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On Ringeisen's isolation game II

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Abstract

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We further develop the theory of Ringeisen's isolation game on a graph, in which two players alternately 'switch' at successive vertices v not previously switched. The switching operation deletes all edges incident with v, and creates new edges between v and those vertices not previously adjacent to it. The game is won when a vertex is first isolated. In this paper we prove the (somewhat surprising) result that with best play such games can be won only either very early or very late, implying that most graphs are nonwinnable by either player.

1. Background

We shall deal exclusively with finite undirected graphs $H = (V, E)$ which are *simple* (no loops, no multiple edges), and set $n = |V| > 1$ throughout. The neighborhood set of a vertex $v \in V$ will be denoted by $N(v) = \{x \in V: (v, x) \in E\}$, ($N(v, H)$ if the underlying graph needs specification); its cardinality, $d_H(v)$, is the *degree* of vertex v.

The operation of *switching* H at $v \in V$ (briefly: 'switching v'), apparently introduced by van Lint and Seidel [12], replaces H by the graph obtained from H by deleting the edges $\{(v, x): x \in N(v)\}$ and adjoining new edges $\{(v, y): y \notin N(v)\}$. This switching operation and its induced equivalence relation have been studied, e.g. by Colbourn and Corneil [7], Mallows and Sloane [13], Taylor [20], and Goldman [9].

In 1974 Ringeisen [15] introduced the *isolation game* $I_n(H)$, describable for our purposes as follows: play begins with the n-vertex graph H. Players 1 and 2, denoted P1 and P2, switch alternately, each time at a vertex not previously used

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for switching. Play ends as soon as one player succeeds in isolating a vertex; otherwise the game is drawn after move n. (Ringeisen's definition of $I_n(H)$ required H to be connected. This condition proved inessential, and so we require only (to avoid trivial 'pre-won' cases) that the graph H be free of isolated vertices.)

For which graphs H is $I_n(H)$ a win for P1 (assuming best play), or a win for P2, or a draw? If a win, how long can the loser postpone defeat? For example [15], for $H = K_n$ (complete), any switch is an immediate win for P1, while for $H = C_n$ (*n-cycle*, $n > 3$) P2 can quickly win. In [15] it is also shown that for $H = K_{q,n-q}$ (complete bipartite, $2 \leq q \leq n-2, n > 4$), neither player has a forced win. Surprisingly, it appears that no further analyses of $I_n(H)$ had been published (Ringeisen, personal communications). The present paper is the fourth in a series (based on [16]) redressing this neglect.

A first obstacle in analyzing $I_n(H)$ is the difficulty of 'tracking' the more-than-local changes in H produced by switching operations. That difficulty was overcome by Theorem 2.1 in [17], a result which we repeat as:

Theorem 1.1. *A play of $I_n(H)$ ends, with v as isolated vertex and S the set of switched vertices, iff S is $N(v)$ or its complement $N(v)^c$.*

Note that the identity of the winning player is determined by the parity of $|S|$, the number of moves in the win, which by the Theorem must be $d_H(v)$ or $n - d_H(v)$. For example, if all vertex-degrees in H are odd then (since n must be even) P2 cannot win $I_n(H)$, while if all degrees are even and n is even, P1 cannot win.

The above theorem is useful because it allows reasoning about the progress of the game to be carried out in terms only of the initial graph: its underlying neighborhood sets and their complements. This proved sufficient in [17] to permit considerable further analysis of $I(H)$; for example, drawing strategies for one player in a large class of graphs were formulated and justified, and a problem closely related to nonwinnability of $I_n(H)$ was shown to be NP-complete. (Other results from [17] are noted later.) However, a general analysis was thwarted by a second obstacle, our inability to find a proof-facilitating recursive structure: the result of a partial play of $I_n(H)$ does not seem to correspond to any $I_m(H')$, a consequence of the 'symmetry-spoiling' presence of each $v \in V$ in the complement of its neighborhood. This motivated imbedding the isolation games in a larger class of games which do admit recursive treatment. Theorem 1.1 suggested such a class, which we describe next.

The *set coincidence game* $G(V, W)$ is played on a finite nonempty set V of *elements*. W is a collection of nonempty subsets of V, the *winning* sets. Players P1 and P2 move alternately, with P1 leading off; at each turn, a player adds a new element to an expanding set S, which was empty at the start of play. If a player's move causes S to coincide with some $w \in W$, then that player *wins* (the opponent loses), and play ends. If V is exhausted (i.e., $S = V$) without a win, then the game is drawn. The fact that both players' choices contribute to building up a *single* set S, rather than individual sets S^1 and S^2, suffices to differentiate $G(V, W)$ from the more-studied 'positional games' of Berge [3], whose more specialized 'types 1 and 2' [2], called 'amoeba games' (weak and strong) in Beck and Csirmaz [1], in turn include most of the 'achievement and avoidance' games of Harary (e.g. [10–11]). On the other hand, the diameter and geodesic achievement games of Buckley and Harary [5–6] are set coincidence games. For an n-vertex graph $H = (V, E)$, Theorem 1.1 asserts precisely that $I_n(H) = G(V, W)$ where W consists of $N(v)$ and $N(v)^c$ for all $v \in V$.

That the games $G(V, W)$ indeed form a class admitting recursive treatment is readily seen. For, consider a partial play of $G(V, W)$ which has not yielded a win, and as above, let S denote the set of elements selected so far (by both players). Then the resultant *continuation game*, denoted $G(V, W, S)$, is readily seen to coincide with the game $G(V - S, W_S)$ where $W_S = \{w - S: w \in W, S \subset w\}$. (This is precisely the notion of 'induced hypergraph'.)

$G(V, W)$ will be called a *forced p-win* if one of the players has a strategy assuring a win in no more than p moves, but the opponent has at least one way to prolong play to a full p moves. (Thus $p \leqslant n = |V|$, and the winner is P1 or P2 according as p is odd or even.) If $G(V, W)$ is a forced p-win for some p, we call it a *forced win*. For example, by Theorem 1.1, $I_n(H)$ is a forced 1-win iff graph H contains a vertex of degree 1 or $n - 1$. It was shown in [17] that for each p there exist connected graphs H for which $I_n(H)$ is a forced p-win (if $p \neq 3$, H can be chosen bipartite), and all forced p-wins $I_n(H)$ for $p \leqslant 3$ were characterized.

