

MC3020 - Comparing Two Population Parameters

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Comparing two population parameters

- Comparing Two Population Proportions.
- Comparing Two Independent Population Means.
- Comparing Two Dependent or Matched Population Means.
- Comparing Two Independent Population Variances.

Comparing Two Population Proportions



Confidence interval for $p_1 - p_2$

Conditions:

1. Let X_1 be the number of successes in n_1 Bernoulli trials having proportion of success p_1 .
2. Let X_2 be the number of successes in n_2 Bernoulli trials having proportion of success p_2 .
3. The two experiments are independent.



The **point estimate** for the difference between the proportions is $p_1 - p_2$,

$$\hat{p}_1 - \hat{p}_2 = \frac{X_1}{n_1} - \frac{X_2}{n_2}$$

The population mean of $\hat{p}_1 - \hat{p}_2$ is,

$$E(\hat{p}_1 - \hat{p}_2) = p_1 - p_2$$

The population variance of $\hat{p}_1 - \hat{p}_2$ is,

$$\frac{p_1(1 - p_1)}{n_1} - \frac{p_2(1 - p_2)}{n_2}$$



By the central limit theorem, for large n_1 and n_2 , $\hat{p}_1 - \hat{p}_2$ has approximate normal distribution with mean $p_1 - p_2$ and variance $\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}$

Then $(1 - \alpha) * 100\%$ confidence interval estimate for $p_1 - p_2$ is computed as

$$(\hat{p}_1 - \hat{p}_2 - E, \hat{p}_1 - \hat{p}_2 + E)$$

where, $E = Z_{\alpha/2} * \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$

Example 1:

It is claimed that in the 2008 Democratic Presidential Nomination Primaries in USA, Senator Barack Obama was preferred by the black voters. To test the claim, a research firm sampled 600 black democrats and found that 384 support the senator and in another sample of 720 non-black democrats 417 support the senator. Construct a 97% confidence interval for the difference between the two populations proportions.



Testing for the difference between two independent population proportions

Case 1: $H_0 : p_1 \geq p_2$ $H_1 : p_1 < p_2$

Case 2: $H_0 : p_1 \leq p_2$ $H_1 : p_1 > p_2$

Case 3: $H_0 : p_1 = p_2$ $H_1 : p_1 \neq p_2$



The corresponding test statistic is

$$Z = \frac{\hat{p}_1 - \hat{p}_2 - (p_1 - p_2)}{\sqrt{\bar{p}(1 - \bar{p}) \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}}$$

where $\bar{p} = \frac{X_1 + X_2}{n_1 + n_2}$

Note that $p_1 - p_2 = 0$ in computations for all three cases above. But in general it is not necessarily zero as if we want to test that one proportion is at least an amount higher than the other then $p_1 - p_2$ is that least amount in proportion, and so on.



Example 2:

It is claimed that in the 2008 Democratic Presidential Nomination Primaries in USA, Senator Barack Obama was preferred by the black voters. To test the claim, a research firm sampled 600 black democrats and found that 384 support the senator and in another sample of 720 non-black democrats 417 support the senator. Test the claim using 5% level of significance.



Exercises

In your course tutorial 5 exercises,

- Question 01
- Question 02



Comparing Two Independent Population Means



Let X_1, X_2, \dots, X_{n_1} be a random sample from a population with mean μ_1 and variance σ_1^2 .

Let Y_1, Y_2, \dots, Y_{n_2} be a random sample from a population with mean μ_2 and variance σ_2^2 . The populations are independent.

The point estimate for the population mean difference $\mu_1 - \mu_2$ is the sample mean difference $\bar{X} - \bar{Y}$.



