Rigid Body Dynamics

Mayuresh Bhattu

Mini Project: Mentor - Anant Joshi

mayubhattu@gmail.com

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Problem Statement

Exploring rigid body dynamics and its use in Advitiy -

- Studying rotational transformation, time derivative with respect to rotating frame
- Understanding Moment of Inertia tensor
- Understanding Angular momentum and torque
- Euler's equation
- Stability of torque free rigid bodies
- Runge Kutta order 4 solver and its implemetation for simulating constant torque rigid bodies

Rotational Transformation between Frames

$$\begin{aligned} \mathbf{u}_{x'} &= c_{x'x}\mathbf{u}_x + c_{x'y}\mathbf{u}_y + c_{x'z}\mathbf{u}_z \\ \mathbf{u}_{y'} &= c_{y'x}\mathbf{u}_x + c_{y'y}\mathbf{u}_y + c_{y'z}\mathbf{u}_z \\ \mathbf{u}_{z'} &= c_{z'x}\mathbf{u}_x + c_{z'y}\mathbf{u}_y + c_{z'z}\mathbf{u}_z \end{aligned} \quad (1)$$

$$\mathbf{C} = \begin{bmatrix} c_{x'x} & c_{x'y} & c_{xz} \\ c_{y'x} & c_{y'y} & c_{y'z} \\ c_{z'x} & c_{z'y} & c_{z'z} \end{bmatrix} \quad (2)$$

$$[C][C]^t = [C]^t[C] = [1] \quad (3)$$

$$\mathbf{r} = x\mathbf{u_x} + y\mathbf{u_y} + z\mathbf{u_z} = x'\mathbf{u_{x'}} + y'\mathbf{u_{y'}} + z'\mathbf{u_{z'}} \quad (4)$$

$$\left\{ \begin{array}{c} x' \\ y' \\ z' \end{array} \right\} = [C] \left\{ \begin{array}{c} x \\ y \\ z \end{array} \right\} \quad (5)$$

Transport Theorem

The following equation, known as the Transport Theorem, is a formula for the derivative of any vector r frame of reference O

$$(\frac{d\mathbf{r}}{dt})_{/O} = (\frac{d\mathbf{r}}{dt})_{/P} + \mathbf{w}_{P/O} \times \mathbf{r} \tag{6}$$

Moment of Inertia tensor

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{7}$$

$$I_{xx} = \sum_{i=1}^{N} m_i(r_i^2 - x_i^2) = \sum_{i=1}^{N} m_i(y_i^2 + z_i^2) \tag{8}$$

$$I_{xy} = -\sum_{i=1}^{N} m_i(x_i y_i) = I_{yx} \tag{9}$$

Following is applicable only to points for which O is stationary in inertial frame or when O coincides with Center of Mass

$$H_x = I_{xx} w_x + I_{xy} w_y + I_{xz} w_z \tag{10}$$

$$\{H\}_O = [I]_O \{w\} \tag{11}$$

Parallel Axis Theorem

When in the shifted frame, A = (a, b, c)

$$[I]_B = [I]_A + M \begin{pmatrix} 2(by_c + cz_c) & -(bx_c + ay_c) & -(cx_c + az_c) \\ -(bx_c + ay_c) & 2(cz_c + ax_c) & -(cy_c + bz_c) \\ -(cx_c + az_c) & -(cy_c + bz_c) & 2(ax_c + by_c) \end{pmatrix} + K \tag{12}$$

$$K = M \begin{pmatrix} b^2 + c^2 & -ab & -ac \\ -ab & c^2 + a^2 & -bc \\ -ac & -bc & a^2 + b^2 \end{pmatrix} \tag{13}$$

$$[I]_B = [I]_C + K \tag{14}$$

Rotational transformation of Inertia tensor components-

$$[I'] = [C][I][C]^t \tag{15}$$

Principal Directions

Those exceptional directions such that, when the body rotates about an axis parallel to principal direction, the angular momentum vector has the same direction.

