UNIVERSITY OF JAFFNA

FACULTY OF ENGINEERING

END SEMESTER EXAMINATION– JUNE 2023

MC4010- DISCRETE MATHEMATICS

Date: 09 - 06 -2023 Duration: TWO Hours

Instructions

1. This paper contains **FIVE (5)** questions:
2. Answer **all** questions in the answer book provided.
3. If you have any doubt as to the interpretation of the wording of a question, make your own decision, but clearly state in the script.
4. This examination accounts for **50%** of module assessment. Total maximum mark attainable is **100**.

Question 1[20 marks]

1. (a) Write the set $E = \{x \in \mathbb{Z} : x^2 - 2x - 8 \leq 0\}$ by listing its elements.

   (b) Determine the power set $\mathcal{P}(A)$ for $A = \{1, 3, 5\}$.

2. Let C be the set of professors in a mathematics department who taught a calculus class and D the set of professors in a mathematics department who taught a discrete mathematics course. Describe each of the following sets in terms of C and D.

   (a) the set of professors in the mathematics department who taught both a calculus course and a discrete mathematics course.

   (b) the set of professors in the mathematics department who taught either a calculus course or a discrete mathematics course.

   (c) the set of professors in the mathematics department who taught neither a calculus course nor a discrete mathematics course.

   (d) the set of professors in the mathematics department who taught a calculus course but not a discrete mathematics course.

   (e) the set of professors in the mathematics department who taught a discrete mathematics course but not a calculus course.

3. For the universal set $U = \{1, 2, \ldots, 10\}$, draw a Venn diagram for three sets A, B and C indicating the locations of the elements of U if A, B and C satisfy all of the conditions (a) - (h):

(a) $A \cap B \cap C = \{5\}$

(b) $\overline{A \cup B \cup C} = \{10\}$

(c) $A - (B \cup C) = \{3\}$

(d) $B - (A \cup C) = \{4\}$

(e) $C - (A \cup B) = \{2\}$

(f) $(B \cup C) - A = \{1, 2, 4, 9\}$

(g) $(A \cup C) - B = \{2, 3, 7, 8\}$

(h) $(A \cup B) - C = \{3, 4, 6\}$

Question 2[20 marks]

1. Let $O(x)$ be the statement "x is in this exam hall", $W(x)$ be "x is a woman", let $M(x)$ be the statement "x is a man", and let $G(x)$ be the statement "x wears glasses". Express each of these statements in terms of O(x), W(x), M(x), G(x), quantifiers, and logical connectives. Let the domain consist of all people.

   (a) Someone in this exam hall wears glasses.

   (b) Not everyone in this exam hall wears glasses.

   (c) No woman in this exam hall wears glasses.

   (d) All men in this exam hall wears glasses.

2. Using logical equivalences, show that each of the following compound propositions is a tautology.

   (a) $[p \wedge (p \to q)] \to q$

   (b) $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$

3. (a) Construct a combinatorial circuit using **inverters**, **OR** gates, and **AND** gates that produces the output $(p \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$ from input bits p, q, and r.

   (b) Find the output of each of these combinatorial circuits.

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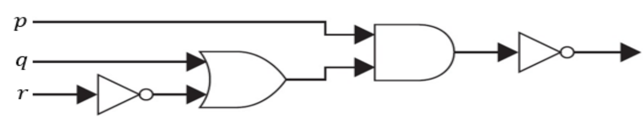

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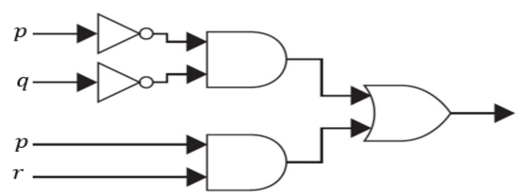

Question 3[20 marks]

1. (a) Show that the function $f : \mathbb{R} \to \{x \in \mathbb{R} : -1 < x < 1\}$ defined by

$$f\left(x\right) = \frac{x}{1+|x|}, \quad \forall x \in \mathbb{R}$$

is a bijective function.