When the two population distributions are normal, the sampling distribution of $\bar{X} - \bar{Y}$ is normal with mean

$$E(\bar{X} - \bar{Y}) = \mu_1 - \mu_2$$

and variance

$$\text{Var}(\bar{X} - \bar{Y}) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$



The confidence interval for the difference between the two-population means is computed as follows:

Case 1:

When the two independent population distributions are normal and the population variances σ_1^2 and σ_2^2 are known, the $(1 - \alpha) * 100$ % confidence interval for $\mu_1 - \mu_2$ is computed as,

$$\left(\bar{X} - \bar{Y} - Z_{\alpha/2} * \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} , \bar{X} - \bar{Y} + Z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \right)$$

Case 2:

When the two independent population distributions are **normal** and the **population variances σ_1^2 and σ_2^2 are unknown and unequal**, the $(1 - \alpha) * 100$ % confidence interval for $\mu_1 - \mu_2$ is computed as,

$$\left(\bar{X} - \bar{Y} - t_{\alpha/2} * \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}, \bar{X} - \bar{Y} + t_{\alpha/2} * \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \right)$$

Where s_1^2 and s_2^2 are the corresponding sample variances, and the degrees of freedom for t is,



$$df = \frac{(A + B)^2}{\frac{A^2}{n_1 - 1} + \frac{B^2}{n_2 - 1}}$$

Where $A = \frac{s_1^2}{n_1}$ and $B = \frac{s_2^2}{n_2}$

Case 3:

When the two independent population distributions are normal and the population variances σ_1^2 and σ_2^2 are unknown but equal, the $(1 - \alpha) * 100$ % confidence interval for $\mu_1 - \mu_2$ is computed as,

$$\left(\bar{X} - \bar{Y} - t_{\alpha/2} * S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}, \bar{X} - \bar{Y} + t_{\alpha/2} * S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right)$$

Where $S_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1 + n_2 - 2}}$ is the pooled standard deviation from the two sample standard deviations, and the degrees of freedom for t is $n_1 + n_2 - 2$.



Case 4:

When the two independent population distributions are **not normal** and the **population variances σ_1^2 and σ_2^2 are known**, and the sample size **n_1 and n_2 are large**, the $(1 - \alpha) * 100$ % confidence interval for $\mu_1 - \mu_2$ is computed as,

$$\left(\bar{X} - \bar{Y} - Z_{\alpha/2} * \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}, \bar{X} - \bar{Y} + Z_{\alpha/2} * \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \right)$$



Case 5:

When the two independent population distributions are **not normal** and the **population variances σ_1^2 and σ_2^2 are unknown**, and the sample size **n_1 and n_2 are large**, the $(1 - \alpha) * 100$ % confidence interval for $\mu_1 - \mu_2$ is computed as,

$$\left(\bar{X} - \bar{Y} - Z_{\alpha/2} * \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}, \bar{X} - \bar{Y} + Z_{\alpha/2} * \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \right)$$

Example 3:



Two different types of drugs 'A' and 'B' are tried on certain patients for increasing weight. Five randomly selected patients were given drug 'A', and 7 randomly selected patients were given drug 'B'. The increases in weight (in pounds) are given below:

Drug 'A': 8 12 13 9 3

Drug 'B': 10 8 12 15 6 8 11

Assume that the population distributions of the measurements are normal with equal variances. Construct a 95% confidence interval for the difference between the two means.

Example 4:



To test effect of a fertilizer on rice production, 64 plots of land having equal areas were chosen. Half of these plots were treated with fertilizer and the other half were untreated. Other conditions were the same. The mean yield of rice on the untreated plots was 4.8 quintals with a standard deviation of 0.4 quintal, while the mean yield on the treated plots was 5.1 quintals with a standard deviation of 0.36 quintal. Construct a 94% confidence interval estimate for the mean difference between the untreated plots and treated plots.

