Topics

Relations

Relations, properties, operations, and applications.

Directed graphs

Directed graphs and representing relations as directed graphs.

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What are relations?

Binary relation

Let A and B be sets.

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A *binary relation* from A to B is a subset of $A \times B$.

A binary relation R is a set of tuples. If R is a relation from a set A to itself, we say that R is a relation **on** A.

Examples of relations we've seen before

$\geq$ on $\mathbb{N}$

$$\{(x, y) : x \geq y \text{ and } x, y \in \mathbb{N}\}$$

$<$ on $\mathbb{R}$

$$\{(x, y) : x < y \text{ and } x, y \in \mathbb{R}\}$$

$=$ on Σ^*

$$\{(x, y) : x = y \text{ and } x, y \in \Sigma^*\}$$

$\subseteq$ on $\mathcal{P}(U)$ for a universe U

$$\{(A, B) : A \subseteq B \text{ and } A, B \in \mathcal{P}(U)\}$$

More examples of relations

$$R_1 = \{(a,1),(a,2),(b,1),(b,3),(c,3)\}$$

$$R_2 = \{(x,y) : x \equiv y \pmod 5\}$$

$$R_3 = \{(c_1,c_2) : c_1 \text{ is a prerequisite for } c_2\}$$

$$R_4 = \{(s,c) : \text{student } s \text{ has taken course } c\}$$

Important properties of relations

Relation properties

Let R be a relation on A.

R is *reflexive* iff $(a, a) \in R$ for every $a \in A$.

R is *symmetric* iff $(a, b) \in R$ implies $(b, a) \in R$.

R is *antisymmetric* iff $(a, b) \in R$ and $a \neq b$ implies $(b, a) \notin R$.

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$\{(1, 2), (2, 1), (1, 3)\}$	none!

Operations on relations

Relation composition

Let R be a relation on A to B.

Let S be a relation on B to C.

The *composition* of R and S, $R \circ S$, is the relation from A to C defined by:

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Since relations are just sets, we can also combine them with the standard set theoretic operators ($\cup$, $\cap$, $\setminus$).

Examples of relational composition

$R \circ S = \{(a, c) : \exists b. (a, b) \in R \text{ and } (b, c) \in S\}$

Consider the relations Parent **and** Sister.

$(a, b) \in \mathrm{Parent}$ iff b is a parent of a

$(a, b) \in \mathrm{Sister}$ iff b is a sister of a

When is $(x, y) \in \mathrm{Parent} \circ \mathrm{Sister}$**?**

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y is an aunt of x.

When is $(x, y) \in \mathrm{Sister} \circ \mathrm{Parent}$**?**

y is a parent of x and x has a sister.

More examples of relational composition

$R \circ S = \{(a, c) : \exists b. (a, b) \in R \text{ and } (b, c) \in S\}$

Use the relations $\text{Parent}, \text{Child}, \text{Sister}, \text{Brother}, \text{Sibling}$ **to express**

$(a, b) \in \text{Uncle}$ iff b is an uncle of a

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Powers of a relation

$R^0 = \{(a, a) : a \in A\}$

The *identity* relation on A.

$R^1 = R^0 \circ R = R$

$R^2 = R^1 \circ R = R \circ R = \{(a, c) : \exists b.\, (a, b) \in R \text{ and } (b, c) \in R\}$

$\vdots$

$R^{n+1} = R^n \circ R$ for $n \geq 0$

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Reflexive transitive closure

Let R be a relation on A.

The *reflexive transitive closure*, R^*, of R is defined by $R^* = \bigcup_{n=0}^{\infty} R^n = R^0 \cup R^1 \cup R^2 \cup \ldots$

Applications of relations

Databases use relations to store and organize data.

Relations are also used to reason about software systems.

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So far, we've only talked about binary relations.

But the same concepts extend to *n-ary relations*, which contain tuples of length $n \geq 1$.

