UNIT – 01 Ordinary Differential Equations of Higher Order

Differential Equation:- A differential equation is an equation involving differentials or differential Coefficients. it is of two type

1. ordinary Diff. Eqⁿ
2. Partial Diff Equation

ORDER OF A DIFFERENTIAL EQUATION:- The order of a differential equation is the order of the highest ordered derivative occuring in the differential equation

DEGREE OF A DIFFERENTIAL EQUATION:- The degree of a differential equation is the degree of the highest ordered derivative present in the differential equation when It is made free from radical signs and fractional Powers

Ex $\frac{d^2y}{dx^2} + \sqrt{1+\left(\frac{dy}{dx}\right)^3} = 0 \Rightarrow \frac{d^2y}{dx^2} = -\sqrt{1+\left(\frac{dy}{dx}\right)^3}$

Squaring both sides as it has radical sign

$\left(\frac{d^2y}{dx^2}\right)^2 = 1+\left(\frac{dy}{dx}\right)^3 \Rightarrow$ $\boxed{\text{clearly degree} = 2,\ \text{Order} = 2}$ Ans

Linear differential Equations with Constant Coefficients

Equation of the form $a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} \cdots a_ny = Q$

where $a_0, a_1, a_2 \cdots a_n$ are all constants and Q is a function of x alone is called L.D. Equation with Constant Coefficients which can be written as

$$\boxed{f(D)y = Q}$$

$$\Big\downarrow \text{Solution}$$

$$\boxed{y = C.F + P.I}$$

where C.f is Complementary function and P.I is Particular Integral.

STEPS OF FINDING AUXILIARY EQUATION

1. Replace y by 1
2. Replace D by m, D^2 by m^2 and so on
3. By Doing so we get an algebric equation in m of degree n called auxiliary equation.

HOW TO FIND C.F (Complementary function)

C.f depends on the nature of roots of A.E

There arises following Cases

Roots	Nature	C.f
1. $m_1, m_2, m_3 \ldots$	Real and distinct	$C.f = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} \ldots$
2. $m, m, m \ldots$	Real and Repeated	$C.f = (c_1 + c_2x + c_3x^2)e^{mx}$
3. $\alpha \pm i\beta$	Real Imaginary Roots	$C.f = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$
4. $\alpha \pm i\beta$, $\alpha \pm i\beta$	Imaginary and Repeated Roots	$C.f = e^{\alpha x}[(c_1+c_2x)\cos\beta x + (c_3+c_4x)\sin\beta x]$
5. $\alpha \pm \sqrt{\beta}$	Irrational Roots	$C.f = e^{\alpha x}[c_1 \cosh\sqrt{\beta}x + c_2 \sinh\sqrt{\beta}x]$
6. $\alpha \pm \sqrt{\beta}$, $\alpha \pm \sqrt{\beta}$	Irrational and Repeated Roots	$C.f = e^{\alpha x}[(c_1+c_2x)\cosh\sqrt{\beta}x + (c_3+c_4x)\sinh\sqrt{\beta}x]$

How To Find P.I. (Particular Integral)

Particular Integral depends on the R.H.S Part (Q) of Given differential Equation There arises following cases

Case I when $Q = e^{ax}$

$$P.I = \frac{1}{f(D)} \cdot Q$$

$$= \frac{1}{f(D)} e^{ax}$$

$$= \frac{1}{f(a)} e^{ax}, \quad f(a) \neq 0$$

if $f(a) = 0$ then case fail

$$\Rightarrow x \frac{1}{f'(D)} \cdot e^{ax}$$

$$= x \cdot \frac{1}{f'(a)} e^{ax}$$

$$f'(a) \neq 0$$

Case-II when $Q = \sin(ax+b)$ or $\cos(ax+b)$

$$P.I = \frac{1}{f(D^2)} \sin(ax+b)$$

Put $D^2 = -a^2$

$$= \frac{1}{f(-a^2)} \sin(ax+b)$$

where $f(-a^2) \neq 0$

if $f(-a^2) = 0$ then Case fail.

Case III when $Q = x^m$

In this case

$$P.I = \frac{1}{f(D)} = x^m$$

We Use Binomial Expansion to find P.I

$$(1-D)^{-1} = 1+D+D^2+D^3 - -$$

$$(1+D)^{-1} = 1-D+D^2-D^3 - -$$

Case IV when $Q = e^{ax} \cdot V$ where V is any function of x

$$P.I = \frac{1}{f(D)} \cdot e^{ax} \cdot V$$

$$= e^{ax} \cdot \frac{1}{f(D+a)} V$$

Case V when Q is any other function of x

$$\frac{1}{D-\alpha} Q = e^{\alpha x} \int e^{-\alpha x} Q \, dx$$

$$\frac{1}{D+\alpha} Q = e^{-\alpha x} \int e^{\alpha x} Q \, dx$$

Q.1:- Solve: $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Sol^n:- Auxiliary equation is

$(m+2)(m-1)^2 = 0 \Rightarrow m = -2, 1, 1$

$\therefore \quad C.F. = C_1 e^{-2x} + (C_2 + C_3 x) e^{x}$

$$P.I. = \frac{1}{(D+2)(D-1)^2} (e^{-2x} + 2\sinh x)$$

$$= \frac{1}{(D+2)(D-1)^2} (e^{-2x} + e^{x} - e^{-x}) \qquad \left[\because \sinh x = \frac{e^x - e^{-x}}{2}\right]$$

Now

$$\frac{1}{(D+2)(D-1)^2} e^{-2x} = \frac{1}{D+2}\left[\frac{1}{(D-1)^2} e^{-2x}\right] = \frac{1}{D+2}\left[\frac{1}{(-2-1)^2} e^{-2x}\right]$$

$$= \frac{1}{9} \cdot \frac{1}{D+2} e^{-2x} \qquad [\text{Case of failure}$$

$$= \frac{x}{9} e^{-2x}$$

$$\frac{1}{(D+2)(D-1)^2} e^{x} = \frac{1}{(D-1)^2}\left[\frac{1}{D+2} e^{x}\right] = \frac{1}{(D-1)^2}\left[\frac{1}{1+2} e^{x}\right]$$

$$= \frac{1}{3} \cdot \frac{1}{(D-1)^2} e^{x} \qquad [\text{Case of failure}$$

$$= \frac{1}{3} x \frac{1}{2(D-1)} e^{x} \qquad [\text{Case of failure}$$

$$= \frac{1}{3} x^2 \cdot \frac{1}{2} e^{x} = \frac{1}{6} x^2 e^{x}$$

$$\frac{1}{(D+2)(D-1)^2} e^{-x} = -\frac{1}{(-1+2)(-1-1)^2} e^{-x} = \frac{1}{4} e^{-x}$$

$$\therefore P.I. = \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^{x} + \frac{1}{4} e^{-x}$$

Hence the complete sol^n is $y = C.F. + P.I.$

$$\Rightarrow \boxed{y = C_1 e^{-2x} + (C_2 + C_3 x) e^{x} + \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^{x} + \frac{1}{4} e^{-x}.}$$

Q2:- Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y + 37\sin 3x = 0$ and find the value of y when $x = \frac{\pi}{2}$ being given that $y = 3$, $\frac{dy}{dx} = 0$, when $x = 0$.

Sol^n:- We have, $(D^2 + 2D + 10)y = -37\sin 3x$

Auxiliary equation is $m^2 + 2m + 10 = 0$

$\Rightarrow m = -1 \pm 3i$

$\therefore$ C.F. $= e^{-x}(C_1\cos 3x + C_2\sin 3x)$

P.I. $= \frac{1}{D^2+2D+10}(-37\sin 3x) = -37\frac{1}{2D+1}\sin 3x$ [replacing D^2 by -9]

$= (-37)\frac{2D-1}{(4D^2-1)}\sin 3x$

$= (-37)\frac{2D-1}{(-37)}\sin 3x$ $[D^2 = -9]$

$= 2D\sin 3x - \sin 3x = 6\cos 3x - \sin 3x$

Hence the general solution is

$y =$ C.F. $+$ P.I.

