F1.

a)

- `Xor = \a.\b.a (b False True) (b True False)`
- `Maj = \a.\b.\c.a (b True (c True False)) (b (c True False) False)`

b)

- `IsZero = \a.a (\x.True False x) True`
- ```
  Invert = \a.a False True
  IsEven = \a.a (\x.Invert x) True
  ```

c)

- ```
  Y = \f.(\x.x x) (\x.f (x x))
  Next = \h.\f.\x.f (h f x)
  Sum = \x.\y.x Next y
  AlmostFib = \f.\x.(Eq x 0) 0 (
    (Eq x 1) 1 (
      (Sum (f (Sub x 1)) (f (Sub x 2)))
    )
  )
  Fib = Y AlmostFib
  AlmostDiv = \f.\n.\c.(IsZero n) c (f (Sub n 3) (Next c))
  Div = Y AlmostDiv
  Div3 = \x.Div x 0
  ```

d)

e)

- ```
  Pred = \n.\f.\x.n (\g.\h.h (g f)) (\u.x) (\u.u);
  Sub = \x.\y.y Pred x
  Eq = \x.\y.(IsZero (Sub x y)) ((IsZero (Sub y x)) True False) False
  Leq = \x.\y.(IsZero (Sub x y) True False)
  ```

e)

f)

- ```
  Less = \n.\m.(IsZero (Sub n m)) (IsZero (Sub m n) False True) False
    Div = \n.\m.\acc.(Less n m) (Pair acc n) (Div (Sub n m) m (Next acc))
  ```

2.

3.

4. **a)**

   - ```
     Len = \list.\cnt.(IsNil list) cnt (Len (Tail list) (Next cnt))
     LenOfList = \list.Len list 0
     ```

   b)

   - ```
     AlmostMap = \f.\first.\second.\acc.(IsNil first) acc ((IsNil second) acc
     (AlmostMap f (Tail first) (Tail second) (Cons (f (Head first) (Head second))
     acc)))

     Map2 = \f.\first.\second.Rev (AlmostMap f first second Nil)
     ```

   c)

   - ```
     AlmostRev = \list.\acc.(IsNil list) acc (AlmostRev (Tail list) (Cons (Head
     list) acc))
     Rev = \x.AlmostRev x Nil
     ```

7. - $(\lambda x.xx)((\lambda y.y)(\lambda z.z))$
     - $((\lambda y.y)(\lambda z.z)) \wedge ((\lambda y.y)(\lambda z.z))$
     - $(\lambda x.x\ x)(\lambda z.z)$
     ```

2. $\overline{2} = \lambda f.\lambda x.f(fx)$

$\overline{2}\,\overline{2} = (\lambda f.\lambda x.f(fx))(\lambda f'.\lambda x'.f'(f'x')) \underset{\beta}{\to} \lambda x.(\lambda f'.\lambda x'.f'(f'x'))((\lambda f'.\lambda x'.f'(f'x'))x) \underset{\beta}{\to}$
$\lambda x.(\lambda f'.\lambda x'.f'(f'x'))(\lambda y'.x(x\,y')) \underset{\beta}{\to} \lambda x.(\lambda x'.(\lambda y'.x(x\,y'))((\lambda y'.x(x\,y'))x')) \underset{\beta}{\to}$
$\lambda x.(\lambda x'.(\lambda x'.x(x\,x'))(x(x\,x'))) \underset{\beta}{\to} \lambda x.\lambda x'.(x(x\,(x(x\,x'))))$

$\overline{2}\,\overline{2}\,\overline{2}$