A (relational) database table is an n-ary relation, e.g., $R \subseteq \text{Student} \times \text{ID} \times \text{GPA}$.

Student	ID	GPA
Einstein	299792458	2.11
Newton	667408310	3.42
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Relations are also used to reason about software systems.

Relational logic (relations + predicate logic) is a language for specifying and automatically checking properties of software systems.

Many applications, including verifying safety critical software, synthesizing memory consistency models, and verifying security and privacy policies.

A filesystem spec

$\text{Root} = \{(\texttt{root})\}$

$\text{Root} \subseteq \text{Dir}$

$\text{contents} \subseteq \text{Dir} \times (\text{File} \cup \text{Dir})$

$(\text{File} \cup \text{Dir}) \subseteq \text{Root} \circ \text{contents}^*$

Directed graphs

Directed graphs and representing relations as directed graphs.

What is a directed graph?

Directed graphs

A *directed graph* $G = (V, E)$ consists of a set of *vertices* V and a set of *edges* $E \subseteq V \times V$, which are ordered pairs of vertices.

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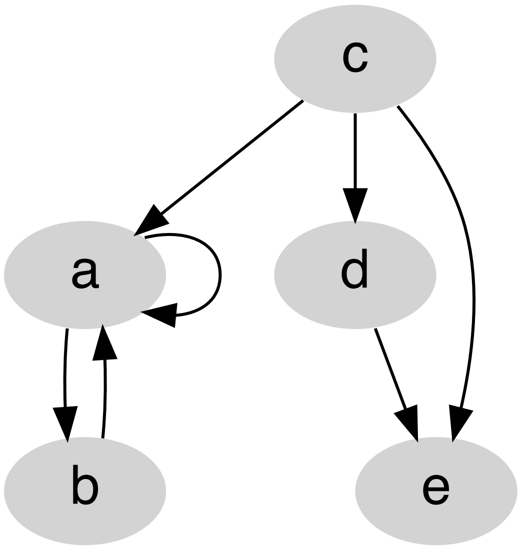
Example directed graph $G = (V, E)$

$V = \{a, b, c, d, e\}$

$E = \{(a, a), (a, b), (b, a), (c, a), (c, d), (c, e), (d, e)\}$



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$V = \{a, b, c, d, e\}$

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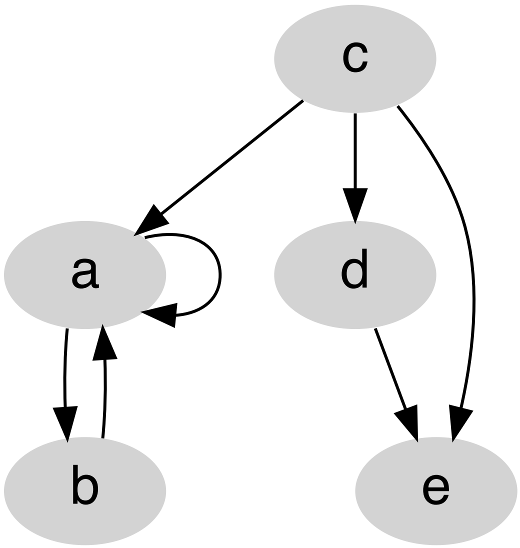
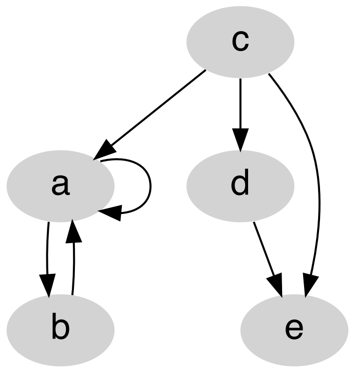
E is just a relation on V!



