Mathematical Physics

Based on Qiao Gu's and Liang's.

Complex Function

Basic Definitions

For a complex function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$, and an interior point $z = x + iy$ in its domain. It's derivable at z if and only if u and v are all differentiable at (x, y) and the function satisfy Cauchy-Riemann condition at that point.

C-R condition:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Also, we have the polar coordinate version C-R condition:

$$\frac{\partial u}{\partial \rho} = \frac{1}{\rho}\frac{\partial v}{\partial \phi}$$

$$\frac{1}{\rho}\frac{\partial u}{\partial \phi} = -\frac{\partial v}{\partial \rho}$$

If function $f(z)$ is differentiable at any point in a neighborhood of z_0, then we call it a holomorphic function at z_0. (equivalent to **analytic** because of some mathematical reason)

Tips

Q: If you know the function is holomorphic, but you only have the $f(x, y)$ form, how to quickly gain the compact $f(z)$ form?

A: There's a clever trick for form-reshaping: let all y in $f(x, y)$ be zero and let x in $f(x, y)$ be z, then you get the answer! You probably wonder why, and the reason behind is the uniqueness of a power series. For an analytic function, its power series' form is unique, so that when z is real ($z = x + i0$), the expression of power series shall remain.

Harmonic functions

The real and imaginary parts of a holomorphic function are all harmonic, which can be easily verified by operating C-R condition.

Taylor Series

Know that the Taylor series is unique, and wherever the function is holomorphic, it can be expanded. Thus the convergent radius must be the distance between the nearest singularity and the central point for expansion. (If not, the small neighborhoods at the fringe could combine and expanded the convergent circle into a larger one, contradiction)

Laurent Series

Laurent theorem's statement: If function $f(z)$ is holomorphic in a disk $\{R_1 < |z - z_0| < R_2\}$, then it could be expressed in the form

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n$$

$$C_n := \frac{1}{2\pi i} \oint_{C_R} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$

The theorem is complicated as you can see, but the theorem itself is not the key point. The key advantage is the uniqueness of Laurent series, which implies that no matter how you get the expanded series' expression, it's unique and correct. Thus you can apply the formula of geometric power series to get the result, etc. Besides, the Laurent theorem is almost the **Residue theorem**.

Integrals

To simplify the framework, you only need to remember the **Residue Theorem**, because Cauchy Formula and Cauchy Theorem are all special case of Residue Theorem.

$$\oint_l f(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res} f(b_j)$$

where b_j are isolated singularities, and the function must be analytic inside of l except at these singularities.

Note that Residue Theorem works as well for essential singularities! The only obstacle is how to calculate the Residue there though.

Jordan's Lemma

If $g(z)$ is continuous outside a sufficient large circle in z-plane and $\lim_{z\to\infty} g(z) = 0$, then for $\alpha > 0$ we have:

$$\lim_{R\to\infty} \int_{\Gamma_R} g(z)e^{i\alpha z}dz = 0$$

where Γ_R is an *arc* with radius R.

Useful Variable Changes

When the integral interval is $0 \to 2\pi$, let $z = e^{ix}$
When the integral interval is $-\infty \to \infty$ then let $z = x$ and apply Jordan's Lemma. ($zf(z) \rightrightarrows 0$)
When the integral interval is $0 \to \infty$, first convert it to $-\infty \to \infty$.
For $\int_0^\infty F(x)\cos mx dx$ or so, first consider $\text{Re}\int_0^\infty F(z)e^{imz}dz$. ($F(z) \rightrightarrows 0$). If you don't convert trigonometrical functions (boundless) into exponential functions, the Residue Theorem is not allowed to use! Because the

validness is guaranteed by Jordan's Lemma which require exponential form.

Fourier Theory

Fourier Integral

$$f(x)=\int_0^\infty [A(\omega)\cos\omega x+B(\omega)\sin\omega x]d\omega$$

where

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x dx$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x dx$$

Q: why is the interval like this?

A: $f(x)=\int_0^\infty\cdots$,comes from the Fourier series $\sum\limits_0^\infty$.

$A(\omega),B(\omega)=\int_{-\infty}^{\infty}$ comes from $\lim\limits_{L\to\infty}\int_{-L}^{L}$.

An Example

Dirichlet integral :

considering

the function below and its Fourier integral

$$
f(x) = \begin{cases} 0, & (|x| \leq 1) \\ 1, & (|x| > 1) \end{cases}
$$

we get

$$
\int_0^\infty \frac{\sin \omega}{\omega} d\omega = \frac{\pi}{2}
$$

This could be derived later by FT more generally.

Fourier Transfom

$$
F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx
$$

$$
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega
$$

Property

- Linear
- $\frac{d}{dx} f(x) \to i\omega F(\omega)$
- $x f(x) \to i \frac{d}{d\omega} F(\omega)$
- $\int f(x) dx \to \frac{F(\omega)}{i\omega}$
- $f(x+\zeta) \to e^{i\omega\zeta} F(x)$
- convolution

The methods to verify properties involving integral and derivative are similar to Laplacian Transform, the detail of it will be shown there.

Two Examples

when f is even

$f(x) * \cos \omega x = F(\omega) \cos \omega x$

$f(x) * \sin \omega x = F(\omega) \sin \omega x$

δ Function

We view δ function as a generalized function.

