MC4010 - Discrete Mathematics

Counting

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2023-11-06

All contents of these slides are agglomerated from recommended reference textbook of this course and the previous year's lecture slides (Which were prepared by Dr P. Kathirgamanathan).

Slides are prepared through RMarkdown Beamer presentation option and executed via RStudio.

Introduction

Suppose that a password on a computer system consists of six, seven, or eight characters. Each of these characters must be a digit or a letter of the alphabet. Each password must contain at least one digit. How many such passwords are there?

The techniques needed to answer this question and a wide variety of other counting problems will be introduced in this section.

Counting problems arise throughout mathematics and computer science. For Example, we must count the successful outcomes of experiments and all the possible outcomes of these experiments to determine probabilities of discrete events. We need to count the number of operations used by an algorithm to study its time complexity.

We will introduce the basic techniques of counting in this section. These methods serve as the foundation for almost all counting techniques.

We first present two basic counting principles, the product rule and the sum rule. Then we will show how they can be used to solve many different counting problems. The product rule applies when a procedure is made up of separate tasks.

Definition (Product rule)

Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Example

A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution:

The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways. By the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

Example

The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solution: :

The procedure of labeling a chair consists of two tasks, namely, assigning to the seat one of the 26 uppercase English letters, and then assigning to it one of the 100 possible integers. The product rule shows that there are $26 \times 100 = 2600$ different ways that a chair can be labeled. Therefore, the largest number of chairs that can be labeled differently is 2600.

Example

There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

Solution: :

The procedure of choosing a port consists of two tasks, first picking a microcomputer and then picking a port on this microcomputer. Because there are 32 ways to choose the microcomputer and 24 ways to choose the port no matter which microcomputer has been selected, the product rule shows that there are $32 \cdot 24 = 768$ ports.

Example

How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene)?

Solution: :

There are 26 choices for each of the three uppercase English letters and ten choices for each of the three digits. Hence, by the product rule there are a total of $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$ possible license plates.

Definition (Sum rule) If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example:

Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Solution: :

There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choose a student who is a mathematics major. Choosing a member of the mathematics faculty is never the same as choosing a student who is a mathematics major because no one is both a faculty member and a student. By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick this representative.

Example:

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Solution: :

The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are $23 + 15 + 19 = 57$ ways to choose a project.

Practice Problems

1. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?
2. How many different three-letter initials with none of the letters repeated can people have?
3. How many strings of four decimal digits
    a) do not contain the same digit twice?
    b) end with an even digit?
4. How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?

The Pigeonhole Principle

Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, a least one of these 19 pigeonholes must have at least two pigeons in it.

To see why this is true, note that if each pigeonhole had at most one pigeon in it, at most 19 pigeons, one per hole, could be accommodated.

This illustrates a general principle called the pigeonhole principle, which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it (see Figure 1). Of course, this principle applies to other objects besides pigeons and pigeonholes.

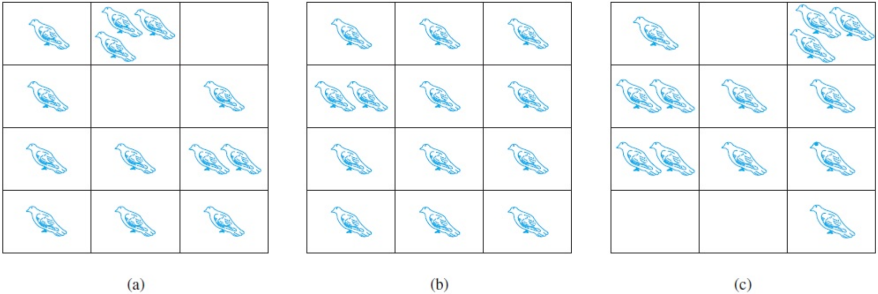

Figure 1: There Are More Pigeons Than Pigeonholes.

Theorem 1 (The Pigeonhole Principle)

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Proof:

We prove the pigeonhole principle using a proof by contraposition. Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction, because there are at least $k+1$ objects.