Representing relations as directed graphs

A relation R on A corresponds to the graph $G = (A, R)$.

$$R = \{(a,a),(a,b),(b,a),(c,a),(c,d),(c,e),(d,e)\}$$
$$A = \{a,b,c,d,e\}$$



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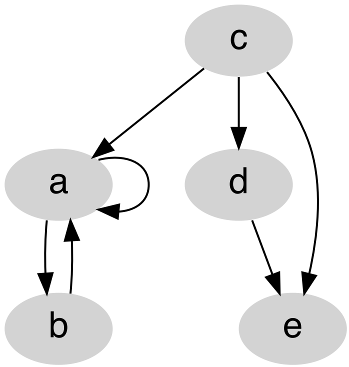

How about a relation R from A to B?

$$R = \{(a,1),(a,2),(b,1),(b,3),(c,3)\}$$
$$A = \{a,b,c\}$$
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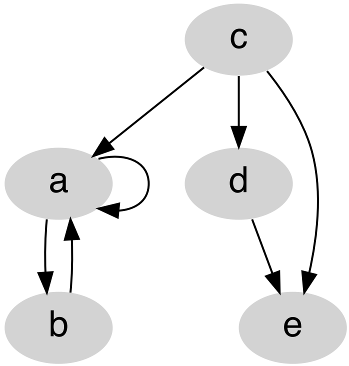
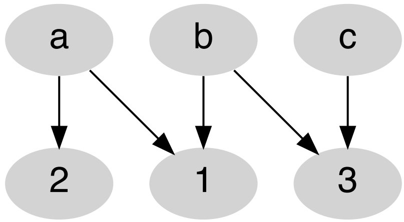


How about a relation R from A to B?

$$R = \{(a, 1), (a, 2), (b, 1), (b, 3), (c, 3)\}$$
$$A = \{a, b, c\}$$
$$B = \{1, 2, 3\}$$
$$G = (A \cup B, R)$$



Paths in graphs

Path in a directed graph

Let $G = (V, E)$ be a directed graph.

A *path* in G is a sequence of vertices $v_0, v_1, \ldots, v_k$ where $(v_i, v_{i+1}) \in E$ for each $0 \leq i < k$.

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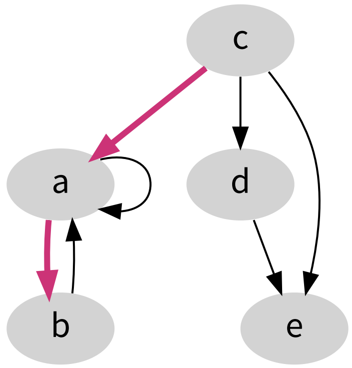
Example paths

Simple path (no v_i repeated): c, a, b



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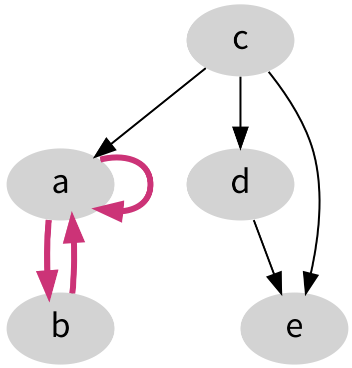

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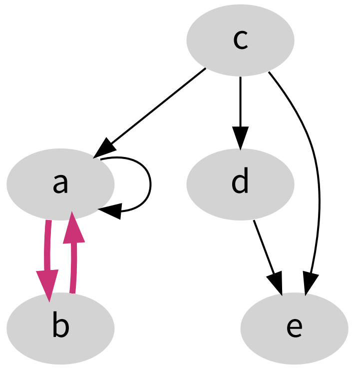

Paths in graphs and relations

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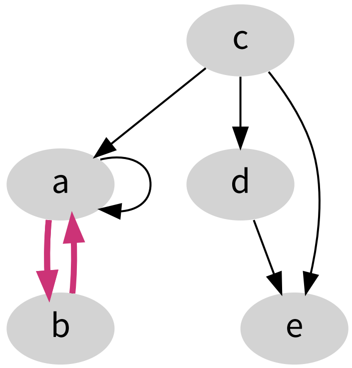

We've defined paths on graphs but the same definition applies to relations, since a relation and its graph representation are interchangeable.