In the form $\int f(x)dy$, what exactly does dy mean? A mathematical perspective is the **measure** σ of a set. Riemann coincidentally defined his integral using the simplest measure namely $x_{i+1} - x_i$, i.e. the measure of interval $\left[x_i, x_{i+1}\right)$ in partition. When $\delta(x)$ is entailed, $\int f(x)d\sigma$ shall be viewed as a Lebesgue Integral where only $f(0)$ gained a measure of 1, any other value weighed 0, such that the integral $= \Sigma f(x_i)\sigma_i = f(0) \times 1 = f(0)$.

Also, we may define it (W) using **inner product**: $\langle L(W), \phi\rangle = \langle W, L^*(\phi)\rangle$, where L represents any linear differential operator, and L^* its adjoint, ϕ is a test function of great property.

Besides, Riesz Representation Theorem guarantees the equality of an inner product and a linear functional, thus **Linear Functional** is also a good choice to describe generalized functions.

The mathematical interpretation shall stop and physical introduction begins.

The convolution of $\delta(x)$ and $f(x)$

$$\delta(x-a) * f(x) = f(x-a)$$

satisfy the eigen-equation of coordinates:

$$x\delta(x-x_0) = x_0\delta(x-x_0)$$

where x is the operator of coordinates.

It's obvious that delta function is orthogonal to each other and the completeness is given by:

$$f(x) = \int_{-\infty}^{\infty} f(\zeta)\delta(\zeta - x)d\zeta$$

where $f(\zeta)$ are the parameters.

its FT & Applications

$$\mathscr{F}[\delta(x)] = 1$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} d\omega$$

or write it as

$$\delta(p-p') = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}(p-p')x} dx$$

which implies the orthogonality of momentum the normalized eigen-vector of momentum is:

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}px}$$

as for its completeness, considering the Fourier transform of $f(x)$:

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} F(\omega) e^{(i/\hbar)(\hbar\omega)x} d(\hbar\omega) \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} F(\omega) e^{(i/\hbar)(\hbar\omega)x} d(\hbar\omega) \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \tilde{F}(p) e^{(i/\hbar)px} dp \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \tilde{F}(p) \psi_p(x) dp \end{aligned}$$

where

$$\tilde{F}(p) = F\left(\frac{p}{\hbar}\right) = \int_{-\infty}^{\infty} f(x) e^{-i\frac{p}{\hbar}} dx$$

Delta function in concrete forms

common ones:

$$V(x) = \lim_{\beta \to \infty} \frac{\sin \beta x}{\pi x}$$

$$G(x) = \lim_{\beta \to 0} \frac{1}{\sqrt{\pi \beta}} \exp(-\frac{x^2}{\beta})$$

$$L(x) = \lim_{\beta \to 0} \frac{1}{\pi} \frac{\beta}{x^2 + \beta^2}$$

Dirichlet Kernel

$$D_m(x) = \frac{1}{2\pi} \frac{\sin(m + \frac{1}{2}x)}{\sin \frac{1}{2}x} = \frac{1}{2\pi}(1 + 2\sum_{k=1}^{m} \cos kx)$$

and

$$\lim_{m \to \infty} D_m(x) = \delta(x)$$

- The fatal loophole in Qiao Gu's proof of Dirichlet theorem is that he apply the property of selection.
  Delta function here corresponds to Good-Kernel in math, which requires three essential aspects to satisfy, while Dirichlet Kernel violates one of them. So Dirichlet Kernel cannot be used to approach a function, i.e. cannot apply the property of selection.

Example of δ function

Though $\sin kx, \cos kx$ aren't absolutely integrable, but we can discuss their Fourier Transform in physics.

$$\begin{aligned}
F_{\sin}(\omega) &= \int_{-\infty}^{\infty} \sin kx e^{-i\omega x} dx \\
&= \frac{1}{2i} \int_{-\infty}^{\infty} (e^{ikx} - e^{-ikx}) e^{-i\omega x} dx \\
&= i\pi[\delta(\omega + k) - \delta(\omega - k)] \\
F_{\cos}(\omega) &= \int_{-\infty}^{\infty} \cos kx e^{-i\omega x} dx \\
&= \frac{1}{2} \int_{-\infty}^{\infty} (e^{ikx} + e^{-ikx}) e^{-i\omega x} dx \\
&= \pi[\delta(\omega + k) + \delta(\omega - k)]
\end{aligned}$$

important transform pairs

$$\mathscr{F}[\frac{1}{1+x^2}] \rightleftharpoons e^{-|x|}$$

Gauss function: $g(x) = \exp(-\frac{ax^2}{2})$

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} \exp(-\frac{ax^2}{2}) e^{-i\omega x} dx \\
&= \ldots \text{(integration by parts)} \\
&= \frac{ia}{\omega} \mathscr{F}[x \exp(-\frac{ax^2}{2})] \\
&= \frac{ia}{\omega} i \frac{d}{d\omega} G(\omega)
\end{aligned}
$$

solve an ode to get $G(\omega) = \sqrt{\frac{2\pi}{a}} \exp(-\frac{\omega^2}{2a})$

How to calculate the FT of the leap function?

$$
u(x) = \begin{cases} 0, & (x < 0) \\ 1, & (x \geq 0) \end{cases}
$$

view it as the $\lim\limits_{\beta \to 0^+} f(x)$

where

$$
f(x) = \begin{cases} 0, & (x < 0) \\ e^{-\beta x}, & (x \geq 0) \end{cases}
$$

hence the FT of $u(t)$ is

$$
U(\omega) = \lim_{\beta \to 0^+} F(\omega)
$$

$$
\begin{aligned}
&= \lim_{\beta \to 0^+} \frac{1}{\beta + i\omega} \\
&= \pi\delta(\omega) + \frac{1}{i\omega}
\end{aligned}
$$

The final step see $L(x)$ there.

