MC4010 - Discrete Mathematics

Induction and Recursion

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All contents of these slides are agglomerated from recommended reference textbook of this course and the previous year's lecture slides (Which were prepared by Dr P. Kathirgamanathan).

Slides are prepared through RMarkdown Beamer presentation option and executed via RStudio.

Introduction

Suppose that we have an infinite ladder, as shown in Figure 1, and we want to know whether we can reach every step on this ladder. We know two things:

- We can reach the first rung of the ladder.
- If we can reach a particular rung of the ladder, then we can reach the next rung.

Can we conclude that we can reach every rung? By (1), we know that we can reach the first rung of the ladder. Moreover, because we can reach the first rung by (2), we can also reach the second rung; it is the next rung after the first rung. Applying (2) again, because we can reach the second rung, we can also reach the third rung. Continuing in this way, we can show that we can reach the fourth rung, the fifth rung, and so on.

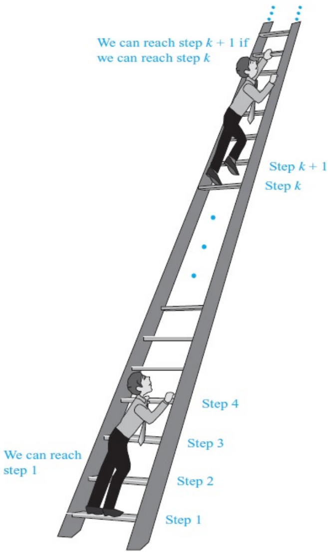

Figure 1: Climbing an Infinite Ladder.

For example, after 100 uses of (2), we know that we can reach the 101st rung. But can we conclude that we are able to reach every rung of this infinite ladder? The answer is yes, something we can verify using an important proof technique called mathematical induction. That is, we can show that $P(n)$ is true for every positive integer n, where $P(n)$ is the statement that we can reach the nth rung of the ladder.

Mathematical induction is an extremely important proof technique that can be used to prove assertions of this type. As we will see in this section and in subsequent sections of this chapter and later chapters, mathematical induction is used extensively to prove results about a large variety of discrete objects.

For example, it is used to prove results about the complexity of algorithms, the correctness of certain types of computer programs, theorems about graphs and trees, as well as a wide range of identities and inequalities.

Mathematical Induction

In general, mathematical induction can be used to prove statements that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function.A proof by mathematical induction has two parts, a basis step, where we show that $P(1)$ is true, and an inductive step, where we show that for all positive integers k, if $P(k)$ is true, then $P(k+1)$ is true.

PRINCIPLE OF MATHEMATICAL INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

To complete the inductive step of a proof using the principle of mathematical induction, we assume that $P(k)$ is true for an arbitrary positive integer k and show that under this assumption, $P(k+1)$ must also be true. The assumption that $P(k)$ is true is called the ***inductive hypothesis***.

Once we complete both steps in a proof by mathematical induction, we have shown that $P(n)$ is true for all positive integers, that is, we have shown that $\forall n P(n)$ is true where the quantification is over the set of positive integers.

In the inductive step, we show that $\forall k(P(k) \rightarrow P(k+1))$ is true, where again, the domain is the set of positive integers.

Expressed as a rule of inference, this proof technique can be stated as

$$(P(1) \land \forall k(P(k) \to P(k+1))) \to \forall n P(n),$$

when the domain is the set of positive integers.

Remark: In a proof by mathematical induction it is not assumed that $P(k)$ is true for all positive integers! It is only shown that if it is assumed that $P(k)$ is true, then $P(k+1)$ is also true. Thus, a proof by mathematical induction is not a case of begging the question, or circular reasoning.

Wayes to remember how induction works!

Thinking of the infinite ladder and the rules for reaching steps can help you remember how mathematical induction works. Note that statements (1) and (2) for the infinite ladder are exactly the basis step and inductive step, respectively, of the proof that $P(n)$ is true for all positive integers n, where $P(n)$ is the statement that we can reach the nth rung of the ladder. Consequently, we can invoke mathematical induction to conclude that we can reach every rung.

Another way to illustrate the principle of mathematical induction is to consider an infinite row of dominoes, labeled $1, 2, 3, \ldots, n, \ldots$, where each domino is standing up. Let $P(n)$ be the proposition that domino n is knocked over. If the first domino is knocked over—i.e., if $P(1)$ is true—and if, whenever the kth domino is knocked over, it also knocks the $(k+1)$st domino over—i.e., if $P(k) \to P(k+1)$ is true for all positive integers k—then all the dominoes are knocked over. This is illustrated in Figure 2.