Relational composition using directed graphs

If R and S are relations on A, compute $R \circ S$ as follows:

Create the digraph $G = (A, R \cup S)$.

Add an edge (a, b) to $R \circ S$ iff there is a path a, v, b in G such that $(a, v) \in R$ and $(v, b) \in S$.

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Example:

$R = \{(1, 2), (2, 1), (1, 3)\}$

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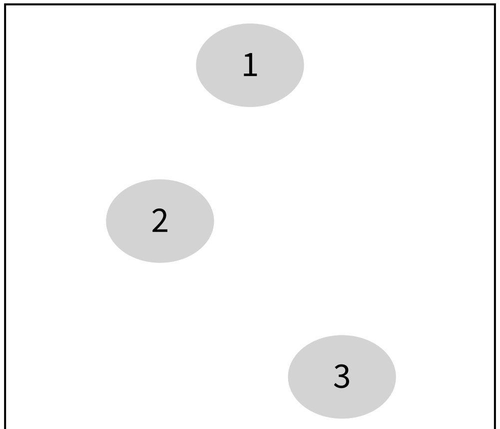

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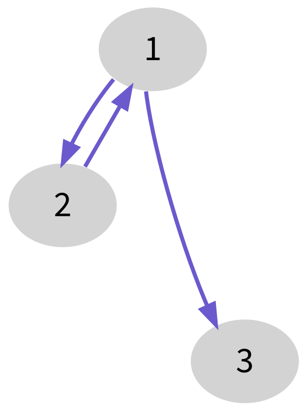

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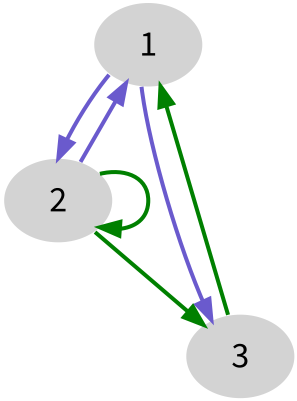

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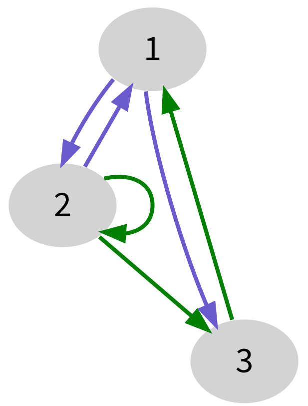

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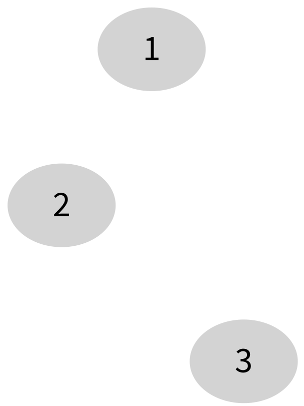

$R \circ S$

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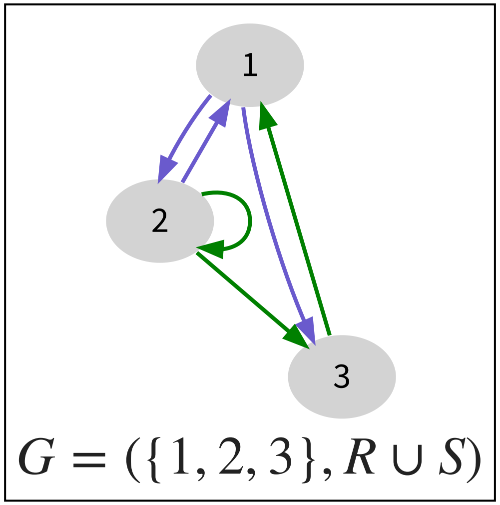
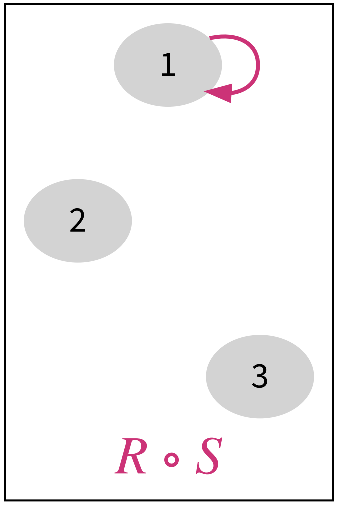