Testing for the difference between two independent population means

Case 1: $H_0 : \mu_1 \geq \mu_2$ $H_1 : \mu_1 < \mu_2$

Case 2: $H_0 : \mu_1 \leq \mu_2$ $H_1 : \mu_1 > \mu_2$

Case 3: $H_0 : \mu_1 = \mu_2$ $H_1 : \mu_1 \neq \mu_2$



Case 1:

When the two independent population distributions are normal and the population variances σ_1^2 and σ_2^2 are known, the test statistic is

$$Z = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$



Case 2:

When the two independent population distributions are normal and the population variances σ_1^2 and σ_2^2 are unknown and unequal, the test statistic is

$$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

where T has a t-distribution with degrees of freedom

$$df = \frac{(A+B)^2}{\frac{A^2}{n_1-1} + \frac{B^2}{n_2-1}}$$

Where $A = \frac{s_1^2}{n_1}$ and $B = \frac{s_2^2}{n_2}$

Case 3:

When the two independent population distributions are normal and the population variances σ_1^2 and σ_2^2 are unknown but equal, the test statistic is

$$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

where T has a t distribution with degrees of freedom $n_1 + n_2 - 2$ and $S_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}}$ is the pooled standard deviation from the two sample standard deviations.



Case 4:

When the two independent population distributions are **not normal** and the **population variances σ_1^2 and σ_2^2 are known**, and the sample size n_1 and n_2 are large, the test statistic is

$$Z = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$



Case 5:

When the two independent population distributions are **not normal** and the **population variances σ_1^2 and σ_2^2 are unknown**, and the sample size n_1 and n_2 are large, the test statistic is

$$Z = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$



Note that $\mu_1 - \mu_2$ for all three cases above. But in general it is not necessarily zero as if we want to test that one mean is at least an amount higher than the other then $\mu_1 - \mu_2$ is that least amount, and so on.



Example 5:



Two different types of drugs 'A' and 'B' are tried on certain patients for increasing weight. Five randomly selected patients were given drug 'A' and 7 randomly selected patients were given drug 'B'. The increases in weight (in pounds) are given below:

Drug 'A': 8 12 13 9 3

Drug 'B': 10 8 12 15 6 8 11

Assume that the population distributions of the measurements are normal. Do the two drugs differ significantly with regard to their effect in increasing weight? Use 0.05 significance level.

Example 6:



To test effect of a fertilizer on rice production, 64 plots of land having equal areas were chosen. Half of these plots were treated with fertilizer and the other half were untreated. Other conditions were the same. The mean yield of rice on the untreated plots was 4.8 quintals with a standard deviation of 0.4 quintal, while the mean yield on the treated plots was 5.1 quintals with a standard deviation of 0.36 quintal. Can we conclude that there is a significant improvement in rice production because of the fertilizer at 4% level of significance?

Example :

An urban economist wanted to determine whether the mean price of a home in Lemont is less than the mean price of a home in Naperville. A random sample of homes sold in each neighborhood results in the following statistics, where the means and standard deviations are in thousands of dollars:

Lemont	Naperville
$n_1 = 50$	$n_2 = 50$
$\bar{x}_1 = 200$	$\bar{x}_2 = 300$
$s_1 = 45$	$s_2 = 75$

Test the claim that housing is less expensive in Lemont than in Naperville at the 5% level of significance.

Exercises

In your course tutorial 5 exercises,

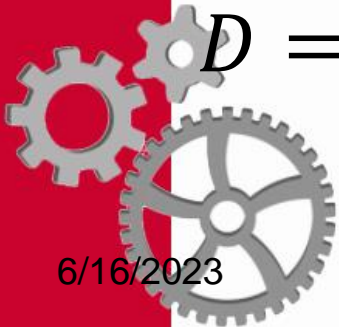
- Question 04
- Question 06



Comparing Two Dependent or Matched Population Means



Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ be pairs of random measurements from two dependent or matched populations. Such situations usually but not exclusively occur when measurements are taken on the same subjects before or after the experimentation. To consider all other factors same except the factor that measured by and , we combine the populations by finding the difference $D = X - Y$.



Let D_1, D_2, \dots, D_n be a random sample from a population with mean μ_D and variance σ_D^2 .