Note that $G(V, W)$ is a forced win for P1 iff either (i) it is a forced 1-win, i.e., W includes a singleton, or (ii) at least one of the continuation games $\{G(V, W, \{v\}): v \in V\}$ is a forced win for its *second* player. Since checking W for singletons can be regarded as trivial, we see that the problem of determining whether forced winnability by P1 holds, can be reduced to the corresponding problem (on a smaller game) for P2. Thus our analyses in [18] and [19] often concentrated on the latter problem, with assurance that no loss of generality could result.

Except where continuation games are involved, the set V of elements affects $G(V, W)$ only via its cardinality n, and the above-mentioned recursive arguments involved induction on n. We therefore often write $G(n, W)$ instead of $G(V, W)$, implicitly assuming $V = \{1, 2, \ldots, n\}$, when no ambiguity is possible.

A natural question is: given any particular $G(n, W)$, is it a forced win for P1 or a forced win for P2 or a draw? In [18] we showed—for a 'tightened' encoding of $G(n, W)$—that the decision problem for forced-winnability by P2 is PSPACE-complete. This complements the known result [8] that positional games of the first type [3] (and therefore the general class of positional games) are complete in PSPACE. For positional games of the second type, it is known [3] that P2 cannot have a forced win; we do not know the status of the decision problem for forced-winnability by P1.

As will be shown, it is fruitful for the analysis of isolation games $I_n(H)$ to consider the combinatorial optimization problem

$$\Pi(n, p): \quad \min\{|W|: G(n, W) \text{ a forced } p\text{-win}\}$$

which might confront a *designer* of set coincidence games required to produce a forced p-win using a limited allowance of winning sets. Note that feasibility of $\Pi(n, p)$ is not in question, since choosing W to consist of all p-sets in V certainly yields a forced p-win. The case $p = n$ is trivial (just take $W = \{V\}$).

Analysis of $\Pi(n, p)$ is aided by visualizing the Hasse diagram of subsets of V as a digraph D_n, with an arc from each node at *level* λ of the digraph (these nodes are just the λ-sets of V) for $\lambda = 0, 1, \ldots, n-1$, to each of the $(\lambda+1)$-sets that contains it. Each play of $G(n, W)$ corresponds to a *path* in D_n, beginning at the root-node (ϕ) of D_n and rising through nodes S_λ at successive levels λ until terminated either by reaching some $w \in W$ (a win) or by winlessly reaching the single level-n node V (a draw). Thus S_λ denotes the 'value' of the expanding set S just after move λ. We use the term 'trajectory' to denote the sequence of subsets of *even* cardinality encountered along a path.

In [18], we showed that in game $G(n, \emptyset)$, the minimum width for any fixed strategy of P2 of the tree of 'attainable' play-trajectories (corresponding to the different strategies for P1) increases rapidly as the tree rises from level to level, until the mid-level $\lceil n/2 \rceil$ is reached. This suggests the intuition that unless $n-p$ is small (p even), an optimal solution W of $\Pi(n, p)$ must place its meager number of winning sets within D_n so as to *limit play* at the lower levels to just a very few trajectories, in the sense that deviations by the winning player P2 are punished by 'losing the win' (permitting the opponent to draw or win), while deviations by P1 are punished by premature loss. Accordingly, in Section 2 of [19], we characterized those feasible solutions W of $\Pi(n, p)$—to be called *p-filters*—which (roughly speaking) minimize $|W|$ subject to the further restriction of limiting play at the first $p-2$ levels to just a single trajectory. The preceding 'intuition' was then formalized by a precise statement of the *Filter Conjecture*: unless $n-p$ is small, the optimal solutions of $\Pi(n, p)$ are precisely the p-filters.

We have not succeeded in proving the Filter Conjecture, and offer its general case as a challenging open problem. Section 3 of [19] contains our (increasingly complicated) verifications of its low-order cases $p = 2, 4, 6$. Fortunately, these cases were adequate to provide most of the induction base for establishing, in Section 4 of [19], the following weaker result: for even $p \geq 8$, unless $n-p$ is small, $n+3$ is a lower bound for the optimal value of $\Pi(n, p)$. As will be shown below, this more limited result is sufficient to permit completing our analysis of the isolation games $I_n(H)$, with the surprising outcome that (apart from a few identified possible exceptions) these games can be forced-won only either very *early* ($p \leq 5$) or very *late* ($p = n-2$). Sharper results should be expected for particular classes of graphs; in a subsequent paper too lengthy for incorporation with the present work, we specialize to *regular* graphs H, with results which in particular complete the analysis begun in [17] of nonwinnability of $I_n(H)$ for several 'classical' graphs given in Bondy and Murty [4].

Before beginning the body of the paper, we remind the reader of the notation S_λ defined above, and introduce the notation W_λ for the family of winning λ-sets in $G(n, W)$. The complement of a set B, with respect to some context-specified superset, will be denoted B^c.

2. The isolation game theorem

We first give three results about set coincidence games proven in [19, cf. Theorems 2.3, 3.3, 4.1, and the proof of Lemma 2.3]. Throughout the next two sections we set $k = \lceil n/2 \rceil$.

