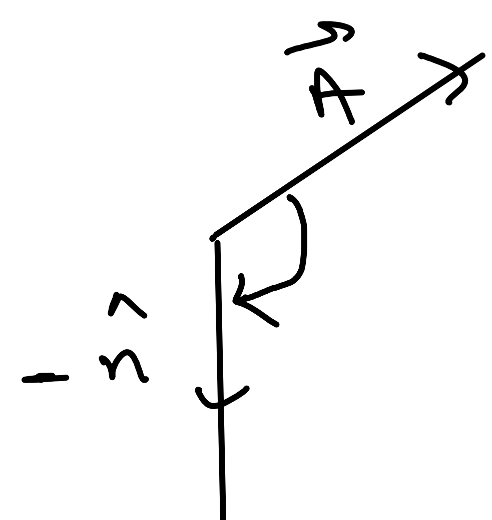
Vector product

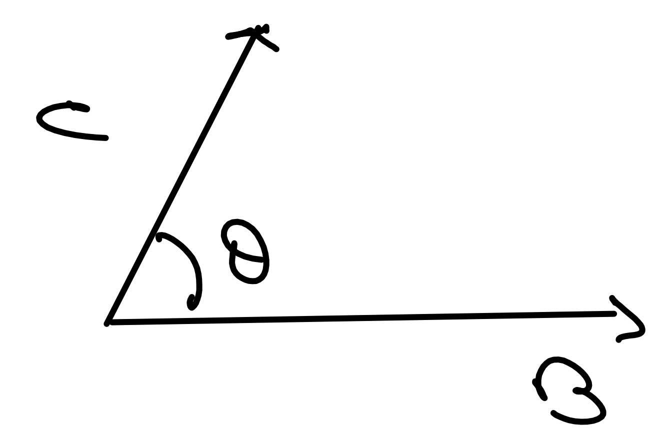
$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$



$$x\vec{B} - y\vec{C}$$

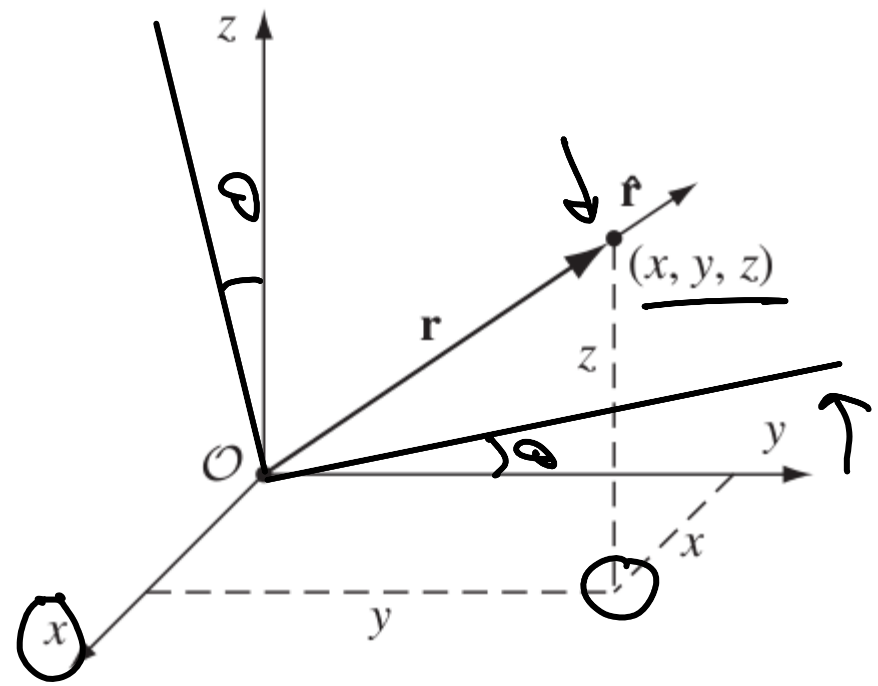






$\vec{A}\times(-\hat{n})$ = on BC plane
(pointed towards right)

Position Vector

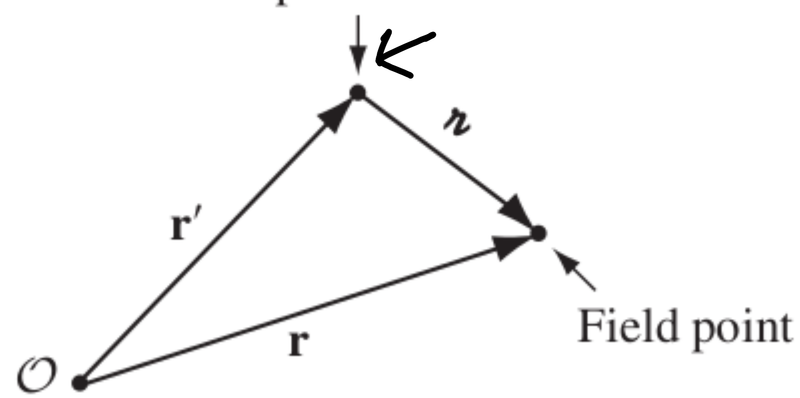


$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2+y^2+z^2}}$$

Separation Vector



$\vec{r} \equiv$ Position vector at which we wish to calculate the field

$\vec{r}\,' \equiv$ Position vector of the source

$\vec{\mathscr{r}} \equiv$ Separation vector.

$= \vec{r} - \vec{r}\,'$

Transformation of Vectors

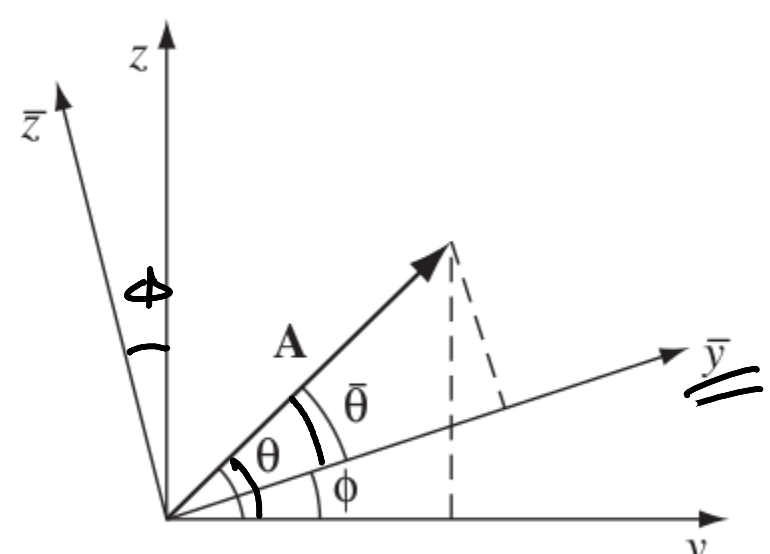
In order for $\vec{A}$ to be called a vector it has to transform "like a vector"

Say, we are going from (x, y, z) to $(\bar{x}, \bar{y}, \bar{z})$

→ Rotation by ϕ relative to (x, y, z) about $x = \bar{x}$

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

$$\vec{A} = \bar{A}_x \hat{\bar{x}} + \bar{A}_y \hat{\bar{y}} + \bar{A}_z \hat{\bar{z}}$$



$$\theta = \phi + \bar{\theta}$$

$$\bar{A}_y = A \cos\bar{\theta} = A \cos(\theta - \phi)$$

$$= A(\cos\theta\cos\phi + \sin\theta\sin\phi)$$

$$= A_y \cos\phi + A_z \sin\phi$$

$$\bar{A}_z = A\sin\bar{\theta} = A\sin(\theta - \phi)$$

$$= A(\sin\theta\cos\phi - \cos\theta\sin\phi)$$

$$= A_z\cos\phi - A_y\sin\phi$$

$$\bar{A}_x = A_x$$

This can be written as

$$\begin{pmatrix} \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$

More generally,

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} \overset{=1}{R_{xx}} & \overset{=0}{R_{xy}} & \overset{=0}{R_{xz}} \\ 0 = R_{yx} & R_{yy} & R_{yz} \\ 0 = R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

In compact form:

$$\vec{A}_i = \sum_{j=1}^{3} R_{ij} A_j \qquad \left\{ \begin{array}{l} j=1 \equiv x \\ j=2 \equiv y \\ j=3 \equiv z \end{array} \right.$$

Link to the Recording:

https://bennettu.sharepoint.com/sites/EPHY105L-Odd2021/Shared%20Documents/General/Recordings/Meeting%20in%20_General_-20210930_134250-Meeting%20Recording.mp4

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Transformation of vectors

For rotation about an arbitrary axis in 3-D

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{zy} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$ i.e.

In compact notation,

$$\bar{A}_i = \sum_{j=1}^{3} R_{ij} A_j$$

$i = 1 \equiv x$

$i = 2 \equiv y$

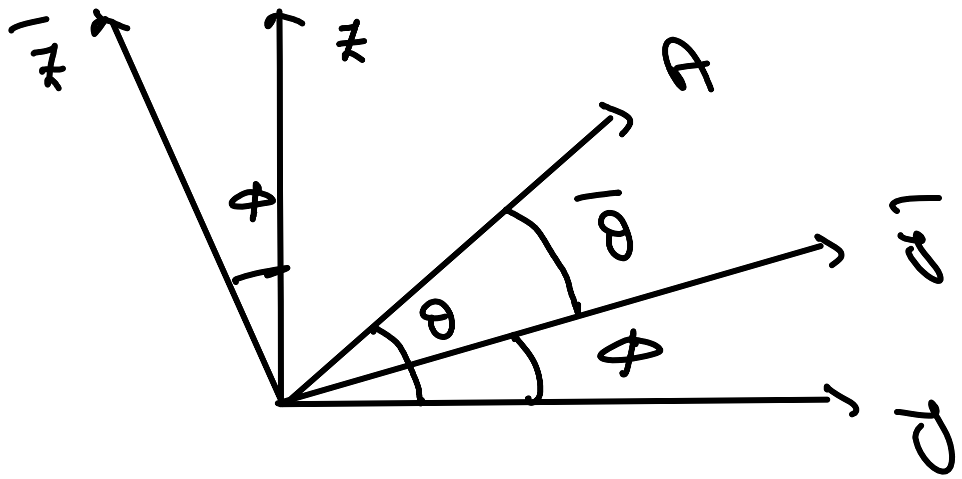
$i = 3 \equiv z$

For rotation along x-axis by an angle ϕ:

$R_{xx} = 1, \quad R_{xy} = 0, \quad R_{xz} = 0$

$R_{yx} = 0, \quad R_{yy} = \cos\phi, \quad R_{yz} = \sin\phi$

$R_{zx} = 0, \quad R_{zy} = -\sin\phi, \quad R_{zz} = \cos\phi$



$$x = \bar{x}$$

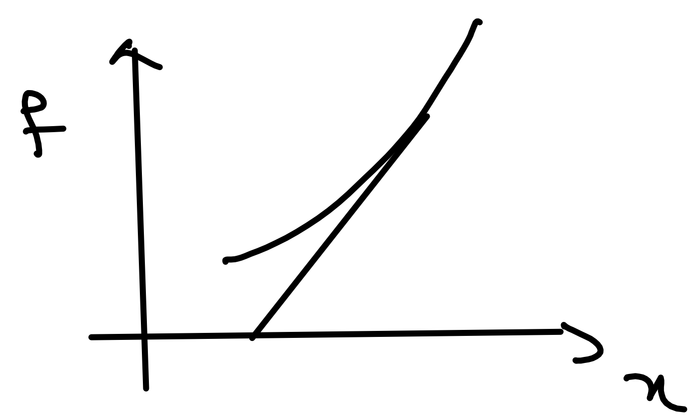
$$\begin{cases} \bar{A}_x = A_x \\ \bar{A}_y = A_y \cos\phi + A_z \sin\phi \\ \bar{A}_z = -A_y \sin\phi + A_z \cos\phi \end{cases}$$

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\begin{aligned} \bar{A}_y = A\cos\bar{\theta} &= A\cos(\theta - \phi) \\ &= A(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ &= A_y\cos\phi + A_z\sin\phi \end{aligned}$$

Gradient

Derivative of a fn. tells us how fast the fn varies with respective coordinate



$$\underline{\underline{df}} = \left(\frac{df}{dx}\right) dx$$

↳ Slope of the graph.

Say, $T = T(x, y, z)$

$$\Rightarrow dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

↳ tells us how T varies when we alter all 3 variables by infinitesimal amounts dx, dy, dz.

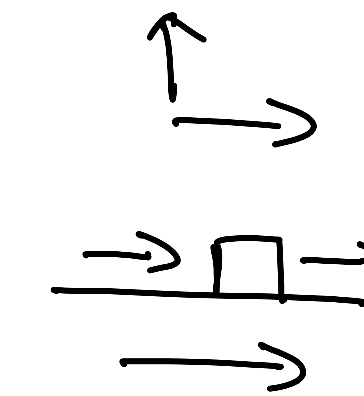
Rewrite dT as:

$$dT = \left(\hat{x}\,\frac{\partial T}{\partial x} + \hat{y}\,\frac{\partial T}{\partial y} + \hat{z}\,\frac{\partial T}{\partial z}\right)\cdot\left(\hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz\right)$$

$$= \underbrace{\left(\vec{\nabla} T\right)}_{\text{Gradient of } T.}\cdot\left(d\vec{l}\right)$$

We can write,

$$dT = |\vec{\nabla} T|\,|d\vec{l}|\cos\theta$$



Then for fixed $|d\vec{l}|$, dT is maximum when $\theta = 0$ or $\cos\theta = 1$

→ The magnitude $|\vec{\nabla} T|$ gives the slope along this maximal direction.

→ $\vec{\nabla} T$ is directed along the direction of maximum increase of T.

Link to the Recording:

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Del Operator

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

↳ Operator, only has meaning when it operates on something.

Operations

$\vec{\nabla} T \equiv$ Gradient

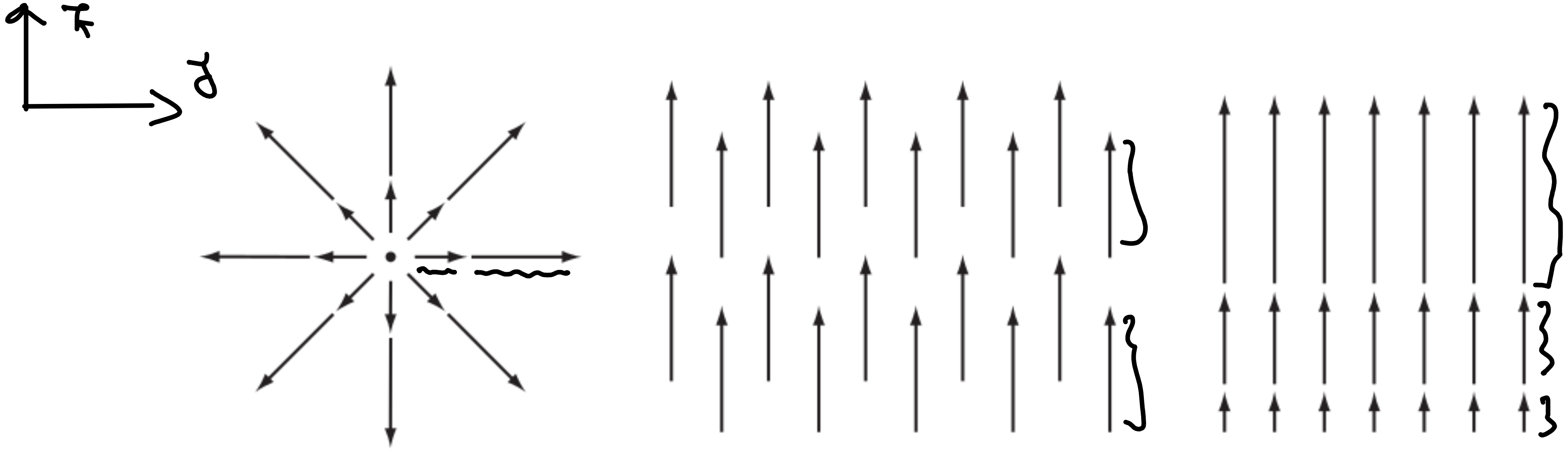
$\vec{\nabla} \cdot \vec{A} \equiv$ Divergence

$\vec{\nabla} \times \vec{A} \equiv$ Curl

Divergence

$\vec{\nabla} \cdot \vec{A} \equiv$ Measurement of how the vector spreads out

D. J. Griffiths



(a) (b) (c)

$\vec{A} = y\,\hat{y} + z\,\hat{z}$

$\vec{\nabla}\cdot\vec{A} \neq 0$

$\vec{A} = c\,\hat{z}$

$\vec{\nabla}\cdot\vec{A} = 0$

$\vec{A} = z\,\hat{z}$

$\vec{\nabla}\cdot\vec{A} \neq 0$

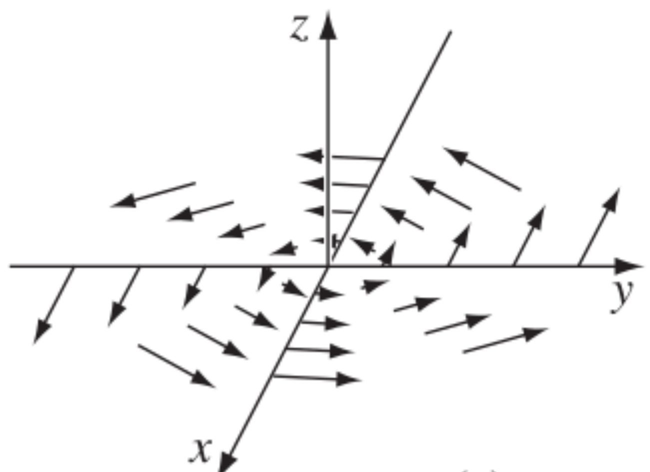
$$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

↳ Scalar quantity.

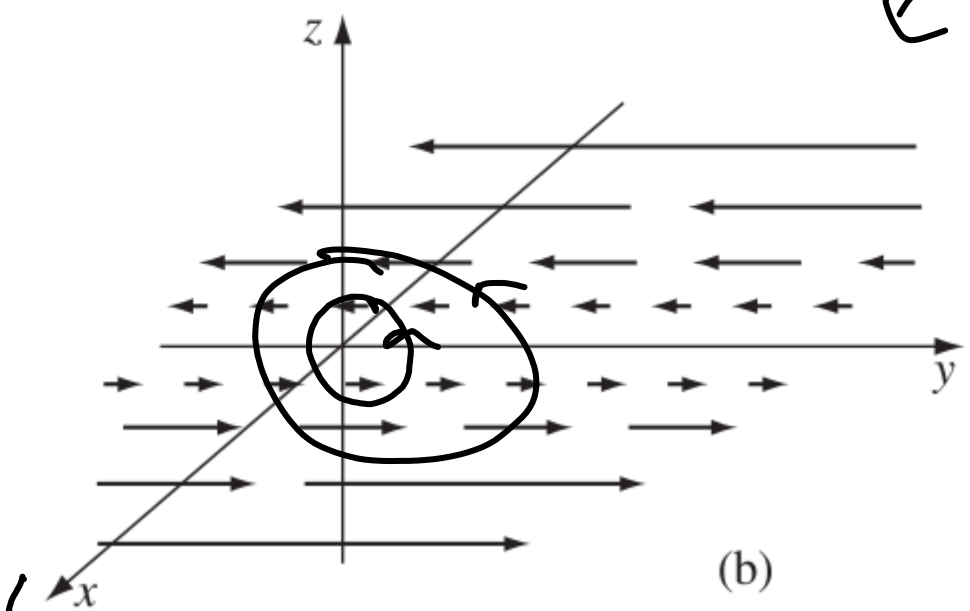
Curl

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \\ 0 & x & 0 \end{vmatrix} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

→ Measurement of how much the vector swirls around a point.