The point estimate for the population mean difference is $\mu_1 - \mu_2$ the sample mean of the differences

$$\bar{D} = \frac{\sum D}{n}.$$

The mean of \bar{D} is

$$E(\bar{D}) = \mu_D$$



The variance of \bar{D} is $V(\bar{D}) = \frac{\sigma_D^2}{n}$

When the population distribution of the differences D is normal, the sampling distribution of \bar{D} is normal with mean $E(\bar{D}) = \mu_D$ and variance $V(\bar{D}) = \frac{\sigma_D^2}{n}$.



The $(1 - \alpha) * 100\%$ confidence interval for the difference between the two dependent population means $\mu_1 - \mu_2 = \mu_D$ is computed as follows,

Case 1

When the population distribution of the differences D is normal and the population variance σ_D^2 is known, the $(1 - \alpha) * 100\%$ confidence interval for $\mu_1 - \mu_2 = \mu_D$ is computed as

$$\left(\bar{D} - Z_{\alpha/2} * \frac{\sigma_D}{\sqrt{n}}, \bar{D} + Z_{\alpha/2} * \frac{\sigma_D}{\sqrt{n}} \right)$$

Case 2 When the population distribution of the differences D is normal and the population variance σ_D^2 is unknown, the $(1 - \alpha) * 100\%$ confidence interval for $\mu_1 - \mu_2 = \mu_D$ is computed as

$$\left(\bar{D} - t_{\alpha/2} * \frac{S_D}{\sqrt{n}}, \bar{D} + t_{\alpha/2} * \frac{S_D}{\sqrt{n}} \right)$$

where the degrees of freedom for t is $n - 1$ and

$$S_D = \sqrt{\frac{\sum (D - \bar{D})^2}{n-1}} = \sqrt{\frac{n \sum D^2 - (\sum D)^2}{n(n-1)}} \quad \text{the sample standard deviation for the differences.}$$



Case 3

When the population distribution of the differences D is not normal, n is large, and the population variance σ_D^2 is known, the $(1 - \alpha) * 100\%$ confidence interval for $\mu_1 - \mu_2 = \mu_D$ is computed as,

$$\left(\bar{D} - Z_{\alpha/2} * \frac{\sigma_D}{\sqrt{n}}, \bar{D} + Z_{\alpha/2} * \frac{\sigma_D}{\sqrt{n}} \right)$$



Case 4

When the population distribution of the differences D is not normal, n is large, and the population variance σ_D^2 is unknown, the $(1 - \alpha) * 100\%$ confidence interval for $\mu_1 - \mu_2 = \mu_D$ is computed as,

$$\left(\bar{D} - Z_{\alpha/2} * \frac{S_D}{\sqrt{n}}, \bar{D} + Z_{\alpha/2} * \frac{S_D}{\sqrt{n}} \right)$$



Note that since the measurements are on the same subjects, if there is no real difference in terms of the factor of interest, the distributions of the differences are often normal. Hence the situation 2 is a very common phenomenon.



Testing for the difference between two dependent population means

$$\begin{aligned}\text{Case 1: } H_0: \mu_1 &\geq \mu_2 && \equiv H_0: \mu_D \geq 0 \\ H_1: \mu_1 &< \mu_2 && H_1: \mu_D < 0\end{aligned}$$

$$\begin{aligned}\text{Case 2: } H_0: \mu_1 &\leq \mu_2 && \equiv H_0: \mu_D \leq 0 \\ H_1: \mu_1 &> \mu_2 && H_1: \mu_D > 0\end{aligned}$$

$$\begin{aligned}\text{Case 3: } H_0: \mu_1 &= \mu_2 && \equiv H_0: \mu_D = 0 \\ H_1: \mu_1 &\neq \mu_2 && H_1: \mu_D \neq 0\end{aligned}$$