Theorem 2.1. *Let $G(n, W)$ be a forced 6-win. If $n \geqslant 10$, then $|W| \geqslant 3k - 5$. If $n \geqslant 11$ and $|W| = 3k - 5$, then $|W_2| = k - 1$ and $|W_4| = |W_6| = k - 2$. If $n = 9$, then $|W| \geqslant 9$, with equality only if $|W_2| = 3$, $|W_4| = 0$ and $|W_6| = 6$.*

Theorem 2.2 (Weak Filter Theorem). *Assume $G(n, W)$ is a forced $2m$-win, with $m \geqslant 4$. If $n \geqslant 2m + 3$, then $|W| \geqslant n + 3$.*

Theorem 2.3. *Let $G^* = G(V, W^*)$ be formed from forced p-win $G = G(V, W)$ by deleting from W all sets at levels $>p$ or differing in parity from p. Then G^* is also a forced p-win.*

Lemma 2.1. *For $I_n(H)$ with $n > 1$, any 1-vertex set $\{v\}$ is completable to at most n winning sets, any 2-vertex set $\{v, w\}$ to at most $n - 1$ winning sets.*

Proof. The winning sets can (without regard to distinctness) be arranged in n pairs $N(u)$, $N(u)^c$, one for each vertex u. Switching v renders incompletable one member of each pair, $N(u)^c$ if u is adjacent to v and $N(u)$ otherwise. If $N(v) = N(w)$ then there are at most $n - 1$ distinct pairs, hence at most $n - 1$ completable winning sets. If not, then some vertex u must be adjacent to one but not both members of $\{v, w\}$, so that switching on $\{v, w\}$ spoils completability of *both* members of the pair $N(u)$, $N(u)^c$, as well as at least one member of each other pair; again at most $n - 1$ winning sets remain completable. $\square$

We now state the main result of this paper, which achieves the goal that motivated [17–19].

Theorem 2.4 (Isolation Game Theorem). *If the isolation game $I_n(H)$ with $n > 2$ is a forced win, then it is one of the following*:

(i) *A forced p-win with $p \leqslant 5$ or $p = n - 2$.*
(ii) *A forced 6-win on* 10, 12 *vertices.*
(iii) *A forced 7-win on* 10, 11 *vertices.*
(iv) *A forced 8-win on* 12 *vertices.*

Proof. Assume that $I_n(H) = G(V, W) = G$ is a forced p-win; recall that W consists of all neighborhood sets $N(v)$ and their complements. We give the main line of the proof below, with the elimination of some more difficult cases deferred to Section 3 (Theorems 3.3–5).

Claim 1. $n \geqslant p + 2$.

Proof. Since at least one winning set has cardinality p, we have $n \geqslant p$. If $p = n$ then some winning set $N(v)$ or $N(v)^c$ would be all of V; the first case is impossible since $v \in N(v)^c$, the second because it implies that v is isolated at the outset of play. And if $p = n - 1$ then the game would have a winning $(n-1)$-set, whose singleton complement is therefore also winning; this implies $p = 1$ and thus $n = 2$, a contradiction. □

We may assume $p > 2$, else (i) of the Theorem would hold. Since G is a forced p-win, there exist two vertices $\{v, w\}$ such that the continuation games

$$G' = G(n-1, W') = G(V, W, \{v\})$$

and

$$G'' = G(n-2, W'') = G(V, W, \{v, w\})$$

are respectively a forced $(p-1)$-win and a forced $(p-2)$-win.

Claim 2. *If $p = 2m + 1$ with $m \geqslant 4$, then $n = 2m + 3$.*

Proof. By Claim 1, $n \geqslant 2m + 3$. Suppose $n > 2m + 3$. By Lemma 2.1, $|W'| \leqslant n$. Since $m \geqslant 4$ and $n - 1 \geqslant 2m + 3$, Theorem 2.2 applies to G' to yield $|W'| \geqslant (n-1) + 3$, giving a contradiction. □

Claim 3. *If $p = 2m$ with $m \geqslant 5$, then $n = 2m + 2$.*

Proof. By Claim 1, $n \geqslant 2m + 2$. Suppose $n > 2m + 2$. By Lemma 2.1, $|W''| \leqslant n - 1$. Since $m - 1 \geqslant 4$ and $n - 2 \geqslant 2(m-1) + 3$, Theorem 2.2 applies to G'' to yield $|W''| \geqslant (n-2) + 3$, giving a contradiction. □

Consequences of Claims 1–3. It follows from Claims 2 and 3 that for all $p \geqslant 9$, the only possible value for n is $p + 2$. This situation falls under (i) of the theorem, as do the cases $p \leqslant 5$. Thus it only remains to consider the values $p = 6, 7, 8$. To aid this analysis, we refer to a vertex v as *doubled up* with respect to p, when the complementary winning sets $N(v)$ and $N(v)^c$ have cardinalities $\leqslant p$ which share the parity (π) of p. Note that if n is odd, or if $n > 2p$, then no vertex can be doubled up and so $I_n(H) = G(n, W)$ can have at most n winning sets on levels $\leqslant p$ of parity π, i.e. the forced p-win $G^* = G(n, W^*)$ of Theorem 2.3 has $|W^*| \leqslant n$.

For $p = 6$, Claim 1 gives $n \geqslant 8$. If $n \geqslant 13$ or $n = 9$, 11, no vertex can be doubled up and so $|W^*| \leqslant n$; the inequality $n \geqslant 3k - 5$ obtained by applying Theorem 2.1 to G^* rules out $n = 11$ and all $n \geqslant 13$. The case $n = 8$ falls under alternative (i) of the theorem and the case $n = 9$ under Theorem 3.3, leaving only the possibilities listed in (ii).

For $p = 7$, Claim 1 gives $n \geqslant 9$, with the case $n = 9$ falling under alternative (i). Theorem 2.1 applied to G', plus Lemma 2.1, rule out all $n \geqslant 12$. Theorem 3.4 eliminates $n = 13$, leaving only the cases listed in (iii).