$\Rightarrow \boxed{y = e^{-x}(C_1\cos 3x + C_2\sin 3x) + 6\cos 3x - \sin 3x}$ —①

Applying the condition $y(0) = 3$ in equation ①, we get

$3 = C_1 + 6 \Rightarrow C_1 = -3$

from ①, $\frac{dy}{dx} = e^{-x}(-3C_1\sin 3x + 3C_2\cos 3x) - e^{-x}(C_1\cos 3x + C_2\sin 3x) - 18\sin 3x - 3\cos 3x$ —②

Applying the condition $\frac{dy}{dx} = 0$ when $x = 0$ in ②

$0 = 3C_2 - C_1 - 3 \Rightarrow C_2 = 0$ $[\because C_1 = -3]$

Substituting the value of C_1 and C_2 in eq^n ①, we get

$y = (6 - 3e^{-x})\cos 3x - \sin 3x$

when $x = \frac{\pi}{2}$, $\boxed{y = -\sin\frac{3\pi}{2} = 1}$

Q3:- Solve the differential equation

$\frac{d^2y}{dx^2} + y = e^{-x} + \sin x\cos x + x^3 + e^x\cos x$

Sol^n:- We have,

$(D^2+1)y = e^{-x} + \sin x\cos x + x^3 + e^x\cos x$

A.E. is $m^2+1 = 0 \Rightarrow m = \pm i$

$\Rightarrow$ C.F. $= C_1\cos x + C_2\sin x$

Now P.I. $= \frac{1}{(D^2+1)}(e^{-x} + \sin x\cos x + x^3 + e^x\cos x)$

Now $\frac{1}{D^2+1}e^{-x} = \frac{1}{(-1)^2+1}e^{-x} = \frac{1}{2}e^{-x}$ [replacing D by -1]

$\frac{1}{D^2+1}\sin x\cos x = \frac{1}{2(D^2+1)}2\sin x\cos x = \frac{1}{2(D^2+1)}\sin 2x$

$= \frac{1}{2(-2^2+1)}\sin 2x$ [replacing D^2 by -4]

$= -\frac{1}{6}\sin 2x$

$\frac{1}{D^2+1}x^3 = (1+D^2)^{-1}x^3$

$= (1 - D^2 + D^4 - D^6 + \text{———})x^3$ $[\because (1+x)^{-1} = 1 - x + x^2 - x^3 \text{———}]$

$= [x^3 - D^2x^3 + D^4x^3 - D^6x^3 \text{———}]$

$= [x^3 - 6x + 0 - 0 \text{———}] = x^3 - 6x$

and $\frac{1}{D^2+1} e^x \cos x = e^x \frac{1}{(D+1)^2+1} \cos x$ [replacing D by $D+1$]

$$= e^x \frac{1}{D^2+2D+2} \cos x$$

$$= e^x \frac{1}{-1+2D+2} \cos x \quad [\text{replacing } D^2 \text{ by } -1]$$

$$= e^x \frac{1}{2D+1} \cos x = e^x \frac{(2D-1)}{4D^2-1} \cos x = e^x \frac{(2D-1)}{-4-1} \cos x$$

$$= \frac{-1}{5} e^x (2D\cos x - \cos x)$$

$$= \frac{-1}{5} e^x (-2\sin x - \cos x) = \frac{e^x}{5} (\sin x + \cos x)$$

Hence the complete solution is

$y = C.F. + P.I.$

$$\boxed{\therefore y = C_1 \cos x + C_2 \sin x + \frac{1}{2} e^{-x} - \frac{1}{6} \sin 2x + x^3 - 6x + \frac{e^x}{5} (\sin x + \cos x)}$$

Q 4:- Find the complete solution of the differential equation

$$\frac{d^2y}{dx^2} + a^2 y = \sec(ax)$$

Sol:- Auxiliary equation is

$m^2 + a^2 = 0 \Rightarrow m = \pm ai$

$\therefore C.F. = C_1 \cos ax + C_2 \sin ax$

$$P.I. = \frac{1}{D^2+a^2} \sec ax = \frac{1}{(D+ia)(D-ia)} \sec ax$$

$$= \frac{1}{2ia} \left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \sec ax = \frac{1}{2ia} \left[\frac{1}{D-ia} \sec ax - \frac{1}{D+ia} \sec ax\right]$$

$$= \frac{1}{2ia} (P_1 - P_2)$$

where $P_1 = \frac{1}{D-ia} \sec ax = e^{iax} \int e^{-iax} \sec ax \, dx$

$$= e^{iax} \int (\cos ax - i \sin ax) \sec ax \, dx$$

$$= e^{iax} \int (1 - i \tan ax) \, dx = e^{iax} \left\{ x + i \left(\frac{\log \cos ax}{a} \right) \right\}$$

$$P_2 = \frac{1}{D+ia} (\sec ax) = e^{-iax} \left\{ x - i \left(\frac{\log \cos ax}{a} \right) \right\}$$

[replacing i by $-i$]

$$P.I. = \frac{1}{2ia} \left[e^{iax} \left\{ x + i \left(\frac{\log \cos ax}{a} \right) \right\} - e^{-iax} \left\{ x - i \left(\frac{\log \cos ax}{a} \right) \right\} \right]$$

$$= \frac{1}{2ia} \left[x (e^{iax} - e^{-iax}) + i \left(\frac{\log \cos ax}{a} \right) (e^{iax} + e^{-iax}) \right]$$

$$= \frac{1}{2ia} \left[2ix \sin ax + \frac{i}{a} (\log \cos ax)(2 \cos ax) \right]$$

$$= \frac{1}{a} \left[x \sin ax + \frac{1}{a} \cos ax \log \cos ax \right]$$

Hence the complete solution is

$$y = C.F. + P.I. = C_1 \cos ax + C_2 \sin ax + \frac{1}{a} \left(x \sin ax + \frac{1}{a} \cos ax \log \cos ax \right)$$

where C_1 and C_2 are arbitrary constants of integration.

Homogeneous Linear Differential Equation (Euler-Cauchy Equations) :-

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \text{———} + a_n y = Q$$

where a_1, a_2 ——— a_n are constants and Q is a function of x, is called "Cauchy homogeneous linear equations."

Steps for Solution :-

(1) Put $x = e^z \Rightarrow z = \log x$ and let $D \equiv \frac{d}{dz}$

(2) Replace $x\frac{d}{dx}$ by D

$x^2 \frac{d^2}{dx^2}$ by $D(D-1)$

$x^3 \frac{d^3}{dx^3}$ by $D(D-1)(D-2)$ and so on-----

(3) By doing so, this type of equation reduces to linear differential equation with constant coefficients.

Q.5 :- Solve $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^{e^x}$ ——(1)

Sol^n :- Putting $x = e^z$, $D \equiv \frac{d}{dz}$, $x\frac{d}{dx} = D$, $x^2\frac{d^2}{dx^2} \equiv D(D-1)$

So from (1)

$$D(D-1)y + 4Dy + 2y = e^{e^z}$$

$$(D(D-1) + 4D + 2)y = e^{e^z} \Rightarrow (D^2 + 3D + 2)y = e^{e^z}$$

Auxiliary equation is

$$m^2 + 3m + 2 = 0 \Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2$$

$$\therefore C.F. = c_1 e^{-z} + c_2 e^{-2z}$$

$$P.I. = \frac{1}{D^2+3D+2} e^{e^z} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^z}$$

$$= \frac{1}{D+1} e^{e^z} - \frac{1}{D+2} e^{e^z} = e^{-z}\int e^{z} e^{e^z} dz - e^{-2z}\int e^{e^z} e^{2z} dz$$

$$= e^{-z} e^{e^z} - e^{-2z}(e^z - 1) e^{e^z} = e^{-2z} e^{e^z}$$

Hence the complete solution is

$$\boxed{y = C.F. + P.I. = c_1 e^{-z} + c_2 e^{-2z} + e^{-2z} e^{e^z} = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{x^2} e^x}$$

where c_1 and c_2 are arbitrary constants of integration.