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{16}$$

To convert a general frame into a principal direction frame, we need to solve the following

$$[I]\{u\} = \lambda\{u\} \tag{17}$$

Directions obtained from solutions of u are the principal directions where,

$$\lambda = H/w \tag{18}$$

Angular momentum of a rigid body

$$\mathbf{H}_B = \sum_{i=1}^{N} \mathbf{b}_i \times m_i \mathbf{v}_i = \sum_{i=1}^{N} \mathbf{b}_c \times m_i \mathbf{v}_i + \sum_{i=1}^{N} \mathbf{c}_i \times m_i \mathbf{v}_i \tag{19}$$

$$\mathbf{H}_C = \sum_{i=1}^{N} \mathbf{c}_i \times m_i \mathbf{v}_i . \tag{20}$$

$$\mathbf{H}_B = \mathbf{b}_c \times \mathbf{P} + \mathbf{H}_c \tag{21}$$

When the moment center is taken as origin of body coordinate system and with zero velocity

$$\mathbf{H}_o = \sum_{i=1}^{N} m_i \mathbf{r}_i \times (\mathbf{w} \times \mathbf{r}_i) \tag{22}$$

Torque on a system of particles

Total torque about B is defined as -

$$\tau_B = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i \tag{23}$$

$$\mathbf{H}_B = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i. \tag{24}$$

$$\tau = \frac{d\mathbf{H}_B}{dt} + \mathbf{V}_B \times \mathbf{P} \tag{25}$$

Thus, torque is rate of change of angular momentum only when moment center is stationary or its velocity is parallel to velocity of the body. A special case of the second condition appears when moment center is COM.

Euler's Equations

For O fixed in body and in inertial space or when O = COM

$$\tau_O = \frac{d\mathbf{H}_0}{dt} \tag{26}$$

$$\tau_O = (\frac{d\mathbf{H}_0}{dt})_{rel} + \mathbf{w} \times \mathbf{H}_O \tag{27}$$