Apply Residue Theorem

The above examples show some Physical methods to deal with special functions. While for a wide range of integrations we are able to calculate them using ***Residue Theorem***, a much more handy way than Qiao Gu's Fourier method in his book.

Usual Steps (for trigonometric and exponential):

1.Adjust the interval to $(-\infty, \infty)$

2.Show proper form like $\int_{-\infty}^{\infty} e^{imx} f(x) dx$

3.If m<0, $\int_{-\infty}^{\infty} e^{-i|m|x} f(x) dx = \int_{-\infty}^{\infty} e^{i|m|y} f(-y) dy$

4.Examine whether $f(z)$ converges uniformly to 0 at ∞.

5.Jordan's Lemma.

For instance (P62):

$$
\begin{aligned}
&\int_0^\infty \frac{\cos\omega t}{\beta^2+\omega^2}d\omega = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos\omega t}{\beta^2+\omega^2}d\omega \\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{\cos\omega t + i\sin\omega t}{\beta^2+\omega^2}d\omega \\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{e^{i\omega t}}{\beta^2+\omega^2}d\omega \quad (\text{when } t>0) \\
&= \pi i\{\mathrm{Res}[\frac{e^{i\omega t}}{\beta^2+\omega^2}, i\beta]\} \\
&= \frac{2\beta}{\pi}e^{-\beta t}
\end{aligned}
$$

for $t<0$, replace $-\omega$ with p:

$$
\begin{aligned}
&= \frac{1}{2}\int_{-\infty}^\infty \frac{e^{-ipt}}{\beta^2+p^2}dp \\
&= \frac{2\beta}{\pi}e^{\beta t}
\end{aligned}
$$

We already apply the **Jordan's Lemma**. Be careful when you meet trigonometric functions, because they're not bounded in $\mathbb{C}$, that's why Jordan's Lemma is an essential. In the above example the even and odd function helps us to achive the form of Jordan's Lemma. When the function you met is neither even nor odd, please write like this:$\int f(x)\cos x = \mathrm{Re}(\int f(x)e^{ix}) = \cdots$

Note again that Jordan's Lemma could only be utilized when m in $\int e^{imx}dx$ **is larger than 0** (which explained the absolute value in the end), and only the singularities **above the x-axis** are needed to be considered.

Another example (P38 in Qiao's book):

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{\sin a\omega}{\omega}e^{i\omega x}d\omega=\int_{-\infty}^{\infty}\frac{e^{i(a+x)\omega}-e^{i(x-a)\omega}}{2i\omega}d\omega\\
&=\frac{1}{2i}\left(\int_{-\infty}^{\infty}\frac{e^{i(a+x)\omega}}{\omega}d\omega-\int_{-\infty}^{\infty}\frac{e^{i(x-a)\omega}}{\omega}d\omega\right)\\
&=\frac{\pi i}{2i}\left(\begin{cases}\mathrm{Res}[\frac{e^{i(x+a)\omega}}{\omega},0], & (x>-a)\\ \mathrm{Res}[-\frac{e^{-i(x+a)\omega}}{\omega},0], & (x<-a)\end{cases}-\begin{cases}\mathrm{Res}[\frac{e^{i(x-a)\omega}}{\omega},0], & (x>a)\\ \mathrm{Res}[-\frac{e^{-i(x-a)\omega}}{\omega},0], & (x<a)\end{cases}\right)\\
&=\begin{cases}0, & |x|>a\\ \pi, & |x|<a\end{cases}
\end{aligned}
$$

Note that the singularity is 0, which stuck in the x-axis, so $2\pi i\times\mathrm{Res}$ is not allowed (only works for singularities above the x-axis). Instead, the multipliers should be replaced by πi.
A drawback of Residue Theorem is that we're not able to calculate the value of such an integral at $x=\pm a$, which could be solved by Dirichlet's Theorem.

Again, the two cases above entail polar points (a kind of singularities) only. For **Essential singularities**, the Residue Theorem works as well. The only extra obstacle is how to calculate the Residue of an essential singularity, and you'd better calculate its ***Laurent series*** stiffly and examine the C_{-1} term.

When the singularity isn't isolated, Residue Theorem is no longer legal to use.

Laplacian Transform

The integral of Laplacian transform converges absolutely and uniformly, such that shifting the order of operators is legal.

LT's Properties

Let's appreciate their symmetry:

$$
\begin{aligned}
\frac{d}{ds}\mathscr{L}[f(t)] &= \mathscr{L}[-tf(t)] \\
\mathscr{L}[\frac{d}{dt}f(t)] &= s\mathscr{L}[f(t)] - f(0^+) \\
\int_s^\infty \mathscr{L}[f(t)]du &= \mathscr{L}[\frac{f(t)}{t}] \\
\mathscr{L}[\int_0^t f(\tau)d\tau] &= \frac{1}{s}\mathscr{L}[f(t)]
\end{aligned}
$$

How to verify these properties smoothly?
(I mean treat them individually, do not quote others results)

- $\frac{d}{ds}\mathscr{L}[f(t)] = \mathscr{L}[-tf(t)]$
  proof

$$
\begin{aligned}
\frac{d}{ds}\mathscr{L}[f(t)] &= \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt \\
&= \int_0^\infty -tf(t)e^{-st}dt \\
&= \mathscr{L}[-tf(t)]
\end{aligned}
$$