Q.6 :- Solve the simultaneous equations:

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = y \quad \text{and} \quad \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 25x + 16e^t$$

Sol^n :- Let $\frac{d}{dt} \equiv D$ then

$$(D^2 - 4D + 4)x - y = 0 \quad \text{——(1)}$$

$$-25x + (D^2 + 4D + 4)y = 16e^t \quad \text{——(2)}$$

Operating (1) by (D^2+4D+4) and adding to (2), we get

$$(D^2-4D+4)(D^2+4D+4)x - 25x = 16e^t$$

$$(D^4 - 8D^2 - 9)x = 16e^t$$

A.E. is $m^4 - 8m^2 - 9 = 0 \Rightarrow (m^2-9)(m^2+1) = 0 \Rightarrow m = \pm i, \pm 3$

$$\therefore C.F. = c_1 e^{3t} + c_2 e^{-3t} + c_3 \cos t + c_4 \sin t$$

$$P.I. = \frac{1}{D^4 - 8D^2 + 9} 16e^t = 8e^t$$

$$\therefore \boxed{x = c_1 e^{3t} + c_2 e^{-3t} + c_3 \cos t + c_4 \sin t + 8e^t} \quad \text{—(3)}$$

$$\frac{dx}{dt} = 3C_1e^{3t} - 3C_2e^{-3t} + C_3(-\sin t) + C_4\cos t + 8e^t$$

$$\frac{d^2x}{dt^2} = 9C_1e^{3t} + 9C_2e^{-3t} - C_3\cos t - C_4\sin t + 8e^t$$

from (1), $y = \frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x$

$$= 9C_1e^{3t} + 9C_2e^{-3t} - C_3\cos t - C_4\sin t + 8e^t$$
$$-4(3C_1e^{3t} - 3C_2e^{-3t} - C_3\sin t + C_4\cos t + 8e^t)$$
$$+4(C_1e^{3t} + C_2e^{-3t} - C_3\cos t + C_4\sin t + 8e^t)$$

$$\Rightarrow \boxed{y = C_1e^{3t} + 25C_2e^{-3t} + (3C_3 - 4C_4)\cos t + (4C_3 + 3C_4)\sin t + 8e^t} \quad \text{--- (4)}$$

Equations (3) & (4) when taken together give the complete solution.

Method! To find the complete solution of $y'' + Py' + Qy = R$ by changing the independent variable.

Steps (i) Compare the given eq^n with $y'' + Py' + Qy = R$ and find P, Q, R.

(ii) Choose z such that $\left(\frac{dz}{dx}\right)^2 = Q$ where Q is taken in such a way that it remains whole square of a function without surd and its negative sign is ignored.

(iii) find $P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$, $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$, $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$

(iv) Reduce eq^n is $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1$ solve this eq^n and get the result. In last replace the value of z in terms of x.

Q-7. Solve by changing the independent variable

$$\frac{d^2y}{dx^2} + (3\sin x - \cot x)\frac{dy}{dx} + 2y\sin^2 x = e^{-\cos x}\sin^2 x$$

Sol On Compare $P = 3\sin x - \cot x$, $Q = 2\sin^2 x$, $R = e^{-\cos x}\sin^2 x$

Choose z s.t

$$\left(\frac{dz}{dx}\right)^2 = \sin^2 x$$

$$\frac{dz}{dx} = \sin x \quad \text{(1)}$$

On differentiation

$$\frac{d^2z}{dx^2} = +\cos x$$

On Integration of (1)

$$\boxed{z = -\cos x}$$

Now $P_1 = \frac{\cos x + (3\sin x - \cot x)\sin x}{\sin^2 x}$

$\boxed{P_1 = 3}$

$$Q_1 = \frac{2\sin^2 x}{\sin^2 x} = 2$$

$$R_1 = \frac{e^{-\cos x}\sin^2 x}{\sin^2 x} = e^{-\cos x} = e^{z}$$

Reduced eq^n is

$$\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = e^{z}$$

A.E is $m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m = -1, -2$, $C.F = C_1e^{-z} + C_2e^{-2z}$

$$P.I = \frac{1}{D^2 + 3D + 2}e^{z} = \frac{1}{1+3+2}e^{z} = \frac{e^z}{6}$$

$$\therefore y = C_1e^{-z} + C_2e^{-2z} + \frac{e^{z}}{6} \Rightarrow y = C_1\ldots$$

Compact Note:

METHOD: To find the complete solution of $y'' + Py' + Qy = R$ by the method of VARIATION OF PARAMETERS

Steps for Solution: (i) find out part of C.F and let them u and v.

(ii) Consider the complete solution as $y = Au + Bv$ —(1)

where $A = \int -\frac{Rv}{W} dx + c_1$, $B = \int \frac{Ru}{W} dx + c_2$

Where $W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = uv_1 - u_1 v$

(iii) Put the values of A & B in eq^n (1) and we get the solution.

Q-9 Use method of variation of parameter to solve

$$\frac{d^2y}{dx^2} + y = \tan x$$

Solution: Given eq^n is $(D^2+1)y = \tan x$

A.E is $m^2 + 1 = 0$
$m^2 = -1 \Rightarrow m = \pm i$, $R = \tan x$

$C.F = c_1 \cos x + c_2 \sin x$

let $u = \cos x$, $v = \sin x$

let $y = Au + Bv$ —(1) be the solution of given eq^n.

Now $A = \int -\frac{Rv}{W} dx + c_1$, $B = \int \frac{Ru}{W} dx + c_2$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \Rightarrow \cos^2 x + \sin^2 x = 1$

$\therefore A = \int \frac{-\tan x \cdot \sin x}{1} dx + c_1$

$= -\int \frac{\sin^2 x}{\cos x} dx + c_1$

$\Rightarrow A = -\int \frac{(1-\cos^2 x)}{\cos x} dx + c_1$

$= -\int (\sec x - \cos x) dx + c_1$

$= \sin x - \log(\sec x + \tan x) + c_1$

$\boxed{A = \sin x - \log(\sec x + \tan x) + c_1}$

$B = \int \frac{\tan x \cdot \cos x}{1} dx + c_2 \Rightarrow B = \int \sin x \, dx + c_2$

$\Rightarrow \boxed{B = c_2 - \cos x}$

Put all values in (1)

$\boxed{y = [\sin x - \log(\sec x + \tan x) + c_1] \cos x + [c_2 - \cos x] \sin x}$

where c_1, c_2 are arbitrary constants. Answer

Question-10 Apply method of variation of parameters to solve

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$$

Solution Put $x = e^z$, $x\frac{dy}{dx} = Dy$, $x^2 \frac{d^2y}{dx^2} = D(D-1)y$

Given eq^n become

$D(D-1)y + 4Dy + 2y = e^{e^z}$

$\Rightarrow (D^2 + 3D + 2)y = e^{e^z}$

A.E is $m^2 + 3m + 2 = 0$

$m^2 + 2m + m + 2 = 0$

$m(m+2) + 1(m+2) = 0$

$(m+1)(m+2) = 0$

$m = -1, -2$

$C.F = c_1 e^{-z} + c_2 e^{-2z}$

Put $e^z = x$, $e^{-z} = \frac{1}{x}$

$\therefore C.F = \frac{c_1}{x} + \frac{c_2}{x^2}$

let $u = \frac{1}{x}$, $v = \frac{1}{x^2}$

Let $y = Au + Bv$ —(1) be the solution

Now $A = \int -\frac{Rv}{W} dx + c_1$

$W = \begin{vmatrix} \frac{1}{x} & \frac{1}{x^2} \\ -\frac{1}{x^2} & -\frac{2}{x^3} \end{vmatrix} \Rightarrow \frac{-2}{x^4} + \frac{1}{x^4}$

$W = -x^{-4}$

$\therefore A = \int + \frac{e^x}{x^2 \cdot x^2 \cdot x^{-4}} dx + c_1$ $\left\{ \because R = \frac{e^x}{x^2} \right.$

$= \int e^x dx + c_1 = e^x + c_1$

$B = \int \frac{Ru}{W} dx + c_2$

$= \int \frac{-e^x}{x^2 \cdot x \cdot x^{-4}} dx + c_2$

$= -\int x e^x dx + c_2$

$= -[x e^x - (1)e^x] + c_2$

$B = c_2 + (1-x)e^x$

Put above values in (1)

$\boxed{y = (e^x + c_1)\frac{1}{x} + (c_2 + (1-x)e^x)\frac{1}{x^2}}$ Ans

Note In given eq^n on comparing $R = \frac{e^x}{x^2}$

UNIT-2

Laplace Transform   Unit : 02

Def^n (Laplace Transform): Let $F(t)$ be a function of t defined for all $t \geq 0$. Then the Laplace transform of $F(t)$, denoted by $L\{F(t)\}$, is defined by

$$\boxed{L\{F(t)\} = f(p) = \int_0^\infty e^{-pt} F(t)\,dt}$$

provided that the integral exists, 'p' is a parameter which may be real or complex.