Relational composition using directed graphs

If R and S are relations on A, compute $R \circ S$ as follows:

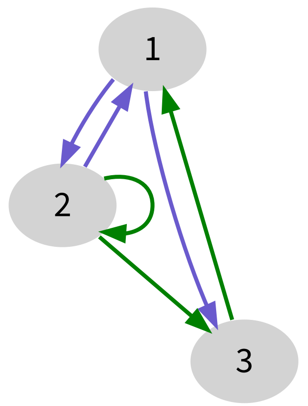
Create the digraph $G = (A, R \cup S)$.

Add an edge (a, b) to $R \circ S$ iff there is a path a, v, b in G such that $(a, v) \in R$ and $(v, b) \in S$.

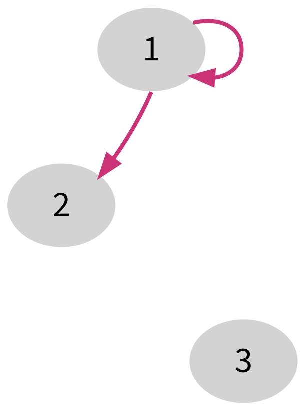
Example:

$R = \{(1, 2), (2, 1), (1, 3)\}$

$S = \{(2, 2), (2, 3), (3, 1)\}$



$G = (\{1, 2, 3\}, R \cup S)$



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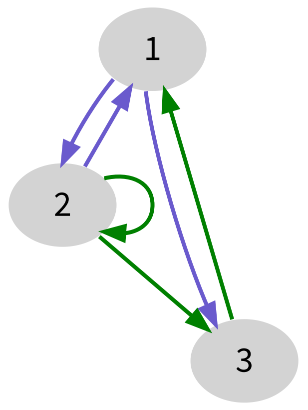

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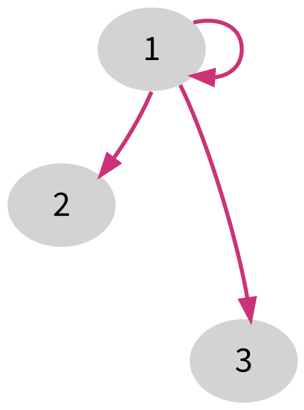

$R \circ S$

Powers of a relation using directed graphs

Length of a path in a graph

The *length* of a path $v_0, \ldots, v_k$ is the number of edges in it, i.e., k.

Powers and paths

Let R be a relation on a set A. There is a path of length n from a to b in R if and only if $(a, b) \in R^n$.

Reflexive transitive closure using directed graphs

Connectivity relation

Let R be a relation on a set A. The *connectivity* relation $R^\star$ consists of the tuples (a, b) such that there is a path (of any length) from a to b in R.

Connectivity and reflexive transitive closure

The reflexive transitive closure R^* of a relation R is its connectivity relation $R^\star$, i.e., $R^\star = R^* = \bigcup_{n=0}^{\infty} R^n$.

Summary

Relations are a fundamental structure in computer science.

- A relation is a set of tuples.
- Relations can be reflexive, (anti)symmetric, transitive.
- We can combine binary relations with the composition $\circ$ operator.

Directed graphs consist of nodes and edges (ordered pairs of nodes).

- Relations can be represented as directed graphs.
- The two representations are interchangeable: relational operations have corresponding graph operations and vice versa.