For $p = 8$, Claim 1 gives $n \geqslant 10$, with $n = 10$ falling under alternative (i). For $n \geqslant 17$ and for all odd $n \geqslant 11$, no vertex can be doubled up and so $|W^*| \leqslant n$, contradicting Theorem 2.2 applied to G^*. To rule out $n = 16$ we observe that the continuation game $G'' = G(n-2, W'')$ is now a forced 6-win on 14 vertices, and so Theorem 2.1 yields $|W''| \geqslant 16$ whereas Lemma 2.1 gives $|W''| \leqslant 15$. Theorem 3.5 rules out $n = 14$, so the only remaining case is that listed under (iv). □

Remark. Several remarks about this theorem seem appropriate here. First, the existential status of its main case, (i), is affirmatively settled for all p by Theorems 2.2–2.5 of [17], even under the further requirement of connectivity for H. Second, the existential status of the exceptional cases (ii)–(iv) is at the moment unsettled (we are skeptical about them), but in any event the main content of the theorem is that with at most a few stipulated exceptions, forced wins in the isolation game either occur very *early*, i.e., in the first five moves or else can be delayed by the opponent until very *late*, i.e., just two moves, before the end of play. Third, the theorem implies that for all $p > 5$, the number of vertices in a forced p-win is at most $p + 6$ (at most $p + 2$ if cases (ii)–(iv) can be ruled out). Theorem 2.6 of [17], which asserts that for each $p \leqslant 5$ there exist forced p-wins $I_n(H)$ with n arbitrarily large (and H connected), shows that in this regard five is indeed the critical value of p. Fourth, we note under 'unfinished business', besides the resolution of the listed exceptional cases, also the conjecture in [17] that no graph H with an even number of vertices can have $I_n(H)$ a forced 5-win.

A final remark stems from Theorem 1.1's identification of the special nature of the winning sets in $I_n(H)$. A player seeking to be the first to make the growing set S of switched vertices coincide with one of these special sets, and confronted with a clever opponent who alternates in choosing the members of S, a priori seems unlikely to succeed. Thus one would expect $I_n(H)$ to be a draw (with best play) for most graphs H. We are indebted to E.R. Scheinerman for detailing the following approach to verifying this expectation.

Let $\delta(H)$ and $\Delta(H)$ denote the smallest and largest vertex degrees in H, Γ_n the set of all graphs on the vertex set $V_n = \{1, 2, \ldots, n\}$, and F_n the subset of Γ_n consisting of graphs yielding forced wins. Furthermore, let

$$G_n = \{H \in \Gamma_n: 6 \leqslant \delta(H) \leqslant \Delta(H) \leqslant n-6\}.$$

By Theorems 1.1 and 2.3, for $n > 13$ an n-vertex graph H can give a forced win only if at least one vertex has degree d or $n-d$ for some $d \leqslant 5$. This implies that $F_n \cap G_n = \emptyset$. Now let P_n be a probability measure on Γ_n. Then the above-mentioned expectation can be expressed as $P_n(F_n) \to 0$ $(n \to \infty)$, and would follow from the statement $P_n(G_n) \to 1$. But this last statement is true for many of the probability models, (i.e., sequences $\{P_n\}$) adopted in random graph theory. As a simple example, if each P_n is formed by treating each vertex pair in V_n independently and making it an edge with probability q $(0 < q < 1)$, and if for any ε in $(0, 1)$ with $q(1+\varepsilon) < 1$ we set

$$H_n = \{H \in \Gamma_n: (1-\varepsilon)qn < \delta(H) \leqslant \Delta(H) < (1+\varepsilon)qn\},$$

then (Palmer [14, Theorem 5.1.4]) it is known that $P_n(H_n) \to 1$. Since $H_n \subset G_n$ for almost all n, this implies the desired conclusion $P_n(G_n) \to 1$.

3. The difficult cases

As in Section 2, we begin by citing useful results about set coincidence games proven in [19, cf. Lemmas 3.1, 3.2 and Theorem 3.2].

Theorem 3.1. *If $G(n, W)$ is a forced p-win with $p > 1$, then* $|W_p| \geqslant \lceil (n-p)/2 \rceil + 1$.

Theorem 3.2. *If $G(n, W)$ is a forced 4-win and $n \geqslant 7$, then $|W| \geqslant 2k-2$. If $|W| = 2k-2$ and $n \geqslant 9$, then $|W_2| = |W_4| = k-1$.*

We now give the resolution of the three difficult cases deferred from the proof of Theorem 2.4; these escape the general combinatorial tools developed previously, and hence require further arguments to exploit their special structures.

Theorem 3.3. *There is no forced 6-win $I_9(H)$.*

Proof. Suppose such a forced 6-win $I_9(H) = G(V, W)$ existed. We use the notation of the proof of the Isolation Game Theorem (Theorem 2.4).

Claim 1. $|W_3| = |W_6| = 6$, $|W_2| = |W_7| = 3$, *all other* $W_i = \emptyset$.

Proof. Applying Theorem 2.1 to Theorem 2.3's $G^* = G(V, W^*)$ gives $|W^*| \geq 9$. Since n is odd, no vertex can be doubled up, so $|W^*| \leq n = 9$. By Theorem 2.1 again, $|W_2^*| = 3$ and $|W_4^*| = 0$, implying $|W_6^*| = 6$. By the definition of G^*, it follows that $|W_2| = 3$, $|W_4| = 0$, $|W_6| = 6$. Since W is closed under complementation, we have $|W_7| = 3$, $|W_5| = 0$, $|W_3| = 6$. This accounts for 18 members of W; since $|W| \leq 2n = 18$ by Theorem 1.1, there are no other members. $\square$

The preceding argument also implies that the 18 sets $\{N(v), N(v)^c: v \in V\}$ are distinct; this will be used implicitly below. Note also that by Claim 1, the only possible vertex-degrees in H are $2, 3, 6, 7$.

As before, some continuation game $G'' = G(7, W'') = G(V, W, \{v, w\})$ corresponding to initial best-play choices $\{v, w\}$ is a forced 4-win on 7 elements. By Theorem 2.3, the same is true of the game $G''^* = G(7, W''^*)$ obtained from G'' by deleting all winning sets on odd levels and on levels >4 of G''. By Theorem 3.2, we have $|W''^*| \geq 6$. Since $W_2''^* \subseteq W_4 = \emptyset$, we have $W''^* = W_4''^* \subseteq W_6$, and since $|W_6| = 6$, equality holds. Thus $\{v, w\}$ must lie in all 6 members of W_6.