The function $f(p)$ is called the Laplace transform.

NOTE: $L\{F(t)\}$ is said to exists if the above integral converges for some value of p otherwise not.

Linearity properties:

If c_1, c_2 are constants and f, g are function of t, then

$$\boxed{L\{c_1 f(t) + c_2 g(t)\} = c_1 L\{f(t)\} + c_2 L\{g(t)\}}$$

Functions of Exponential Order: A function $F(t)$ is said to be of exponential order as $t \to \infty$, if there exist constants M and b and a fixed value t_0 of t such that

$$|F(t)| < Me^{bt}, \text{ for } t \geq t_0$$

Existence Theorem:

If $F(t)$ is piecewise continuous for $t \geq 0$ and is of exponential order b, then

$$L\{F(t)\} = f(p) \text{ exists for } p > b.$$

Laplace Transform of some elementary functions

① $L\{1\} = \frac{1}{p}$, $p > 0$

Pf ⇒ $L\{1\} = \int_0^\infty e^{-pt}(1)\,dt = \left[\frac{e^{-pt}}{-p}\right]_0^\infty = \left(\frac{-1}{p}\right)(0-1)$

$\boxed{L\{1\} = \frac{1}{p}}$, $p > 0$

② $L\{t^n\} = \frac{n!}{p^{n+1}}$ or $\frac{\Gamma(n+1)}{p^{n+1}}$

Pf $L\{t^n\} = \int_0^\infty e^{-pt} t^n\,dt = \int_0^\infty e^{-x}\left(\frac{x}{p}\right)^n \frac{dx}{p}$

$= \frac{1}{p^{n+1}}\int_0^\infty e^{-x} x^n\,dx$

$= \frac{1}{p^{n+1}}\int_0^\infty e^{-x} x^{(n+1)-1}\,dx = \frac{\Gamma(n+1)}{p^{n+1}}$

Let $pt = x$
⇒ $p\,dt = dx$
⇒ $dt = \frac{dx}{p}$

$\boxed{L\{t^n\} = \Gamma(n+1)/p^{n+1}}$ $\boxed{L\{t^n\} = \frac{n!}{p^{n+1}}, n \in \mathbb{N}}$

Ques Find the Laplace transform of
$7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3 + 5\sin 3t + 2$

Ans. $L(7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3 + 5\sin 3t + 2)$

$\Rightarrow 7L\{e^{2t}\} + 9L\{e^{-2t}\} + 5L\{\cos t\} + 7L\{t^3\} + 5L\{\sin 3t\} + 2L\{1\}$

$\Rightarrow \frac{7}{p-2} + \frac{9}{p+2} + \frac{5p}{p^2+1} + 7\cdot\frac{3!}{t^4} + 5\cdot\frac{3}{p^2+9} + 2\cdot\frac{1}{p}$

$\Rightarrow \frac{7}{p-2} + \frac{9}{p+2} + \frac{5p}{p^2+1} + \frac{42}{t^4} + \frac{15}{p^2+9} + \frac{2}{p}$

Ques Find the Laplace transform of

$$F(t) = \begin{cases} \cos t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$$

Ans. $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$

$= \int_0^\pi e^{-pt} F(t)\,dt + \int_\pi^\infty e^{-pt} F(t)\,dt$

$= \int_0^\pi e^{-pt} \cos t\,dt + \int_\pi^\infty e^{-pt}(0)\,dt$

$= \left[\frac{e^{-pt}}{(-p)^2+1}\left[-p\cos t + \sin t\right]\right]_0^\pi + 0$

$\boxed{L\{F(t)\} = \left[\frac{e^{-pt}}{p^2+1}\left[-p\cos t + \sin t\right]\right]_0^\pi}$

$\boxed{\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]}$

Que. find the Laplace transform cosh at cos bt (2016)

A. We have to find Laplace transform of

$$\cosh at \cos bt = \left(\frac{e^{at}+e^{-at}}{2}\right)\cos bt$$

$$= \frac{1}{2}\left[e^{at}\cos bt + e^{-at}\cos bt\right]$$

$$= I_1 + I_2 \quad \text{———} \quad (1)$$

Now

$$L\{\cos bt\} = \frac{p}{p^2+b^2} = f(p)$$

then

$$L\{e^{at}\cos bt\} = f(p-a) = \frac{p-a}{(p-a)^2+b^2}$$

$$L\{e^{-at}\cos bt\} = f(p+a) = \frac{p+a}{(p+a)^2+b^2}$$

[first shifting property]

Now using (1)

$$L\{\cosh at \cos bt\} = \frac{1}{2}\left[L\{e^{at}\cos bt\} + L\{e^{-at}\cos bt\}\right]$$

$$= \frac{1}{2}\left[\frac{p-a}{(p-a)^2+b^2} + \frac{p+a}{(p+a)^2+b^2}\right]$$

Second Translation Property or Heaviside's Shifting Theorem.

If $L\{f(t)\} = f(p)$ and $G(t) = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$

then $\boxed{L\{G(t)\} = e^{-ap} f(p)}$

(ii) $F(t) = \begin{cases} \cos\left(t-\frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$

A. $L\{\cos t\} = \frac{p}{p^2+1} = f(p)$

So $L\{F(t)\} = e^{-(2\pi/3)p} f(p) = \frac{\left(e^{-\left(\frac{2\pi}{3}\right)p}\right) p}{(p^2+1)}$

Change of Scale Property

If $L\{F(t)\} = f(p)$ then $\boxed{L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)}$

Que. If $L\{F(t)\} = \frac{p^2-p+1}{(2p+1)^2(p-1)}$, Show that (2014, 18)

$$L\{F(2t)\} = \frac{p^2-2p+4}{4(p+1)^2(p-2)}$$

A. $L\{F(t)\} = \frac{p^2-p+1}{(2p+1)^2(p-1)} = f(p)$

Now

$$L\{F(2t)\} = \frac{1}{2} f\left(\frac{p}{2}\right) \quad \left[\text{Using Change of Scale property}\right]$$

$$L\{F(2t)\} = \frac{1}{2}\left[\frac{\left(\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)+1}{\left(2\left(\frac{p}{2}\right)+1\right)^2\left(\frac{p}{2}-1\right)}\right] = \frac{1}{2}\left[\frac{\frac{p^2}{4}-\frac{p}{2}+1}{(p+1)^2\left(\frac{p-2}{2}\right)}\right]$$

$$L\{F(2t)\} = \frac{1}{2}\left[\frac{\frac{p^2-2p+4}{4}}{(p+1)^2\left(\frac{p-2}{2}\right)}\right] = \frac{p^2-2p+4}{4(p+1)^2(p-2)}$$

Divison by t

If $L\{F(t)\} = f(p)$ then $\boxed{L\{\frac{1}{t}F(t)\} = \int_p^\infty f(p)\,dp}$

Ques Find the Laplace transform of

(i) $\int_0^t e^{-t}\cos t\,dt$ (2022)

(ii) $\int_0^t \frac{\sin t}{t}\,dt$ (2022)

(iii) $\int_0^t e^{t}\frac{\sin t}{t}\,dt$ (2014)

Sol (i) $L\{\cos t\} = \frac{p}{p^2+1}$

$L\{e^{-t}\cos t\} = \frac{p+1}{(p+1)^2+1} = f(p)$

Now

$L\left(\int_0^t e^{-t}\cos t\,dt\right) = \frac{1}{p}f(p)$

(by Laplace transform of Integral)

$L\left(\int_0^t e^{-t}\cos t\,dt\right)$

$= \frac{1}{p}\left[\frac{p+1}{(p+1)^2+1}\right]$

$= \frac{p+1}{p(p^2+2p+2)}$

(ii) $L\{\sin t\} = \frac{1}{p^2+1}$

$L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty \frac{1}{p^2+1}\,dp$

$= (\tan^{-1}p)_p^\infty$

$= (\tan^{-1}\infty - \tan^{-1}p)$

$= \frac{\pi}{2} - \tan^{-1}p$

$L\left\{\frac{\sin t}{t}\right\} = \cot^{-1}p = f(p)$

Now

$L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{p}f(p)$

$= \frac{\cot^{-1}p}{p}$

Q Find the following integral.