- $\mathscr{L}[\frac{d}{dt}f(t)] = s\mathscr{L}[f(t)] - f(0^+)$
  proof

$$\begin{aligned}\mathscr{L}[\frac{d}{dt}f(t)] &= \int_0^\infty e^{-st}df(t) \\ &= \left[f(t)e^{-st}\right]_0^\infty - \int_0^\infty -sf(t)e^{-st}dt \\ &= s\mathscr{L}[f(t)] - f(0^+)\end{aligned}$$

- $\int_s^\infty \mathscr{L}[f(t)]du = \mathscr{L}[\frac{f(t)}{t}]$
  proof

$$\begin{aligned}\int_s^\infty \mathscr{L}[f(t)]du &= \int_s^\infty \left(\int_0^\infty f(t)e^{-ut}dt\right)du \\ &= \int_0^\infty \left(\int_s^\infty f(t)e^{-ut}du\right)dt \\ &= \int_0^\infty \frac{1}{t}f(t)e^{-st}dt \\ &= \mathscr{L}[\frac{f(t)}{t}]\end{aligned}$$

- $\mathscr{L}[\int_0^t f(\tau)d\tau] = \frac{1}{s}\mathscr{L}[f(t)]$
  proof

$$
\begin{aligned}
\mathscr{L}[\int_0^t f(\tau)d\tau] &= \int_0^\infty e^{-st}\left(\int_0^t f(\tau)d\tau\right)dt \\
&= -\frac{1}{s}\int_0^\infty (\int_0^t f(\tau)d\tau)d(e^{-st}) \\
&= \frac{1}{s}\int_0^\infty e^{-st}d\left(\int_0^t f(\tau)d\tau\right) \\
&= \frac{1}{s}\int_0^\infty e^{-st}f(t)dt \\
&= \frac{1}{s}\mathscr{L}[f(t)]
\end{aligned}
$$

Conclusion:
If the integral and derivative sign is in $\mathscr{L}[\ ]$,the method is Integration by Parts;
If the integral and derivative sign is out of $\mathscr{L}[\ ]$,the method is Changing the Order of integral operators.

Translations:(easy to derive)

$$
\begin{aligned}
e^{-st_0}\mathscr{L}[f(t)] &= \mathscr{L}[f(t-t_0)] \\
\mathscr{L}[e^{s_0 t}f(t)] &= \mathscr{L}[f(t)](s-s_0)
\end{aligned}
$$

Warning: Laplacian Transform focus on the **right side of y-axis**, i.e. $f(x) = f(x)u(x)$. Functions like $f(t-1)$may cause misunderstandings. A better way to avoid such consequence is to apply Convolution Property instead of Translation in practice.

Convolution:

$$\mathscr{L}[f(t)] \times \mathscr{L}[g(t)] = \mathscr{L}[f * g(t)]$$

E.g.

$$\begin{aligned}
&\mathscr{L}^{-1}\{e^{-st_0}\mathscr{L}[f(t)]\} = \mathscr{L}^{-1}[e^{-st_0}] * f(t) \\
&= \delta(t - t_0) * f(t) = \int_0^t \delta(t - t_0 - \tau) f(\tau) d\tau \\
&= \begin{cases} f(t - t_0) & (t - t_0 > 0) \\ 0 & (t - t_0 < 0) \end{cases}
\end{aligned}$$

If we simply copied the conclusion of *Translation*, we happily got $f(t - t_0)$, whose $(0, t_0)$ part is actually wrong.

Partial Differential Equation

The Classes of PDE

We only discuss PDE with two variables.

Basic PDE Example

What matters is to find the characteristics of a PDE (**constant** coefficient):

$$\mathcal{L}(u) = 0$$

where

$$\mathcal{L} = a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2}$$

If we find the characteristic, then the equation become **single variable equation** of the characteristic.
To achieve this, we factorize the above equation:

$$\mathcal{L} = \left(\frac{b+\sqrt{\Delta}}{c^{1/2}}\frac{\partial}{\partial x} + c^{1/2}\frac{\partial}{\partial y}\right)\left(\frac{b-\sqrt{\Delta}}{c^{1/2}}\frac{\partial}{\partial x} + c^{1/2}\frac{\partial}{\partial y}\right)$$

where $\Delta := b^2 - ac$
Note that the two factors are commutative, hence if $\Delta \neq 0$, the two factors implies two characteristics separately.
If $\Delta = 0$, there's only one characteristic line.
And the characteristic is given by

$$\frac{dy}{dx} = \frac{b \pm \sqrt{\Delta}}{a}$$

Because any x, y with this relation (i.e. $y(x)$) would imply $\mathcal{L} = 0$ and the $u(x, y)$ actually becomes $u(x, y(x))$, single valued.

To achieve the **canonical form** of the second order PDE, we import such variables: $\xi = \sqrt{c}x - \frac{b}{\sqrt{c}}y; \eta = \frac{1}{\sqrt{c}}y.$

$$\mathcal{L} = \left(\frac{b+\sqrt{\Delta}}{c^{1/2}}\sqrt{c}\frac{\partial}{\partial\xi} + c^{1/2}\frac{-b}{\sqrt{c}}\frac{\partial}{\partial\xi} + c^{1/2}\frac{1}{\sqrt{c}}\frac{\partial}{\partial\eta}\right) \times$$
$$\left(\frac{b-\sqrt{\Delta}}{c^{1/2}}\sqrt{c}\frac{\partial}{\partial\xi} + c^{1/2}\frac{-b}{\sqrt{c}}\frac{\partial}{\partial\xi} + c^{1/2}\frac{1}{\sqrt{c}}\frac{\partial}{\partial\eta}\right)$$
$$= \left(\sqrt{\Delta}\frac{\partial}{\partial\xi} + \partial\eta\right)\left(-\sqrt{\Delta}\frac{\partial}{\partial\xi} + \partial\eta\right) = -\Delta\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}$$