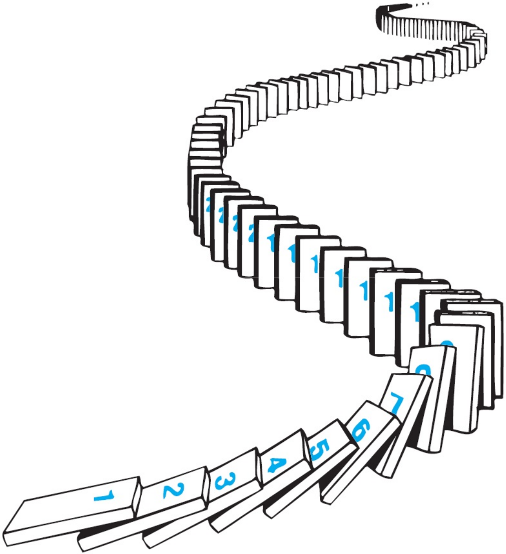

Figure 2: Illustrating How Mathematical Induction Works Using Dominoes.

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

Example 1

Show that if n is a positive integer, then

$$1 + 2 + \cdots\cdots n = \frac{n(n+1)}{2}$$

Solution: Let $P(n)$ be the proposition that the sum of the first n positive integers, $1 + 2 + \cdots n = n(n+1)/2$, is $n(n+1)/2$. We must do two things to prove that $P(n)$ is true for $n = 1, 2, 3, \cdots$ Namely, we must showt hat $P(1)$ is true and that the conditional statement $P(k)$ implies $P(k+1)$ is true for $k = 1, 2, 3, \cdots$

BASIS STEP: $P(1)$ is true, because $1 = 1(1+1)/2$. (The left-hand side of this equation is 1 because 1 is the sum of the first positive integer. The right-hand side is found by substituting 1 for n in $n(n+1)/2$.)

Solution:

INDUCTIVE STEP:: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$1 + 2 + \cdots + k = k(k+1)/2.$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

is also true. When we add $k+1$ to both sides of the equation in $P(k)$, we obtain

$$
\begin{aligned}
1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

This last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step.

We have completed the basis step and the inductive step, so by mathematical induction we know that $P(n)$ is true for all positive integers n. That is, we have proven that $1 + 2 + \cdots + n = n(n+1)/2$ for all positive integers n.

Example 2

Use mathematical induction to show that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all non-negative integers n.

Solution:

Example 3

Sums of Geometric Progressions Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term a and common ratio r:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1}; \quad \text{when} \quad r \neq 1,$$

where n is a non-negative integer.

Solution:

Example 4

Use mathematical induction to prove the inequality

$$n < 2^n$$

for all positive integers n.

Solution:

Practice problems

Let $P(n)$ be the statement that $n! < n^n$, where n is an integer greater than 1.

1. What is the statement P(2)?
2. Show that P(2) is true, completing the basis step of the proof.
3. What is the inductive hypothesis?
4. What do you need to prove in the inductive step? Complete the inductive step.
5. Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

Practice problems

Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ for the positive integer n.

1. What is the statement $P(1)$? Show that $P(1)$ is true, completing the basis step of the proof.
2. What is the inductive hypothesis?
3. What do you need to prove in the inductive step?
4. Complete the inductive step, identifying where you use the inductive hypothesis.
5. Explain why these steps show that this formula is true whenever n is a positive integer.

Practice problems

1. Use mathematical induction to show that

$$n^2 > 3n$$

for all $n \geq 4$.

2. Use mathematical induction to show that

$$1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$$

for all non-negative integers n.

Practice problems

An Inequality for Harmonic Numbers: The harmonic numbers $H_j, j = 1, 2, 3, \cdots$, are defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}$$

For instance,

$$H4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

Use mathematical induction to show that

$$H_{2^n} \geq 1 + \frac{n}{2},$$

whenever n is a non negative integer.

Strong Induction

STRONG INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that the proposition $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \to P(k+1)$ is true for all positive integers k.

Strong Induction is sometimes called the second principle of mathematical induction or complete induction.

Strong induction and the infinite ladder

To better understand strong induction, consider the infinite ladder example in Figure 1. Strong induction tells us that we can reach all rungs if

1. we can reach the first rung, and
2. for every integer k, if we can reach all the first k rungs, then we can reach the $(k+1)$th rung.

That is, if $P(n)$ is the statement that we can reach the nth rung of the ladder, by strong induction we know that P(n) is true for all positive integers n, because (1) tells us P(1) is true, completing the basis step and (2) tells us that $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ implies $P(k+1)$, completing the inductive step.

Strong vs Mathematical Induction

- We can always use strong induction instead of mathematical induction. But there is no reason to use it if it is simpler to use mathematical induction.
- In fact, the principles of mathematical induction, strong induction, and the well- ordering property (this is study later on) are all equivalent.
- Sometimes it is clear how to proceed using one of the three methods, but not the other two.

Example

Show that if n is an integer greater than 1, then n can be written as the product of primes.

Solution:

Let $P(n)$ be the proposition that n can be written as a product of primes.