(a)



(b)

$\vec{A} = -y\hat{x} + x\hat{y}$

$\vec{\nabla} \times \vec{A} \neq 0$

$\vec{A} = x\hat{y}$

$\vec{\nabla} \times \vec{A} \neq 0$

Product Rules

$$\vec{\nabla}(f+g) = \vec{\nabla}f + \vec{\nabla}g$$

$$\vec{\nabla}\cdot(\vec{A}+\vec{B}) = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{B}$$

$$\vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B}$$

$$\rightarrow \quad \vec{\nabla}(kf) = k\vec{\nabla}f + f\vec{\nabla}k$$

$$\rightarrow \quad \vec{\nabla}\cdot(k\vec{A}) = k(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}k)$$

$$\vec{\nabla}\times(k\vec{A}) = k(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla}k)$$

Accordingly, there are *six* product rules, two for gradients:

(i) $$\nabla(fg) = f\nabla g + g\nabla f,$$

(ii) $$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A},$$

two for divergences:

(iii) $$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f),$$

(iv) $$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}), \leftarrow$$

and two for curls:

(v) $$\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f),$$

(vi) $$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}).$$

Second Derivative

1. $\vec{\nabla} \cdot (\vec{\nabla} T) \equiv$ Divergence of a gradient
2. $\vec{\nabla} \times (\vec{\nabla} T) \equiv$ Curl of a gradient
3. $\vec{\nabla} (\vec{\nabla} \cdot \vec{A}) \equiv$ Gradient of a divergence
4. $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) \equiv$ Divergence of a curl
5. $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) \equiv$ Curl of a curl

① $\vec{\nabla} \cdot (\vec{\nabla} T) = \nabla^2 T$

$$= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

↳ Laplacian of a scalar.

② $\vec{\nabla} \times (\vec{\nabla} T) = 0$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{\partial T}{\partial x} & \dfrac{\partial T}{\partial y} & \dfrac{\partial T}{\partial z} \end{vmatrix} = \hat{x}\left(\cancel{\frac{\partial^2 T}{\partial y \partial z}} - \cancel{\frac{\partial^2 T}{\partial z \partial y}} \right) + \hat{y}\left(\cancel{\frac{\partial^2 T}{\partial x \partial z}} - \cancel{\frac{\partial^2 T}{\partial z \partial x}} \right) + \hat{z}\left(\cancel{\frac{\partial^2 T}{\partial x \partial y}} - \cancel{\frac{\partial^2 T}{\partial y \partial x}} \right)$$

$$= \underline{0}\,\hat{x} + \underline{0}\,\hat{y} + \underline{0}\,\hat{z}$$

$$= \text{Null vector.}$$

③ $\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{A}}_{=T})$ (not important)

$$= \hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}$$

(4) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \frac{\partial}{\partial x}\left(\cancel{\frac{\partial A_z}{\partial y}} - \cancel{\frac{\partial A_y}{\partial z}}\right) + \frac{\partial}{\partial y}\left(\cancel{\frac{\partial A_x}{\partial z}} - \cancel{\frac{\partial A_z}{\partial x}}\right) + \frac{\partial}{\partial z}\left(\cancel{\frac{\partial A_x}{\partial y}} - \cancel{\frac{\partial A_y}{\partial x}}\right)$$

$$= 0$$

(5) $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

$\equiv$ (3)

$$\nabla^2 \vec{A} = \hat{x}(\nabla^2 A_x) + \hat{y}(\nabla^2 A_y) + \hat{z}(\nabla^2 A_z)$$

Link to the recording:

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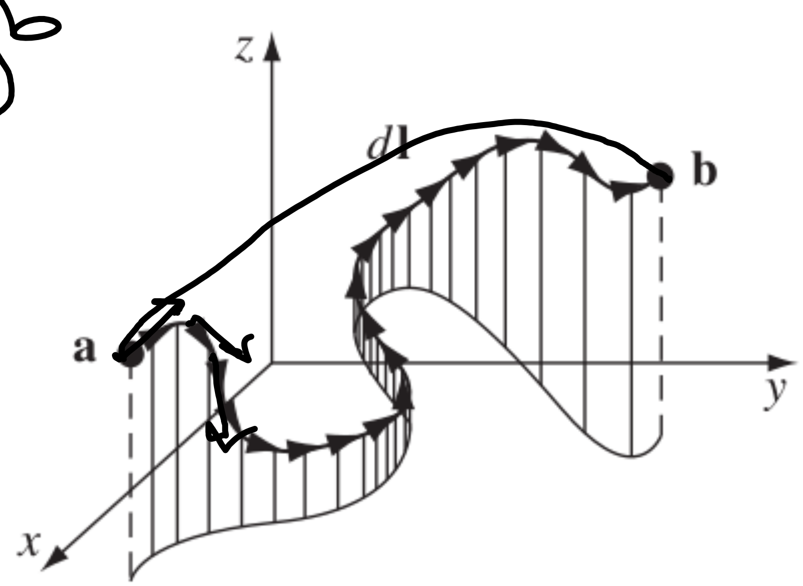
Integral Calculus

Line Integral

Integrals of form: $\int_a^b \vec{v}\cdot \underline{\underline{d\vec{l}}}$ ←



→ integration to be carried out along the shown path from A to B.

(i) Closed integral : $\oint \vec{v}\cdot d\vec{l}$

$$= \int_a^b \underline{\vec{v}\cdot d\vec{l}} + \int_b^a \underline{\vec{v}\cdot d\vec{l}}$$

(i) One simple example : $W = \int \vec{F}\cdot d\vec{l}$ (= work done)

* Usually the line integral depends on the path taken, but for a class of vectors the integral only depends on the end points.

→ A force that has this property is called "conservative force."

For such vectors,

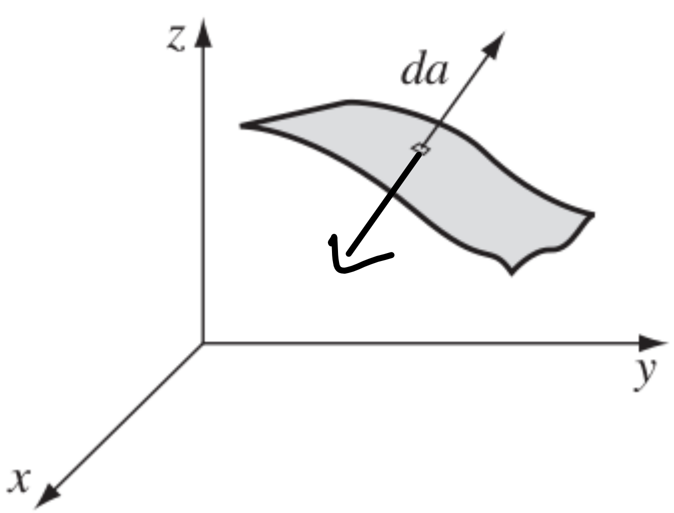
$$\oint = \int_a^b (\vec{F} \cdot d\vec{l}) + \int_b^a (\vec{F} \cdot d\vec{l})$$

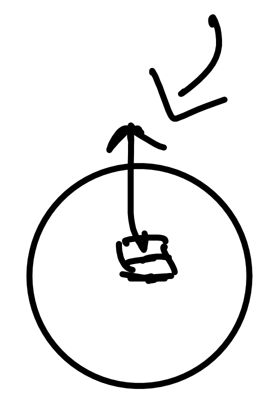
$$= \int_a^b (\vec{F} \cdot d\vec{l}) - \int_a^b (\vec{F} \cdot d\vec{l}) = 0$$

Surface Integral

Integral is of form: $\int_S \vec{v} \cdot d\vec{a}$ ←

closed surface $\equiv \oint_S \vec{v} \cdot d\vec{a}$





$d\vec{a} \equiv$ infinitesimal surface area element

→ integral is taken over a specified surface area.

⊕ There are two directions perpendicular to any surface.

→ if surface is closed $d\vec{a}$ is always outward.

Flux: $\vec{v}$ describes flow of a fluid.
(mass per unit area)

Then $\int \vec{v} \cdot \underline{\underline{d\vec{a}}}$ $\Rightarrow$ total mass per unit time passing through the surface.

$\Rightarrow$ **Flux.**

Volume Integral

Integral is of the form: $\int_V T \, d\tau$

$d\tau \equiv$ infinitesimal volume element

$T \equiv$ Any scalar.

In Cartesian coordinate system
$d\tau = dx \, dy \, dz$

Ex: if $T \equiv$ density of some substance,

$\int T \, d\tau \equiv$ Total mass.

Fundamental theorem for gradients

Let us consider a line integral.

Here the vector, $\vec{v} = \vec{\nabla} T$

Then $\underline{dT} = (\vec{\nabla} T) \cdot \underline{d\vec{l}}$

$\rightarrow$ many infinitesimal steps taken from a to b.

Total change in 'T'

$$\int_a^b dT = \int_a^b (\vec{\nabla} T) \cdot d\vec{l} = T_b - T_a$$

(independent of path)

Corollary

(1) $$\int_a^b (\vec{\nabla} T) \cdot d\vec{l}$$ is independent of path

(2) $$\oint (\vec{\nabla} T) \cdot d\vec{l} = 0$$

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<u>Fundamental Theorem</u> states that the integral of a derivative over a region is equal to the value of the function at the boundary.

⊛ <u>fundamental Theorem of gradient</u>

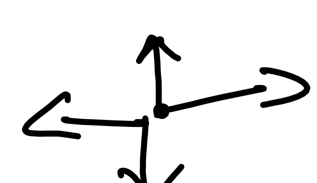
$$\int_a^b (\vec{\nabla} T) \cdot d\vec{l} = T(b) - T(a)$$

$$\Leftrightarrow \oint (\vec{\nabla} T) \cdot d\vec{l} = 0$$

⊛ <u>Fundemental Theorem of Divergence</u>

Statement : $\int_V (\vec{\nabla} \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$

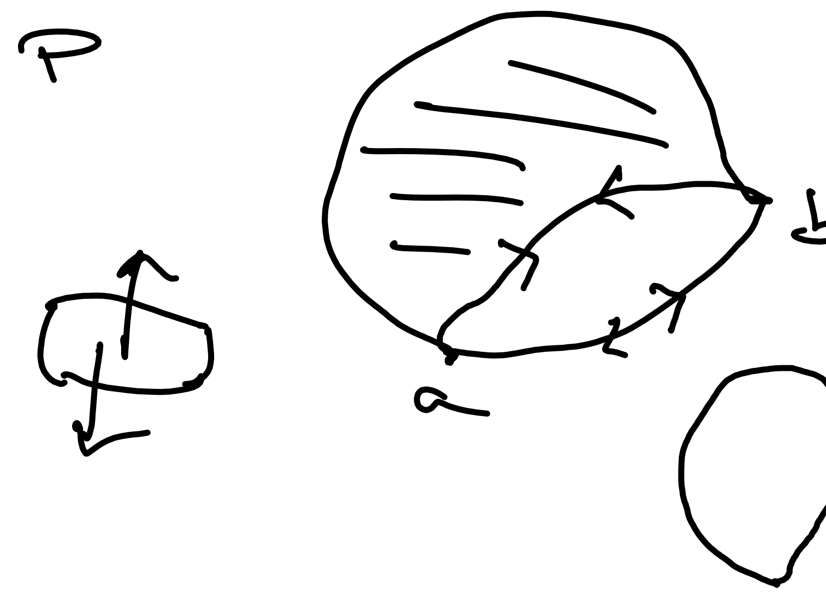
≡ Gauss' Theorem / Divergence theorem.

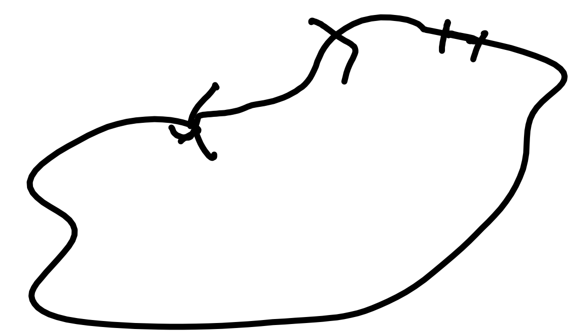


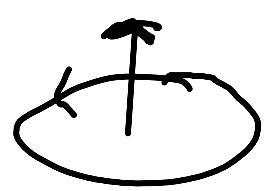
⊗ Fundamental theorem for curls

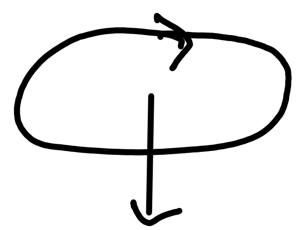
Statement: $\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint_P \vec{v} \cdot d\vec{l}$

$\equiv$ Stoke's theorem.









→ The consistency (of sign) of Stoke's Theorem is given by the right-hand rule.

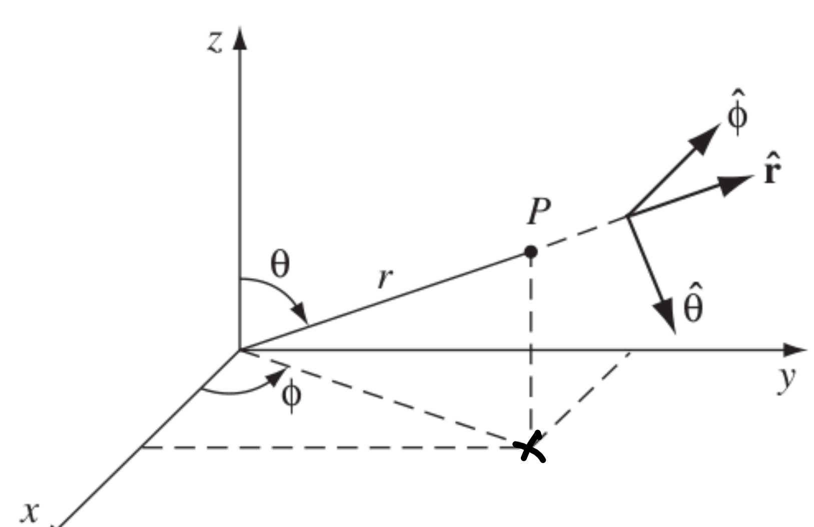
⊗ Stoke's theorem states that $\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a}$ is equal to the line integral of $\vec{v}$ around the boundary.

<u>Corollary</u>: ⊛ $\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a}$ depends only on the boundary line, not on the particular surface chosen.

⊛ $\oint_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = 0$ for any closed surface since the boundary shrinks to a single point and line integral vanishes.

Curvilinear Coordinates

⊛ Spherical-Polar coordinates



Three coordinates ≡ (r, θ, ϕ)

$r \to$ Distance from origin

$\theta \to$ Polar angle $\equiv$ angle measured from z - axis.

$\phi \to$ Azimuthal angle $\equiv$ angle around from x-axis.

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

⊗ Unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ constitute an orthogonal basis just like $\hat{x}, \hat{y}, \hat{z}$

Any vector can be written as

$$\vec{A} = \underset{\text{radial}}{\underset{\downarrow}{A_r}} \hat{r} + \underset{\text{Polar}}{\underset{\downarrow}{A_\theta}} \hat{\theta} + \underset{\text{azimuthal}}{\underset{\downarrow}{A_\phi}} \hat{\phi}$$

The unit vectors:

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

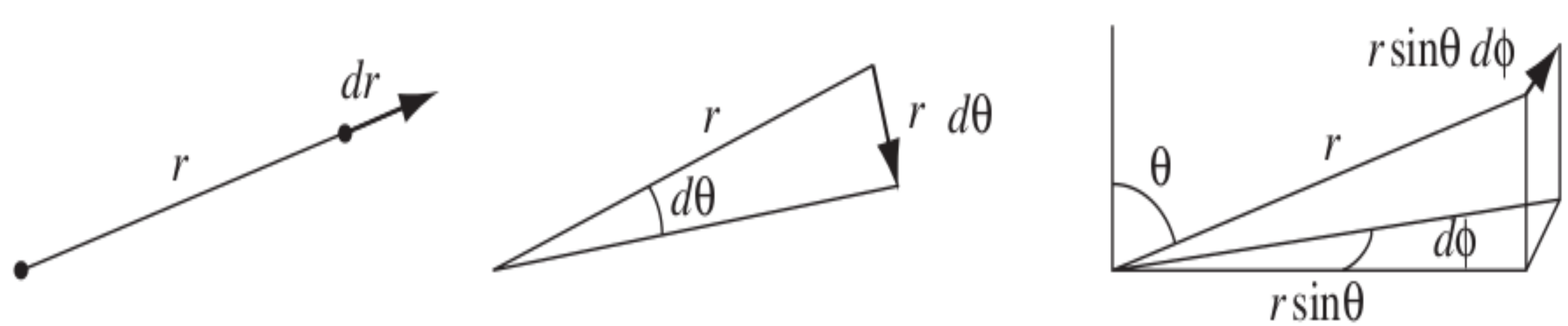
$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

to get:

$$\vec{r} = r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$$

$$\hat{r} = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|}, \quad \hat{\theta} = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|}, \quad \hat{\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|}$$

⊗ Infinitesimal displacement along $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$



$dl_r = dr$

$dl_\theta = r\, d\theta$

$dl_\phi = r \sin\theta\, d\phi$

→ Infinitesimal displacement

$$d\vec{l} = dr\, \hat{r} + r\, d\theta\, \hat{\theta} + r \sin\theta\, d\phi\, \hat{\phi}$$

→ Infinitesimal volume

$$d\tau = dl_r\, dl_\theta\, dl_\phi$$

$$= r^2 \sin\theta\, dr\, d\theta\, d\phi$$

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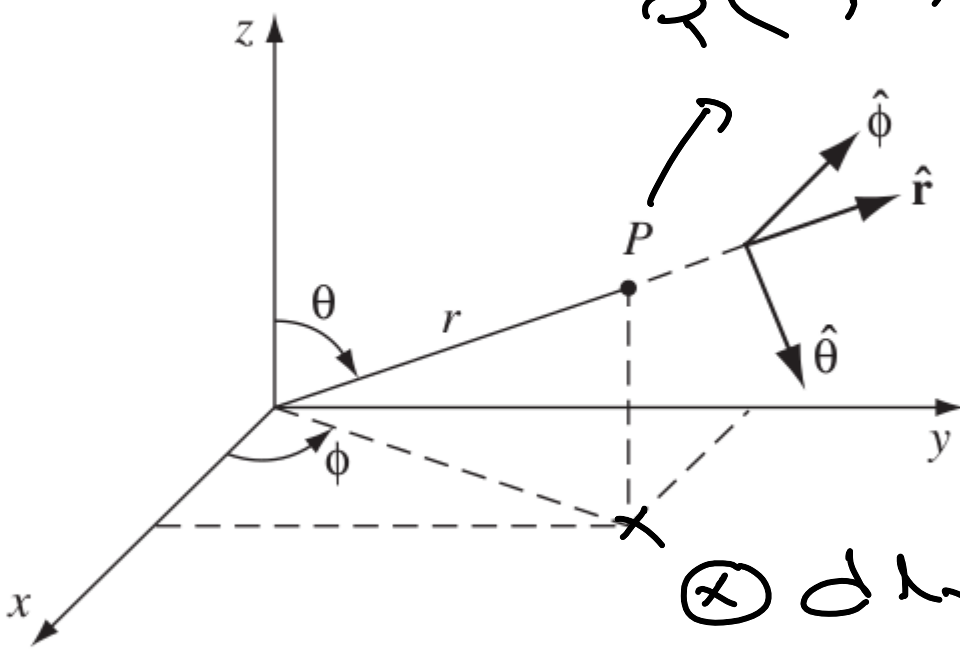
Spherical polar coordinates

$P(r,\theta,\phi)$



$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$\circledast$ $\vec{A} = A_r\,\hat{r} + A_\theta\,\hat{\theta} + A_\phi\,\hat{\phi}$

$\circledast$ $dl_r = dr,\quad dl_\theta = r\,d\theta,\quad dl_\phi = r\sin\theta\,d\phi$

→ Infinitesimal displacement:

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

→ Infinitesimal volume:

$$d\tau = dl_r\,dl_\theta\,dl_\phi$$

$$= r^2\sin\theta\,\underline{dr}\,\underline{d\theta}\,\underline{d\phi}$$

$$V = \int d\tau = \int_0^R\int_0^\pi\int_0^{2\pi} r^2\sin\theta\,dr\,d\theta\,d\phi = \underline{\frac{4}{3}\pi R^3}$$

<u>Limits</u>

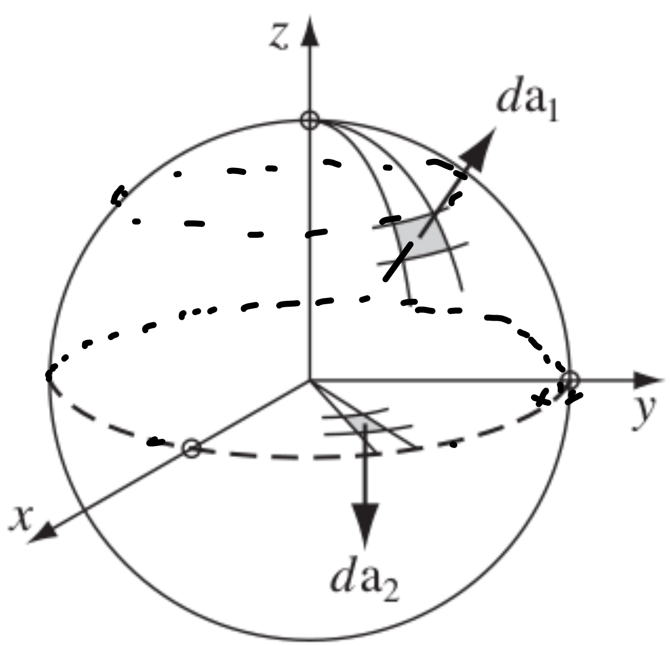
$r \equiv$ Ranging from 0 to ∞

$\theta \equiv$ Ranging from 0 to π

$\phi \equiv$ Ranging from 0 to 2π

→ Infinitesimal surface

→ Depends on geometry.



