Department of Inter Disciplinary Studies,
Faculty of Engineering,
University of Jaffna, Sri Lanka
MC 4010 : Assignment 3 Answers

40 minutes **22-11-2023**

Important instructions:

- **Answer all the questions (1-4).**
- **If you have any doubt as to the interpretation of the wording of a question, make your own decision, but clearly state in the script.**
- **If it is determined that you have violated any policies during this exam, you will receive a score of zero for this assignment, without any exceptions or considerations.**

1. In a discrete mathematics class within the engineering faculty at University of Jaffna,there are 10 students, including a pair of students—Alice and Bob. Unfortunately, due to the eye infection spread on the premises, many students were unable to participate in the class that week. The lecturer randomly selects six students to participate in a special problem-solving activity during the class. Your task is to calculate the number of ways the selected engineering students can be arranged under different conditions:

   (a) Determine the number of arrangements when Alice must be part of the selected group of 6 engineering students.

   (b) Calculate the arrangements when both Alice and Bob must be together in the selected group of 6 engineering students.

   (c) Find the arrangements when exactly one of the pair (Alice or Bob) is included in the selected group of 6 engineering students.

   (d) Calculate the number of ways to form a group of 3 students, including Alice and Bob, from the remaining students who attended the class that week.

   Solutions:

   (a) Alice must be part and Alice can be arranged in$= {}^6P_1 = 6$
   Other can be arranged $= {}^9P_5 = 15120$
   total number of arrangements $= {}^6P_1 \times^9 P_5 = 90720$

   (b) Alice, Bob must be part can be arranged $= {}^6P_2 = 30$
   Other can be arranged $= {}^8P_4 = 1680$
   total number of arrangements $= {}^6P_2 \times^8 P_4 = 50400$

   (c) only Alice $= {}^6P_1 \times^8 P_5 = 40320$
   similarly only bob $= {}^6P_1 \times^8 P_5 = 40320$
   arrangements when exactly one of the pair $= 40320 + 40320 = 80640$

   (d) number of ways to form a group of 3 students, including Alice and Bob
   $=1 \times 1 \times^8 C_1 = 8$

2. In a computer engineering lab, there are a dozen USB drives, half of which are designed for data storage (Data Drives) and half for software development (Dev Drives). A computer engineering student randomly selects USB drives in the dark because of an unscheduled power failure. Your task is to answer the following questions:

   (a) How many USB drives must the student take out to be sure that they have at least two USB drives of the same type (either Data Drive or Dev Drive)?

   (b) How many USB drives must the student take out to be sure that they have at least two USB drives designed for software development?

Solutions

(a) $\dfrac{(m-1)}{n} + 1 = 2$

$\dfrac{(m-1)}{2} + 1 = 2$

$m - 1 = 2$

$m = 3$

(b) 8 USB drives (Worst case 6 can be Data drives So the remaining 2 will be Dev drive.)

3. (a) Let n and k be positive integers with $n \geq k$. Then prove that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

   (b) Suppose that k and n are integers with $1 \leq k < n$. Prove the **hexagon identity**

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

   which relates terms in Pascal's triangle that form a hexagon.

   (c) Let n be a positive integer. Then prove that

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

Solutions

(a) We consider $\binom{n}{k-1} + \binom{n}{k}$

$$\begin{aligned}
&= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!}{(n-k)!(n-k+1)(k-1)!} + \frac{n!}{(k-1)!k(n-k)!} \\
&= \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{n-k+1} + \frac{1}{k}\right) \\
&= \frac{n!(n+1)}{(k-1)!(n-k)!k(n-k+1)} \\
&= \frac{(n+1)!}{k!(n-k+1)!} \\
&= \frac{(n+1)!}{k!(n+1-k)!} \\
&= \binom{n+1}{k}
\end{aligned}$$

(b) We consider LHS

$$\begin{aligned}
LHS &= \binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} \\
&= \left(\frac{(n-1)!}{(n-k)!(k-1)!}\right)\left(\frac{n!}{(n-k-1)!(k+1)!}\right)\left(\frac{(n+1)!}{(n-k+1)!k!}\right) \\
&= \left(\frac{(n-1)!}{(n-k-1)!k!}\right)\left(\frac{n!}{(n-k+1)!(k-1)!}\right)\left(\frac{(n+1)!}{(n-k)!(k+1)!}\right) \\
&= \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1} \\
&= RHS
\end{aligned}$$

Therefore we have shown that the hexagon identity holds.

(c) Let x and y be variables, and let n be a non-negative integer. The binomial theorem states that for any positive integers n and k,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

here put $x = 1$ and $y = 2$

$$\begin{aligned}
(1+2)^n &= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k \\
3^n &= \sum_{k=0}^{n} \binom{n}{k} 2^k \qquad [\because 1^{n-k} = 1]
\end{aligned}$$

$$\therefore \quad \sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

4. Nine women and eleven men are on the E-week committee formed for the E-week 2023 program in the Faculty of Engineering at the University of Jaffna.

(a) How many ways are there to select a sub committee of seven members of the E-week committee if at least one woman must be on the committee?

(b) How many ways are there to select a sub committee of seven members of the E-week committee if at least one woman and at least one man must be on the committee?

Solutions:

(a) The total number of subcommittee of seven numbers is

$$\begin{aligned}
\binom{20}{7} &= \frac{20!}{7!(20-7)!} \\
&= \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13!}{7!13!} \\
&= \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \\
&= 19 \times 17 \times 16 \times 15 \\
&= 77520
\end{aligned}$$

The number of subcommittee with no women is

$$\binom{11}{7} = \frac{11!}{7!\ 4!} = \frac{8 \times 9 \times 10 \times 11}{4 \times 3 \times 2 \times 1} = 330$$

So the number of subcommittee with at least one woman is $77520 - 330 = 77190$

(b) The total number of subcommittee of seven numbers is $\binom{20}{7} = 77520$

The number of subcommittee with only woman is

$$\begin{aligned} \binom{9}{7} &= \frac{9!}{7!\ 2!} \\ &= \frac{9 \times 8}{2 \times 1} \\ &= 36 \end{aligned}$$

The number of subcommittee with only man is $\binom{11}{7} = 330$

Therefore the number of subcommittee with at least one woman and at least one man is $77520 - 330 - 36 = 77154$

Some useful formulas

1. Let x and y be variables, and let n be a non-negative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

2. $P(n,r) = nPr = \dfrac{n!}{(n-r)!}$

3. $C(n,r) = nCr = \dfrac{n!}{(n-r)!r!}$