Case 1: When the population distribution of the differences D is normal and the population variance σ_D^2 is known, the test statistic is

$$Z = \frac{\bar{D} - \mu_D}{\sigma_D / \sqrt{n}}$$



Case 2 When the population distribution of the differences D is normal and the population variance σ_D^2 is unknown, the statistic is

$$T = \frac{\bar{D} - \mu_D}{S_D / \sqrt{n}}$$

where T has a t distribution with $(n - 1)$ degrees of freedom



Case 3 When the population distribution of the differences D is not normal, n is large, and the population variance σ_D^2 is known, the test statistic is

$$Z = \frac{\bar{D} - \mu_D}{\sigma_D / \sqrt{n}}$$



Case 4 When the population distribution of the differences D is not normal, n is large, and the population variance σ_D^2 is unknown, the test statistic is

$$Z = \frac{\bar{D} - \mu_D}{S_D / \sqrt{n}}$$



Note that $\mu_1 - \mu_2 = \mu_D = 0$ for all three cases above. But in general it is not necessarily zero as if we want to test that one mean is at least an amount higher than the other then $\mu_1 - \mu_2$ is that least amount, and so on.



Example 7:

To compare the demand for two different entrees, the manager of a cafeteria recorded the number of purchases for each entrée on seven consecutive days. The data are shown in the next table: Since the cafeteria demands depend on days of the week, the data are considered to be related. Assume that the differences of the measurements follow a normal distribution.

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
A	420	374	434	395	637	594	679
B	391	343	469	412	538	521	625

- a) Construct a 95% confidence interval for the mean difference.
- b) Test the hypothesis that the demand for item A is higher than the demand for item B. Use 5% level of significance.



Example:

A test preparation company claims that its SAT preparation course improves SAT math scores. The company administers the SAT to 9 randomly selected students and determines their scores. The same students then participate in the course. Upon completion, they retake the SAT. The results are presented below:

Before:	436	431	270	463	528	377	397
	413	525					
After:	443	429	287	501	522	380	402
	450	548					

Test the claim that the preparatory course improves SAT math scores at the 10% level of significance. (Assume that the differences between the scores have an approximate normal distribution.)

Exercises

In your course tutorial 5 exercises,

- Question 03



Comparing Two Independent Population Variances



Let X_1, X_2, \dots, X_{n_1} be a random sample from a population with mean μ_1 and variance σ_1^2 .

Let Y_1, Y_2, \dots, Y_{n_2} be a random sample from a population with mean μ_2 and variance σ_2^2 . The populations are independent.

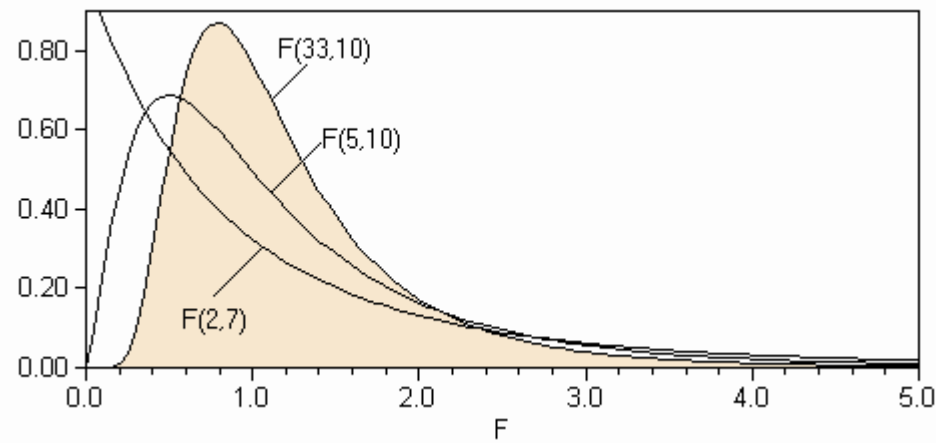


As the variances cannot be negative, we consider the ratio of the two variances instead of their difference in making comparison between them.

The point estimate for the ratio σ_1^2 / σ_2^2 is s_1^2 / s_2^2 where S_1^2 and S_2^2 are the respective sample variances.