⊛ A surface element on the outer surface :

$$\underline{\underline{d\vec{a}_1}} = \underline{dl_\theta}\,\underline{dl_\phi}\,\hat{r}$$

$$= r^2 \sin\theta\, d\theta\, d\phi\ \hat{r}$$

⊛ Surface lies on the xy plane.

→ $\theta \equiv$ const.

$$d\vec{a}_2 = dl_r\, dl_\phi\, \hat{\theta} = r\, dr\, d\phi\ \hat{\theta}$$

(x) Now, we can write vector derivatives:

$$\vec{\nabla} T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial T}{\partial \theta}\left(\frac{\partial \theta}{\partial x}\right) + \frac{\partial T}{\partial \phi}\left(\frac{\partial \phi}{\partial x}\right)$$

⋮

Gradient:

$$\underline{\underline{\nabla T}} = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}. \tag{1.70}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}. \tag{1.71}$$

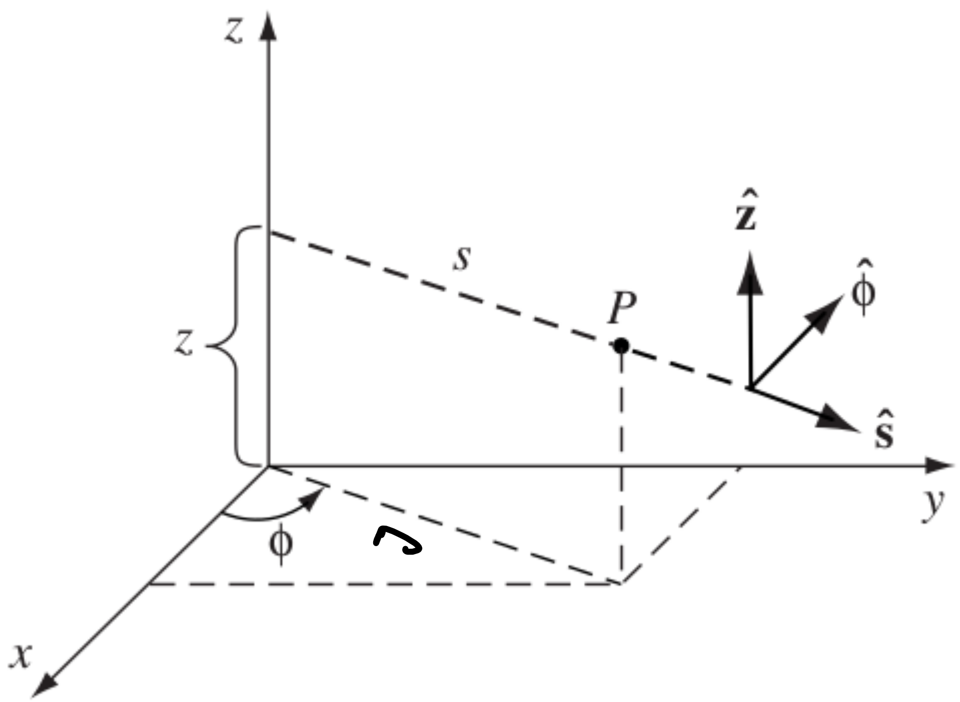
Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}}$$
$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}. \tag{1.72}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}. \tag{1.73}$$

Cylindrical Coordinates



$$P \equiv P(s, \phi, z)$$

$\phi \equiv$ Same as spherical polar

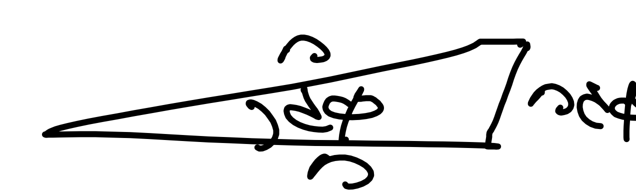
$z \equiv$ Same as Cartesian

$s \equiv$ Distance from z-axis.

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases}$$

$$\left.\begin{aligned} \hat{s} &= \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \\ \hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \\ \hat{z} &= \hat{z} \end{aligned}\right\}$$

⊕ Infinitesimal displacements:



$$dl_s = ds$$

$$dl_\phi = s\,d\phi \quad , \quad dl_z = dz.$$

$\Rightarrow d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$

⊕ Volume element, $d\tau = s\,ds\,d\phi\,dz$

→ Ranges: $s \equiv$ Ranging from 0 to ∞

$\phi \equiv$ Ranging from 0 to 2π

$z \equiv$ Ranging from $-\infty$ to $+\infty$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\,\hat{\mathbf{z}}. \tag{1.79}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}. \tag{1.80}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}. \tag{1.81}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}. \tag{1.82}$$

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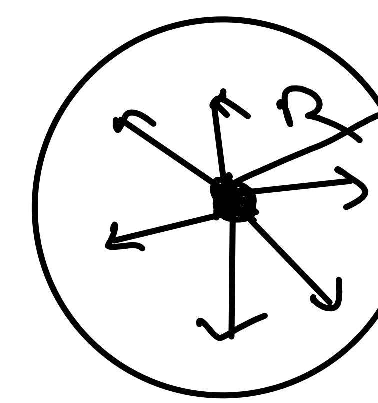
Volume integral

Volume integral : $\int_V T \, d\tau$

Fundamental th : $\int_V (\vec{\nabla} \cdot \vec{v}) \, d\tau = \oint_S \vec{v} \cdot d\vec{a}$

$\int \vec{v} \cdot d\vec{a} =$ flux.



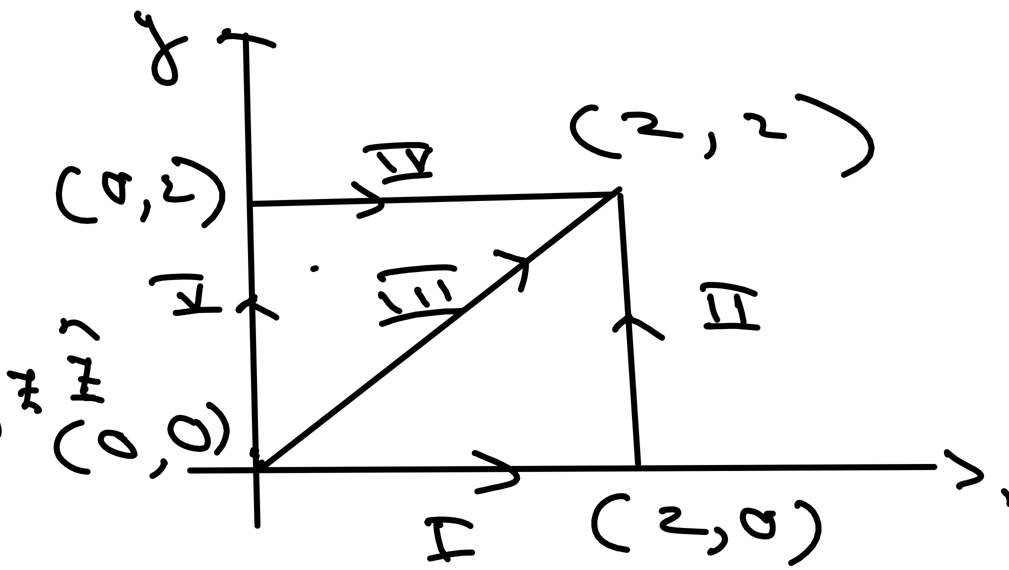
$\int d\tau = \int_0^{l} dx \int_0^{l} dy \int_0^{l} dz$

$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$\int_V d\tau = \int_0^{R} \int_0^{\pi} \int_0^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi = \frac{4}{3}\pi R^3$

Line integral

$$\int (\vec{F} \cdot d\vec{l})$$



$$\vec{F} = x^2 \hat{x} + y^2 x \hat{y} + xyz \hat{z}$$

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

Along I $\quad d\vec{l} = dx\,\hat{x}$

$$\vec{F} \cdot d\vec{l} = x^2\, dx$$

$$\int_{I} x^2\, dx = \int_0^2 x^2\, dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$$

Along II $\quad d\vec{l} = dy\,\hat{y}$

$$\vec{F} \cdot d\vec{l} = y^2 x\, dy$$

$$\int_{II} y^2 x\, dy = \int_0^2 2y^2\, dy = 2 \left.\frac{y^3}{3}\right|_0^2 = \frac{16}{3}$$

$\mathrm{I} + \mathrm{II}$ $\quad \int = \frac{24}{3} = 8$

$y = 2x$

$dy = 2dx$

III $\quad d\vec{l} = dx\,\hat{x} + dy\,\hat{y}$

$\vec{F} \cdot d\vec{l} = x^2\,dx + y^2 x\,dy$

$\begin{cases} (0,0) \\ \downarrow \\ (2,2) \end{cases}$

$\int (x^2\,dx + y^2 x\,dy)$

$\cancel{= x^2\,dx}$

$\cancel{+ 4x^3 \cdot 2\,dx}$

$\cancel{2x^3\,dx + 8x^3\,dx}$

$y = mx + \cancel{c}^{\,0}$

$= \int_0^2 (x^2\,dx + x^3\,dx)$

$y = x$ $\quad = \frac{y_2 - y_1}{x_2 - x_1}\,x$

$= \left.\frac{x^3}{3}\right|_0^2 + \left.\frac{x^4}{4}\right|_0^2$

$dy = dx$ $\quad = x$

$= \frac{8}{3} + \frac{16}{4}$

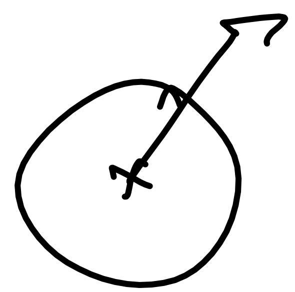
$= \frac{8}{3} + 4$

Surface integral

On the surface of a sphere,

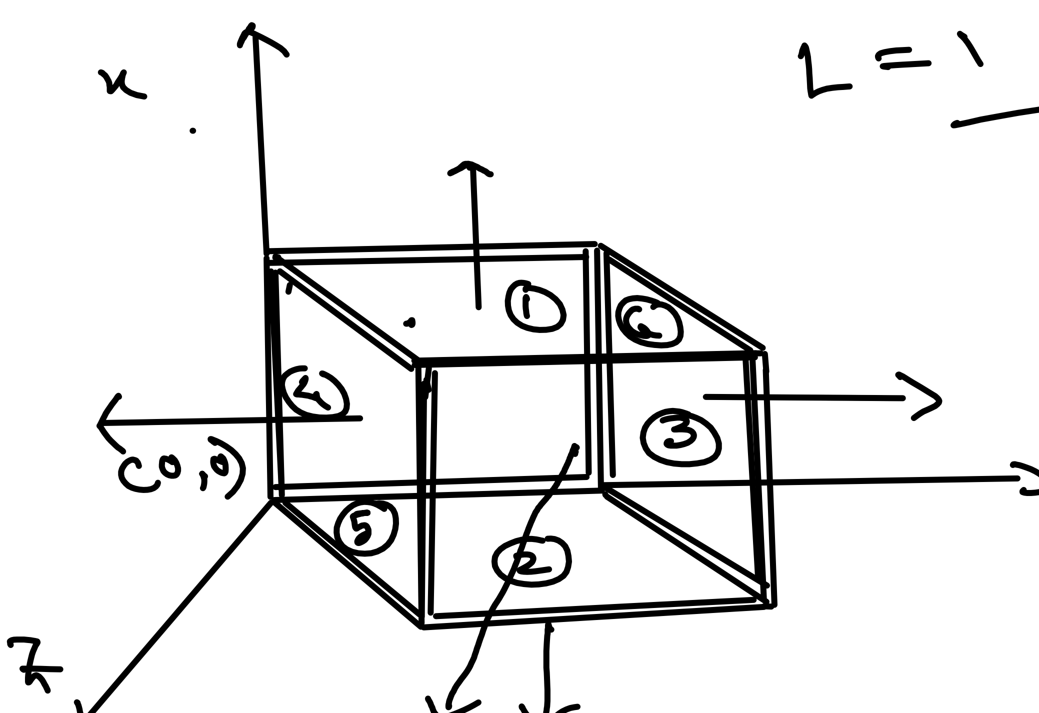
$$d\vec{a} = dl_\theta \, dl_\phi \, \hat{r}$$

$$= r^2 \sin\theta \, d\theta \, d\phi$$



$$\vec{F} = \frac{A}{r^2} \hat{r}$$

$$\int \vec{F} \cdot d\vec{a} = \int \frac{A}{r^2} r^2 \sin\theta \, d\theta \, d\phi$$



L = 1 unit

$$\vec{A} = x^2 \hat{x} + y^2 x \hat{y} + xyz \hat{z}$$

$$d\vec{a}_1 = dy \, dz \, \hat{x} \quad (x = 1)$$

$$d\vec{a}_2 = - dy \, dz \, \hat{x} \quad (x = 0)$$

$$d\vec{a}_3 = dx \, dy \, \hat{z} \quad (z = 1)$$

$$\oint \vec{A} \cdot d\vec{a}$$

$$= \int \vec{A} \cdot d\vec{a}_1 + \int \vec{A} \cdot d\vec{a}_2 + \int A \cdot d\vec{a}_3 + \int \vec{A} \cdot d\vec{a}_4 + \int \vec{A} \cdot d\vec{a}_5 + \int \vec{A} \cdot d\vec{a}_6$$

$$d\vec{a}_4 = -dx\, dy\, \hat{z} \quad (z=0)$$

$$d\vec{a}_5 = \underline{-dx\, dz\, \hat{y}} \quad (y=0)$$

$$d\vec{a}_6 = dx\, dz\, \hat{y} \quad (y=1)$$

$$\int \vec{A} \cdot d\vec{a}_1 = \int x^2\, dy\, dz = 1 \int_0^L dy \int_0^L dz$$

$$= L^2$$

$$\int \vec{A} \cdot d\vec{a}_5 = -\int y^2 x\, dx\, dz = 0$$

$$\left(\oint_S \vec{A} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{A})\, d\tau \right)$$

$$\int d\tau = \int_0^L dx \int_0^L dy \int_0^L dz$$

$$\vec{\nabla} \cdot \vec{A} = 2x + 2yx + xy$$

$$\vec{A} = x^2\,\hat{x} + y^2 x\,\hat{y} + xyz\,\hat{z}$$

$$\vec{\nabla}\cdot\vec{A} = \left(\underline{\hat{x}}\frac{\partial}{\partial x} + \underline{\hat{y}}\frac{\partial}{\partial y} + \underline{\hat{z}}\frac{\partial}{\partial z}\right)\cdot\left(x^2\,\underline{\hat{x}} + y^2 x\,\hat{y} + xyz\,\underline{\hat{z}}\right) =$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2 x) + \frac{\partial}{\partial z}(xyz)$$

$$= 2x + 2yx + xy$$

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The Electric field

Electric charge is quantised. Charge of any system can be written as $q = ne$

$e \simeq 1.602 \times 10^{-19}\,C \equiv$ charge carried by one electron.

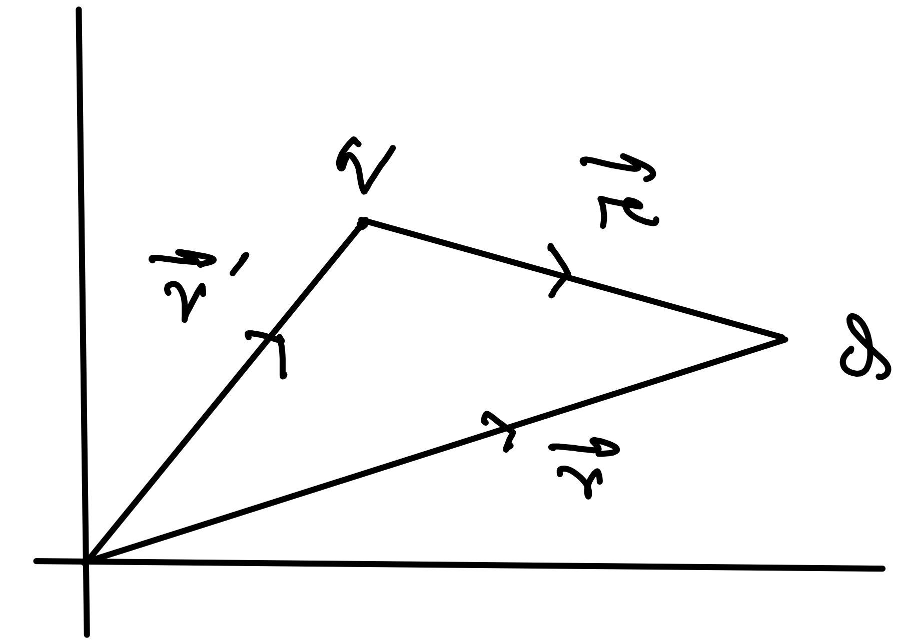
$n \equiv$ A multiple, "almost always" an integer (exception: quarks)

Electrostatics $\equiv$ Study of electric charges which are stationary.

Coulomb's law

Force on a test q due to a single

point charge q at rest and distance 'r' away:



$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q\,p}{r^2}\,\hat{r}$$

$$\vec{r} = \vec{p} - \vec{r}'$$

ϵ_0 = Permittivity of free space. In SI unit, $\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}$

Electric Field

Say, we have several such point charges $q_1, q_2, \ldots, q_n$ at distances $r_1, r_2, r_3, \ldots, r_n$ away from p.