Claim 2. *The vertices in $N(v)$ have degree 6 or 7; those in $N(v)^c$ have degree 2 or 3.*

Proof. For $u \in N(v)$, $d_H(u) = 2$ implies that P2 could follow P1's initial switch on v with a win on move 2 by switching the other neighbor of u, while if $d_H(u) = 3$ then v could not lie in member $N(u)^c$ of W_6. For $u \in N(v)^c$, $d_H(u) = 7$ implies that P2 could win on move 2 by switching the other nonneighbor of u, while if $d_H(u) = 6$ then v could not lie in member $N(u)$ of W_6. $\square$

Consequence of Claims. It suffices to rule out the two cases $d_H(v) = 2, 3$. Let q be the number of degree-6 vertices in $N(v)$.

First suppose v has degree three, contributing $N(v)$ to W_3 and $N(v)^c$ to W_6. The degree-6 members of $N(v)$ contribute q more of the 6 members of W_6, leaving $5 - q$ to be contributed by degree-3 vertices in $N(v)^c - \{v\}$. The $3 - q$ degree-7 members of $N(v)$ contribute $3 - q$ of the 3 members of W_2, leaving q to be contributed by degree-2 vertices in $N(v)^c - \{v\}$. Summing all vertex degrees yields

$$2q + 3(1 + (5 - q)) + 6q + 7(3 - q) = 39 - 2q,$$

whose oddness is impossible.

Finally, suppose $d_H(v) = 2$, so that $q \leq 2$. Let $N(v) = \{u, u'\}$. Since $W_4 = \emptyset$, P1 could assure without premature loss that $\{v, u\} \subset S_3$, as well as $S_1 = \{v\}$. By Theorem 3.1 applied to $G(V, W, S_3)$, S_3 and thus u must lie in at least 3 members of W_6. Similarly, u' must lie in at least 3 members of W_6.

Since $N(v) \cup \{v\}$ contributes only q to the quota $6 = |W_6| = |W_3|$, there must be exactly $6 - q$ degree-3 vertices $\{u_1, \ldots, u_{6-q}\}$ in the 6-set $N(v)^c - \{v\}$, which must therefore contain exactly q degree-2 vertices $\{v_1, \ldots, v_q\}$. Note that u (u') cannot be adjacent to more than 3 vertices u_j, since this would exclude it from more than 3 members $N(u_j)^c$ of the 6-set W_6. Thus u (u') must be adjacent to at least $d_H(u)-5$ $(d_H(u')-5)$ of the vertices $\{v_1, \ldots, v_q\}$.

If $q<2$ so that $\max\{d_H(u), d_H(u')\}=7$, then the last sentence would imply at least 2 vertices in $\{v_1, \ldots, v_q\}$, a contradiction. So assume $q=2$. Then the degree-6 vertex u cannot be adjacent to more than 2 vertices u_j (since it would be excluded from member $N(u)$ of W_6 as well as each such $N(u_j)^c$), and so must be adjacent to both of the degree-2 vertices $\{v_1, v_2\}$. Similarly, u' is adjacent to both of the degree-2 vertices $\{v_1, v_2\}$. It follows that $N(v_1)=N(v_2)=\{u, u'\}$, contradicting the distinctness of all neighbor-sets.

Theorem 3.4. *There is no forced 7-win $I_{13}(H)$.*

Proof. Suppose $G=G(13, W)=I_{13}(H)$ is such a 7-win. For suitable $v, w \in V$, $G'=G(12, W')=G(V, W, \{v\})$ is a forced 6-win on 12 elements, while $G''=G(11, W'')=G(V, W, \{v, w\})$ is a forced 5-win on 11 elements.

Claim 1. $|W_3|=|W_{10}|=5$, $|W_5|=|W_6|=|W_7|=|W_8|=4$, *and all other* $W_i=\emptyset$. *Also*, $v\in(\bigcap W_3)\cap(\bigcap W_5)\cap(\bigcap W_7)$, $v\in V-(\bigcup W_6)\cup(\bigcup W_8)\cup(\bigcup W_{10})$, *and* $W'=W_2'\cup W_4'\cup W_6'$.

Proof. By Lemma 2.1, $|W'|\leq 13$. By Theorem 2.1 applied to G', it follows first that $|W'|\geq 13$ and then, since equality holds, that $W'=W_2'\cup W_4'\cup W_6'$ has $|W_2'|=5$ and $|W_4'|=|W_6'|=4$, inducing $5, 4, 4$ members $\{v\}\cup w'$ $(w'\in W')$ of W_3, W_5, W_7 respectively. Taking complements yields $5, 4, 4$ v-free members of W_{10}, W_8, W_6 respectively. This accounts for 26 members of W; since Theorem 1.1 yields $|W|\leq 26$, there are no other members, and the stated results follow. □

Claim 2. *The vertices in* $N(v)$ *have degrees in* $\{3, 5, 7\}$; *those in* $N(v)^c$ *have degrees in* $\{6, 8, 10\}$.

Proof. For $u\in N(v)$, $N(u)$ is a winning set that contains v, so by Claim 1 it lies in $W_3\cup W_5\cup W_7$. For $u\in N(v)^c$, $N(u)$ is a winning set that does not contain v, so by Claim 1 it lies in $W_4\cup W_6\cup W_8$. □

Claim 3. *No vertex appears together with v in more than one $w_3\in W_3$.*

Proof. This says that no vertex lies in more than one $w_2'\in W_2'$. Its demonstration alludes to some more detailed definitions and results from [19]: since $|W'|=13$, by Theorem 3.3 of [19] W' must be a '6-filter', and hence by definition '6-directive', which by definition implies the existence of a 2-set $Q_2\subseteq V-\{v\}$ such that for each $u\in V-\{v\}-Q_2$, $G(V-\{v\}, W', \{u\})$ is a forced 1-win. Thus each element u of the 10-set $V - \{v\} - Q_2$ lies in at least one member of W_2'; since W_2' consists of 5 2-sets, the result follows. □

Claim 4. *H contains either $5 - I(d_H(v) = 10)$ degree-3 vertices and no degree-10 vertices except possibly for v, or else no degree-3 vertices and $5 - I(d_H(v) = 10)$ degree-10 vertices except possibly for v.*

Proof. Let x denote a generic degree-3 vertex, y a generic degree-10 vertex other than v. The 5 members of W_3 consist of all $N(x)$, all $N(y)^c$ and of $N(v)^c$ in case $d_H(v) = 10$. Since the proof of Claim 1 yields distinctness of all 26 sets $N(u)$ and $N(u)^c$, there are $5 - I(d_H(v) = 10)$ vertices of types x and y together. By Claim 2, $x \in N(v)$ and $y \in N(v)^c - \{v\}$.