(i) $\int_0^\infty t^3 e^{-t}\sin t\,dt$

(ii) $\int_0^\infty \frac{e^{-t}\sin^2 t}{t}$

Ans (i) $L\{\sin t\} = \frac{1}{p^2+1}$

$L\{t^3\sin t\} = -\frac{d^3}{dp^3}\left[\frac{1}{p^2+1}\right]$

$= \frac{d^2}{dp^2}\left[\frac{2p}{(p^2+1)^2}\right]$

$= -\frac{d}{dp}\left[\frac{2(1-3p^2)}{(p^2+1)^3}\right]$

$= \frac{24p(p^2-1)}{(p^2+1)^4}$

$L\{t^3\sin t\} = \frac{24p(p^2-1)}{(p^2+1)^4}$

$\int_0^\infty e^{-pt}t^3\sin t\,dt = \frac{24p(p^2-1)}{(p^2+1)^4}$

by defn of Laplace Transn

Now put $p=1$ then

$\int_0^\infty e^{-t}t^3\sin t\,dt = \frac{24(1-1)}{(1+1)^4}$

$= 0$

(ii) $L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$

$= \frac{1}{2}L\{1-\cos 2t\}$

$= \frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right]$

$= \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4}\right]$

Now

$L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{2}\int_p^\infty \left(\frac{1}{p} - \frac{p}{p^2+4}\right)$

$= \frac{1}{2}\left[\log p - \frac{1}{2}\log(p^2+4)\right]_p^\infty$

$= \frac{1}{4}\left\{\log\left[\frac{p^2}{p^2+4}\right]\right\}_p^\infty$

$= \frac{1}{4}\left[\lim_{p\to\infty}\log\left[\frac{p^2}{p^2+4}\right] - \log\left[\frac{p^2}{p^2+4}\right]\right]$

$= \frac{1}{4}\left[\lim_{p\to\infty}\log\left[\frac{1}{1+\frac{4}{p^2}}\right] + \log\left[\frac{p^2+4}{p^2}\right]\right]$

$= \frac{1}{4}\left[0 + \log\left[\frac{p^2+4}{p^2}\right]\right]$

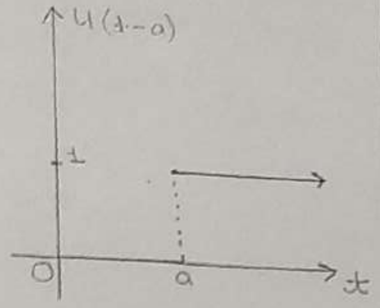
$= \frac{1}{4}\log\left[\frac{p^2+4}{p^2}\right]$

Unit Step function (or Heaviside's Unit step function)

The unit step function $u(t-a)$ is defined as

$$u(t-a) = \begin{cases} 0, & \text{for } t<a \\ 1, & \text{for } t \geq a \end{cases}$$

where $a \geq 0$



As a particular case,

$$u(t) = \begin{cases} 0 & , \text{for } t<0 \\ 1 & , \text{for } t \geq 0 \end{cases}$$

The product $F(t)\,u(t-a) = \begin{cases} 0 & , \text{for } t<a \\ F(t) & , \text{for } t \geq a \end{cases}$

The function $F(t-a)\,u(t-a)$ represents the graph of $F(t)$ shifted through a distance a to the right.

Laplace transform of Unit step function

$$L\{u(t-a)\} = \int_0^{\infty} e^{-pt}\,u(t-a)\,dt$$

$$= \int_0^{a} e^{-pt}(0)\,dt + \int_a^{\infty} e^{-pt}(1)\,dt$$

$$= 0 + \left[\frac{e^{-pt}}{-p}\right]_a^{\infty} = \frac{1}{p}e^{-ap}$$

(iii) $\sin t\; u(t-\pi)$

$\sin t = \sin[(t-\pi)+\pi]$

$\sin t = -\sin(t-\pi)$

$L\{\sin t\; u(t-\pi)\}$

$= L\{-\sin(t-\pi)\,u(t-\pi)\}$

$= -L\{\sin(t-\pi)\,u(t-\pi)\}$

On comparing $\sin(t-\pi)u(t-\pi)$ with $F(t-a)\,u(t-a)$ we get

$a=\pi$, $F(t) = \sin t$

$L\{\sin t\; u(t-\pi)\} = -e^{-\pi p}L\{\sin t\}$

$= \dfrac{-e^{-\pi p}}{p^2+1}$

(iv) $e^{-3t}u(t-2)$

$L\{u(t-2)\} = \frac{1}{p}e^{-2p} = f(p)$

$\left(\because L\{u(t-a)\} = \frac{1}{p}e^{-ap}\right)$

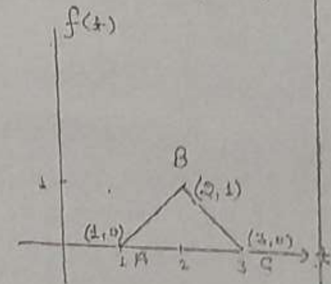
$L\{e^{-3t}u(t-2)\}$

$= f(p+3)$

$= \dfrac{1}{(p+3)}\,e^{-2(p+3)}$

$\left(\because L\{e^{-at}F(t)\} = f(p+a)\right)$

Exn Express the function shown in the diagram in terms of unit step function and obtain its Laplace transform (2015)



Ans Equation of line AB

$f(t) = t-1 \quad , \quad 1<t<2$

Equation of line BC

$f(t) = 3-t \quad , \quad 2<t<3$

So $f(t) = \begin{cases} t-1 & , \; 1<t<2 \\ 3-t & , \; 2<t<3 \end{cases}$

So $f(t) = (t-1)[u(t-1) - u(t-2)] + 1[u(t-2)]$

$f(t) = (t-1)\,u(t-1) - (t-2)\,u(t-2)$.

By Second Shifting theorem

$$L\{f(t)\} = L\{(t-1)u(t-1)\} - L\{(t-2)u(t-2)\}$$
$$= \frac{e^{-p}}{p^2} - \frac{e^{-2p}}{p^2} = \left(\frac{e^{-p} - e^{-2p}}{p^2}\right)$$

<u>Periodic function</u> [Laplace transform of periodic function]

If $f(t)$ is a periodic function with period T

i.e. $f(t+T) = f(t)$ then

$$\boxed{L\{f(t)\} = \frac{1}{1-e^{-pT}}\int_0^T e^{-pt} f(t)\,dt}$$

<u>Ques</u> Draw the graph and find the Laplace transform of the triangular wave function of period $2c$ given by

$$f(t) = \begin{cases} t, & 0 < t \le c \\ 2c - t, & c < t < 2c \end{cases} \quad (2014, 22)$$

Ans Here period $\boxed{T = 2c}$

$$L\{f(t)\} = \frac{1}{1-e^{-pT}}\int_0^T e^{-pt} f(t)\,dt$$
$$= \frac{1}{1-e^{-2cp}}\int_0^{2c} e^{-pt} f(t)\,dt$$

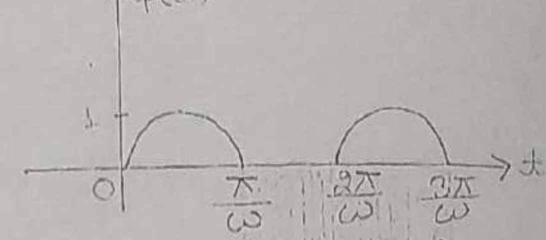
<u>Ques</u> Find the Laplace transform of the following rectified semi-wave function defined by

$$f(t) = \begin{cases} \sin(\omega t), & 0 < t \le \pi/\omega \\ 0, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

or

Find the Laplace transform of following periodic function.