$$\tau_O = (I\dot{\mathbf{w}}) + \mathbf{w} \times (I\mathbf{w}) \tag{28}$$

$$\tau_1 = (I_1\dot{w_1}) + (I_3 - I_2)w_2w_3 \tag{29}$$

Torque-free motion

Assuming two of the principal moments of inertia to be same

$$I_1 = I_2 \tag{30}$$

$$I_1 \dot{w_1} = (I_1 - I_3) w_3 w_2 \tag{31}$$

$$I_1 \dot{w_2} = -(I_1 - I_3) w_3 w_1 \tag{32}$$

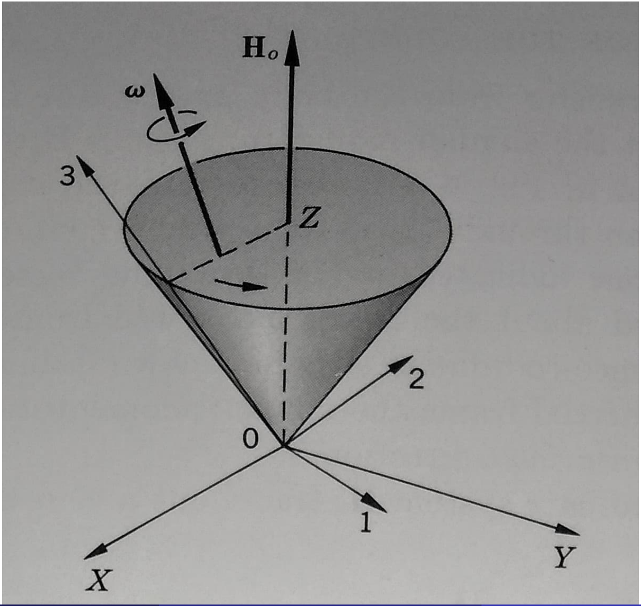
$$I_3 \dot{w_3} = 0 \tag{33}$$

$$w_1 = w_0 cos(\gamma - st) \tag{34}$$

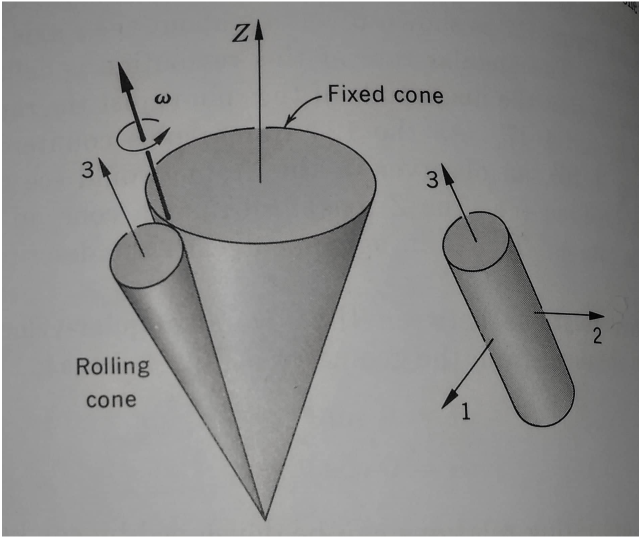
$$w_2 = w_0 sin(\gamma - st) \tag{35}$$

$$w_3 = constant \tag{36}$$

Torque-free motion



Torque-free motion



Stability of Torque-free motion

Initially

$$\mathbf{w} = w_0 \hat{\mathbf{k}} \tag{37}$$

Now let's disturb the motion slightly

$$w_x = \delta w_x, w_y = \delta w_y, w_z = w_0 + \delta w_z \tag{38}$$

$$A\delta \dot{w}_x + (C - B) w_0 \delta w_y = 0 \tag{39}$$

$$A\delta \ddot{w}_x + (C - B) w_0 \delta \dot{w}_y = 0 \tag{40}$$

$$\delta \ddot{w} + k\delta w = 0 \tag{41}$$

if k is positive,

$$\delta w \propto e^{\pm i\sqrt{kt}} \tag{42}$$

Runge Kutta Method

$$\dot{y} = f(y, t) \tag{43}$$

$$y(t_0) = y_0 \tag{44}$$

$$y_{n+1} = y_n + 1/6(k_1 + 2k_2 + 2k_3 + k_4) \tag{45}$$

$$t_{n+1} = t_n + h \tag{46}$$

$$k_1 = hf(t_n, y_n) \tag{47}$$

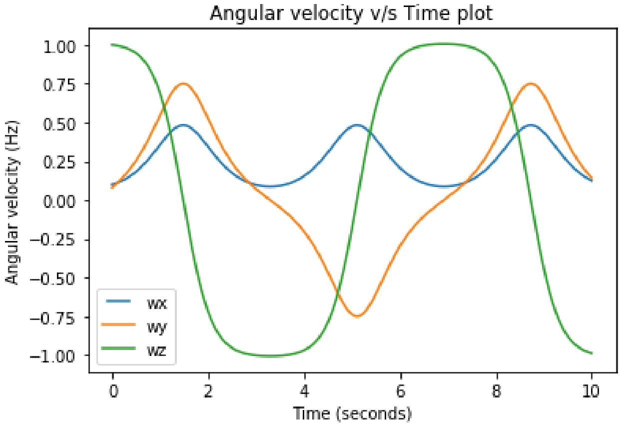
$$k_2 = hf(t_n + h/2, y_n + k_1/2) \tag{48}$$

$$k_1 = hf(t_n + h/2, y_n + k_2/2) \tag{49}$$

$$k_1 = hf(t_n + h, y_n + k) \tag{50}$$

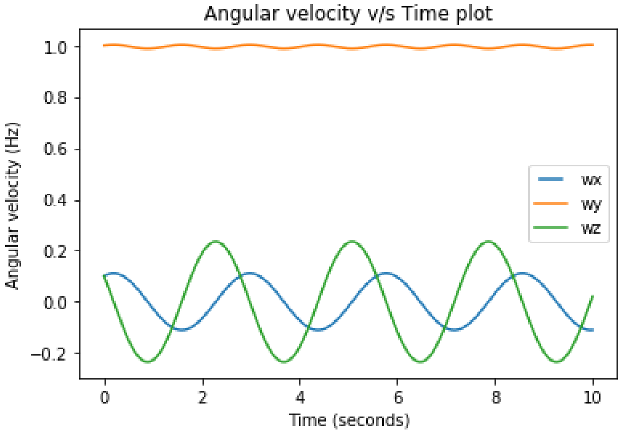
Simulations - intermediate axis

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}, w_0 = \begin{pmatrix} 0.1 \\ 0.08 \\ 1 \end{pmatrix}, M = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{51}$$



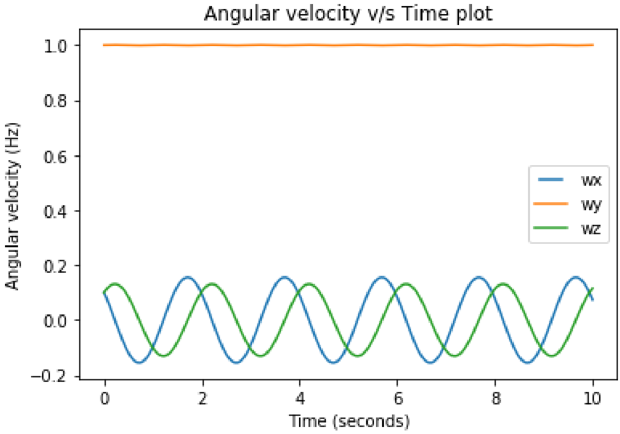
Simulations - major axis

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}, w_0 = \begin{pmatrix} 0.1 \\ 1 \\ 0.1 \end{pmatrix}, M = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{52}$$



Simulations - minor axis

$$\mathbf{I} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, w0 = \begin{pmatrix} 0.1 \\ 1 \\ 0.1 \end{pmatrix}, M = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{53}$$



Thank you