```
λ> (λf.λx.f (f (f (f x)))) (λf.λx.f (f x))
λ > ((λf. (λx. (f (f (f (f x)))))) (λf. (λx. (f (f x)))))
α > ((λf. (λx. (f (f (f (f x)))))) (λX0. (λX1. (X0 (X0 X1)))))
β > (λx. ((λX0. (λX1. (X0 (X0 X1)))) ((λX0. (λX1. (X0 (X0 X1)))) ((λX0. (λX1. (X0 (X0
X1)))) ((λX0. (λX1. (X0 (X0 X1)))) x)))))
β > (λx. ((λX0. (λX1. (X0 (X0 X1)))) ((λX0. (λX1. (X0 (X0 X1)))) ((λX0. (λX1. (X0 (X0
X1)))) (λX1. (x (x X1)))))))
α > (λx. ((λX0. (λX1. (X0 (X0 X1)))) ((λX0. (λX1. (X0 (X0 X1)))) ((λX0. (λX1. (X0 (X0
X1)))) (λX2. (x (x X2)))))))
β > (λx. ((λX0. (λX1. (X0 (X0 X1)))) ((λX0. (λX1. (X0 (X0 X1)))) (λX1. ((λX2. (x (x
X2))) ((λX2. (x (x X2))) X1))))))
β > (λx. ((λX0. (λX1. (X0 (X0 X1)))) ((λX0. (λX1. (X0 (X0 X1)))) (λX1. ((λX2. (x (x
X2))) (x (x X1)))))))
α > (λX3. ((λX0. (λX1. (X0 (X0 X1)))) ((λX0. (λX1. (X0 (X0 X1)))) (λX1. ((λX2. (X3
(X3 X2))) (X3 (X3 X1)))))))
β > (λX3. ((λX0. (λX1. (X0 (X0 X1)))) ((λX0. (λX1. (X0 (X0 X1)))) (λX1. (X3 (X3 (X3
(X3 X1))))))))
α > (λX3. ((λX0. (λX1. (X0 (X0 X1)))) ((λX0. (λX1. (X0 (X0 X1)))) (λX4. (X3 (X3 (X3
(X3 X4))))))))
β > (λX3. ((λX0. (λX1. (X0 (X0 X1)))) (λX1. ((λX4. (X3 (X3 (X3 (X3 X4))))) ((λX4. (X3
(X3 (X3 (X3 X4))))) X1)))))
β > (λX3. ((λX0. (λX1. (X0 (X0 X1)))) (λX1. ((λX4. (X3 (X3 (X3 (X3 X4))))) (X3 (X3
(X3 (X3 X1))))))))
α > (λX5. ((λX0. (λX1. (X0 (X0 X1)))) (λX1. ((λX4. (X5 (X5 (X5 (X5 X4))))) (X5 (X5
(X5 (X5 X1))))))))
β > (λX5. ((λX0. (λX1. (X0 (X0 X1)))) (λX1. (X5 (X5 (X5 (X5 (X5 (X5 (X5 (X5
X1))))))))))))
α > (λX5. ((λX0. (λX1. (X0 (X0 X1)))) (λX6. (X5 (X5 (X5 (X5 (X5 (X5 (X5 (X5
X6))))))))))))
β > (λX5. (λX1. ((λX6. (X5 (X5 (X5 (X5 (X5 (X5 (X5 (X5 X6))))))))) ((λX6. (X5 (X5 (X5
(X5 (X5 (X5 (X5 (X5 X6))))))))) X1))))
β > (λX5. (λX1. ((λX6. (X5 (X5 (X5 (X5 (X5 (X5 (X5 (X5 X6))))))))) (X5 (X5 (X5 (X5
(X5 (X5 (X5 (X5 X1)))))))))))
α > (λX7. (λX1. ((λX6. (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 X6))))))))) (X7 (X7 (X7 (X7
(X7 (X7 (X7 (X7 X1)))))))))))
β > (λX7. (λX1. (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7
X1))))))))))))))))))
>>> (λX7. (λX1. (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7 (X7
X1))))))))))))))))))
```

b)

Здесь и далее $m \equiv \overline{m}$ в зависимости от контекста и $\underset{\beta}{=} \equiv =$.

Индукцией покажем, что $nm = m^n$. Действительно,

$$nm = (\text{Inc } (n-1))m = (\lambda f.\lambda x.(n-1)f(fx))m = \lambda x.(n-1)m(mx)$$

По предположению индукции

$$\lambda x.((n-1)m)(mx) = \lambda x.m^{n-1}(mx) = \overline{m^n}$$

Осталось показать, что для любых a и b верно.

$$\lambda f.a(bf) \underset{?}{=} \overline{ab}$$

Покажем это индукцией по a.

$$\lambda f.a(bf) = \lambda f.(\lambda x.(bf)^a x) = \lambda f.(\lambda x.(bf)^{a-1}((bf)x)) = \lambda f.\left(\lambda x.f^{b(a-1)}f^b x\right) = \overline{ab}$$

5.

$S = \lambda x.\lambda y.\lambda z.xz(yz)$
$K = \lambda x.\lambda y.x$
$I = \lambda x.x$

a)
$\lambda x.x\ x = SII$:
$\lambda x.\lambda y.\lambda z.xz(yz)\ I\ I \underset{\beta}{\to} \lambda y.\lambda z.Iz(yz)\ I \underset{\beta}{\to} \lambda z.Iz(Iz) \underset{\beta}{\to} \lambda z.z(Iz) \underset{\beta}{\to} \lambda z.z\ z$
$\Omega = (\lambda x.x\ x)\ (\lambda x.x\ x) = (SII)\ (SII)$

b)
$F = \lambda x.\lambda y.y \Leftrightarrow \lambda y.\lambda x.x = KI$:
$\lambda x.\lambda y.x\ \lambda x.x \underset{\beta}{\to} \lambda y.\lambda x.x$
$\overline{1} = \lambda f.\lambda x.f\ x = SF$:
$\lambda y.\lambda f.\lambda x.yx(fx)\ \lambda x'.\lambda y'.y' \underset{\beta}{\to} \lambda f.\lambda x.(\lambda x'.\lambda y'.y')x(fx) \underset{\beta}{\to} \lambda f.\lambda x.(\lambda y'.y')(fx) \underset{\beta}{\to} \lambda f.\lambda x.fx$

c)
$\lambda x.\lambda y.\lambda z.y = KK$:
$\lambda x'.\lambda x.x'\ \lambda y.\lambda z.y \underset{\beta}{\to} \lambda x.\lambda y.\lambda z.y$

7

$(\lambda x.xx)((\lambda y.y)(\lambda z.z)) \underset{\beta}{\to} (\lambda x.xx)(\lambda z.z)$

$(\lambda x.xx)((\lambda y.y)(\lambda z.z)) \underset{\beta}{\to} ((\lambda y.y)(\lambda z.z))((\lambda y.y)(\lambda z.z))$

8

a)

рассмотрим $\vdash A : \varphi$

доказательство типа A было получена по одному из правил:

Определение

Просто-типизированное лямбда-исчисление (по Карри). Типы: $\tau ::= \alpha | (\tau \to \tau)$.
Язык: $\Gamma \vdash A : \varphi$

$$\frac{}{\Gamma, x : \varphi \vdash x : \varphi}\ x \notin \Gamma \qquad \frac{\Gamma, x : \varphi \vdash A : \psi}{\Gamma \vdash \lambda x.A : \varphi \to \psi}\ x \notin \Gamma \qquad \frac{\Gamma \vdash A : \varphi \qquad \Gamma \vdash B : \varphi \to \psi}{\Gamma \vdash BA : \psi}$$

если по правилу 1, то это атомарная переменная, не имеет подвыражений

если по правилу 2, то имеет вид $A = \lambda x.B$, и B имеет тип

если по правилу 3, то $A = BC$, и B, и C имеют тип

b)

пусть у Y или Ω есть тип

тогда у всех его подвыражений должен быть тип

в какой-то момент будет использовано правило 2:

$$\frac{\Gamma \vdash x : t_1 \to t_2 \quad \Gamma \vdash x : t_1}{\Gamma \vdash x\,x : t_2}$$

получается, у x бесконечный тип $t_1 = t_1 \to t_2 = (t_1 \to t_2) \to t_2 ...$

c)