(Half-wave rectifier)



Ans Here $f(t)$ is a periodic function with period

So $\boxed{T = 2\pi/\omega}$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-pT}}\int_0^T e^{-pt} f(t)\,dt$$
$$= \frac{1}{1-e^{-2\pi p/\omega}}\int_0^{2\pi/\omega} e^{-pt} f(t)\,dt$$
$$= \frac{1}{1-e^{-2\pi p/\omega}}\left[\int_0^{\pi/\omega} e^{-pt}\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-pt}(0)\,dt\right]$$
$$= \frac{1}{1-e^{-2\pi p/\omega}}\int_0^{\pi/\omega} e^{-pt}\sin\omega t\,dt + 0$$

Inverse Laplace Transforms :-

If $L\{F(t)\} = f(p)$, then $F(t)$ is called the inverse Laplace transform of $f(p)$ and is denoted by –

$$\boxed{L^{-1}\{f(p)\} = F(t)}$$

Here L^{-1} denotes the inverse Laplace transform operator.

for e.g. since $L\{e^{5t}\} = \frac{1}{p-5}$

$$\therefore L^{-1}\left\{\frac{1}{p-5}\right\} = e^{5t}$$

The inverse Laplace transform given below follow at once from the results of Laplace transforms given earlier:

1. $L^{-1}\left\{\frac{1}{p}\right\} = 1$
2. $L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$
3. $L^{-1}\left\{\frac{1}{p^n}\right\} = \frac{t^{n-1}}{\lfloor n-1}$, if n is a positive integer. $\left(\text{otherwise} = \frac{t^{n-1}}{\Gamma(n)}\right)$
4. $L^{-1}\left\{\frac{1}{(p-a)^n}\right\} = e^{at}\frac{t^{n-1}}{\lfloor n-1}$
5. $L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{1}{a}\sin at$
6. $L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$
7. $L^{-1}\left\{\frac{1}{p^2-a^2}\right\} = \frac{1}{a}\sinh at$
8. $L^{-1}\left\{\frac{p}{p^2-a^2}\right\} = \cosh at$

$$\Rightarrow L^{-1}\left\{\frac{1}{p^{3/2}} + \frac{4}{p-2} + \frac{2p-18}{p^2-9}\right\} = 2\sqrt{\frac{t}{\pi}} + 4e^{2t} + 2\cosh 3t - 6\sinh 3t$$

Ans. $\left(\because \Gamma\frac{3}{2} = \frac{\sqrt{\pi}}{2}\right)$

(iii):- $L^{-1}\left\{\frac{p^3}{p^4-a^4}\right\} = L^{-1}\left[p\left\{\frac{p}{(p^2-a^2)(p^2+a^2)}\right\}\right]$

$$= L^{-1}\left\{\frac{p}{2}\left(\frac{1}{p^2-a^2} + \frac{1}{p^2+a^2}\right)\right\} \quad \text{(using property of partial fraction)}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{p}{p^2-a^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{p}{p^2+a^2}\right\}$$

$$= \frac{1}{2}\cosh at + \frac{1}{2}\cos at$$

$$\Rightarrow L^{-1}\left\{\frac{p^3}{p^4-a^4}\right\} = \frac{1}{2}(\cosh at + \cos at)$$ Ans.

First Translation or Shifting Property :->

If $L^{-1}\{f(p)\} = F(t)$, then $\boxed{L^{-1}\{f(p-a)\} = e^{at}F(t)}$

we know that

$$f(p) = \int_0^\infty e^{-pt}F(t)\,dt \quad \text{(By definition)}$$

$$\therefore f(p-a) = \int_0^\infty e^{-(p-a)t}F(t)\,dt$$

$$\Rightarrow f(p-a) = \int_0^\infty e^{-pt}.e^{at}F(t)\,dt$$

$$\Rightarrow f(p-a) = L\{e^{at}F(t)\}$$

$$\Rightarrow \boxed{L^{-1}\{f(p-a)\} = e^{at}F(t)}$$

(iii) :→ $\dfrac{p^2+2a^2}{p^4+4a^4} = \dfrac{1}{2}\left[\dfrac{1}{(p-a)^2+a^2} + \dfrac{1}{(p+a)^2+a^2}\right]$ (using Partial fraction Property)

$$\therefore L^{-1}\left\{\frac{p^2+2a^2}{p^4+4a^4}\right\} = \frac{1}{2}L^{-1}\left[\frac{1}{(p-a)^2+a^2} + \frac{1}{(p+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{a}e^{at}\sin at + \frac{1}{a}e^{-at}\sin at\right]$$

$$= \frac{1}{a}\left(\frac{e^{at}+e^{-at}}{2}\right)\sin at$$

$$L^{-1}\left\{\frac{p^2+2a^2}{p^4+4a^4}\right\} = \frac{1}{a}\sin at\cosh at \quad \underline{\text{Ans.}}$$

Second Translation or Shifting Property :–

If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{e^{-ap}f(p)\} = G(t), \text{ where } G(t) = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$$

Note :- We may write $G(t)$ in terms of Heaviside's unit step function as $F(t-a)\,U(t-a)$ or $F(t-a)\,H(t-a)$

∴ the above theorem can be restated as

If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{e^{-ap}f(p)\} = F(t-a)\,H(t-a)$$

(iii) :→

Let

$$\frac{1}{p(p^2+1)} = \frac{A}{p} + \frac{Bp+C}{p^2+1}$$

$\Rightarrow 1 = A(p^2+1) + p(Bp+C)$

$\Rightarrow 1 = Ap^2 + A + Bp^2 + pC$

$\Rightarrow 1 = (A+B)p^2 + pC + A$

Equating coefficients of p^2, p & constant, we get

$A+B=0 \Rightarrow \boxed{B=-1}$

$\boxed{C=0}$ & $\boxed{A=1}$

$$\therefore \frac{1}{p(p^2+1)} = \frac{1}{p} - \frac{p}{p^2+1}$$

$$L^{-1}\left\{\frac{1}{p(p^2+1)}\right\} = L^{-1}\left\{\frac{1}{p} - \frac{p}{p^2+1}\right\}$$

$$= L^{-1}\left\{\frac{1}{p}\right\} - L^{-1}\left\{\frac{p}{p^2+1}\right\}$$

$$\boxed{L^{-1}\left\{\frac{1}{p(p^2+1)}\right\} = 1 - \cos t} \quad \underline{\text{Ans.}}$$

(iv) Let $\dfrac{p-1}{p^2(p-7)} = \dfrac{A}{p} + \dfrac{B}{p^2} + \dfrac{C}{(p-7)}$

$\Rightarrow (p-1) = Ap(p-7) + B(p-7) + Cp^2$

$\Rightarrow p-1 = Ap^2 - 7Ap + Bp - 7B + Cp^2$

$\Rightarrow p-1 = (A+C)p^2 + (B-7A)p - 7B$

Comparison gives

* Inverse Laplace Transform of Derivatives :-

If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{f^n(p)\} = L^{-1}\left[\frac{d^n}{dp^n}\{f(p)\}\right] = (-1)^n t^n F(t).$$

we have $L\{t^n F(t)\} = (-1)^n \left\{\frac{d^n}{dp^n} f(p)\right\} = (-1)^n f^n(p)$

$$\therefore L^{-1}\{f^n(p)\} = (-1)^n t^n F(t).$$

Question No. Find the inverse Laplace transform of

(i) $\log\left(1+\frac{1}{p^2}\right)$ (2015, 2022), (ii) $\log\left(\frac{p+1}{p-1}\right)$ (2012)

(iii) $\cot^{-1}\left(\frac{p+3}{2}\right)$ (2013), (iv) $L^{-1}\left[\log\left(\frac{p^2+4p+5}{p^2+2p+5}\right)\right]$ (2014)

Solution (i) :→ let $L^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\} = F(t)$ (say)

$$\therefore L^{-1}\left[\frac{d}{dp}\left\{\log\left(1+\frac{1}{p^2}\right)\right\}\right] = (-1)^1 t^1 F(t)$$

$$\Rightarrow L^{-1}\left[\frac{1}{\left(1+\frac{1}{p^2}\right)} \times \left(\frac{-2}{p^3}\right)\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{-2p^2}{p^3(p^2+1)}\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{-2}{p(p^2+1)}\right] = -tF(t) \quad \text{------ (1)}$$

let $\frac{-2}{p(p^2+1)} = \frac{A}{p} + \frac{Bp+C}{p^2+1}$

(iii) :→ let $L^{-1}\left\{\cot^{-1}\left(\frac{p+3}{2}\right)\right\} = F(t)$ (say)

$$\therefore L^{-1}\left[\frac{d}{dp}\left\{\cot^{-1}\left(\frac{p+3}{2}\right)\right\}\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{-1}{1+\left(\frac{p+3}{2}\right)^2} \times \frac{1}{2}\right] = -tF(t) \quad \left(\because \frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}\right)$$

$$\Rightarrow L^{-1}\left[\frac{-2}{4+(p+3)^2}\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{-2}{(p+3)^2+2^2}\right] = -tF(t)$$

$$\Rightarrow -e^{-3t}\sin 2t = -tF(t) \quad \text{(using first shifting property)}$$

$$\Rightarrow \boxed{F(t) = \frac{e^{-3t}\sin 2t}{t}} \text{ Ans.}$$

(iv) :→ let $L^{-1}\left[\log\left(\frac{p^2+4p+5}{p^2+2p+5}\right)\right] = F(t)$ (say)

$$\therefore L^{-1}\left[\frac{d}{dp}\left\{\log(p^2+4p+5) - \log(p^2+2p+5)\right\}\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{(2p+4)}{(p^2+4p+5)} - \frac{(2p+2)}{(p^2+2p+5)}\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{(2p+4)}{p^2+4p+4+1} - \frac{(2p+2)}{p^2+2p+1+4}\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{2p+4}{(p+2)^2+1} - \frac{2p+2}{(p+1)^2+4}\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{2(p+2)}{(p+2)^2+1} - \frac{2(p+1)}{(p+1)^2+2^2}\right] = -tF(t)$$