Systematic Process

The equation is:

$$\mathcal{L}(u) = g$$

where:

$$\mathcal{L} = a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x\partial y} + c\frac{\partial^2}{\partial y^2} + d\frac{\partial}{\partial x} + e\frac{\partial}{\partial y} + f$$

Attention: a to g are all functions of x and y, they're **not constant** generally. Replacing the variables by:

$$\xi = \xi(x,y) \quad \eta = \eta(x,y)$$

Apply *Chain Rule* and calculate, we got:

$$\left(A\frac{\partial^2}{\partial\xi^2} + 2B\frac{\partial^2}{\partial\xi\partial\eta} + C\frac{\partial^2}{\partial\eta^2} + D\frac{\partial}{\partial\xi} + E\frac{\partial}{\partial\eta} + F\right)(u) = G$$

where A and C have the same form:

$$
a(\frac{\partial W}{\partial x})^2 + 2b\frac{\partial W}{\partial x}\frac{\partial W}{\partial y} + c(\frac{\partial W}{\partial y})^2
$$

In A's case $W = \xi$, in C's case $W = \eta$.

Suppose the solution is $W(x, y) = \text{const}$, then by **Implicit Function** theorem, $y' = -\frac{\partial W/\partial x}{\partial W/\partial y}$, so that the equation above will be:

$$
ay'^2 - 2by' + c = 0 \qquad \therefore y' = \frac{b \pm \sqrt{\Delta}}{a}
$$

If you successfully solve the last differential equation we got $\xi(x, y) = \text{const}$ and $\eta(x, y) = \text{const}$, which are exactly two characteristics.

1. If there are two solutions and they are real, that's typical *hyperbola* type equation, hence we get:

$$
\frac{\partial^2 u}{\partial \xi \partial \eta} = \dots
$$

   or you may as well adjust it to $\frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial^2 u}{\partial \beta^2} = \dots$ by letting $\alpha = \frac{1}{2}(\xi + \eta)$ and $\beta = \frac{1}{2}(\xi - \eta)$.

2. If there's only one solution (i.e. *parabolic*), then another characteristic can be chosen by yourself, as long as it's independent with the previous one. The one you chose randomly namely is η, then the equation be like:

$$
\frac{\partial^2 u}{\partial \eta^2} = \dots
$$

3. If the solutions $\notin \mathbb{R}$, then extract the W 's **real** and **imaginary** part to be ξ and η separately. (it can be proved by simply plugging in that such arrangement is indeed the appropriate solution) And we get *elliptical* equation:

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = \dots$$

D'Almbert Method

To solve the vibrating equation:

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial t^2} = 0$$

written as

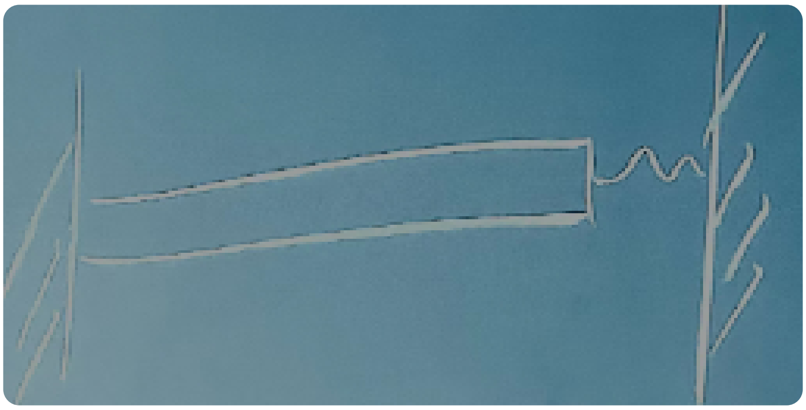
$$(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x})(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x})u = 0$$

The general solution would be $f_1(x+at) + f_2(x-at)$.

$$u(x,t) = \frac{1}{2}(\phi(x+at) + \phi(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)d\xi$$

Boundary Condition

Mechanical Example



Consider the example of a rod oscillates longitudinal with a spring sticking to its the end.

Analysis the small segment in Rod's part: $u(x)$ represents the small displacement at the point x, and note *rightward* as the positive direction. Calculate the new length of the segment:

$$(x + \Delta x + u(x + \Delta x)) - (x + u(x)) = \Delta x + \frac{\partial u}{\partial x}\Delta x$$

Such that the extension is $\frac{\partial u}{\partial x}\Delta x$. From Hook's Law, the **relative extension** is proportional to the quotient of the force and sectional area:

$$\frac{\frac{\partial u}{\partial x}\Delta x}{\Delta x} = \frac{F}{SY} \quad \therefore F = SY\frac{\partial u}{\partial x}$$

Analysis the force to get Dynamic Equation:

$$SY\left(\frac{\partial u}{\partial x}\mid_{x+\Delta x} - \frac{\partial u}{\partial x}\mid_{x}\right) = \frac{\partial^2 u}{\partial t^2}\rho S\Delta x$$

