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MC 4010 - Tutorial 03

1. Let $P(n)$ be the statement that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n^4 + 2n^3 + n^2}{4}\right)$ for the positive integer n.

   a) What is the statement $P(1)$?
   b) Show that $P(1)$ is true, completing the basis step of the proof.
   c) What is the inductive hypothesis?
   d) What do you need to prove in the inductive step?
   e) Complete the inductive step, identifying where you use the inductive hypothesis.
   f) Explain why these steps show that this formula is true whenever n is a positive integer.

2. Prove that

$$\sum_{j=0}^{n} \left(\frac{-1}{2}\right)^{j} = \frac{1}{3}\left(\frac{2^{n+1} + (-1)^n}{2^n}\right)$$

   whenever n is a non negative integer.

3. Suppose that we want to prove that

$$\prod_{j=1}^{n} \left(\frac{2j-1}{2j}\right) < \frac{1}{\sqrt{3n}} \text{ for all positive integers } n.$$

   (a) Show that if we try to prove this inequality using mathematical induction, the basis step works, but the inductive step fails.
   (b) Show that mathematical induction can be used to prove the stronger inequality

$$\prod_{j=1}^{n} \left(\frac{2j-1}{2j}\right) < \frac{1}{\sqrt{3n+1}}$$

   for all integers greater than 1, which, together with a verification for the case where $n = 1$, establishes the weaker inequality we originally tried to prove using mathematical induction.

4. Let f_n is the n^{th} Fibonacci number. Show that $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ when n is a positive integer.

5. (a) Find the general solution of the recurrence relation $a_{n+2} + a_{n+1} - 12a_n = 0,\ n \geq 0$ satisfying the initial conditions $a_0 = 1,\ a_1 = 1$.
   (b) Find the general solution of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$
   (c) Find the general solution of the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$, with initial conditions $a_0 = 1,\ a_1 = -2$ and $a_2 = -1$.

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6. a) Prove the binomial theorem by using mathematical induction.

   b) Suppose that k and n are integers with $1 \leq k < n$. Prove the **hexagon identity**

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

   which relates terms in Pascal's triangle that form a hexagon.

7. Use mathematical induction to show that

$$\sum_{j=1}^{n} \cos(jx) = \left[\cos\frac{(n+1)x}{2}\right]\frac{\sin(\frac{nx}{2})}{\sin(\frac{x}{2})}$$

   whenever n is a positive integer and $\sin(\frac{x}{2}) \neq 0$.

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