Division By p :-

If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\left\{\frac{f(p)}{p}\right\} = \int_0^t F(u)\,du$$

Also, $L^{-1}\left\{\frac{f(p)}{p^2}\right\} = \int_0^t \int_0^t F(u)\,du\,du$

$$\cdots \cdots \cdots$$

$$L^{-1}\left\{\frac{f(p)}{p^n}\right\} = \int_0^t \int_0^t \cdots \int_0^t F(u)\,du\,du \cdots du \quad (n\text{-times})$$

Question No: Find the inverse Laplace transform of

(i). $\frac{1}{p(p+1)^3}$ (ii) $\frac{1}{p\sqrt{p+4}}$ (iii) $\frac{1}{p(p^2+a^2)}$

Solution(i):- Since $L^{-1}\left[\frac{1}{(p+1)^3}\right] = \frac{e^{-t}t^2}{2!}$ (By first shifting property)

$$\therefore L^{-1}\left[\frac{1}{p(p+1)^3}\right] = \frac{1}{2}\int_0^t t^2 e^{-t}\,dt$$

$$= \frac{1}{2}\left[t^2(-e^{-t}) - 2te^{-t} + 2(-e^{-t})\right]_0^t$$

$$= \frac{1}{2}\left[(-t^2-2t-2)e^{-t}+2\right]$$

$$\boxed{L^{-1}\left[\frac{1}{p(p+1)^3}\right] = 1 - e^{-t}\left(1+t+\frac{t^2}{2}\right)}$$

Ans.

Convolution Theorem :- (2016, 2018)

If $L^{-1}\{f(p)\} = F(t)$ and $L^{-1}\{g(p)\} = G(t)$, then

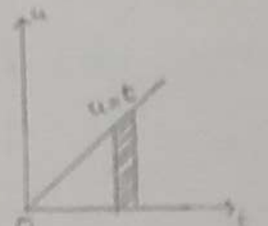
$$L^{-1}\{f(p)\,g(p)\} = F * G = \int_0^t F(u)\,G(t-u)\,du$$

Proof:- Let $\phi(t) = \int_0^t F(u)\,G(t-u)\,du$

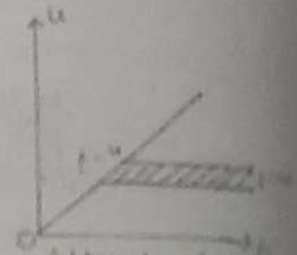
Then

$$L\{\phi(t)\} = \int_0^\infty e^{-pt}\left[\int_0^t F(u)\,G(t-u)\,du\right]dt$$

$$\Rightarrow L\,\phi(t) = \int_0^\infty \int_0^t e^{-pt} F(u)\,G(t-u)\,du\,dt$$



Before changing the order of integration



After changing the order of integration

we get, $L\,\phi(t) = \int_{u=0}^\infty \int_{t=u}^\infty e^{-pt} F(u)\,G(t-u)\,dt\,du$

$$= \int_0^\infty e^{-pu} F(u)\left[\int_u^\infty e^{-p(t-u)} G(t-u)\,dt\right]du$$

$$= \int_0^\infty e^{-pu} F(u)\left[\int_0^\infty e^{-pv} G(v)\,dv\right]du$$

on putting $t-u=v$

$$= \frac{1}{4}\int_0^t [\sin(u+t) + \sin(3u-t)]\,du$$

$$= \frac{1}{4}\left[-\cos(u+t) - \frac{\cos(3u-t)}{3}\right]_0^t$$

$$= \frac{1}{4}\left[\left(-\cos 2t - \frac{\cos 2t}{3}\right) - \left(-\cos t - \frac{\cos t}{3}\right)\right]$$

$$= \frac{1}{4}\left[-\frac{4}{3}\cos 2t + \frac{4}{3}\cos t\right]$$

$$\therefore L^{-1}\left\{\frac{p}{(p^2+1)(p^2+4)}\right\} = \frac{1}{3}(\cos t - \cos 2t) \quad \text{Ans.}$$

(ii): $\frac{p}{(p^2+4)^2} = \frac{1}{(p^2+4)} \cdot \frac{p}{(p^2+4)}$

Let $f(p) = \frac{1}{p^2+4}$ and $g(p) = \frac{p}{(p^2+4)}$

$\therefore F(t) = L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{p^2+4}\right\} = \frac{1}{2}\sin 2t$

and $G(t) = L^{-1}\{g(p)\} = L^{-1}\left\{\frac{p}{p^2+4}\right\} = \cos 2t$

Now $F(u) = \frac{1}{2}\sin 2u$, $G(t-u) = \cos 2(t-u)$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{p}{(p^2+4)^2}\right\} = \int_0^t \frac{1}{2}\sin 2u \cdot \cos 2(t-u)\,du$$

$$= \frac{1}{4}\int_0^t 2\sin 2u \cdot \cos 2(t-u)\,du$$

$$= \frac{1}{4}\int_0^t [\sin 2t + \sin(4u-2t)]\,du$$

Applications to Differential Equations

1. Solution of Ordinary differential equations with constant coefficients

The advantage of using Laplace transform method is that it yields the particular solution directly without the necessity of first finding the general solution and then evaluating the arbitrary constant.

Steps to solve Ode:

1. Take Laplace transform of both sides of the given differential equation, using initial conditions. This gives an algebraic equation.
2. Solve the algebraic equation to get y in terms of p.
3. Use inverse Laplace transform of both sides. This gives y as a function of t which is the desired solution.

$\boxed{\text{Remember:}}$ $L\{f^n(t)\} = p^n L\{f(t)\} - p^{n-1}f(0) - p^{n-2}f'(0) - \dots - p f^{(n-2)}(0) - f^{(n-1)}(0)$

Example: Using Laplace transform, find the solution of the initial value problem:

$$\frac{d^2y}{dt^2} + 9y = 6\cos 3t \,;\; y(0) = 2,\; y'(0) = 0 \qquad [2016]$$

Sol. The given differential equation is:

$$y'' + 9y = 6\cos 3t \qquad \text{——} (1)$$

Taking Laplace transform on both sides of eqn (1), we get

$$L(y'') + 9L(y) = 6L(\cos 3t)$$

$$\Rightarrow \{p^3-3p^2+3p-1\}\bar{y} - p^2+3p-1 = \frac{2}{(p-1)^3}$$

$$\Rightarrow [p-1]^3\bar{y} - p^2+3p-1 = \frac{2}{(p-1)^3}$$

$$\Rightarrow [p-1]^3\bar{y} = p^2-3p+1+\frac{2}{(p-1)^3}$$

$$\Rightarrow \bar{y} = \frac{(p-1)^2}{(p-1)^3} - \frac{p}{(p-1)^3} + \frac{2}{(p-1)^6}$$

$$= \frac{1}{p-1} - \frac{(p-1)+1}{(p-1)^3} + \frac{2}{(p-1)^6}$$

$$= \frac{1}{p-1} - \frac{1}{(p-1)^2} - \frac{1}{(p-1)^3} + \frac{2}{(p-1)^6} \quad \text{——} ②$$

Taking inverse Laplace transform on both sides of eqn ②, we get

$$y = e^t - te^t - \frac{t^2}{2}e^t + \frac{t^5 e^t}{60}$$

$$\therefore \boxed{y = \left(1-t-\frac{t^2}{2}+\frac{t^5}{60}\right)e^t.}$$

Examples 3 Solve by Laplace transform:

$$\frac{d^2y}{dt^2} + y = t\cos 2t,\ t>0$$

given that $y = \frac{dy}{dt} = 0$ for $t=0$. [2016]

Sol. The given equation is:

$$y'' + y = t\cos 2t \quad \text{——} ①$$

Taking Laplace transform on both sides of eqn ①, we get

$$L(y'') + 1L(y) = L(t\cos 2t)$$

2. Solution of simultaneous ordinary differential equations
Laplace transform technique can also be used in solving two or more simultaneous ordinary differential equations.