Hence the equation is:

$$\frac{Y}{\rho}\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

The Boundary Condition shall be given by analyzing the small segment at the end of the rod, adjust the Dynamic Equation above:

$$F_{x+\Delta x}+SY\left(-\frac{\partial u}{\partial x}\mid_x\right)=\frac{\partial^2 u}{\partial t^2}\rho S\Delta x$$

$$F_{L^+}+SY\left(-\frac{\partial u}{\partial x}\mid_{L-\Delta x}\right)=\frac{\partial^2 u}{\partial t^2}\rho S\Delta x$$

$$-ku\mid_L+SY\left(-\frac{\partial u}{\partial x}\mid_{L-\Delta x}\right)=\frac{\partial^2 u}{\partial t^2}\rho S\Delta x$$

Now let $\Delta x\to 0$, to achieve the final goal:

$$-ku\mid_L+\frac{S}{Y}\left(-\frac{\partial u}{\partial x}\mid_L\right)=0$$

When k is large enough, the $\frac{S}{Y}\left(-\frac{\partial u}{\partial L}\mid_L\right)$ part is $o(k)$ and is legal to be ignored, corresponding the situation with fixed end (the first type, *Dirichlet Boundary Condition*).
When k is small enough, the first section $-ku\mid_L$ could be ignored and thus verified the Boundary Condition of free end (the second type, *Neumann Boundary Condition*).
In conclusion, the boundary condition exhibits a lower order of small infinitesimal quantity. At the end of the rod, the Δx could be regarded as zero, which differs from the interior of the rod.

Thermodynamical Example

To understand BC, you shall first understand where the equation comes from, for a small fragment of the rod:

$$\left(-\kappa S\Delta t\frac{\partial u}{\partial x}\right)|_x-\left(-\kappa S\Delta t\frac{\partial u}{\partial x}\right)|_{x+\Delta x}=c(S\Delta x\rho)\Delta u$$

which can be simplified to

$$\kappa\frac{\partial^2 u}{\partial x^2}=c\rho\frac{\Delta u}{\Delta t}$$

For a rod's temperature distribution, the Dirichlet BC means the temperature was shown directly. If it states that some hot stream $f(t)$ flows out, that's the Neumann BC, and the general equation at the right end should be:

$$\left(-\kappa S\Delta t\frac{\partial u}{\partial x}\right)|_L-\Delta tSf(t)=c(S\Delta x\rho)\Delta u$$

Eliminate Δt and let $\Delta x\to 0$,

$$-\kappa\frac{\partial u}{\partial x}|_{x=L}=f(t)$$

or more generally

$$-\kappa\frac{\partial u}{\partial n}=f(t)$$

where at the left end, $\frac{\partial u}{\partial n}=-\frac{\partial u}{\partial x}$, while at the right end $\frac{\partial u}{\partial n}=\frac{\partial u}{\partial x}$, the normal $\vec{n}$ is pointing outward of the rod!

When the end is adiabatic, the BC becomes $\frac{\partial u}{\partial n} = 0$.

There's another possibility that the boundary obeys Newton's cooling law, which corresponds to the *spring* in above mechanical example, because the $\text{force} \propto \Delta\text{displacement}$ corresponds to $\text{flow} \propto \Delta\text{temperature}$.

thus the BC:

$$-\kappa \frac{\partial u}{\partial n} |_{\text{somewhere}} = h(u |_{\text{somewhere}} - \theta)$$

Different Types, Different Methods

Homogenous BC and Homogenous Eq

The easiest one, *Separate two variables* to get an eigen-value problem.

Homogenous BC and Non-Homogenous Eq

There are two ways to deal with it:

Fourier method

Suppose:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

where $X_n(x)$ is the eigen-function which we already know, corresponding to BC, like $\sin(\frac{n\pi}{l}x)$. Plug the above formula into the original Eq to derive $T_n(t)$'s differential equation and solve that ode.

Momentum Theorem

To implement Momentum Theorem, we have to convert the original Eq to the following form, with **all zero** *Initial Condition*, if it's a Vibrating Equation:

$$u_{tt} - a^2 u_{xx} = f(x,t)$$

$$u\mid_{x=0} = 0\ ;\ u\mid_{x=l} = 0$$

$$u\mid_{t=0} = 0\ ;\ u_t\mid_{t=0} = 0$$

Then we only need to solve:

$$v_{tt} - a^2 v_{xx} = 0$$

$$v\mid_{x=0} = 0\ ;\ v\mid_{x=l} = 0$$

$$v\mid_{t=\tau} = 0\ ;\ v_t\mid_{t=\tau} = f(x,\tau)$$

If it's the thermodynamical equation

$$u_t - a^2 u_{xx} = f(x,t)$$

$$u\mid_{x=0} = 0\ ;\ u\mid_{x=l} = 0$$

$$u\mid_{t=0} = 0$$

Then we only need to solve:

$$v_t - a^2 v_{xx} = 0$$

$$v \,|_{x=0} = 0 \,;\; v \,|_{x=l} = 0$$

$$v \,|_{t=\tau} = f(x, \tau)$$

and the final result is:

$$u(x,t) = \int_0^t v(x,t,\tau)d\tau$$

> *Understanding:*
> *The $f(x,t)$ shall be divided into discrete impulses, exerted on different strings (in an ensemble), and the result is the sum of all strings in this ensemble.*

Non-Homogenous BC and Homogenous Eq

The goal is to regain homogenous BC and convert it to the last type of equation.

Let $u = v + w$, where w satisfy homogenous BC. To achieve this, suppose $v = A(t)x + B(t)$, and adjust two coefficients to satisfy the non-homogenous BC. As a side-effect ,w has to satisfy the non-homogenous equations but we already know how to solve it.