If any x and y were adjacent, this would imply $\{v, y\} \subseteq N(x) \cap N(y)^c$, contradicting Claim 3. So no such adjacency is possible. If any y exists, it can be non-adjacent to only one vertex outside $\{v, y\}$, so the last sentence implies that at most one x exists. If any x exists with at least two different y's, say y_1 and y_2, then the next-to-last sentence gives $\{v, x\} \subseteq N(y_1)^c \cap N(y_2)^c$, again contradicting Claim 3. So if both an x and a y exist, then only one of each exists, shortfalling the quota $5 - I(d_H(v) = 10)$. Hence exactly one of the two types must be absent, and the result follows. □

Claim 5. *Vertex w, and another vertex q with $\{v, w, q\} \notin W_3$, are adjacent to every vertex of degree 5 or 7, but to no vertex of degree 6 or 8.*

Proof. G necessarily has a continuation game $G''' = G(10, W''') = G(V, W, \{v, w, q\})$ which is a forced 4-win on 10 elements. Thus $\{v, w, q\} \notin W_3$. By Theorem 3.2, we have $|W'''| \geq 8$. Since W' has no winning sets on odd-levels or levels >6 (cf. Claim 1), W''' has no winning sets on odd levels or levels >4. Thus $W''' = W_2''' \cup W_4'''$, inducing at least 8 members of $W_5 \cup W_7$, which by Claim 1 is in fact an 8-set with $|W_5| = |W_7| = 4$. It follows that equality holds: $|W_2'''| = |W_4'''| = 4$ and each member of $W_5 \cup W_7$ is induced by a member of $W_2''' \cup W_4'''$, and therefore contains $\{w, q\}$. That $\{w, q\} \subseteq \bigcap(W_5 \cup W_7)$ is exactly the content of the claim. □

Consequences of Claims 1–5. By Claim 2, $d_H(v) \in \{10, 8, 6\}$. We eliminate each of these possibilities in turn.

Case 1: $d_H(v) = 10$.

Since winning set $N(v)^c$ is a 3-set, avoidance of premature loss requires P2 to choose w in $N(v)$. By Claim 2, $d_H(w) \in \{3, 5, 7\}$. The second and third possibilities are ruled out by Claim 5. Now suppose $d_H(w) = 3$. By Claim 4, 10-set $N(v)$ contains exactly 4 degree-3 vertices, hence 6 of degree 5 or 7. By Claim 5, w must be adjacent to each of these 6 vertices, exceeding its degree.

Case 2: $d_H(v) = 8$.

If there were any degree-10 vertices, then by Claims 4 and 2 the 4-set $N(v)^c - \{v\}$ would have to contain 5 of them, which is impossible. So all vertices in $N(v)^c - \{v\}$ have degree 6 or 8. By Claim 5, w and q must lie in $N(v)^c - \{v\}$. If either one of them had degree 8, then since it is nonadjacent to v and (Claim 5) to all other members of $N(v)^c - \{v\}$, its neighborhood would have to be the 8-set $N(v)$, contradicting the distinctness of all neighborhoods implied by Claim 1 ($|W| = 26$ for graph H with 13 vertices). Therefore $d_H(w) = d_H(q) = 6$. By Claim 4, $N(v)$ consists of vertices $\{v_1, v_2, v_3, v_4, v_5\}$ of degree 3, plus vertices $\{u_1, u_2, u_3\}$ of degree 5 or 7. By Claim 5, the neighborhood of each of w and q must consist of $\{u_1, u_2, u_3\}$ plus a 3-set from $\{v_1, v_2, v_3, v_4, v_5\}$. These two 3-sets must overlap in at least one vertex v_r. But then $\{v, w, q\}$ is the winning set $N(v_r)$, contradicting Claim 5.

Case 3: $d_H(v) = 6$.

By Claim 5, w and q must lie in the 6-set $N(v)^c - \{v\}$. We consider the two subcases defined by Claim 4. In the first of them, all vertices in $N(v)^c - \{v\}$ have degree 6 or 8. If either of w or q had degree 6, then since it is nonadjacent to v and (Claim 5) to all other vertices in $N(v)^c - \{v\}$, its neighborhood must be the 6-set $N(v)$, contradicting distinctness of neighborhoods. Thus $d_H(w) = d_H(q) = 8$, but the only source of neighbors, 6-set $N(v)$, is too small to satisfy that degree.

In the second subcase, the 6-set $N(v)^c - \{v\}$ consists of 5 degree-10 vertices plus a single vertex of degree 6 or 8; the 6-set $N(v)$ consists of vertices of degree 5 or 7. If $d_H(w) = 10$, then after initial choices $\{v, w\}$, P1 could immediately complete $N(w)^c$, a contradiction. So $d_H(w) \in \{6, 8\}$, while $d_H(q) = 10$. By Claim 5, q is adjacent to all 6 vertices in $N(v)$, but not to w (nor v); to fulfill its degree, it must be adjacent to the other 4 degree-10 vertices in $N(v)^c - \{v\}$. But then $\{v, w, q\} = N(q)^c \in W_3$, contradicting Claim 5. This completes the proof. □

Theorem 3.5. *There is no forced* 8*-win* $I_{14}(H)$.

Proof. Suppose $G = I_{14}(H) = G(V, W)$ is a forced 8-win. As before, with $\{v, w\}$ the first 2 moves of a best play execution, we know that the continuation game $G'' = G(12, W'') = G(V, W, \{v, w\})$ is a forced 6-win on 12 elements. By Theorem 2.3, the same is true of the game $G''^* = G(12, W''^*)$ obtained from G'' by deleting all winning sets at odd levels and levels >6. By Theorem 2.1, we have $|W''^*| \geq 13$. By Lemma 2.1, $|W''| \leq 13$. Since $W''^* \subset W''$, equality holds: $G'' = G''^*$.