The pigeonhole principle is also called the Dirichlet drawer principle, after the nineteenth century German mathematician G. Lejeune Dirichlet, who often used this principle in his work.

Following examples show how the pigeonhole principle is used.

Examples

1. Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.
2. In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.
3. How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Solution:

There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

Theorem 2 (The Generalized Pigeonhole Principle)

If N objects are placed into k boxes, then there is at least one box containing at least $\lfloor N/k \rfloor$ objects.

Where $\lfloor . \rfloor$ is defined as a floor function.

Following Examples illustrate how the generalized pigeonhole principle is applied.

Example:

Among 100 people there are at least $\lfloor 100/12 \rfloor = 9$ who were born in the same month.

Example:

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Solutions:

The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lfloor N/5 \rfloor = 6$. The smallest such integer is $N = 5 \times 5 + 1 = 26$. If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.

Example:

What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form $NXX - NXX - XXXX$, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

Solutions:

There are eight million different phone numbers of the form $NXX - XXXX$. Hence, by the generalized pigeonhole principle, among 25 million telephones, at least $\lfloor 25,000,000/8,000,000 \rfloor = 4$ of them must have identical phone numbers. Hence, at least four area codes are required to ensure that all 10-digit numbers are different.

Permutations and Combinations

Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters.

Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected does not matter.

For example, in how many ways can we select three students from a group of five students to stand in line for a picture? How many different committees of three students can be formed from a group of four students? In this section we will develop methods to answer questions such as these.

Permutations

A permutation of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an r-permutation.

The number of $r-$permutations of a set with n elements is denoted by $P(n, r)$. We can find $P(n, r)$ using the product rule.

Theorem:

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

$r-$ permutations of a set with n distinct elements.

Proof:

We will use the product rule to prove that this formula is correct. The first element of the permutation can be chosen in n ways because there are n elements in the set. We will use the product rule to prove that this formula is correct. The first element of the permutation can be chosen in n ways because there are n elements in the set. There are $n-1$ ways to choose the second element of the permutation, because there are $n-1$ elements left in the set after using the element picked for the first position.

Similarly, there are $n-2$ ways to choose the third element, and so on, until there are exactly $n-(r-1)=n-r+1$ ways to choose the rth element. Consequently, by the product rule, there are

$$n(n-1)(n-2)\cdots(n-r+1)$$

$r-$permutations of the set.

Corollary 1

If n and r are integers with $0 \leq r \leq n$, then $P(n, r) = \frac{n!}{(n-r)!}$

Examples:

1. How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

2. Suppose that there are eight runners in a race. The winner receives a gold medal, the second place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

3. Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Combinations

An $r-$ combination of elements of a set is an unordered selection of r elements from the set. Thus, an $r-$ combination is simply a subset of the set with r elements.

The number of $r-$ combinations of a set with n distinct elements is denoted by $C(n, r)$. Note that $C(n, r)$ is also denoted by $\binom{n}{r}$ and is called a binomial coefficient.

Theorem:

The number of $r-$ combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

Examples

1. How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another University?
2. A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?
3. A professor writes 40 discrete mathematics true/false questions. Of the statements in these questions, 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?
4. How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

Practice problems

1. Seven women and nine men are on the faculty in the Interdisciplinary department at a University.
   a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?
   b) How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?
2. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?
3. Two women (namely W_1, W_2) and three men (namely M_1, M_2, M_3)are on the faculty in the Interdisciplinary department at a University.How many ways are there to select a committee of three members of the department if at least one woman must be on the committee?

Corollary Let n and r be non-negative integers with $r \leq n$. Then

$$C(n, r) = C(n, n - r)$$

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Examples:

Thirteen people on a softball team show up for a game.

a) How many ways are there to choose 10 players to take the field?
b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?
c) Of the 13 people who show up, three are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

Solutions

a) This is just a matter of choosing 10 players from the group of 13, since we are not told to worry about what positions they play; therefore the answer is $C(13, 10) = 286$.

b) This is the same as part (a), except that we need to worry about the order in which the choices are made, since there are 10 distinct positions to be filled. Therefore the answer is $P(13, 10) = 13!/3! = 1,037,836,800$.

c) There is only one way to choose the 10 players without choosing a woman, since there are exactly 10 men. Therefore (using part (a)) there are $286 - 1 = 285$ ways to choose the players if at least one of them must be a woman.