Then the total force on p:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n$$

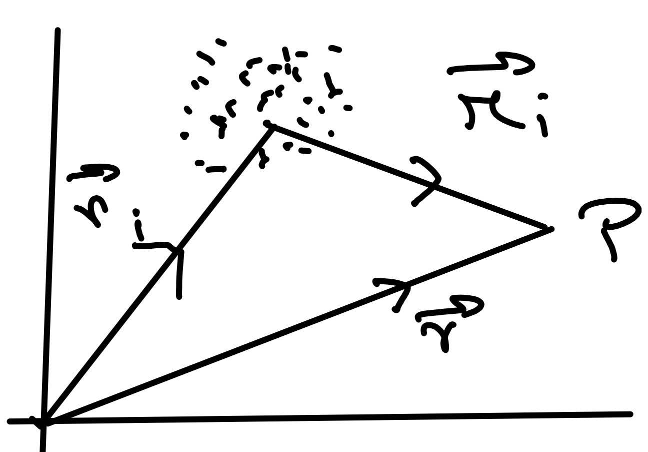
$$= \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q}{r_1^2}\hat{r}_1 + \frac{q_2 q}{r_2^2}\hat{r}_2 + \cdots + \frac{q_n q}{r_n^2}\hat{r}_n\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{q_1}{r_1^2}\hat{r}_1 + \frac{q_2}{r_2^2}\hat{r}_2 + \cdots + \frac{q_n}{r_n^2}\hat{r}_n\right)$$

$$\vec{F} = q\vec{E}$$

↳ electric field

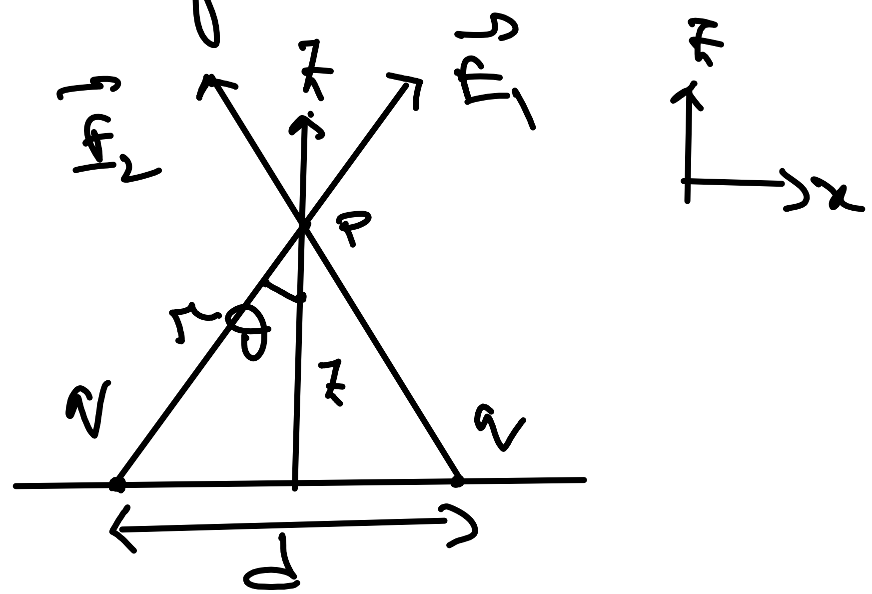
where, $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i$



↳ Electric field of the source charges.

Physically, $E(\vec{r})$ is force per unit charge.

Ex



$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

x-components cancel and z-components add up.

$$E_z = 2 \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \cos\theta$$

$$r = \left[z^2 + \left(\frac{d}{2}\right)^2 \right]^{1/2}$$

$$\cos\theta = \frac{z}{r}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{2qz}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^{3/2}} \hat{z}$$

Far away from charge, $z \gg d$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{2q}{z^2} \hat{z}$$

Continuous charge distribution

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \, dq$$

- charge spread over a line

$$dq = \lambda \, dl'$$

dl' ↳ small line element of source

λ ↳ charge per unit length

- charge spread over a surface

$$dq = \sigma \, da'$$

da' ↳ infinitesimal surface

σ ↳ charge per unit surface

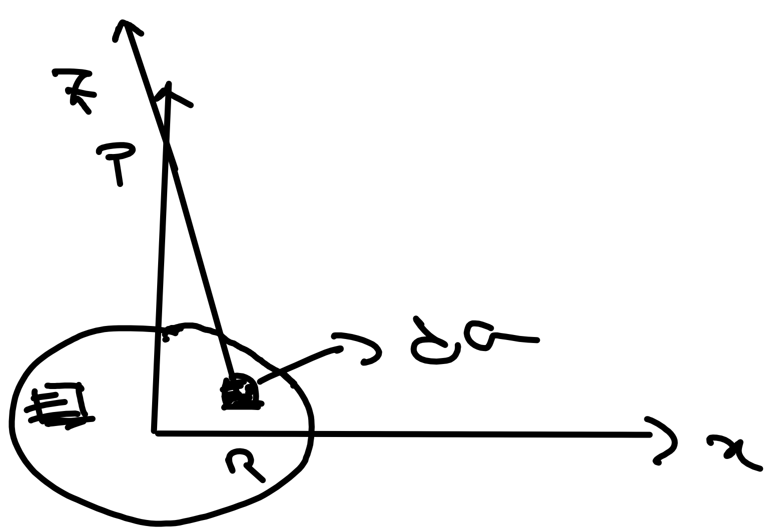
(+) charge spread over a volume

$$dq = \rho \, d\tau$$

$d\tau$ ↳ infinitesimal vol.

ρ ↳ charge density.

Ex



(i) uniform surface charge distn. (σ)

$$da = r \, dr \, d\phi$$

$$= 2\pi r \, dr$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int_0^R \frac{(2\pi r \, dr)\,\sigma}{(z^2 + r^2)^{3/2}} \, z \, \hat{z}$$

$$= \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \left(\frac{1}{z} - \frac{1}{\sqrt{R^2 + z^2}} \right) \hat{z}$$

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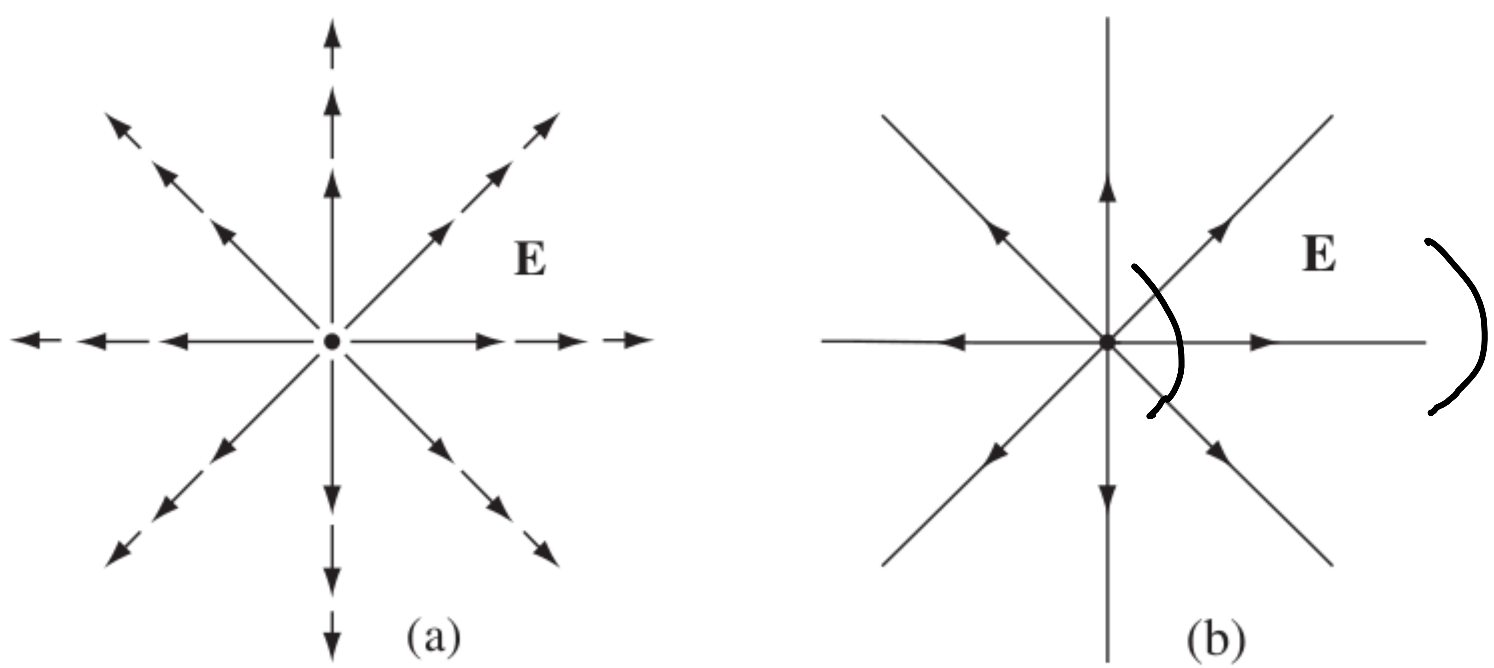
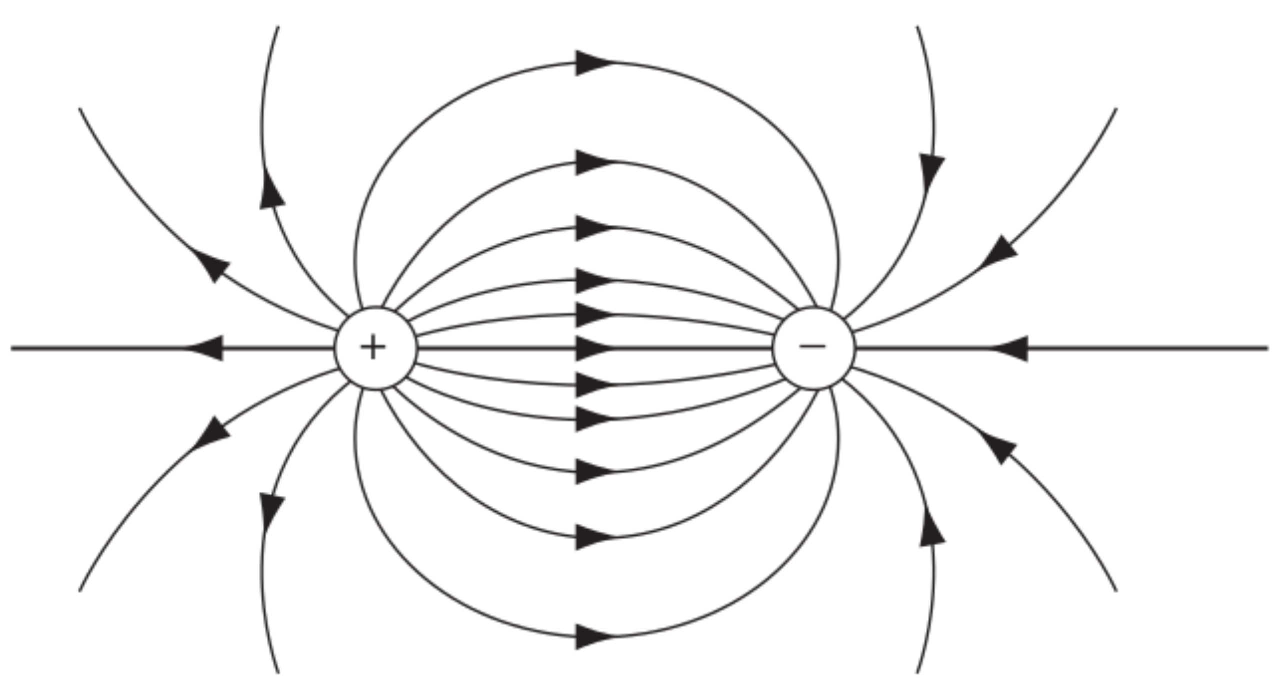
Divergence of Electric Field



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

$$\sim \frac{1}{r^2}$$

→ The field strength falls as r increases

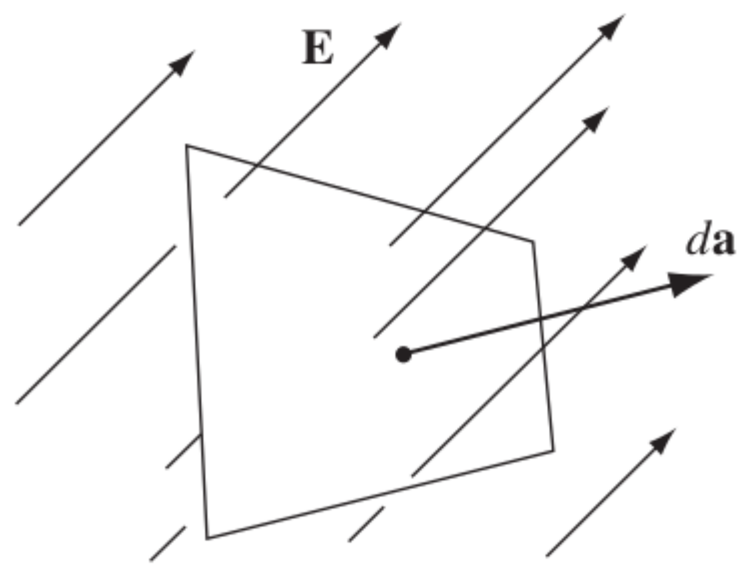


→ Field lines originate at positive charge and terminate at negative charge.

Flux of $\vec{E}$

flux through a surface

$$\Phi_E = \int_S \vec{E} \cdot d\vec{a}$$



↳ measures the "number of field lines" crossing through S.

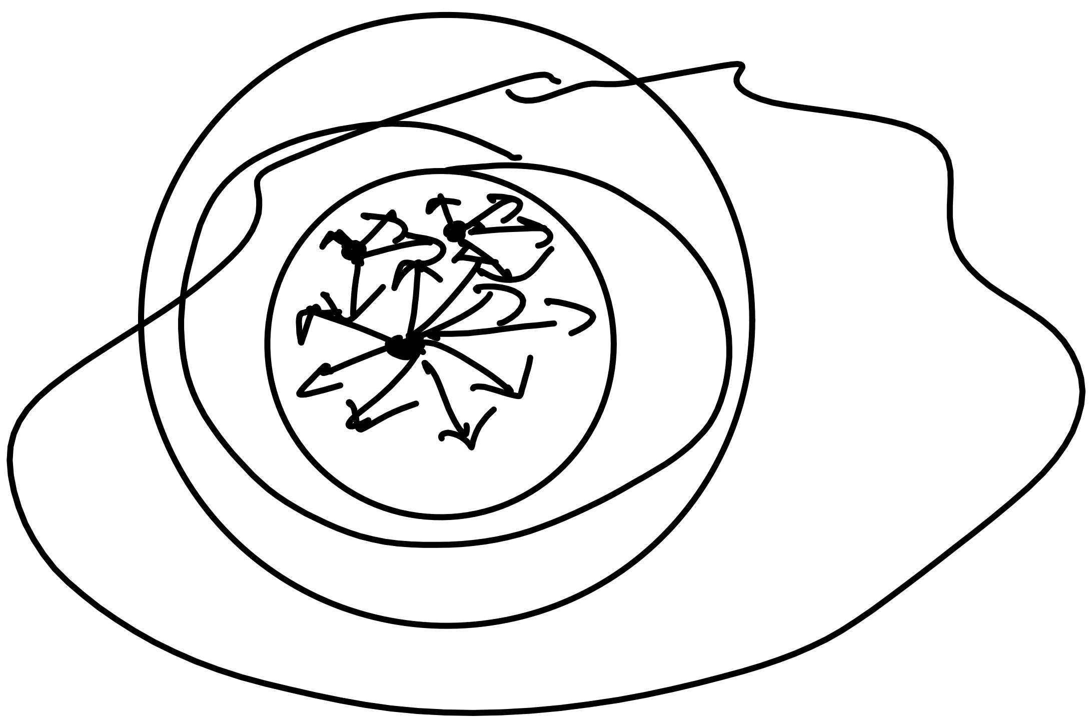
→ flux through any closed surface enclosing the charge is a measure of the electric charge.

⇒ Essence of Gauss' law

** A charge outside does not contribute since the field lines simply pass through the surface.

(x) A point charge q (at the origin) enclosed by a spherical surface.

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{4\pi\epsilon_0} \int \left(\frac{q}{r^2}\right)\hat{r} \cdot (r^2 \sin\theta \, d\theta \, d\phi)\,\hat{r}$$



$$= \frac{q}{4\pi\epsilon_0} \int_0^{\pi}\int_0^{2\pi} \sin\theta \, d\theta \, d\phi$$

$$= \frac{q}{\epsilon_0} \Rightarrow \text{independent of } r$$

→ The number of field lines crossing does not depend on the radius of the sphere.

→ the surface can be of any shape.

(x) flux through any surface enclosing the charge $= \frac{q}{\epsilon_0}$

→ If we have a collection of 'n' charges,

$$\vec{E} = \sum_{i=1}^{n} \vec{E}_i$$

flux through a surface enclosing all charges

$$\oint \vec{E} \cdot d\vec{a} = \sum_{i=1}^{n} \oint \vec{E}_i \cdot d\vec{a}$$

$$= \sum_{i=1}^{n} \frac{q_i}{\epsilon_0}$$

(x) for any closed surface

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc}.$$

$\vec{E}$ → net electric field

Q_{enc} → charge enclosed

Quantitative statement of Gauss' law.

---

* We can write

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{E})\, d\tau = \frac{1}{\epsilon_0} Q_{enc}.$$

if the volume charge density of the system $= \rho$

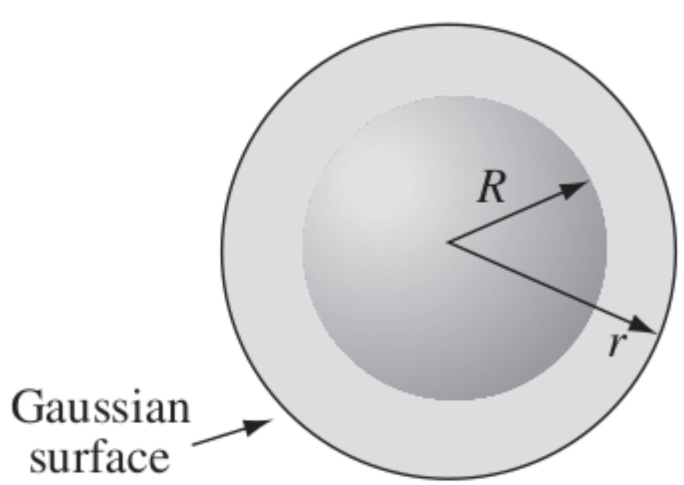
$$Q_{enc} = \int_V \rho \, \underline{d\tau}$$

Hence, $\int_V (\vec{\nabla} \cdot \vec{E})\, d\tau = \frac{1}{\epsilon_0} \int_V \rho \, d\tau$

$\Rightarrow$ $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ $\Rightarrow$ Gauss' law in differential form.

(*) When symmetry permits, this is a very simple way to calculate $\vec{E}$.

Ex:



Electric field outside a uniformly charged solid sphere of radius R and total charge q.

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc.}}{\epsilon_0} = \frac{q}{\epsilon_0}$$

$\rightarrow$ $\vec{E}$ points radially outward, so does $d\vec{a}$

$$\oint_S \vec{E} \cdot d\vec{a} = \oint_S |\vec{E}| \, da$$

$\rightarrow$ The magnitude of $\vec{E}$ is const. over the Gaussian surface.

Hence,
$$\oint_S |\vec{E}| \, da = |\vec{E}| \oint_S da$$
$$= |\vec{E}| \, 4\pi r^2$$

$$\Rightarrow |\vec{E}| \, 4\pi r^2 = \frac{q}{\epsilon_0}$$

$$\Rightarrow \vec{E} = \frac{q}{4\pi \epsilon_0 r^2} \, \hat{r}$$

⊗ Gauss' law is always true, but not always useful

→ 'ρ' has to be uniform

→ Gaussian surface has to be symmetric.

* Different kind of symmetry:
  * Spherical
  * Cylindrical
  * Planer

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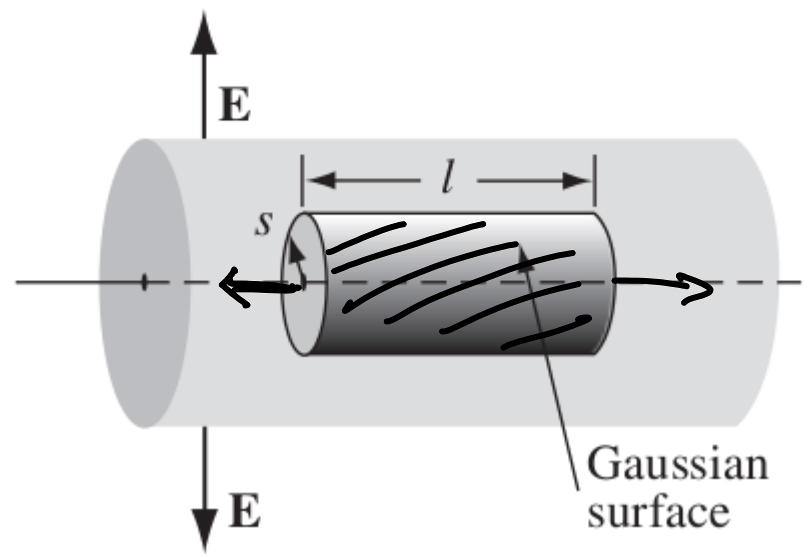
Examples of application of Gauss' law (cont'd.)