Since $W'' = W''^*$, $\{v, w\}$ is not completable to any winning sets of G on odd levels or levels >8. And since $|W''| = 13$, application of Theorem 2.1 to G'' yields $|W''_2| = 5$, $|W''_4| = |W''_6| = 4$. The complements of the members of W_4 induced by W''_2 yield 5 members of W_{10}. The complements of the members of W_6 induced by W''_4 yield 4 members of W_8, which are distinct from the 4 members induced by W''_6 since the latter contain $\{v, w\}$ which is disjoint from the former. Similarly, the complements of the members of W_8 induced by W_6'' yield 4 members of W_6 which are distinct from the 4 members induced by W_4''.

We now have $|W_4| = |W_{10}| \geq 5$, $|W_6| = |W_8| \geq 8$, and so have accounted for 26 members of W (13 complementary pairs). By Theorem 1.1, $|W| \leq 2n = 28$, so at most one complementary pair of winning sets has been omitted. In particular, $|W_2| \leq 1$. If such an omitted pair exists then it must (by the fact of its omission) have a different one of $\{v, w\}$ in each of its two members.

For what follows, we recall from [19] the following definition: a family F of $(\lambda+2)$-sets *covers* a λ-set Q, if every single-element extension of Q lies in some member of F. (If $Q = S_\lambda$, $F = W_{\lambda+2}$, and $W_{\lambda+1} = \emptyset$, this is equivalent to the player of move λ being assured of a win at move $\lambda + 2$.)

Claim 1. *W_4 cannot cover any 2-set Q_2.*

Proof. Since $|W_2''| = 5$, we have $|\bigcup W_2''| \leq 10$, and so the 12-set $V - \{v, w\}$ contains a 2-set $Q_2'' = \{q_2, q_2'\}$ disjoint from every $w_2'' \in W_2''$. For a contradiction, assume W_4 covers a 2-set Q_2. The following division into cases is based on the possible sources ($\{v, w\}$, Q_2'', or neither) of the two elements of Q_2.

Case 1. Suppose $Q_2 = \{v, w\}$. But $Q_2 \cup \{q_2\}$ cannot lie in any set $\{v, w\} \cup w_2''$ since $Q_2'' \cap w_2'' = \emptyset$, and cannot lie in an omitted-pair member of W_4 since such a member could not contain $\{v, w\}$.

Case 2. Now suppose $Q_2 \cap Q_2'' \neq \emptyset$. For each of the 12 vertices $y \in V - Q_2$, $Q_2 \cup \{y\}$ cannot lie in any $\{v, w\} \cup w_2''$ since $Q_2'' \cap w_2'' = \emptyset$, and so would have to lie in the omitted-pair member of W_4. But this is more than such a member—a 4-set—can hold.

Case 3. Next assume $Q_2 \subseteq V - (Q_2'' \cup \{v, w\})$. Then $Q_2 \cup \{q_2\}$ cannot lie in any $\{v, w\} \cup w_2''$ since $Q_2'' \cap w_2'' = \emptyset$, and cannot lie in the omitted-pair member of W_4 unless that member is $Q_2 \cup \{q_2, v\}$ or $Q_2 \cup \{q_2, w\}$, in which case $Q_2 \cup \{q_2'\}$ would lie in no member of W_4.

Case 4. Finally, suppose $Q_2 = \{v, x\}$ with $x \in V - (Q_2'' \cup \{v, w\})$. (The argument is similar if $Q_2 = \{w, x\}$.) Since $Q_2 \cup \{q_2\}$ does not lie in any $\{v, w\} \cup w_2''$, it must lie in the omitted-pair member of W_4. Similarly for $Q_2 \cup \{q_2'\}$. Thus the omitted-pair member of W_4 exists and is $\{v, x, q_2, q_2'\}$. Thus for each of the 9 choices of $y \in V - (Q_2'' \cup \{v, w, x\})$, $Q_2 \cup \{y\}$ must lie in some $\{v, w\} \cup w_2''$, i.e. $\{x, y\} \in W_2''$. But this contradicts $|W_2''| = 5$. □

Claim 2. *W_6 cannot cover any nonwinning 4-set Q_4.*

Proof. For a contradiction, suppose W_6 covers Q_4. We consider each possible value of $|\{v, w\} \cap Q_4|$ in turn.

Case 1: $\{v, w\} \subseteq Q_4$.

Then Q_4 lies neither in the omitted-pair member of W_6 (if such exists, it does not include $\{v, w\}$) nor in any member of W_6 of the form $(\{v, w\} \cup w_6'')^c$ where

$w_6'' \in W_6''$. Thus for each of the 10 vertices $x \in V - Q_4$, $Q_4 \cup \{x\}$ must lie in some member of W_6 of the form $\{v, w\} \cup w_4''$ with $w_4'' \in W_4''$, i.e. $w_4'' = (Q_4 - \{v, w\}) \cup \{x, y\}$ for some $y \in V - Q_4$. But this contradicts the fact $|W_4''| = 4$.

Case 2: $\{v, w\} \cap Q_4 = \emptyset$.

For each of the 8 choices of $x \in V - (Q_4 \cup \{v, w\})$, disjointness from $\{v, w\}$ prevents $Q_4 \cup \{x\}$ from lying in any member of W_6 of the form $\{v, w\} \cup w_4''$ with $w_4'' \in W_4''$. Since W_6 contains at most 1 omitted-pair member, only for one choice of x could $Q_4 \cup \{x\}$ lie in that member, which would have to be $Q_4 \cup \{x, v\}$ or $Q_4 \cup \{x, w\}$. So for at least 7 choices, $Q_4 \cup \{x\}$ must lie in some $(\{v, w\} \cup w_6'')^c$ with $w_6'' \in W_6''$,, i.e. $(\{v, w\} \cup w_6'')^c = Q_4 \cup \{x, y\}$ for some $y \in V - (Q_4 \cup \{v, w, x\})$, so that $w_6'' = Q_4^c - \{v, w, x, y\}$. Thus each of the 4 members w_6'' of W_6'' must be involved in such a selection; 3 would not suffice. This implies that $Q_4 \cap (\bigcup W_6'') = \emptyset$. But since G'' is a forced 6-win, it follows from Theorem 2.2 of [19] that every element of $V - \{v, w\}$, hence of Q_4, must lie in at least one member of W_6''. So a contradiction has been reached.