Binomial Coefficients and Identities

As we remarked in Slide 22, the number of r-combinations from a set with n elements is often denoted by $\binom{n}{r}$. This number is also called a binomial coefficient because these numbers occur as coefficients in the expansion of powers of binomial expressions such as $(a+b)^n$. We will discuss the binomial theorem, which gives a power of a binomial expression as a sum of terms involving binomial coefficients.

The binomial theorem gives the coefficients of the expansion of powers of binomial expressions. A binomial expression is simply the sum of two terms, such as $x + y$. (The terms can be products of constants and variables, but that does not concern us here.)

Binomial Theorem

Theorem

Let x and y be variables, and let n be a non-negative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{r} x^{n-1} y + \cdots \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Examples

1. What is the expansion of $(x+y)^4$?
2. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?
3. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?
4. What is the coefficient of x^8y^9 in the expansion of $(3x+2y)^{17}$?
5. What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x-3y)^{200}$?

1. Let n be a non-negative integer. Then prove that

   (a). $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$

   (b). $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

2. Let n be a positive integer. Then prove that

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

Pascal's Identity and Triangle

The binomial coefficients satisfy many different identities. We introduce one of the most important of these now.

Theorem (Pascal's Identity)

Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Pascal's identity is the basis for a geometric arrangement of the binomial coefficients in a triangle, as shown in Figure 2.

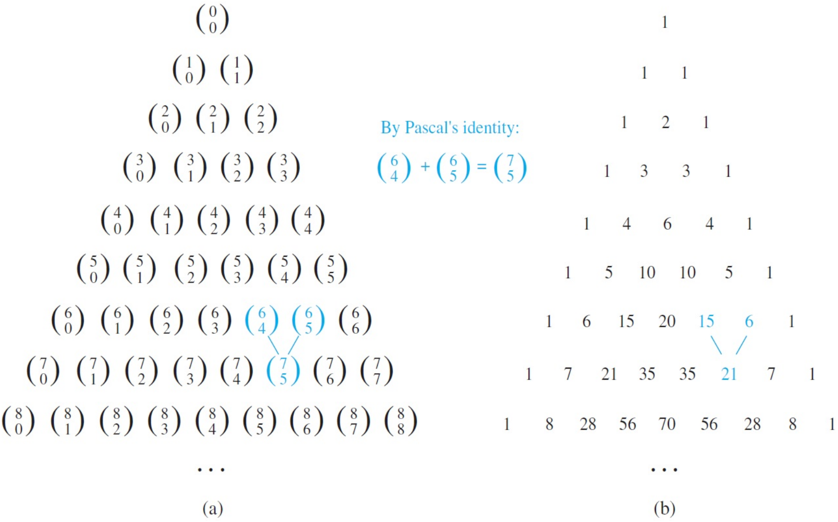

Figure 2: Pascal's Triangle.

The n th row in the triangle consists of the binomial coefficients

$$\binom{n}{k}, \quad k = 0, 1, 2, \cdots n$$

This triangle is known as **Pascal's triangle**. Pascal's identity shows that when two adjacent binomial coefficients in this triangle are added, the binomial coefficient in the next row between these two coefficients is produced.

Practice Problems

1. Prove the binomial theorem using mathematical induction.
2. Suppose that k and n are integers with $1 \leq k < n$. Prove the **hexagon identity**

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

which relates terms in Pascal's triangle that form a hexagon.

Don't hesitate to contact us if you have any questions about this course's teaching contents.

Also don't forget to check out the course page and Microsoft Team folder,

- course page https://mayooran1987.github.io/MC4010_E21/
- Microsoft Team folder link