BASIS STEP: $P(2)$ is true since 2 itself is prime.

INDUCTIVE STEP: The inductive hypothesis is $P(j)$ is true for all integers j with $2 \leq j \leq k$. To show that $P(k+1)$ must be true under this assumption, two cases need to be considered:

If $k+1$ is prime, then $P(k+1)$ is true.

Otherwise, $k+1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < k+1$. By the inductive hypothesis a and b can be written as the product of primes and therefore $k+1$ can also be written as the product of those primes.

Well-Ordering Property.

- Well-ordering property: Every nonempty set of non-negative integers has a least element.
- The well-ordering property is one of the axioms of the positive integers.
- The well-ordering property can be used directly in proofs.
- The well-ordering property can be generalized.
- Definition: A set is well ordered if every subset has a least element.
  - N is well ordered under $\leq$.
  - The set of finite strings over an alphabet using lexicographic ordering is well ordered.

Recursion

- Sometimes it is difficult to define an object explicitly. However, it may be easy to define this object in terms of itself. This process is called recursion.
- For instance, the picture shown in Figure 3 is produced recursively. First, an original picture is given. Then a process of successively superimposing centered smaller pictures on top of the previous pictures is carried out.
- Recursion can be used to define:
  - Functions
  - Sequences
  - Sets
  - Problems
  - Algorithms

A Recursively Defined Picture.

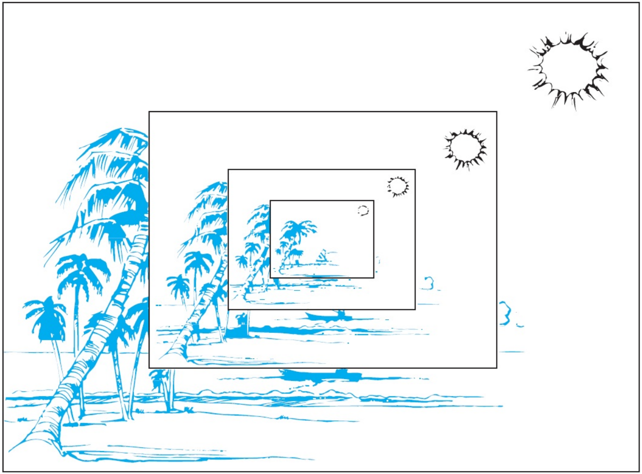

Figure 3: A Recursively Defined Picture.

Recursion vs Induction

- Using recursion, we define an object (e.g., a function, a predicate or a set) over an infinite number of elements by defining large size objects in terms of smaller size ones.
- In induction, we prove all members of an infinite set have some predicate P by proving the truth for large size objects in terms of smaller size ones.

Recursively Defined Functions

Definition: A recursive or inductive definition of a function consists of two steps.

BASIS STEP: Specify the value of the function at zero.

RECURSIVE STEP: Give a rule for finding its value at an integer from its values at smaller integers.

A function $f(n)$ is the same as a sequence $a_0, a_1, \cdots$, where a_i, where $f(i) = a_i$.

Recursively Defined Functions

- One way to define a function $f : \mathbb{N} \to S$ (for any set S) or series $a_n = f(n)$ is to:
  - Define $f(0)$.
  - For $n > 0$, define $f(n)$ in terms of $f(0), \cdots f(n-1)$.
- Example.: Define the series $a_n = 2^n$ recursively:
  - Let $a_0 = 1$.
  - For $n > 0$, let $a_n = 2a_{n-1}$.

Examples

1. Suppose f is defined by: $f(0) = 3$,

$$f(n+1) = 2f(n) + 3$$

Find $f(1), f(2), f(3), f(4)$

Solution:

$f(1) = 2f(0) + 3 = 2 \times 3 + 3 = 9$ and $f(2) = 2f(1) + 3 = 2 \times 9 + 3 = 21$

$f(3) = 2f(2)+3 = 2\times 21+3 = 45$ and $f(4) = 2f(3)+3 = 2\times 45+3 = 93$

2. Give a recursive definition of the factorial function $n!$:

Solution:

$f(0) = 1$ and $f(n+1) = (n+1) \times f(n)$

3. Give a recursive definition of:

$$\sum_{k=0}^{n} a_k$$

Solution:

The first part of the definition is

$$\sum_{k=0}^{0} a_k = a_0$$

The second part is

$$\sum_{k=0}^{n+1} a_k = a_0 = \left(\sum_{k=0}^{n} a_k\right) + a_{n+1}$$

Fibonacci Numbers

The Fibonacci numbers are defined as follows:

$$f_0 = 0$$

$$f_1 = 1$$

$$f_n = f_{n-1} + f_{n-2}$$

Find f_2, f_3, f_4, f_5 .

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- course page https://mayooran1987.github.io/MC4010_E21/
- Microsoft Team folder link