② Cylindrical symmetry:



→ Long cylinder carrying charge density, $\rho = ks$.

$\vec{E}_{inside} = ?$

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc.}}{\epsilon_0}$$

The enclosed charge:

$$Q_{enc.} = \int_V \rho \, d\tau = \int_0^l \int_0^{2\pi} \int_0^s (ks')(s' \, ds' \, d\phi \, dz)$$

$$= 2\pi k l \int_0^s s'^2 \, ds'$$

$$= \frac{2}{3} \pi k l s^3$$

Looking at the symmetry:

$$\int \vec{E} \cdot d\vec{a} = |\vec{E}| \int da = |\vec{E}|\, 2\pi s l$$

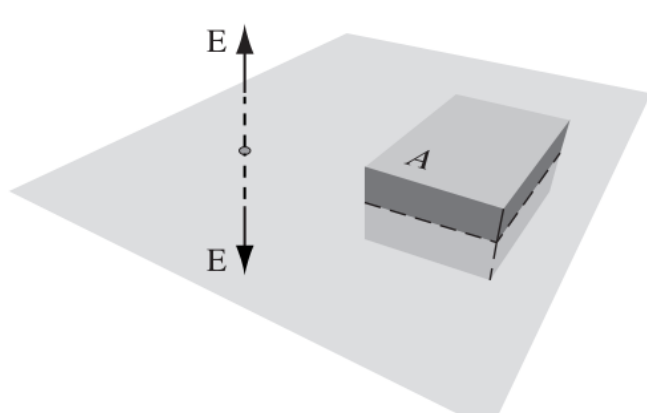
⊗ The two ends contribute nothing since

$\vec{E} \perp d\vec{a}$.

Hence, $$|\vec{E}|\, 2\pi s l = \frac{1}{\epsilon_0} \frac{2}{3} \pi k l s^3$$

$$\Rightarrow \quad \vec{E} = \frac{1}{3\epsilon_0} k s^2 \hat{s}$$

⊗ Infinite plane carrying uniform charge density σ.



→ Gaussian pillbox

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc.}}{\epsilon_0}$$

$$Q_{enc.} = \sigma A$$

→ Area of enclosed surface (equal to the area of the lid)

$\Rightarrow \vec{E}$ points outward.

from top and bottom surfaces,

$$\int \vec{E} \cdot d\vec{a} = 2A|\vec{E}|$$

$$\Rightarrow 2A|\vec{E}| = \frac{\sigma A}{\epsilon_0}$$

$$\Rightarrow \vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n}$$

$\hookrightarrow$ unit vector pointing outward from surface.

Curl of $\vec{E}$

Take a point charge at origin

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

Calculate line integral from point 'a' to 'b'

$$\int_a^b \vec{E} \cdot d\vec{l}$$

In spherical polar coordinate:

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$\Rightarrow \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}\,dr$$

Then $$\int_a^b \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

$\rightarrow$ for a closed path, $r_a = r_b$

$$\Rightarrow \oint \vec{E} \cdot d\vec{l} = 0$$

Apply Stoke's theorem: $\int_s (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = \oint \vec{E} \cdot d\vec{l}$

$\Rightarrow \vec{\nabla} \times \vec{E} = 0 \equiv$ True for only electrostatics.

Electric Potential

The line integral is independent of path:

Define a fn: $V(\vec{r}) = - \int_{O}^{r} \vec{E} \cdot d\vec{l}$

Electric potential

O → reference point

⊛ Potential difference:

$$V(b) - V(a) = - \int_{O}^{b} \vec{E} \cdot d\vec{l} + \int_{O}^{a} \vec{E} \cdot d\vec{l}$$

$$= - \int_{O}^{b} \vec{E} \cdot d\vec{l} - \int_{a}^{O} \vec{E} \cdot d\vec{l}$$

$$= - \int_a^b \vec{E} \cdot d\vec{l}$$

Use fundamental theorem of gradient

$$V(b) - V(a) = \int_a^b (\vec{\nabla} V) \cdot d\vec{l}$$

$$\Rightarrow \int_a^b (\vec{\nabla} V) \cdot d\vec{l} = - \int_a^b \vec{E} \cdot d\vec{l}$$

$\equiv$ True for any two arbitrary points 'a' and 'b'

$$\Rightarrow \underline{\vec{E} = - \vec{\nabla} V}$$

(x) Changing the reference point can change the potential.

$$V'(\vec{r}) = - \int_{O'}^{r} \vec{E} \cdot d\vec{l} = - \int_{O'}^{O} \vec{E} \cdot d\vec{l} - \int_{O}^{r} \vec{E} \cdot d\vec{l} = \underline{\underline{K + V(\vec{r})}}$$

$k \equiv$ const. independent of r.

(x) Potential diff. being a physical quantity should be independent of reference point.

$$V'(b) - V'(a) = V(b) - V(a)$$

(x) Obeys superposition rule. For a collection of charges,

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots$$

$$\Rightarrow V = V_1 + V_2 + \cdots$$

(x) Unit : $\frac{N \cdot m}{C}$ or $\frac{J}{C}$ or volt

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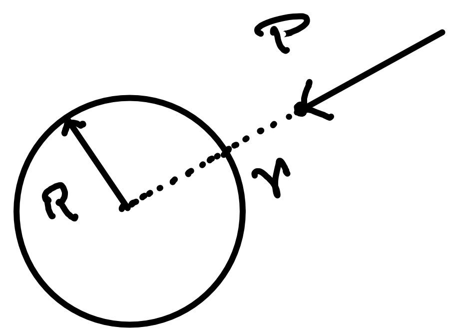
Electric potential

Ex: 

Find potential inside and outside of a spherical shell carrying a uniform charge density

$q \equiv$ Total charge

→ Reference point $(O) \equiv \infty$

From Gauss' law, field outside,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

Field inside the sphere, $\vec{E} = 0$

→ For point outside the sphere $(r \geq R)$

$$V(r) = -\int_O^r \vec{E}\cdot d\vec{l} = -\frac{1}{4\pi\epsilon_0}\int_\infty^r \frac{q}{r'^2}\,dr'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

For points inside the the sphere $(r \leq R)$

$$V(r) = -\int_{0}^{r} \vec{E} \cdot d\vec{l}$$

$$= -\frac{1}{4\pi\epsilon_0} \int_{\infty}^{R} \frac{q}{r'^2} \, dr' - \int_{R}^{r} (0) \, dr'$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{R} \equiv \text{Potential inside}$$

is non-zero although the electric field is zero.

⊛ Poisson's & Laplace's eqn.

$$\left.\begin{aligned} \vec{E} &= -\vec{\nabla} V \\ \vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \times \vec{E} &= 0 \end{aligned}\right\}$$ How do they look like in terms of V ?

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{\nabla}V) = -\nabla^2 V = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \nabla^2 V = -\frac{\rho}{\epsilon_0} \equiv \text{Poisson's eq.}$$

→ If there is no charge distribution,

$$\nabla^2 V = 0 \equiv \text{Laplace's eq.}$$

$$\vec{\nabla}\times\vec{E} = \vec{\nabla}\times(-\vec{\nabla}V) = 0$$

<u>Potential of a charge distribution</u>

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r^2}\hat{r}$$

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$\vec{E}\cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr$$

Setting the reference point at ∞, the potential for a point charge q at origin

$$V(r) = -\int_{-\infty}^{r} \frac{1}{4\pi\epsilon_0} \frac{q}{r'^2}\, dr'$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

$\mathscr{r} \equiv$ distance between q and 'r' then,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}}$$

→ From superposition principle,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i}$$

(for a collection of charges)

[Superposition: if there is a collection of charges: $\vec{E} = \vec{E}_1 + \vec{E}_2 + \dots + \vec{E}_n$

$V = V_1 + V_2 + \dots + V_n$]

→ If we have a continuous charge distribution,

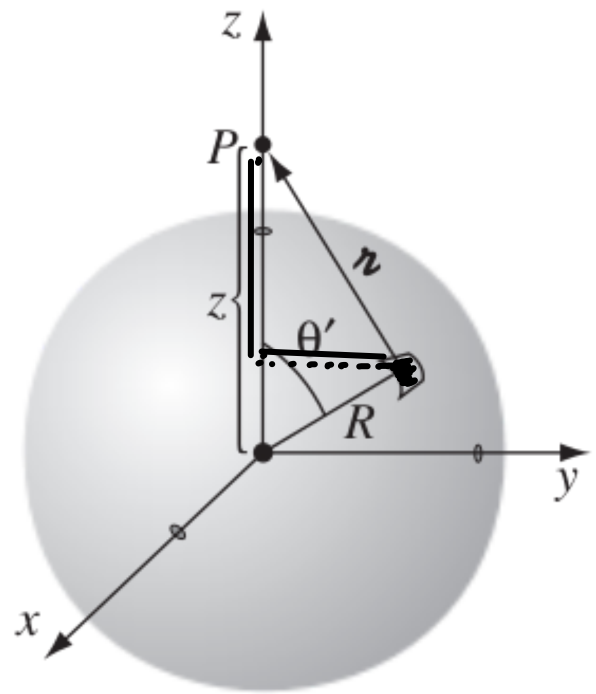
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}}\, dq$$

⊗ volume charge distribution: $dq = \rho\, d\tau'$

⊗ surface " " $dq = \sigma\, da'$

⊗ line " " $dq = \lambda\, dl'$

Ex: Potential of a uniformly charged spherical shell of radius R.



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{r}\, da'$$

$$r^2 = R^2 + z^2 - 2Rz\cos\theta'$$

(cosine law)

$$r^2 = (R\sin\theta')^2 + (z - R\cos\theta')^2$$

$$= R^2(\sin^2\theta' + \cos^2\theta') + z^2 - 2zR\cos\theta'$$

$$= R^2 + z^2 - 2Rz\cos\theta'$$

On the surface of a sphere, element of a surface area $= R^2\sin\theta'\, d\theta' d\phi$

$$\Rightarrow 4\pi\epsilon_0 V(z) = \sigma \int_0^{2\pi}\int_0^{\pi} \frac{R^2\sin\theta'\, d\theta'\, d\phi}{\left[R^2 + z^2 - 2Rz\cos\theta'\right]^{1/2}}$$

$$R^2 + z^2 - 2Rz\cos\theta' = x$$

$$\Rightarrow \quad 2Rz\sin\theta'\,d\theta' = dx$$

$$\Rightarrow 4\pi\epsilon_0 V(z) = 2\pi R^2\sigma \left(\frac{1}{Rz}\left[R^2 + z^2 - 2Rz\cos\theta'\right]^{1/2} \right)\Bigg|_0^{\pi}$$

$$= \frac{2\pi R\sigma}{z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right]$$

$$\Rightarrow \quad V(z) = \frac{R\sigma}{2\epsilon_0 z}\left((R+z) - (z-R) \right)$$

for points outside, $(z > R)$ $\quad V(z) = \dfrac{R^2\sigma}{\epsilon_0 z}$

For points inside, $\quad z < R$

$$V(z) = \frac{R\sigma}{2\epsilon_0 z}\Big[(R+z) - (R-z)\Big]$$

$$= \frac{R\sigma}{\epsilon_0}$$

$$q = 4\pi R^2 \sigma \quad \Big| \quad q = \int \sigma \, da'$$

In terms of r and q

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \qquad (r \geq R)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{R} \qquad (r \leq R)$$

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Work done to move a charge

We have a stationary configuration of charges. We want to move a test charge 'q' from point 'a' to 'b'.

Electric force on test charge q:

$$\vec{F} = q\vec{E}$$

→ The force we have we have to exert in opposition to this force $= -q\vec{E}$

$$\text{Work done, } W = \int_a^b \vec{F} \cdot d\vec{l}$$

$$= -q \int_a^b \vec{E} \cdot d\vec{l}$$

$$= q\left(V(b) - V(a)\right)$$

→ independent of the path taken.

$\Rightarrow$ Electrostatic force is "conservative".

$$W = q[V(r) - V(\infty)] = qV(r)$$

reference point $\equiv \infty$

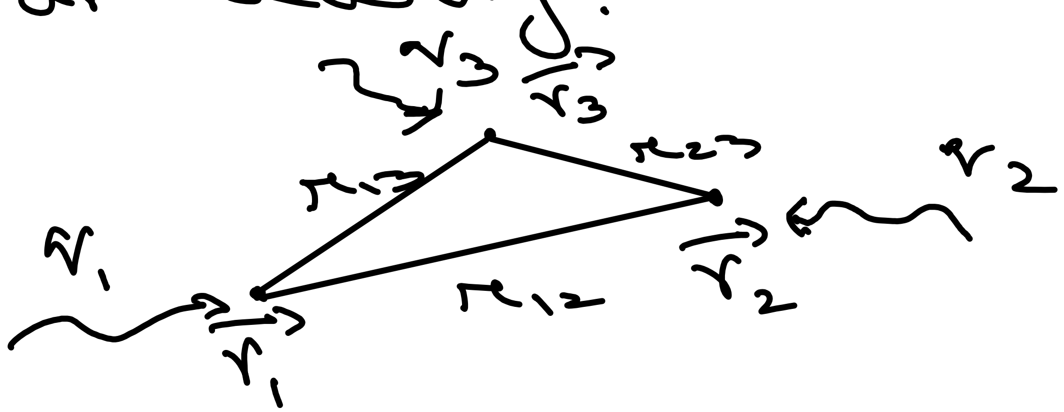
$\rightarrow$ Potential difference betn. two points

$$V(b) - V(a) = \frac{W}{q}$$

Energy of a point charge distribution

$\rightarrow$ To calculate how much work it takes to assemble a collection of point charges

We are bringing the charges one by one from far away.



$\rightarrow$ For q_1, we have to do no work since there is no electric field beforehand.

$\Rightarrow W_1 = 0$

→ in order to bring in q_2,

$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \frac{q_1}{r_{12}}$$

→ similarly, to bring in q_3,

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}} \right)$$

→ for q_4,

$$W_4 = \frac{1}{4\pi\epsilon_0} q_4 \left(\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}} \right)$$

The total work done to complete the charge configuration:

$$W = \frac{1}{4\pi\epsilon_0} \left(\underset{ij}{\frac{q_1 q_2}{r_{12}}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_3}{r_{23}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}} \right)$$

In general, for 'n' number of charges:

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{q_i q_j}{r_{ij}}$$

OR

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \frac{q_i q_j}{r_{ij}}$$

$$\Rightarrow W = \frac{1}{2} \sum_{i=1}^{n} q_i \underbrace{\left(\sum_{j\neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_{ij}} \right)}$$

↳ Potential at point $\vec{r}_i$ due to all other charges.

Hence, $W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i)$

⇒ Represents the energy stored in

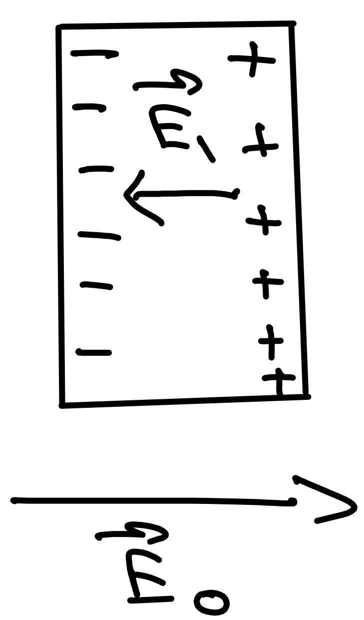
the system as potential energy.

Conductors

A perfect conductor contains unlimited number of free charges.

⊗ Basic electrostatic property:

→ $\vec{E} = 0$ inside any conductor.



A conductor is put in an external electric field. The induced charges produce an electric field $\vec{E}_1$ inside the conductor such that $\vec{E}_1$ is oriented opposite to the direc^n. of $\vec{E}_0$

Then, $\vec{E}_1$ tends to cancel $\vec{E}_0$. The charges continue to flow inside until an equilibrium is reached when $\vec{E}_1$ cancels $\vec{E}_0$ exactly inside the conductor.

$$\text{Inside}, \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \vec{E} = 0 \Rightarrow \rho = 0$$

→ Inside the conductor, charge density is zero ⇒ equal number of positive and negative charges.

⊗ Any net charge can only reside on the surface.

→ Conductors are equipotential

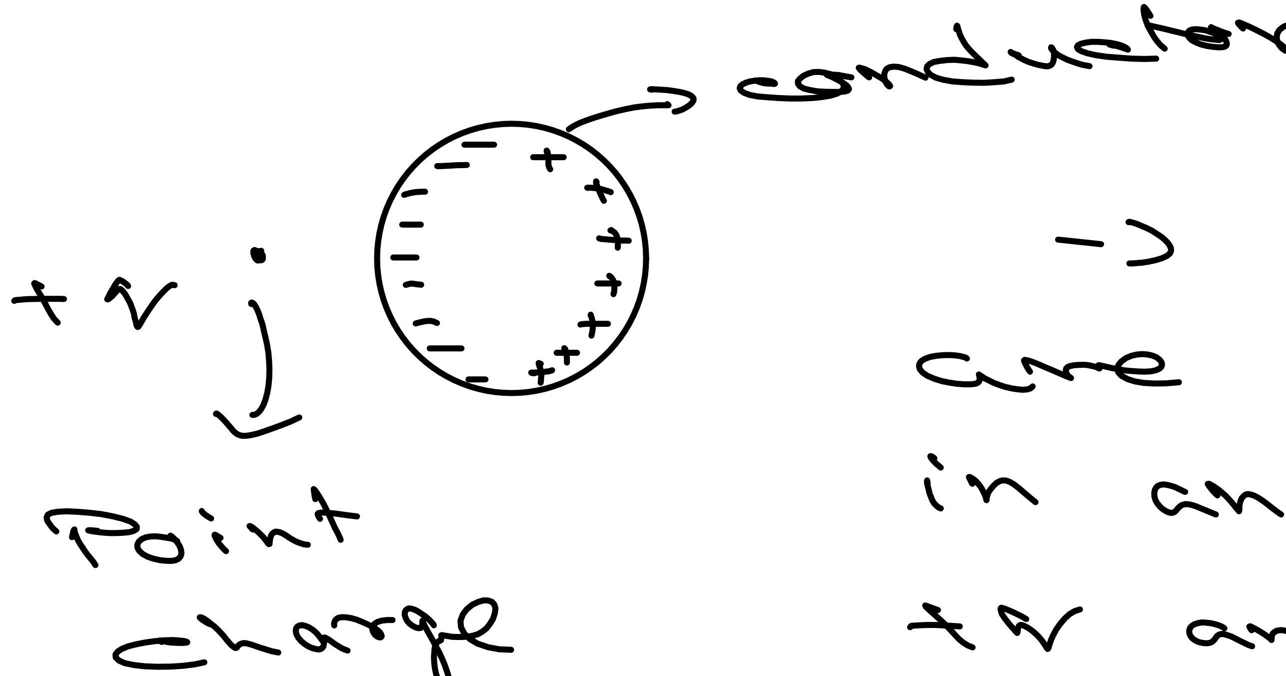
for any two points on the surface of the conductor, $V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{l} = 0$

$$\Rightarrow \quad V(b) = V(a)$$

⊗ The electric field $\vec{E}$ is perpendicular to the surface.