(b) Define the relation $\sim$ on $\mathbb{Z}^+$ by $x \sim y$ if and only if $x - y$ is divisible by 2.

i. Show that $\sim$ is an equivalence relation.

ii. Describe the equivalence classes $[0]\,,[1]$.

2. Students' Union, University of Jaffna arranged a cricket tournament among five Faculties. Each of them **a**- Engineering, **b** - Agriculture, **c** -Technology, **d** - Science, **e** - Arts compete with every other faculty. A relation R on the set $S = \{a, b, c, d, e\}$ is defined by: xRy if and only if x beat y. The following set is defined the elements of this relation .

$$R = \{(a, b), (b, e), (a, e), (e, a), (a, c), (c, b), (c, d), (d, e), (b, d)\}$$

In this set, each element represents a pair (x, y), indicating that faculty x beat faculty y in the cricket tournament.

(a) Prepare the points table for above relation (Total Matches, Win, Loss) for each faculty.

(b) Draw the directed graph for above relation.

(c) Identify the reflexive, symmetry, anti-symmetry, and transitive properties of this relation.

Question 4[20 marks]

1. The traffic Police officer's radar speed gun causes grief for many drivers who use the Jaffna-Kandy (A9) road. A particular location between Ariviyal Nagar and Iranaimadu junction in Kilinochchi catches 12 speeding motorists per hour. This figure is an average of all hours between 9am and 5pm.

(a) How many motorists would you expect to be caught speeding by the Police officer in any given 10 minutes period? And which distribution would best be used to model the number of speeding motorists in a 10 minutes period?

(b) What proportion of 10 minutes period would you expect the Police officer's radar speed gun to remain unused to deliver a fine?

(c) What is the probability that the Police officer catches at-least three motorists in 15 minutes period?

2. Dylan is getting married tomorrow, at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year, so the prior probability of rain is just 5/365. Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 5% of the time. What is the probability that it will rain on the day of Dylan's wedding?

3. Let X be a random variable on a sample space S. Show that $Var(aX+b) = a^2 Var(X)$ whenever a and b are real numbers.

Question 5[20 marks]

1. Suppose that a password for a computer system in computer lab of Faculty of Engineering at Ariviyal Nagar, Kilinochchi must have at least 8, but no more than 12, characters, where each character in the password is a lowercase English letter, an uppercase English letter, a digit, or one of the eight special characters $\star, >, <, !, +, @, \%$ and $=$.

   (a) How many different passwords are available for this computer system?

   (b) How many of these passwords contain at least one occurrence of at least one of the eight special characters?

   (c) Using your answer to part (a), determine how long it takes a hacker to try every possible password, assuming that it takes two nanoseconds (one nanosecond is one-billionth (10^{-9}) of a second) for a hacker to check each possible password.

2. Use proof by mathematical induction method (provide an appropriate induction steps to get full marks) to show that

   (a) for any positive integer n, $13 \times 7^{n-1} + 5^{2n-1}$ is divisible by 18.

   (b) for any positive integer n, $\displaystyle\sum_{k=1}^{n} \frac{k+4}{k(k+1)(k+2)} = \frac{n(3n+7)}{2(n+1)(n+2)}$

Some useful formulas:

1. If X follows Poisson distribution with parameter λ. Then, the probability mass function is given by $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, 2, \cdots\cdots$

2. $\mu = E(X) = \displaystyle\sum_{\text{for all x}} xP(X=x)$ and $\sigma^2 = Var(X) = E(X^2) - (E(X))^2$
   or $Var(X) = E(X-\mu)^2 = \displaystyle\sum_{\text{for all x}} (x-\mu)^2 P(X=x)$

3. Let us consider that a sample space S is divided into two mutually exclusive partitions S_1 and S_2. An event H has occurred, and $P(S_1|H)$ can be written as

$$P(S_1|H) = \frac{P(S_1)P(H|S_1)}{P(S_1)P(H|S_1) + P(S_2)P(H|S_2)}$$

End of Examination