It is known that when the populations are normal, s_1^2/s_2^2 follows an F-distribution with degrees of freedom $n_1 - 1$ and $n_2 - 1$. The table for the F-distribution (Table 5) is given in the Appendix II.



Then $(1 - \alpha) * 100\%$ confidence interval for the σ_1^2 / σ_2^2 is computed as,

$$\frac{S_1^2}{S_2^2} * \frac{1}{F_{\alpha/2, n_1 - 1, n_2 - 1}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2}{S_2^2} * F_{\alpha/2, n_2 - 1, n_1 - 1}$$

Where $F_{\alpha/2, n_2 - 1, n_1 - 1}$ is the percentile value in the F-distribution such that the right side area is for degrees of freedom $n_2 - 1$ and $n_1 - 1$. And $F_{\alpha/2, n_1 - 1, n_2 - 1}$ is the percentile value in the F distribution such that the right side area is $\alpha/2$ for degrees of freedom $n_1 - 1$ and $n_2 - 1$.



Testing for difference between two independent population variances,

Case 1:

$$H_0 : \sigma_1^2 \geq \sigma_2^2 \quad H_1 : \sigma_1^2 < \sigma_2^2$$

Case 2:

$$H_0 : \sigma_1^2 \leq \sigma_2^2 \quad H_1 : \sigma_1^2 > \sigma_2^2$$

Case 3:

$$H_0 : \sigma_1^2 = \sigma_2^2 \quad H_1 : \sigma_1^2 \neq \sigma_2^2$$



Under the assumption that the null hypothesis H_0 is true, the test statistic

$$F = \frac{S_1^2}{S_2^2}$$

has F-distribution with degrees of freedom $n_1 - 1$ and $n_2 - 1$.



Example 7.8

Let us consider the final scores of the Author's two different sections of the elementary statistics courses:

Section 1: 38, 88, 91, 84, 97, 78, 51, 90, 72, 73, 73, 55, 83, 72, 97, 33, 78, 91, 93, 65, 86, 81, 87, 81, 28, 74

Section 2: 64, 36, 87, 73, 72, 43, 90, 81, 79, 43, 77, 89, 91, 72, 75, 68, 78, 72, 81, 72, 35, 72, 93, 74, 85.

Assume that the samples are from independent normal populations.



Solution: To construct a 90% confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$, we obtain $\frac{s_1^2}{s_2^2} *$

$$\frac{1}{F_{\alpha/2, n_1 - 1, n_2 - 1}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2}{s_2^2} * F_{\alpha/2, n_2 - 1, n_1 - 1}$$

$$\frac{19.12^2}{16.48^2} * \frac{1}{F_{0.05, 26 - 1, 25 - 1}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{19.12^2}{16.48^2} *$$

$$F_{0.05, 25 - 1, 26 - 1}$$

$$\frac{19.12^2}{16.48^2} * \frac{1}{1.9750} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{19.12^2}{16.48^2} * 1.9643$$

$$0.6815 < \frac{\sigma_1^2}{\sigma_2^2} < 2.6440$$

To test the claim that the first section has higher variance compared to the second section using 5% level of significance, the hypotheses can be written as

$$H_0 : \sigma_1^2 \leq \sigma_2^2 \quad H_1 : \sigma_1^2 > \sigma_2^2$$

The test statistic

$$F = \frac{S_1^2}{S_2^2} = \frac{19.12^2}{16.48^2} = 1.3461$$



The 5% critical value for degrees of freedom 25 and 24 is 1.975.

So, we fail to reject H_0 at 0.05 level and conclude that the Section 1 variance is not significantly higher than the Section 2 variance.

Similarly, $p\text{-value} = P(F > 1.3461) > 0.05$

So we fail to reject at 0.05 level. Same conclusion!

Exercises

In your course tutorial 5 exercises,

- Question 05



Don't hesitate to contact us if you have any questions about this course's teaching contents. Also, don't forget to check out the course page and Microsoft Team folder,

- course page Link:
<https://mayooran1987.github.io/MC3020/>
- Course page's QR code