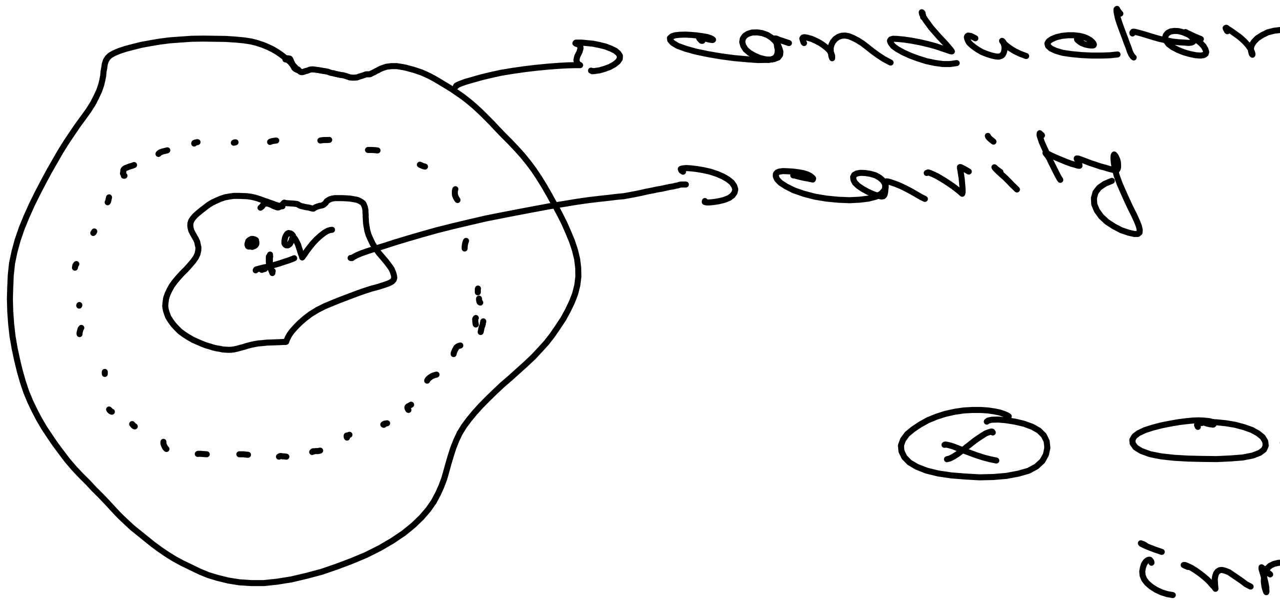
⊗ The electrostatic energy is at its min^m when the charge is spread over the surface.

Induced Charges –



→ Negative induced charges are closer to +q resulting in an attractive force bet^n +q and the conductor.

⊗ +q charge inside the cavity of a conductor



- Inside the cavity

  $\vec{E} \neq 0$

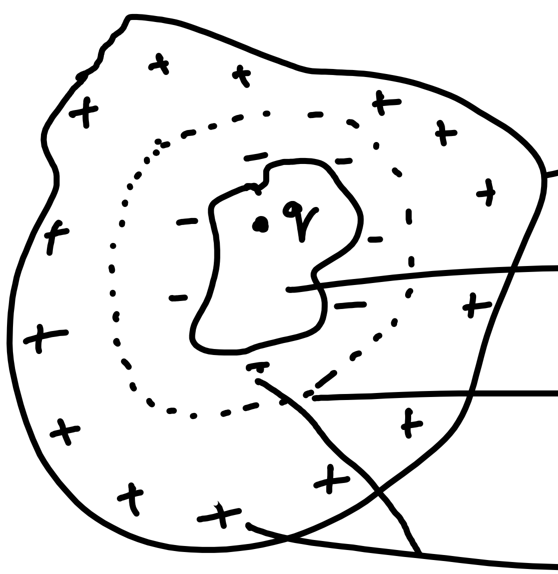
- Outside the cavity, but inside the conductor

  $\vec{E} = 0$

- Outside the conductor,

  $\vec{E} \neq 0$

Cavity inside a conductor



→ The total charge induced on the cavity wall is equal and opposite to the charge inside.

For the Gaussian surface,

$$\oint \vec{E} \cdot d\vec{a} = 0 \Rightarrow \text{net enclosed charge} = 0$$

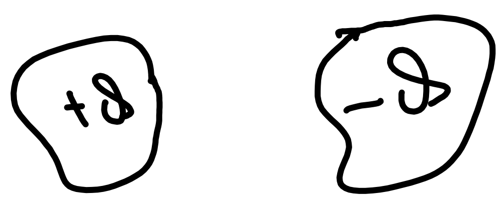
$$\Rightarrow q_{induced} = -q$$

• Field outside the conductor

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

Capacitor

→ Two conductors with



charges +q and −q.

The potential difference,

$$V = V_+ - V_- = -\int \vec{E} \cdot d\vec{l}$$

$$\vec{E} \propto q$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{r^2} \hat{r} \, d\tau$$

→ Increase ρ ⇒ We increase q

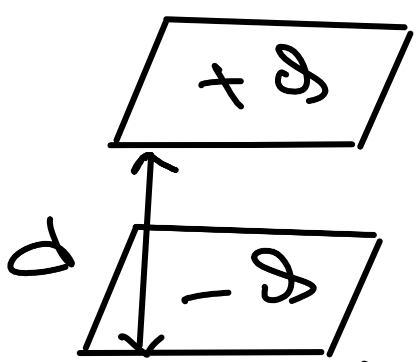
(x) V is also proportional to q.

→ The proportionality constant is called the capacitance.

$$C = \frac{q}{V}$$

↳ Determined by the sizes, shapes and separation betn. two conductors

Ex: Parallel plate capacitor



A → Area of the plates

σ → charge density (surface)

→ upper plate has charge $+q$ and lower one has $-q$

$$\vec{E} = \frac{q}{\epsilon_0 A} \Rightarrow V = \frac{q}{A\epsilon_0} d$$

↳ separation betn. the plates

$$\Rightarrow C = \frac{A\epsilon_0}{d}$$

* Work done to charge up a capacitor to Q:

→ at some point the charge on the plate $= +q$

$\Rightarrow$ Potential diff. $= \frac{q}{C}$

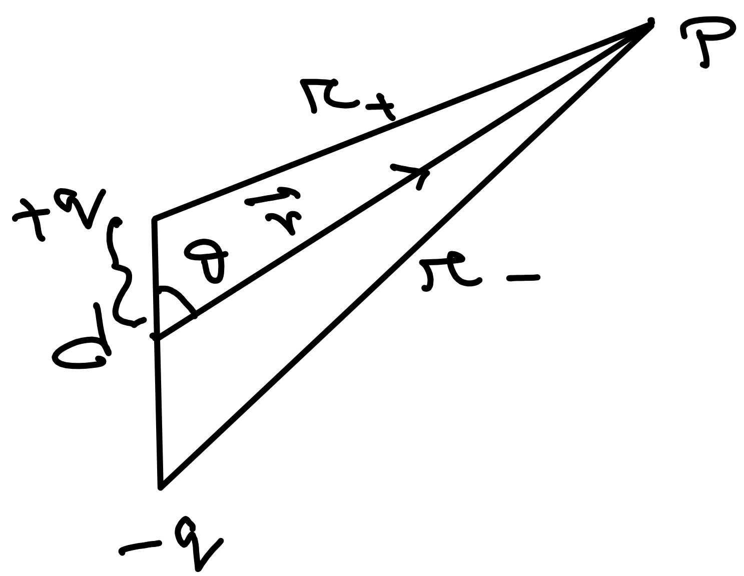
In order to bring dq (infinitesimal amount of charge) to the plate:

$$dW = \frac{q}{C}\,dq$$

The total work done

$$W = \int_0^Q \frac{q}{C}\,dq = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$

Electric Dipole



An electric dipole with two equal and opposite charges ($\pm q$) separated by a distance 'd'.

Potential due to the dipole?

$r_- \equiv$ distance of P from $-q$

$r_+ \equiv$ " " " " $+q$

$\vec{r} \equiv$ Position vector of P.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

$$r_\pm^2 = r^2 + \left(\frac{d}{2}\right)^2 \mp r d \cos\theta$$

$$= r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

$\frac{d^2}{4r^2} << 1$ (negligible) in the limit $r >> d$

$$\frac{1}{r_\pm} \simeq \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2}$$

$$\simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$ (neglecting higher order terms)

Then, $\frac{1}{r_+} - \frac{1}{r_-} \simeq \frac{d}{r^2}\cos\theta$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q\,d\cos\theta}{r^2}$$

$\rightsquigarrow$ The potential, $V(\vec{r}) \sim \frac{1}{r^2}$

$\rightarrow$ The potential falls off more rapidly than the potential of a point charge.

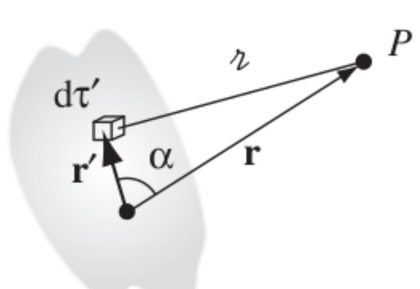
The monopole and dipole terms

The most dominant contribution in multipole expansion comes from the monopole term.

$$V_{mon.}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \qquad \Big| \; Q = \int \rho \, d\tau$$

→ If the total charge $Q = 0$, then the most dominant contribution comes from the dipole term.

$$V_{dip.}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r' \cos\alpha \, \rho(\vec{r}\,') \, d\tau'$$



α = Angle betn. $\vec{r}\,'$ and $\vec{r}$

$$r' \cos\alpha = \hat{r} \cdot \vec{r}\,'$$

$$\Rightarrow V_{dip.}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\,\hat{r} \cdot \int \vec{r}\,' \rho(\vec{r}\,') \, d\tau'$$

This integral (independent of $\vec{r}$) is called the <u>dipole moment</u> of the distribution.

$$\vec{p} = \int \vec{r}\,' \rho(\vec{r}\,')\, d\tau'$$

$$V_{dip.}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}$$

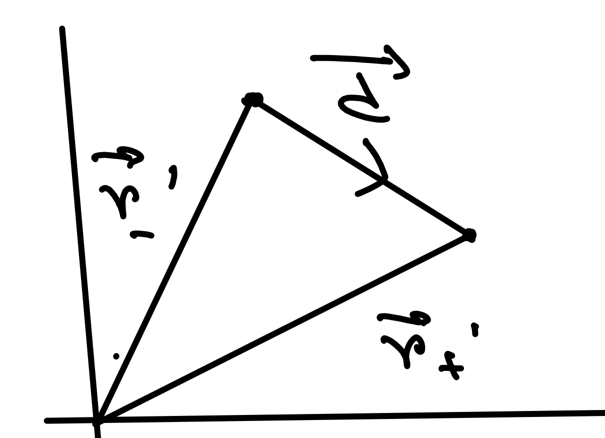
(x) For a collection of point charges,

$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r}_i\,'$$

(x) For a physical dipole with $\pm q$ charges

$$\vec{p} = q\left(\vec{r}_+\,' - \vec{r}_-\,'\right)$$

$$= q\,\vec{d}$$



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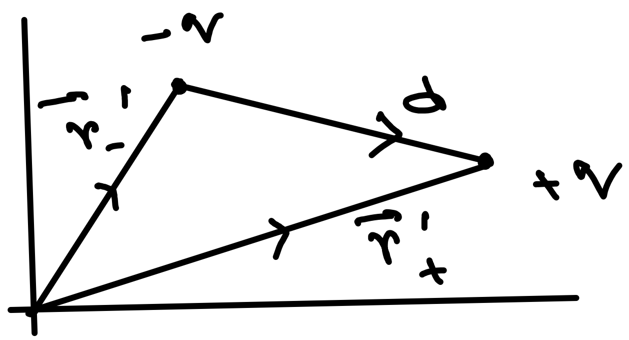
Electric Dipole (contd.)



→ Physical charges with equal and opposite charges $\pm q$,

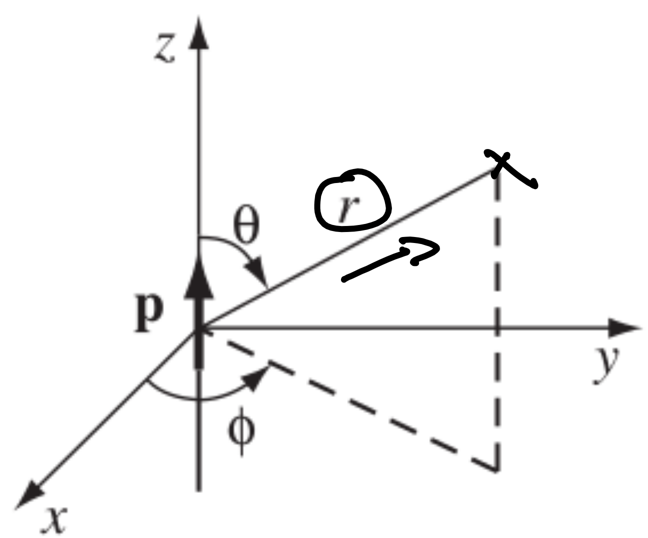
dipole moment, $\vec{p} = q(\vec{r}'_+ - \vec{r}'_-)$
$= q\vec{d}$

→ valid for the condition, $r \gg d$.

Dipole approximation ≡ for a perfect dipole, $d \to 0 \Rightarrow$ then q simultaneously has to increase, $q \to \infty$.

(*) A physical dipole becomes a pure dipole in the limit $d \to 0$ & $q \to \infty$ with the product $p = qd$ kept fixed and finite.

Electric field due to a dipole



$\vec{p}$ is sitting at the origin and points in z-direc^n.

$$V_{dip}(r,\theta) = \frac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_0 r^2}$$

$$\vec{E} = -\vec{\nabla} V$$

$$= \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

To get the electric field,

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

Then, the electric field is given by

$$\vec{E}_{dip.}(r, \theta) = \frac{p}{4\pi \epsilon_0 r^3} \left(2\cos\theta \, \hat{r} + \sin\theta \, \hat{\theta} \right)$$

(x) We can rewrite it as

$$\vec{E}_{dip.}(\vec{r}) = \frac{1}{4\pi \epsilon_0 r^3} \left[3 (\vec{p} \cdot \hat{r}) \hat{r} - \vec{p} \right]$$

$$\vec{p} = \underbrace{(\vec{p} \cdot \hat{r})} \hat{r} + \underbrace{(\vec{p} \cdot \hat{\theta})} \hat{\theta}$$ [$\vec{p} = p \hat{z}$]

$$= p\cos\theta \, \hat{r} - p\sin\theta \, \hat{\theta}$$

[the unit vector,

$$\hat{\theta} = \cos\theta \cos\phi \, \hat{x} + \cos\theta \sin\phi \, \hat{y} \underbrace{- \sin\theta} \, \hat{z}$$]

Now, $3 (\vec{p} \cdot \hat{r}) \hat{r} - \vec{p}$

$$= 3p\cos\theta\,\hat{r} - p\cos\theta\,\hat{r} + p\sin\theta\,\hat{\theta}$$

$$= 2p\cos\theta\,\hat{r} + p\sin\theta\,\hat{\theta}$$

* Electric field due to a dipole falls off $\sim \frac{1}{r^3}$

  Electric field due to a monopole falls off $\sim \frac{1}{r^2}$

* The expression is valid for ideal dipoles and physical dipoles under approximation, $r \gg d$.

Dielectrics

Unlike conductors, in a dielectric material, electrons cannot move about freely. They can move about a bit within the atoms and (or) molecules.

$\Rightarrow$ if their movement is absolutely restricted $\equiv$ Insulators (small dielectric constants)

$\Rightarrow$ if the charged particles can move a little bit $\equiv$ Dielectrics (larger dielectric constants)

(*) Material is made up of neutral atoms (non-polar molecules) $\rightarrow$ in an external electric field → tiny dipoles are induced pointing in the same direcⁿ as the field.

* If material is made of polar molecules → permanent dipoles in the system → external electric field exerts a torque on them → aligns them along the direcⁿ. of the electric field.

→ Net effect is a lot of tiny dipoles pointing along the direcⁿ. of the field

⇒ <u>Polarisation.</u>

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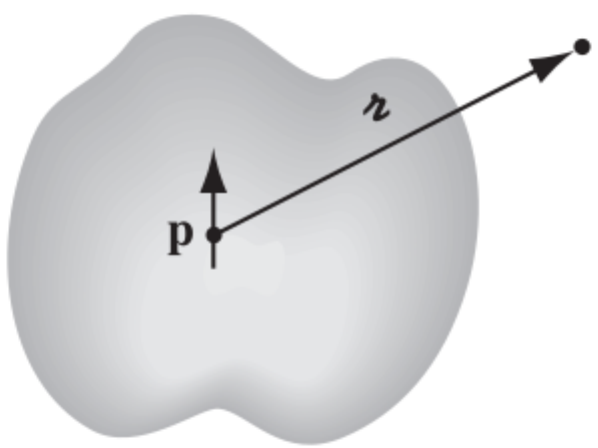
Polarization ($\vec{P}$)

$\vec{P} \equiv$ Dipole moment per unit volume

Field due to a Polarized object



* We have a polarized material. All the dipoles are pointing in the same direction.

For a single dipole $\vec{p}$,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}$$

$\vec{r} \equiv$ vector from the dipole to the reference point.

Dipole moment : $\vec{p} = \underline{\vec{P}\, d\tau'}$ for each infinitesimal volume element $d\tau'$.

→ The total potential,

$$V = \frac{1}{4\pi\epsilon_0} \int_V \frac{\vec{p}(\vec{r}') \cdot \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \int_V \vec{P} \cdot \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) d\tau'$$

$$\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\Rightarrow V = \frac{1}{4\pi\epsilon_0}\left[\int_V \vec{\nabla}' \cdot \left(\frac{\vec{P}}{\mathscr{r}}\right) d\tau' - \int_V \frac{1}{\mathscr{r}} (\vec{\nabla} \cdot \vec{P}) \, d\tau'\right]$$

Using divergence theorem,

$$V = \underbrace{\frac{1}{4\pi\epsilon_0} \oint_S \frac{1}{\kappa} \vec{P} \cdot d\vec{a}'}_{\text{Potential due to a surface charge distribution}} - \underbrace{\frac{1}{4\pi\epsilon_0} \int_V \frac{1}{\kappa} (\vec{\nabla}' \cdot \vec{P})\, d\tau'}_{\text{Potential due to a volume charge distribution}}$$

charge density: $\underline{\sigma_b \equiv \vec{P} \cdot \hat{n}}$

$\underline{\rho_b \equiv -\vec{\nabla}' \cdot \vec{P}}$

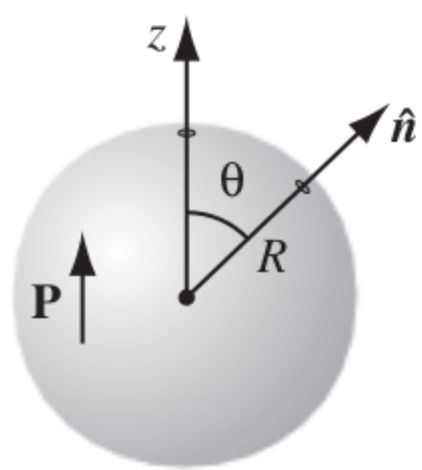
Hence,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \oint_S \frac{\sigma_b}{\kappa}\, da' + \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_b}{\kappa}\, d\tau'$$

→ Potential of a polarised object is the same as that produced by a volume and a surface charge densities.

These are called <u>bound</u> <u>charges</u>.

<u>Ex:</u>



Electric field produced by a uniformly polarized sphere of radius 'R'.

z-axis coincides with $\vec{P}$

→ if $\vec{P}$ is uniform, $\rho_b = 0$

The surface bound charge density,

$$\sigma_b = \vec{P} \cdot \hat{n} = P\cos\theta$$

We need to find the potential of a system with surface charge density $P\cos\theta$

$$V(r,\theta) = \frac{P r \cos\theta}{3\epsilon_0} \quad \text{for } r \leq R$$

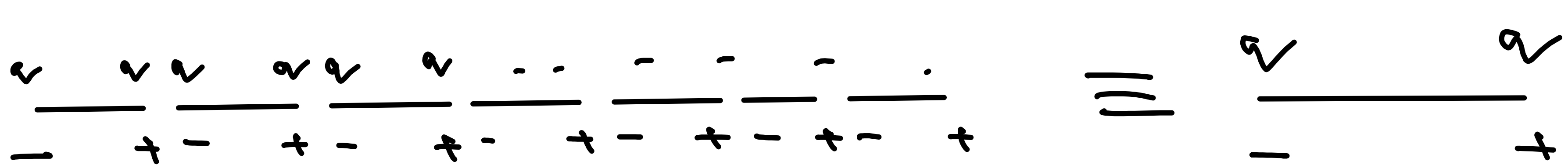
$$= \frac{P R^3 \cos\theta}{3\epsilon_0 r^2} \quad \text{for } r \geq R$$

We know, $z = r\cos\theta$

For $r < R$,

$$\vec{E} = -\vec{\nabla} V = -\frac{P}{3\epsilon_0}\hat{z} = -\frac{\vec{P}}{3\epsilon_0}$$

Physical interpretation of bound charge.