Case 3: $|\{v, w\} \cap Q_4| = 1$.

Let $\{v, w\} = \{u, u'\}$ with $u \in Q_4$ and $u' \in V - Q_4$. For each of the 10 choices of $x \in V - Q_4$, because $u \in Q_4$, $Q_4 \cup \{x\}$ cannot lie in a member of W_6 of the form $(\{v, w\} \cup w_6'')^c$ with $w_6'' \in W_6''$, and so must lie either:

(i) in the omitted-pair member $\bar{w}_6$ of W_6, or

(ii) in a member of W_6 of the form $\{u, u'\} \cup w_4''$ with $w_4'' \in W_4''$.

For $x \neq u'$ to satisfy (ii) requires $w_4'' = (Q_4 - \{u\}) \cup \{x\}$, so since $|W_4''| = 4$, at least 5 of the 9 choices of $x \in V - Q_4 - \{u'\}$ must satisfy (i). Thus $\bar{w}_6$ exists, and contains Q_4. But then (i) can hold only for the 2 members x of $\bar{w}_6 - Q_4$, a contradiction. □

Consequences of Claims. We will complete the proof in the style of [19], by developing contradictory upper and lower bounds on the quantity

$$I_{2468} = 2\,|W_2| + 4\,|W_4| + 6\,|W_6| + 8\,|W_8| = \sum_{u \in V} \sum_{w \in W_2 \cup W_4 \cup W_6 \cup W_8} I(u \in w).$$

For the upper bound note that the members of $W_2 \cup W_4 \cup W_6 \cup W_8$ are the 5 in W_4 and 8 in W_6 and 8 in W_8 accounted for just before Claim 1, plus a possible contribution, from the omitted pair, of 1 to W_2 or 1 to W_4 or 1 each (a complementary pair) to W_6 and W_8. Thus the omitted pair can contribute at most 14 to the first expression for I_{2468}, so that

$$I_{2468} \leq 2\cdot 0 + 4\cdot 5 + 6\cdot 8 + 8\cdot 8 + 14 = 146.$$

Now, the 13 members of $W'' = W_2'' \cup W_4'' \cup W_6''$ contribute 13 members $\{\{v, w\} \cup w'': w'' \in W''\}$ to $W_4 \cup W_6 \cup W_8$, and v lies in all of these, thus contributing at least 13 to the second expression for I_{2468}. Since $|W_2| \leq 1$, there are at least 12 vertices $u \in V - \bigcup W_2$; by Claims 1 and 2, choice of any of these by P1 on the first move can be prolonged to an 8-move play, so that u can play the role of v. It follows that $I_{2468} \geq 12 \cdot 13 = 156$, contradicting the upper bound. $\square$

Remark. Analysis along the same lines of the remaining exceptional cases of the Isolation Game Theorem is further complicated by the presence of additional omitted pairs of the type introduced in the proof of Theorem 3.5, together with more extensive doubling up of vertices. We leave their resolution as an open question.

References

[1] J. Beck and L. Csirmaz, Variations on a game, J. Combin. Theory Ser. A 33 (1982) 297–315.
[2] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1973).
[3] C. Berge, Sur les jeux positionelles, Cahiers Centre Études Rech. Opér. 18 (1976) 91–107.
[4] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (North-Holland, Amsterdam, 1973).
[5] F. Buckley and F. Harary, Diameter avoidance games for graphs, Bull. Malaysian Math. Soc. 1 (1984) 29–33.
[6] F. Buckley and F. Harary, Closed geodetic games for graphs, Congr. Numer. 47 (1985) 131–8.
[7] C.J. Colbourn and D.G. Corneil, On deciding switching equivalence of graphs, Discrete Appl. Math. 2 (1980) 181–184.
[8] S. Even and R.E. Tarjan, A combinatorial problem which is complete in polynomial space, J. Assoc. Comput. Mach. 23 (1976) 710–719.
[9] A.J. Goldman, Eulerian graphs, switching classes and the isolation game, Tech. Rep. 381, Dept. of Math. Sciences, The Johns Hopkins University, 1983.
[10] F. Harary, Achievement and avoidance games for graphs, Ann. Discrete Math. 13 (1982) 111–119.
[11] F. Harary, An achievement game on a toroidal board, in M. Borowiecki, J.W. Kennedy and M.M. Sysło, eds., Lecture Notes in Mathematics, Vol. 1018 (Springer, Berlin, 1983).
[12] J.H. van Lint and J.J. Seidel, Equilateral point sets in elliptic geometry, Nederl. Akad. Wetensch. Proc. Ser. A 69 (1966) 355–348.
[13] C.L. Mallows and N.J.A. Sloane, Two-graphs, switching classes and Euler graphs are equal in number, SIAM J. Appl. Math. 28 (1975) 876–880.
[14] E.M. Palmer, Graphical Evolution, an Introduction to the Theory of Random Graphs (Wiley Interscience, New York, 1985).
[15] R.D. Ringeisen, Isolation: a game on a graph, Math. Mag. 47 (1974) 132–138.
[16] A.G. Robinson, The isolation game and its generalization, Doctoral Dissertation, The Johns Hopkins University, May 1986.
[17] A.G. Robinson and A.J. Goldman, On Ringeisen's isolation game, Discrete Math. 80 (1990) 297–312.
[18] A.G. Robinson and A.J. Goldman, The set coincidence game: complexity, attainability, and symmetric strategies, J. Comput. System Sci. 39 (1989) 376–387.
[19] A.G. Robinson and A.J. Goldman, On the set coincidence game, Discrete Math. 84 (1990) 261–263.
[20] R. Taylor, Switchings constrained to 2-connectivity in simple graphs, SIAM J. Algebraic and Discrete Methods 3 (1) (1982) 114–121.