Poisson function

The form is:

$$\Delta u = f(x, y, z)$$

non-homogenous equations, unfortunately not able to solve by Momentum theorem. Thus we have to develop new method:
Guess! Extract a particular solution out, $u = v + w$ transferring the equation into a homogenous one $\Delta v = 0$. Separate variables......

Note that if the Poisson function is a two dimensional problem with variables x and y, the BC must be fine or else we cannot determine its eigen-function (as you extract a particular solution out and worsen the BC).

While if the variables are ρ and ϕ, the solution w you extract must have the sysmetric form in terms of x and y, in order to combine it to $x^2 + y^2 = \rho^2$, by the way, you will probably encounter *Euler* type ode then.

tips: when encounter Euler type ode, let $\rho = e^t \ldots$. The general solution would be like:

$$v(\rho, \phi) = \sum_{m=1}^{\infty} \rho^m (A_m \cos m\phi + B_m \sin m\phi)$$
$$+C_0 + D_0 \ln \rho + \sum_{m=1}^{\infty} \rho^{-m} (C_m \cos m\phi + D_m \sin m\phi)$$

Solve ODE a Hard Way

Sturm-Liouville Problem

Consider the operator:

$$\mathcal{L}y = \frac{d}{dx}\left(p_0(x)\frac{dy}{dx}\right) + p_2(x)y$$

It is *Hermitian* in differential equation sense, because for two functions of x denote as v and u:

$$\int_a^b v^*(\mathcal{L}u)dx = \int_a^b (\mathcal{L}v)^* u dx + \left[v^* p_0(x)\frac{du}{dx} - \frac{dv^*}{dx}p_0(x)u\right]_a^b$$

where the $[\dots]$ term would vanish when the boundary condition is appropriate.

An Hermitian operator enable us to gain orthogonal, complete basis consisting of eigen-functions with real eigenvalues.

Besides, for an arbitrary ode:

$$\mathcal{L}y = \frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y$$

multiply by $e^{\int a(\xi)d\xi}$ (denote as $w(x)$).

$$w(x)\mathcal{L}y = \frac{d}{dx}\left[e^{\int a(\xi)d\xi}\frac{dy}{dx}\right] + \left[b(x)e^{\int a(\xi)d\xi}\right]y$$

successfully transform into a S-L type. But note that the orthogonality might be different:

$$\int_a^b v^*(w(x)\mathcal{L}u)dx = \int_a^b (w(x)\mathcal{L}v)^* u dx + \left[v^* w(x)\frac{du}{dx} - \frac{dv^*}{dx}w(x)u\right]_a^b$$

Legendre Equation

l th order Legendre Eq:

$$(1-x^2)y'' - 2xy' + l(l+1) = 0$$

Frobenius method by making a series expansion, note that at $x_0 = 0$ the expansion has only positive terms, i.e. *Taylor expansion*:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

plug it in and get the recursion relation:

$$a_{k+2} = \frac{(k-l)(k+l+1)}{(k+2)(k+1)} a_k$$

The general solution:

$$y(x) = D_0 y_0(x) + D_1 y_1(x)$$

A sad fact is that neither y_0 nor y_1 is convergent, which can be verified by applying Gaussian Criteria:

> *Gaussian Criteria*
> *the infinite series is* $\sum u_k$, *denote*
>
> $$G := \lim_{k\to\infty} k\left(\frac{u_k}{u_{k+1}} - 1 - \frac{1}{k}\right)$$
>
> *If* $G < 1$, *the series converges; else if* $G > 1$, *it diverges.*
> *Proof: The theorem could be verified by consider* $\sum \frac{1}{k\ln k}$

Besides the linear combination of these two diverges too.

The only solution to fit the real physical situation (avoid infinity) is to convert the infinite series into a *polynomial* (legendre polynomial).

when $l = 2n$, $y_0(x)$ is an $2n$ th order polynomial, while $y_1(x)$ is still infinite, thus we assert $a_1 = 0$ to discard it. The solution shall be $y(x) = a_0 y_0(x)$.

when $l = 2n+1$, $y_1(x)$ is an $2n+1$ th order polynomial. Then we abandon $y_0(x)$ to get the solution $y(x) = a_1 y_1(x)$.

Bessel Equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

The simple case

where $2\nu \notin \mathbb{Z}$

suppose

$$y=\sum_{k=0}^{\infty} a_k z^{s+k}$$

In convention we select $a_0 \neq 0$

the x^s term's coefficient is $(s^2-\nu^2)a_0$, which yields $s=\nu$ or $s=-\nu$

thus the recursion relation is:

$$a_k=\frac{-1}{(s+k+\nu)(s+k-\nu)}a_{k-2}$$

Besides, the x^{s+1} term's coefficient implies $a_1=0$, thus $a_{2k+1}=0$

A great property of Bessel Eq's solution is that its convergent radii is ∞.

select

$$a_0=\frac{1}{2^\nu}\Gamma(\nu+1)$$

and denote the solution now as $J_\nu(x)$ and $J_{-\nu}(x)$ corresponding to $s=\pm\nu$

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}$$

$$J_{-\nu}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-\nu+1)} \left(\frac{x}{2}\right)^{2m-\nu}$$

They are linear independent, which enables us to form its general solution:

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x)$$

The 1/2 case

when $\nu = \frac{1}{2}$, the above formula also satisfies.

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

General solution:

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x)$$

The tough case

ν is an integer, which implies $J_\nu(x)$ and $J_{-\nu}(x)$ is linear independent.