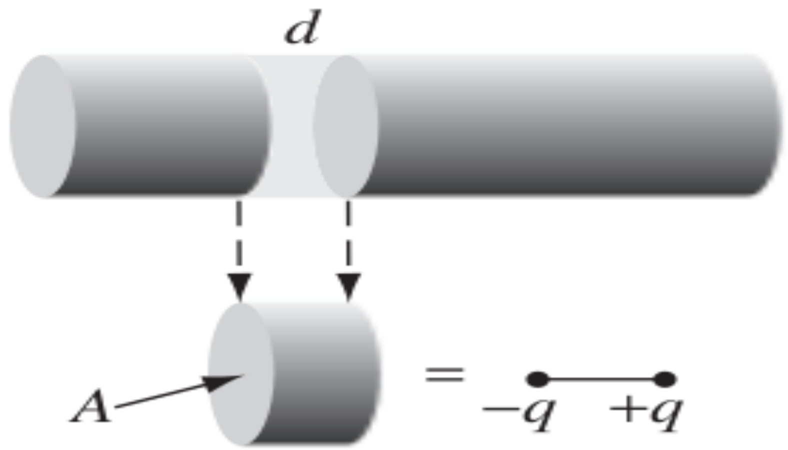


$\longrightarrow$ Bound Charges.

↳ String of dipoles made up of same amount of +ve and -ve charges.

To calculate the amount of bound charges resulting from a polarization:

(1) A tube of dielectric parallel to $\vec{P}$



$\vec{P}\ (\hat{x})$

On the small chunk, the dipole moment

$= P(Ad)$

↳ length of the chunk

Cross-section of the sides

In terms of charges at the end, dipole moment $= qd$

$A = A_{end} \cos\theta$

→ The charges on the end are still the same, but

$$\sigma_b = \frac{q}{A_{end}} = \frac{q\cos\theta}{A} = P\cos\theta$$

Then the bound charge

$$p\,V = q\,d$$

$$\Rightarrow \quad q = \frac{pV}{d} = \frac{pAd}{d}$$

$$\Rightarrow \underline{q = pA}$$

$$\sigma_b = \frac{q}{A} = p = \vec{p}\cdot\hat{n}$$

$$= \vec{p}\cdot\hat{n}$$

The net effect of polarisation is to create a bound charge density

$\boxed{\sigma_b = \vec{p}\cdot\hat{n}}$ over the surface of the material.

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Gauss' law in presence of Dielectrics

Polarisation $\rightarrow$ accumulation of bound charges

$$\rho_b = -\vec{\nabla}\cdot\vec{P} \qquad \sigma_b = \vec{P}\cdot\hat{n}$$

$\rightarrow$ These bound charges give rise to electric field.

Total field = field due to bound charges + field due to free charges

Within a dielectric,

$$\rho = \rho_b + \rho_f$$

$\rho_f \rightarrow$ free charge density

Using Gauss' law,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \epsilon_0 (\vec{\nabla} \cdot \vec{E}) = \rho = \rho_b + \rho_f$$

$$= -\vec{\nabla} \cdot \vec{P} + \rho_f$$

$$\Rightarrow \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f$$

$$\Rightarrow \vec{\nabla} \cdot \vec{D} = \rho_f$$

$\rightarrow \vec{D} = \epsilon_0 \vec{E} + \vec{P} \equiv$ <u>Electric</u> <u>displacement</u>

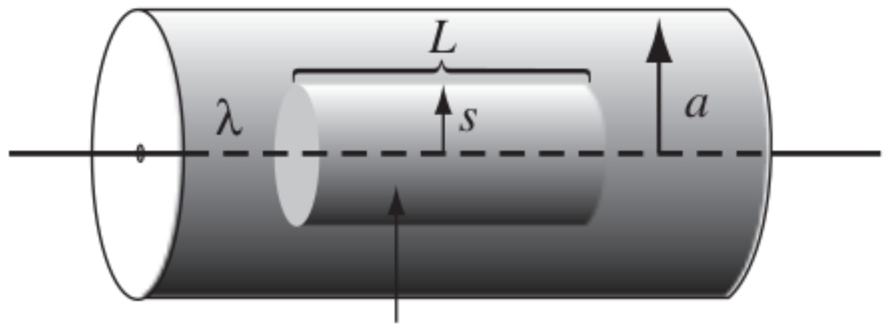
Hence, $\oint \vec{D} \cdot d\vec{a} = \int \rho_f \, d\tau$

Gauss' law for dielectrics.

$$= (Q_f)_{enc.}$$

(total free charge enclosed)

Ex:



A long straight wire carrying uniform line charge density λ, surrounded by a dielectric medium upto a radius 'a'.

→ Gaussian surface has radius ≡ s
length ≡ L

$$\oint \vec{D} \cdot d\vec{a} = (Q_f)_{enc.}$$

$$\Rightarrow \quad D(2\pi s L) = \lambda L$$

$$\Rightarrow \quad D = \frac{\lambda}{2\pi s}$$

$$\vec{D} = \frac{\lambda}{2\pi s}\,\hat{s}$$

Outside, $\vec{P} = 0$

from, $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$= \epsilon_0 \vec{E}$

$\Rightarrow \vec{E} = \dfrac{\vec{D}}{\epsilon_0} = \dfrac{\lambda}{2\pi\epsilon_0 s}\hat{s}$ for $s > a$

For, $s < a$, we need the knowledge of $\vec{P}$ in order to calculate $\vec{E}$.

· <u>Linear Dielectrics</u>

A class of dielectrics for which the induced polarisation is proportional to the electric field.

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$\chi_e \equiv$ Proportionality const. (Electric susceptibility)

'ϵ_0' is brought outside to make χ_e dimensionless.

(x) Any material obeying $\vec{P} = \epsilon_0 \chi_e \vec{E}$ is called a linear dielectric.

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$= \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E}$$

$$= \epsilon_0 (1 + \chi_e) \vec{E}$$

$$= \epsilon \vec{E}$$

$\rightarrow \epsilon = \epsilon_0 (1 + \chi_e) \equiv$ Permittivity of a material

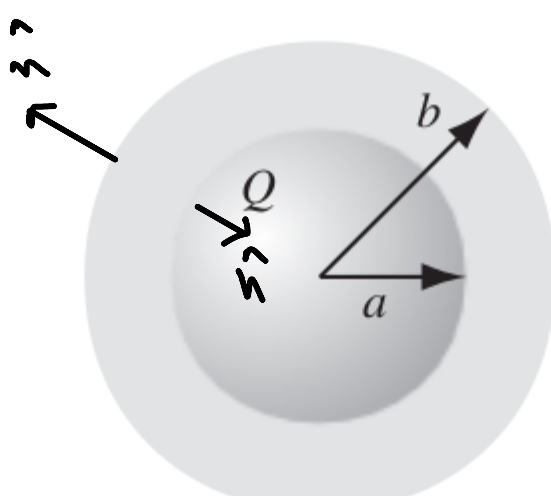
(*) In vacuum, there is no material,

$\chi_e = 0 \Rightarrow \epsilon = \epsilon_0 \equiv$ Permittivity of free space.

(*) $\epsilon_r = 1 + \chi_e \equiv$ Relative permittivity or Dielectric const.

$= \dfrac{\epsilon}{\epsilon_0}$

Ex:



A metal sphere of radius 'a' carrying a total charge 'Q'. It is surrounded by a dielectric material of permittivity 'ϵ'. Calculate the potential at the center.

→ In the region, $r > a$

$$\oint \vec{D} \cdot d\vec{a} = q$$

$$\Rightarrow \quad \vec{D} = \frac{q}{4\pi r^2} \hat{r}$$

Inside the sphere, $r < a$

$$\vec{D} = 0 = \vec{E} = \vec{P}$$

In the region, $a < r < b$

$$\vec{E} = \frac{q}{4\pi \epsilon r^2} \hat{r}$$

In the region, $r > b$

$$\vec{E} = \frac{q}{4\pi \epsilon_0 r^2} \hat{r}$$

Potential at center,

$$V = -\int_{\infty}^{0} \vec{E} \cdot d\vec{l}$$

$$= -\int_{\infty}^{b} \frac{q}{4\pi\epsilon_0 r^2}\,dr - \int_{b}^{a} \frac{q}{4\pi\epsilon r^2}\,dr - \int_{a}^{0} (0)\,dr$$

$$= \frac{q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right)$$

Polarization,

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$= \frac{\epsilon_0 \chi_e q}{4\pi\epsilon r^2}\,\hat{r}$$ (in the dielectric)

Bound charges:

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$$

$$\sigma_b = \vec{P}\cdot\hat{n} = \frac{\epsilon_0 \chi_e\, q}{4\pi\epsilon b^2} \quad \text{(at outer surface)}$$

$$= -\frac{\epsilon_0 \chi_e\, q}{4\pi\epsilon a^2} \quad \text{(at inner surface)}$$

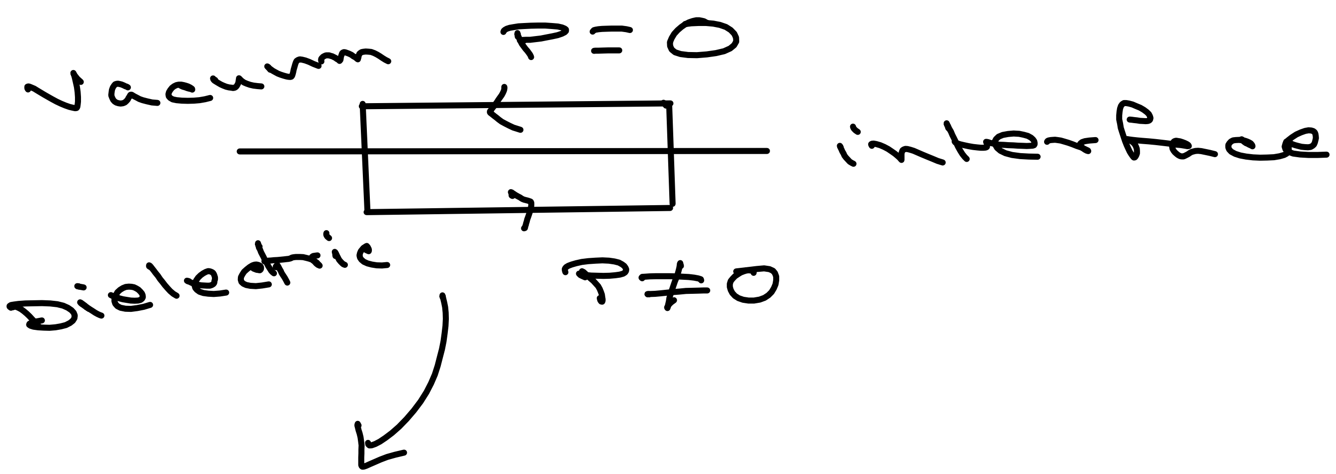
(*) $\hat{n}$ points outward with respect to the dielectric $\Rightarrow$ $\left.\begin{aligned} \hat{n} &\equiv \hat{r} \text{ at } r=b \\ &\equiv -\hat{r} \text{ at } r=a \end{aligned}\right\}$

For a linear dielectric,

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$\vec{D} = \epsilon \vec{E} \quad , \quad \epsilon = \underbrace{\epsilon_r}_{\text{Dielectric const.}} \epsilon_0$$

(*) Does this mean $\vec{\nabla} \times \vec{P} = 0$?



→ In order for $\vec{\nabla} \times \vec{P} = 0$ we should have $\oint \vec{P} \cdot d\vec{l} = 0$

Around this loop

$$\oint \vec{P} \cdot d\vec{l} \neq 0 \implies \vec{\nabla} \times \vec{P} \neq 0$$

→ If the space is completely filled with dielectrics, $\vec{\nabla} \cdot \vec{D} = \rho_f$ & $\vec{\nabla} \times \vec{D} = 0$

$\Rightarrow \vec{D}$ can be obtained from free charge density

$$\vec{D} = \epsilon_0 \vec{E}_{vac.}$$

↳ the field free charge will produce in absence of any dielectric.

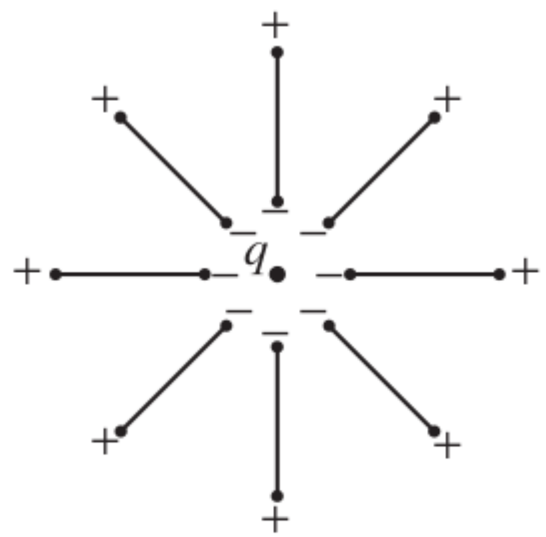
$$\epsilon_r = \frac{\epsilon}{\epsilon_0}$$

$$\vec{E} = \frac{1}{\epsilon} \vec{D}$$

$$= \frac{1}{\epsilon_r} \vec{E}_{vac.}$$

↳ the factor by which the field is reduced in presence of homogeneous linear dielectric.

Ex. If a free charge 'q' is embedded in a large dielectric,

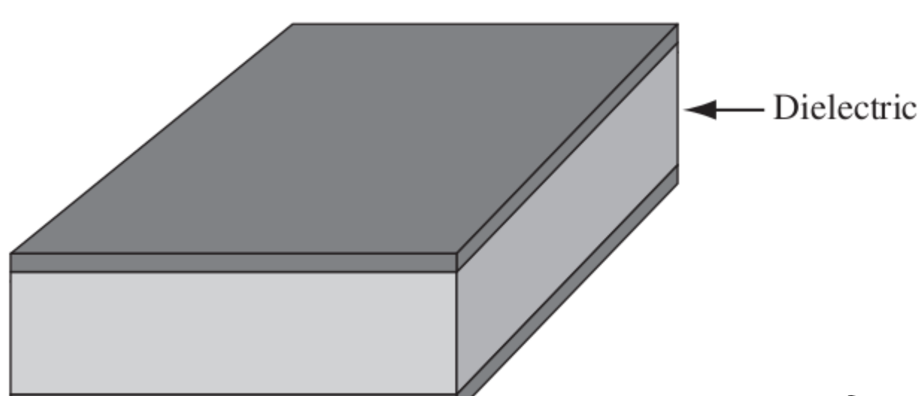


$$\vec{E} = \frac{1}{4\pi\epsilon}\frac{q}{r^2}\hat{r}$$

→ The free charge is shielded from all sides due to the polarisation.

⇒ This shielding effect reduces the electric field.

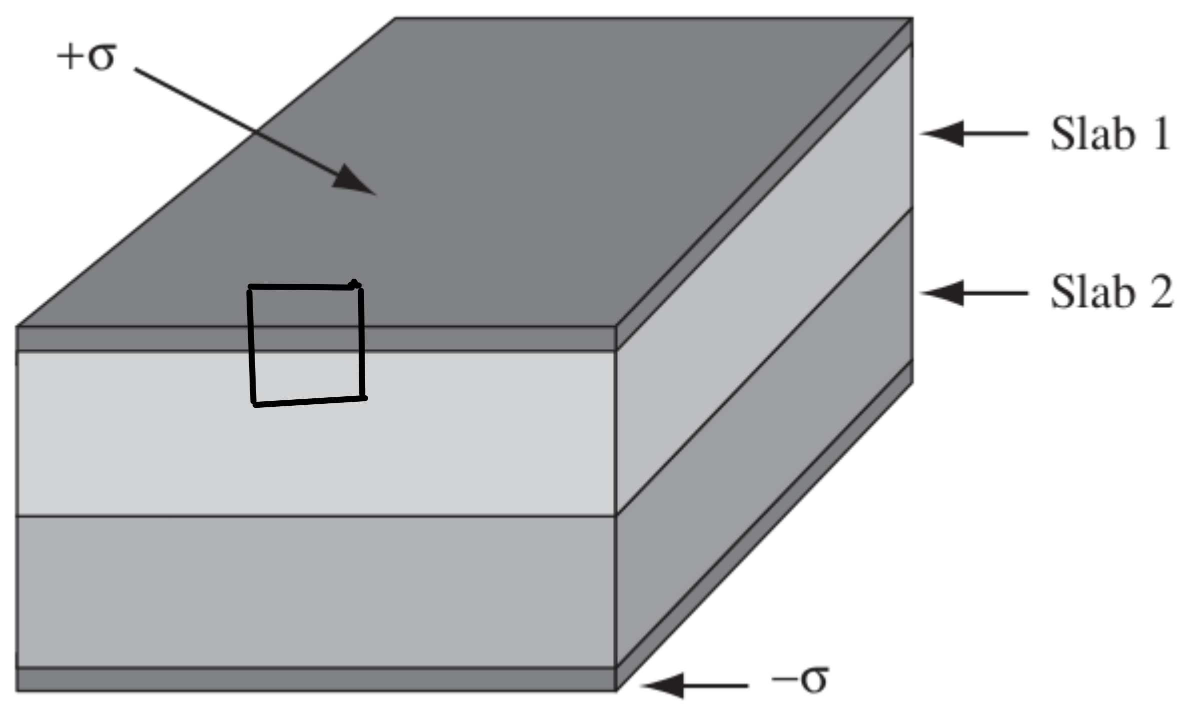
(*) This property can be used to enhance capacitance of a system.



→ A parallel plate Capacitor filled with dielectric material

→ The dielectric reduces $\vec{E}$ and ΔV (potential diff.) by a factor $\frac{1}{\epsilon_r}$.

The capacitance $\left(\frac{Q}{V}\right)$ is increased by a factor, $C = \epsilon_r C_{vac}$.



$\vec{E}_x$: $\hat{z}$

$\vec{D}, \vec{E}, \vec{P}$

$\underline{d = 2a}$

We have a parallel plate capacitor with charge density $\pm\sigma$ on the plates. The space betn. the plates is filled with two dielectric slabs with thickness 'a'

→ Slab 1 has dielectric const = 2

Slab 2 - - - - - = 1.5

(1) $\vec{D}$ in each slab:

$$\oint \vec{D} \cdot d\vec{a} = (Q_f)_{enc}.$$

$$\Rightarrow \quad D A = \sigma A$$

$$\Rightarrow \quad D = \sigma$$

Then, $\quad \vec{D} = -\sigma \hat{z}$

$\vec{D} = 0$ inside metal plates.