```
newtype Mu a = Mu (Mu a -> a)
y f = (\h -> h $ Mu h) (\x -> f . (\(Mu g) -> g) x $ x)


newtype OmegaType = OmegaType { apply :: OmegaType -> OmegaType }

omega :: OmegaType -> OmegaType
omega x = apply x x
```

бесконечный тип
9a

$$\Gamma = \{n : (\alpha \to \alpha) \to (\alpha \to \alpha), f : (\alpha \to \alpha), x : \alpha\}$$

$$\cfrac{\cfrac{\cfrac{}{\Gamma \vdash x : \alpha} \quad \cfrac{}{\Gamma \vdash f : \alpha \to \alpha}}{\Gamma \vdash f\ x : \alpha} \quad \cfrac{\cfrac{}{\Gamma \vdash n : (\alpha \to \alpha) \to (\alpha \to \alpha)} \quad \cfrac{}{\Gamma \vdash f : \alpha \to \alpha}}{\Gamma \vdash n\ f : (\alpha \to \alpha)}}{\cfrac{\Gamma = \{f, n, x\} \vdash n\ f(f\ x) : \alpha}{\cfrac{\{f, n\} \vdash \lambda x.n\ f(f\ x) : (\alpha \to \alpha)}{\cfrac{\{n\} \vdash \lambda f.\lambda x.n\ f(f\ x) : (\alpha \to \alpha) \to (\alpha \to \alpha),}{\vdash \lambda n.\lambda f.\lambda x.n\ f(f\ x) : ((\alpha \to \alpha) \to (\alpha \to \alpha)) \to (\alpha \to \alpha) \to (\alpha \to \alpha)}}}}$$