The first $J_\nu(x)$ still holds, the purpose is to find another solution which is linear independent with it. *Neumann function*:

$$Y_\nu(x) = \lim_{\alpha\to\nu} \frac{J_\alpha(x)\cos\alpha x - J_{-\alpha}(x)}{\sin\alpha x}$$

thus the solution takes the form of:

$$y(x) = AJ_\nu(x) + BY_\nu(x)$$

Appendix

Derive the Laplacian $\Delta = \nabla \cdot \nabla$ in a smarter way:

first consider the *gradient*:

$$\begin{aligned}\nabla u &= \lim_{\text{Displacement}} \vec{e}_r \frac{\Delta u}{\Delta r} + \vec{e}_\theta \frac{\Delta u}{r\Delta\theta} + \vec{e}_\phi \frac{\Delta u}{r\sin\theta\Delta\phi} \\ &= \vec{e}_r \frac{\partial u}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial u}{\partial\theta} + \vec{e}_\phi \frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi}\end{aligned}$$

Then consider the partial derivative of the three orthonormal vectors $\vec{e}_r$ $\vec{e}_\theta$ $\vec{e}_\phi$:

$$\frac{\partial\vec{e}_r}{\partial r} = \frac{\partial\vec{e}_\theta}{\partial r} = \frac{\partial\vec{e}_\phi}{\partial r} = 0$$

which is obvious because when θ and ϕ are fixed, the three vectors won't change their directions.

$$\frac{\partial \vec{e}_r}{\partial \theta} = \vec{e}_\theta$$

$$\frac{\partial \vec{e}_\theta}{\partial \theta} = -\vec{e}_r$$

$$\frac{\partial \vec{e}_\phi}{\partial \theta} = 0$$

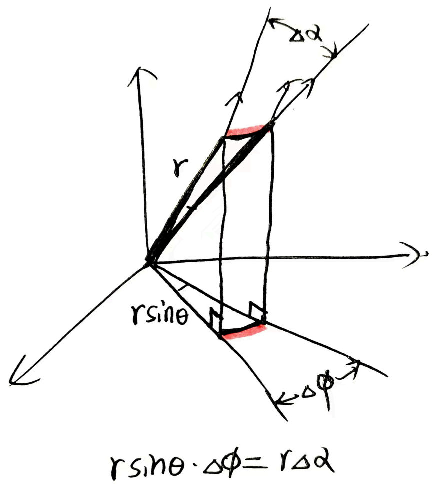
Which is natural too, because the first two equation is the same as what we learnt earlier in the *polar coordinate* case. And the last one could be observed directly from the figure, unchanged.

$$\frac{\partial \vec{e}_r}{\partial \phi} = \sin\theta \vec{e}_\phi$$

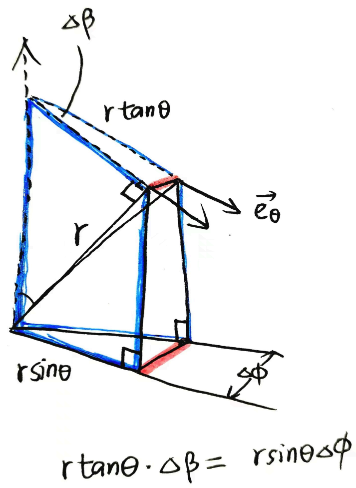
$$\frac{\partial \vec{e}_\theta}{\partial \phi} = \cos\theta \vec{e}_\phi$$

$$\frac{\partial \vec{e}_\phi}{\partial \phi} = -\sin\theta \vec{e}_r - \cos\theta \vec{e}_\theta$$

The first equation could be verified by drawing the infinitesimal displacement angle. The above $\vec{e}_r$ shift angle $\Delta\alpha$, which satisfy $r\Delta\alpha = r\sin\theta\Delta\phi$, i.e. $\Delta\alpha = \sin\theta\Delta\phi$:

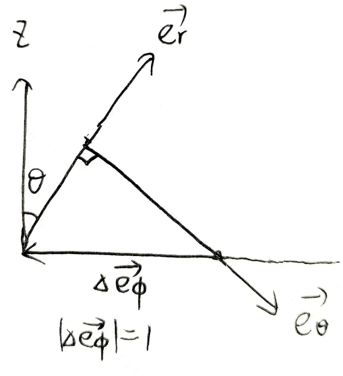


for the second equation, much the same:



as for $\vec{e}_\phi$, $\Delta\vec{e}_\phi$ points inward to the origin (similar to polar coordinate case), thus the direction is a linear combination of $\vec{e}_r$ and $\vec{e}_\phi$:

$$\sin\theta\vec{e}_r + \cos\theta\vec{e}_\theta$$



thus the Laplacian:

$$\nabla \cdot \nabla = \nabla \cdot \left(\vec{e}_r \frac{\partial u}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial u}{\partial \theta} + \vec{e}_\phi \frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi} \right)$$

$$= \left([\vec{e}_r \frac{\partial}{\partial r}] + [\vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta}] + [\vec{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}] \right)$$
$$\cdot \left(\vec{e}_r \frac{\partial u}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial u}{\partial \theta} + \vec{e}_\phi \frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi} \right)$$

$$= [\frac{\partial^2 u}{\partial r^2}] + [\frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}]$$
$$+ [\frac{1}{r \sin \theta} (\sin \theta \frac{\partial u}{\partial r} + \cos \theta \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial^2 u}{\partial \phi^2})]$$

where three $[\dots]$ linked the corresponding terms to three dot products in divergence.

□