(c) $\vec{E}$ in each slab:

$$\vec{D} = \epsilon \vec{E}$$

for slab 1 $\Rightarrow |\vec{E}_1| = \dfrac{\sigma}{\epsilon_1}$

slab 2 $\Rightarrow |\vec{E}_2| = \dfrac{\sigma}{\epsilon_2}$

$$\left.\begin{array}{l} \epsilon_1 = 2\epsilon_0 \\ \epsilon_2 = 1.5\epsilon_0 \end{array}\right\} \begin{array}{l} |\vec{E}_1| = \sigma/2\epsilon_0 \\ |\vec{E}_2| = \sigma/1.5\epsilon_0 = 2\sigma/3\epsilon_0 \end{array}$$

⊗ Polarisation:

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$\Rightarrow \quad |\vec{P}| = \frac{\epsilon_0 \chi_e \sigma}{\epsilon_0 \epsilon_r}$$

$$= \frac{\chi_e}{\epsilon_r} \sigma$$

$$(\chi_e = \epsilon_r - 1)$$

$$\Rightarrow \quad |\vec{P}| = \frac{\epsilon_r - 1}{\epsilon_r} \sigma = \left(1 - \frac{1}{\epsilon_r}\right) \sigma$$

→ for slab ①, $|\vec{P}_1| = \frac{\sigma}{2}$

$$|\vec{P}_2| = \frac{\sigma}{3}$$

⊗ Potential difference,

$$\Delta V = -\int_b^a \vec{E} \cdot d\vec{l}$$

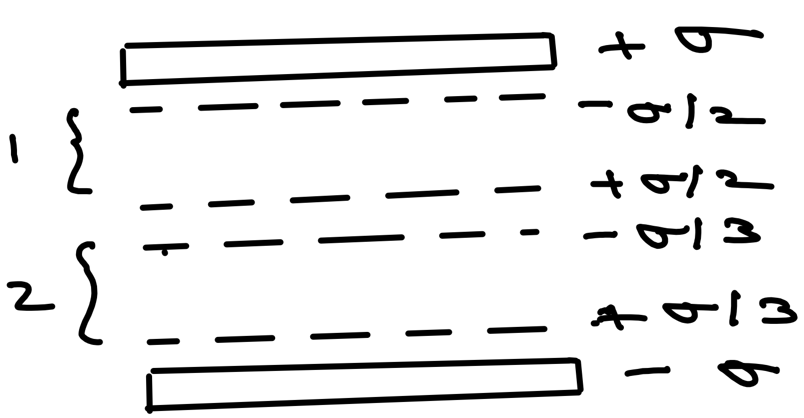
$$= E_1 a + E_2 a = \frac{\sigma a}{\epsilon_0}\left(\frac{1}{2} + \frac{2}{3}\right)$$

$$= \frac{7\sigma a}{6\epsilon_0}$$

(f) Bound chargen:

Constant polarisation $\Rightarrow \rho_b = 0$

The bound surface charge density:



$$\sigma_b = -P_1 \quad (\text{top of slab } 1)$$

$$= +P_1 \quad (\text{bottom of slab } 1)$$

$$= -P_2 \quad (\text{top of slab } 2)$$

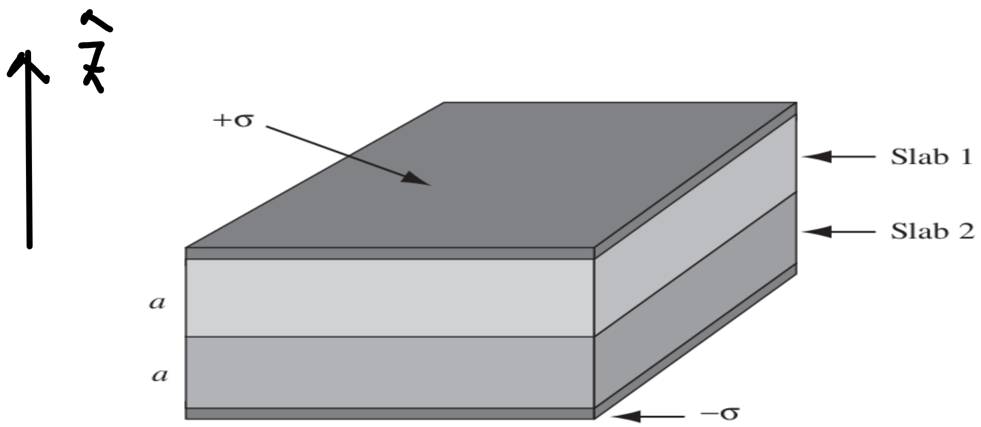
$$= +P_2 \quad (\text{bottom of slab } 2)$$

$$\sigma_b = \vec{P} \cdot \hat{n}$$

$$\vec{P}_1 = \frac{\sigma}{2}(-\hat{z})$$

$$\vec{P}_2 = \frac{\sigma}{3}(-\hat{z})$$

Ex (contd)



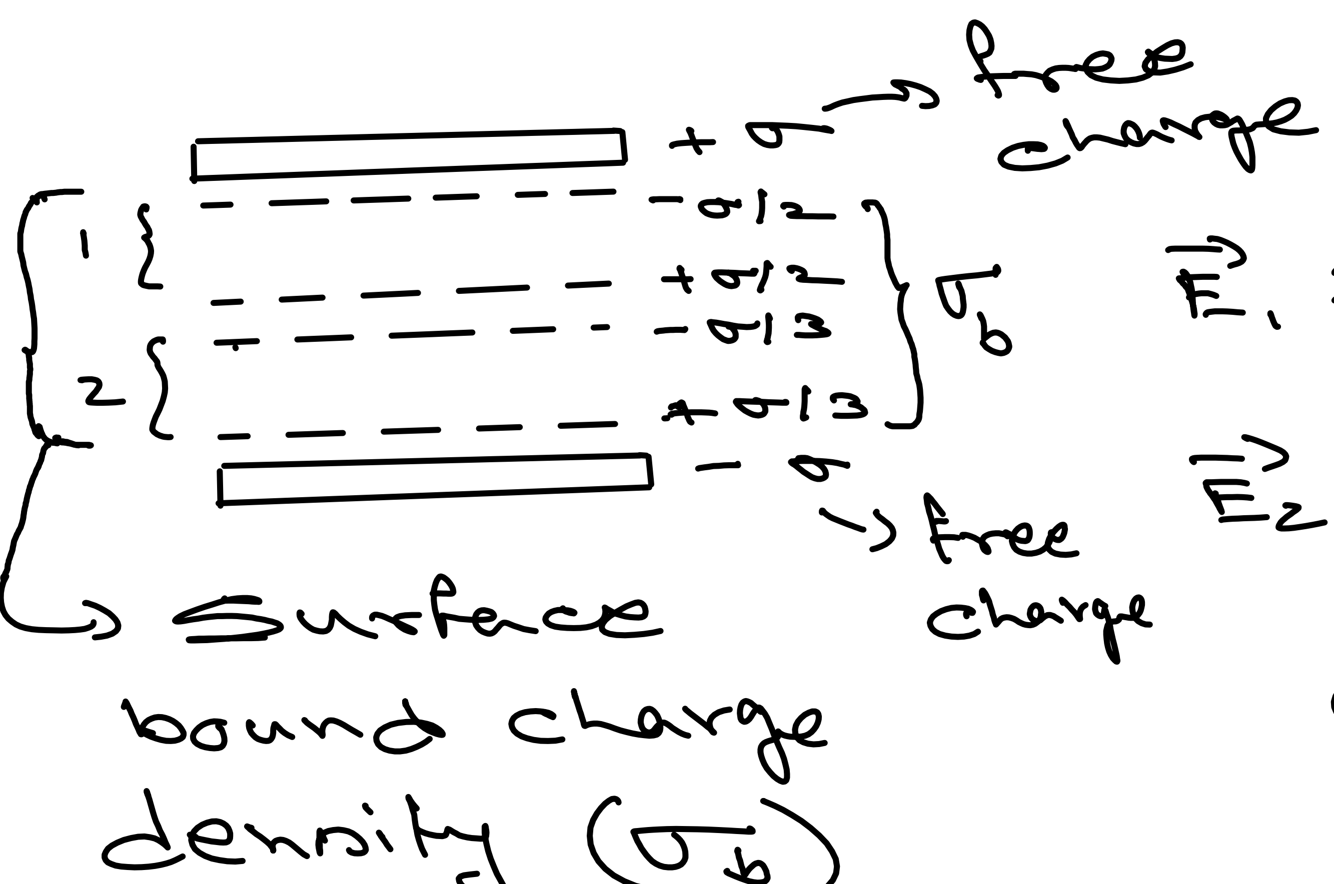
Space betn. two capacitor plates filled with dielectric slab 1 and slab 2 with dielectric const = 2 and 1.5 respectively.

We found:

$$\vec{D} = \sigma(-\hat{z})$$

$$\vec{E}_1 = \frac{\sigma}{2\epsilon_0}(-\hat{z}) \quad ; \quad \vec{P}_1 = \frac{\sigma}{2}(-\hat{z})$$

$$\vec{E}_2 = \frac{2\sigma}{3\epsilon_0}(-\hat{z}) \quad ; \quad \vec{P}_2 = \frac{\sigma}{3}(-\hat{z})$$



⊗ Volume bound charge density

$$\rho_b = 0$$

⊗ Potential difference, $\Delta V = \dfrac{7\sigma a}{6\epsilon_0}$

$$\sigma = \frac{q}{A} \qquad = \frac{7qa}{6A\epsilon_0}$$

q → total charge on plate

A → Area of plate

* Capacitance, $C = \frac{Q}{\Delta V} = \frac{6A\epsilon_0}{7a}$ ✓

* Check the electric fields from knowledge of bound and free charges:

<u>Slab 1</u>: Total surface charge above $= \sigma - \frac{\sigma}{2} = \frac{\sigma}{2}$

" " " below $= -\frac{\sigma}{2}$

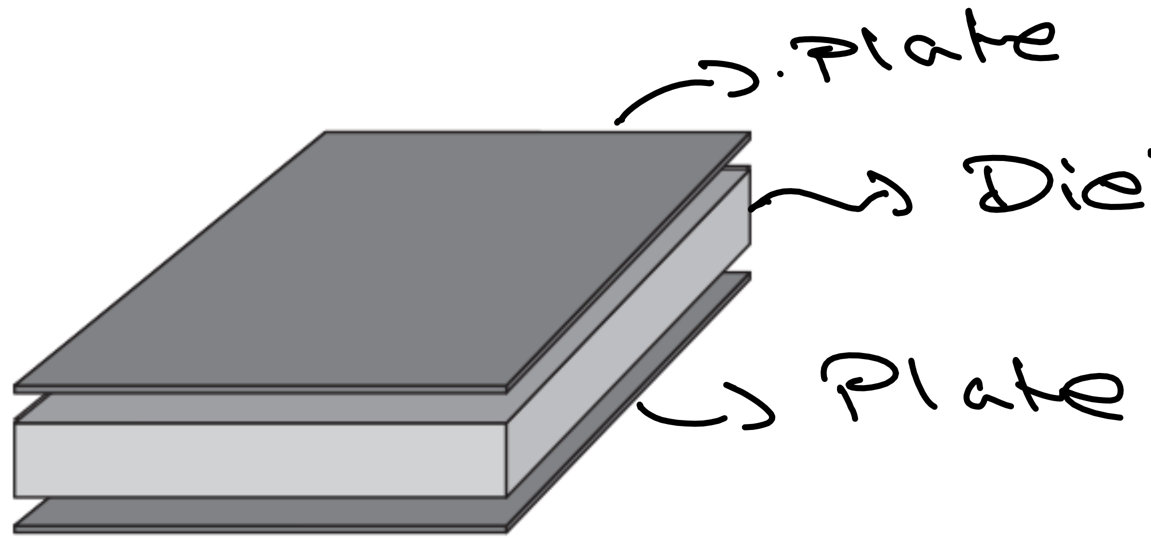
<u>Slab 2</u>: " " " above $= \frac{2\sigma}{3}$

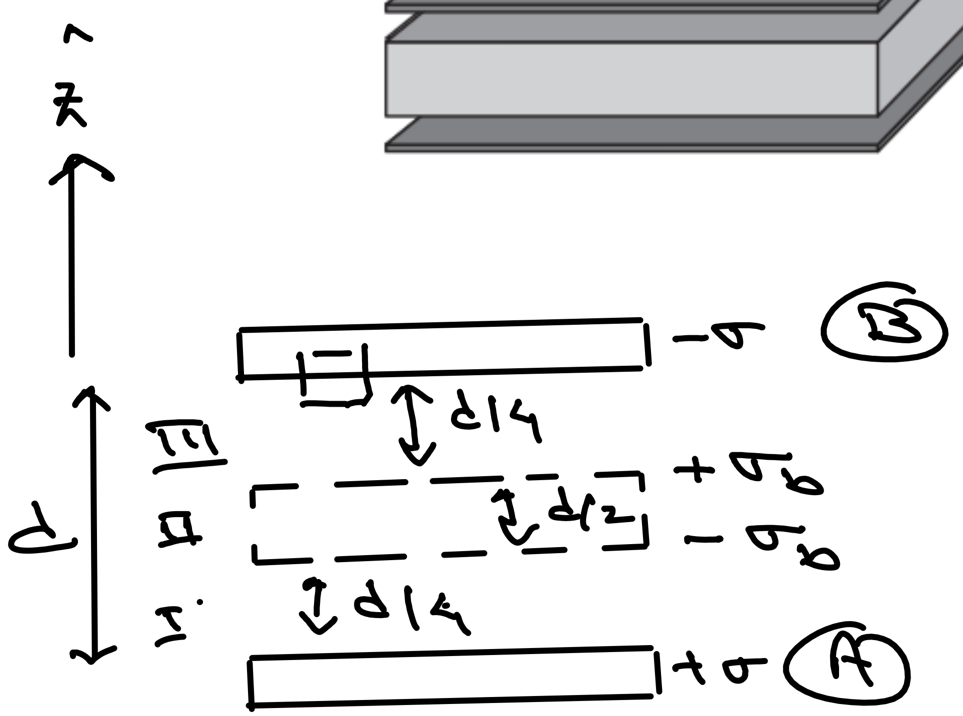
" " " below $= \frac{\sigma}{3} - \sigma = -\frac{2\sigma}{3}$

Then, the electric fields:

$$\begin{cases} \vec{E}_1 = \frac{\sigma}{2\epsilon_0}(-\hat{z}) \\ \vec{E}_2 = \frac{2\sigma}{3\epsilon_0}(-\hat{z}) \end{cases}$$

Ex:





* The space bet^n. the two plates is <u>half filled</u> with a dielectric slab.

→ Spacing bet^n the two plates = d

⇒ Thickness of the dielectric slab = $\frac{d}{2}$

* Electric Displacement, ($\vec{D}$)

$$\oint \vec{D} \cdot d\vec{a} = (Q_f)_{enc.}$$

$$\Rightarrow |\vec{D}| A = \sigma A$$

$$\Rightarrow \vec{D} = \sigma \hat{z} \quad \text{(in all regions)}$$

* $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{z}$ (I)

$= \frac{\sigma}{\epsilon}\hat{z}$ (II)

$= \frac{\sigma}{\epsilon_0}\hat{z}$ (III)

* Polarization: $\vec{P} = 0$ (I)

$= 0$ (III)

in region (II)

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$= \frac{\epsilon_0 \chi_e \sigma}{\epsilon}\hat{z}$$

$$= \frac{\epsilon_0 \chi_e \sigma}{\epsilon_0(1+\chi_e)}\hat{z} = \frac{\chi_e \sigma}{1+\chi_e}\hat{z}$$

(*) Bound charge densities:

$$\rho_b = 0 \quad \text{(polarization is uniform)}$$

$$\sigma_b = \vec{P} \cdot \hat{n} = \frac{\chi_e \sigma}{1+\chi_e} \hat{z} \cdot \hat{z} \quad \text{(top plane of dielectric)}$$

$$= \frac{\chi_e \sigma}{1+\chi_e}$$

$$\sigma_b = \frac{\chi_e \sigma}{1+\chi_e} \hat{z} \cdot (-\hat{z}) \quad \text{(bottom plane of dielectric)}$$

$$= -\frac{\chi_e \sigma}{1+\chi_e}$$

(*) $$\Delta V = -\int_B^A \vec{E} \cdot d\vec{l} = \int_A^B \vec{E} \cdot d\vec{l}$$

$$= \underbrace{\int \frac{\sigma}{\epsilon_0}\, dz}_{\mathrm{I}} + \underbrace{\int \frac{\sigma}{\epsilon}\, dz}_{\mathrm{II}} + \underbrace{\int \frac{\sigma}{\epsilon_0}\, dz}_{\mathrm{III}}$$

$$= \frac{\sigma}{\epsilon_0} \times \frac{d}{4} + \frac{\sigma}{\epsilon} \times \frac{d}{2} + \frac{\sigma}{\epsilon_0} \times \frac{d}{4}$$

$$= \frac{\sigma}{\epsilon_0} \frac{d}{2} + \frac{\sigma}{\epsilon} \frac{d}{2}$$

$$= \frac{\sigma d}{2} \left(\frac{1}{\epsilon_0} + \frac{1}{\epsilon} \right)$$

$$= \frac{Q d}{2 A \epsilon_0} \left(1 + \frac{\epsilon_0}{\epsilon} \right)$$

$Q \equiv$ Total charge on plate

$A \equiv$ Total area of plate

$\sigma = Q/A$

* Capacitance, $C = \dfrac{Q}{\Delta V}$

$$= \frac{2A\epsilon_0}{d\left(1+\frac{\epsilon_0}{\epsilon}\right)}$$

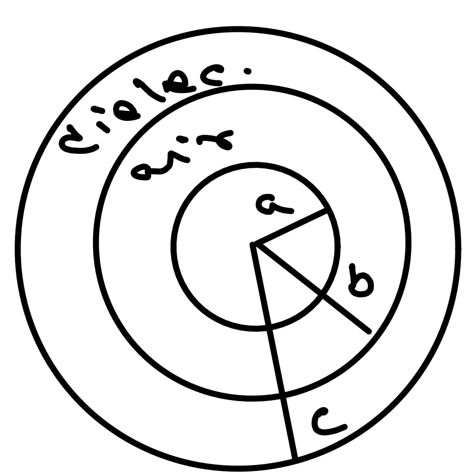
$$\epsilon_r = \frac{\epsilon}{\epsilon_0}$$

$$= \frac{2\epsilon_0 A}{d} \frac{\epsilon}{\epsilon+\epsilon_0}$$

$$= \frac{2\epsilon_0 A}{d}\left(\frac{\epsilon_r}{1+\epsilon_r}\right)$$

→ Capacitance in absence of dielectric $= \frac{\epsilon_0 A}{d}$

Ex:



Coaxial cable consisting of copper wire (radius = a). surrounded by copper tube (radius = b). The space in betn. is filled with dielectric (const = ϵ_r)

↳ partially from b to c

$q \equiv$ Charge on a length 'l' on the copper wire,

$$\oint \vec{D} \cdot d\vec{a} = q$$

$$\Rightarrow \vec{D} = \frac{q}{2\pi r l} \hat{r}$$

Here, $r \equiv$ Arbitrary radial distance

$\hat{r} \equiv$ Radial unit vector.

$$\vec{E} = \frac{\vec{D}}{\epsilon_0} \quad (a < r < b)$$

$$= \frac{q}{2\pi r l \epsilon_0} \hat{r}$$

$$\vec{E} = \frac{q}{2\pi r l \epsilon} \hat{r} \quad (b < r < c)$$

$$\Delta V = -\int_c^a \vec{E} \cdot d\vec{l} = \int_a^b \frac{\lambda}{2\pi \epsilon_0 r l} \, dr$$

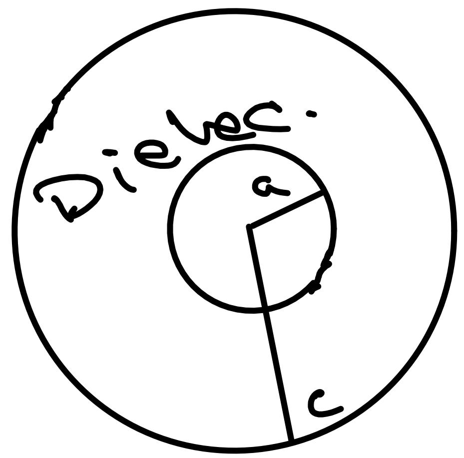
$$+ \int_b^c \frac{\lambda}{2\pi \epsilon r l} \, dr$$

$$= \frac{\lambda}{2\pi \epsilon_0 l} \left[\ln\left(\frac{b}{a}\right) + \frac{\epsilon_0}{\epsilon} \ln\left(\frac{c}{b}\right) \right]$$

Capacitance per unit length:

$$\Rightarrow \frac{C}{l} = \frac{\lambda}{V l} = \frac{2\pi \epsilon_0}{\ln\left(\frac{b}{a}\right) + \left(\frac{1}{\epsilon_r}\right) \ln\left(\frac{c}{b}\right)}$$

Ex:



The space betn. is fully filled with dielectric.

$$\vec{E} = \frac{q}{2\pi r l \epsilon}\hat{r}$$

$(a < r < c)$

Linear charge density on wire $= \lambda$

$q = \lambda l$

$$= \frac{\lambda}{2\pi r \epsilon}\hat{r}$$

$$\Delta V = \int_a^b \vec{E}\cdot d\vec{l} = \int_a^b \frac{\lambda}{2\pi r \epsilon}\, dr$$

$$= \frac{\lambda}{2\pi \epsilon}\ln\left(\frac{b}{a}\right)$$

Capacitance, $C = \dfrac{q}{\Delta V} = \dfrac{2\pi \epsilon l}{\ln\left(\frac{b}{a}\right)}$