This process is illustrated as follows:

Examples 1 Solve the simultaneous equations:

$$\frac{dx}{dt} - y = e^t, \quad \frac{dy}{dt} + x = \sin t$$

given that $x(0) = 1$, $y(0) = 0$. [2010-2011]

Sol. Taking Laplace transform of the given equations, we get

$$[p\bar{x} - x(0)] - \bar{y} = \frac{1}{p-1} \qquad \text{where } \bar{x} = L(x),\ \bar{y} = L(y)$$

i.e., $p\bar{x} - 1 - \bar{y} = \frac{1}{p-1}$ $[\because x(0)=1]$

i.e., $p\bar{x} - \bar{y} = \frac{p}{p-1}$ ——①

and $[p\bar{y} - y(0)] + \bar{x} = \frac{1}{p^2+1}$

i.e., $\bar{x} + p\bar{y} = \frac{1}{p^2+1}$ ——② $[\because y(0)=0]$

Solving ① and ② for $\bar{x}$ and $\bar{y}$, we have

$$\bar{x} = \frac{p^2}{(p-1)(p^2+1)} + \frac{1}{(p^2+1)^2}$$

$$\bar{x} = \frac{1}{2}\left[\frac{1}{p-1} + \frac{p}{p^2+1} + \frac{1}{p^2+1}\right] + \frac{1}{(p^2+1)^2}$$

Examples 2 Solve the simultaneous equations:

$$\frac{d^2x}{dt^2} + 5\frac{dy}{dt} - x = t, \quad 2\frac{dx}{dt} - \frac{d^2y}{dt^2} + 4y = 2$$

given that when $t=0$, $x=0$, $y=0$, $\frac{dx}{dt}=0$, $\frac{dy}{dt}=0$. [2022]

Sol. Let $L\{x(t)\} = \bar{x}(p)$ and $L\{y(t)\} = \bar{y}(p)$

then, taking Laplace transform of given equations, we get

$$\{p^2\bar{x} - px(0) - x'(0)\} + 5\{p\bar{y} - y(0)\} - \bar{x} = \frac{1}{p^2}$$

and $2\{p\bar{x} - x(0)\} - \{p^2\bar{y} - py(0) - y'(0)\} + 4\bar{y} = \frac{2}{p}$

Using the given initial conditions, these equations reduce to

$$(p^2-1)\bar{x} + 5p\bar{y} = \frac{1}{p^2} \quad \text{——} (1)$$

and $2p\bar{x} - (p^2-4)\bar{y} = \frac{2}{p}$ —— (2)

Eliminating $\bar{y}$ between (1) and (2), we find that

$$\{(p^2-1)(p^2-4) + 10p^2\}\bar{x} = \frac{p^2-4}{p^2} + 10$$

$\therefore$ $\bar{x} = \frac{11p^2-4}{p^2(p^2+1)(p^2+4)}$

$$= \frac{-1}{p^2} + \frac{5}{p^2+1} - \frac{4}{p^2+4}$$

Taking inverse Laplace transform, we get

$$\boxed{x = -t + 5\sin t - 2\sin 2t} \quad \text{——} (3)$$

Examples 4 The co-ordinates (x,y) of a particle moving along a plane curve at any time t are given by $\frac{dy}{dt} + 2x = \sin 2t$, $\frac{dx}{dt} - 2y = \cos 2t$; $(t>0)$. It is given that at $t=0$, $x=1$ and $y=0$. Show using transforms that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$. [2017]

Sol. The given equations are

$$\frac{dy}{dt} + 2x = \sin 2t \quad \text{——} (1)$$

$$\frac{dx}{dt} - 2y = \cos 2t \quad \text{——} (2)$$

Above equation may be rewritten as

$$2x + Dy = \sin 2t$$

$$Dx - 2y = \cos 2t, \quad \text{where } D \equiv \frac{d}{dt}$$

Taking Laplace transform of eqn. (1) on both sides, we get

$$2\bar{x} + p\bar{y} - y(0) = \frac{2}{p^2+4}, \quad \text{where } \bar{x} = L(x),\ \bar{y} = L(y)$$

$$\Rightarrow 2\bar{x} + p\bar{y} = \frac{2}{p^2+4} \quad \text{——} (3) \quad [\because y(0)=0]$$

Again, taking Laplace transform of eqn. (2) on both sides, we get

$$p\bar{x} - x(0) - 2\bar{y} = \frac{p}{p^2+4}$$

$5y^2 = 5\sin^2 2t$

$4xy = 4\left[\left(\frac{1}{2}\right.\right.$ [illegible]

$= -($ [illegible] $)$

$\therefore\ 4x^2 + 5y^2 + 4xy =$ [illegible]

Hence, the result.

UNIT-3

Unit-3 Sequence and Series

Q-1 Define convergent, Divergent, Oscillatory sequence with example.

Sol A sequence is a function $f: N \to S$, whose domain is set of Natural Numbers N whereas the range may be any set S

Convergent Sequence → A sequence $\{a_n\}$ is said to be convergent if $\lim_{n\to\infty} a_n$ is finite

Ex. $\langle \frac{1}{2^n} \rangle = \{ \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4} \cdots \}$

$a_n = \frac{1}{2^n}$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0$ (finite)

$\Longrightarrow$ Convergent sequence

Divergent sequence → A sequence $\{a_n\}$ is said to be divergent if $\lim_{n\to\infty} a_n$ is not finite i.e. $\lim_{n\to\infty} a_n = \infty$ OR $-\infty$.

Ex. $\langle n^2 \rangle = \{1, 2^2, 3^2, 4^2 \cdots\}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty \Longrightarrow$ Divergent seq

Oscillatory Sequence If a seq $\{a_n\}$ neither converges to finite number nor diverges to ∞ or $-\infty$ it is called Oscillatory sequence.

Ex. $\langle (-1)^n \rangle = \{-1, 1, -1, 1 \cdots\}$

$$\left.\begin{aligned}\lim_{n\to\infty} a_{2n} &= \lim_{n\to\infty} (-1)^{2n} = 1 \\ \lim_{n\to\infty} a_{2n-1} &= \lim_{n\to\infty} (-1)^{2n-1} = -1\end{aligned}\right\} \text{limit finite but not unique.}$$

$\Rightarrow$ sequence does not exist

Thus $\lim_{n\to\infty} a_n$ does not exist $\Rightarrow$ sequence does not cgt. Hence Oscillatory sequence and Oscillates b/w -1 & 1

Q-2 (A) Discuss the convergence of following sequence

(a) $\langle a_n \rangle = \langle \frac{n+1}{n} \rangle$

(b) $\langle a_n \rangle = \langle \frac{n^2+1}{n+1} \rangle$

Sol (a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}}$

$= 1$ (finite)

$\Rightarrow$ seq is cgt

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+1}{n+1}$

$= \lim_{n\to\infty} \frac{n^{\cancel{2}}(1+\frac{1}{n^2})}{\cancel{n}(1+\frac{1}{n})} = \lim_{n\to\infty} \frac{n(1+\frac{1}{n^2})}{1+\frac{1}{n}}$

$= \infty$ (Infinite)

$\Rightarrow$ seq is dgt.

(B) Define Bounded sequence with example

Ans:— A sequence is said to be bounded if $\exists$ two real numbers k and K $(k \le K)$ such that $\boxed{k \le a_n \le K}$ $\forall\ n \in N$

OR $|a_n| \le M$ $\forall\ n \in N$, $M \in R$

Ex. $\langle \frac{1}{n} \rangle = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \dots\}$

$0 < \frac{1}{n} \le 1$ $\forall\ n \in N$

$\Rightarrow \langle \frac{1}{n} \rangle$ is bounded sequence

If sequence is not bounded then seq is called Unbounded sequence.

$\langle n \rangle = \{1, 2, 3, 4, 5 \dots\}$.

$1 \le n$ i.e $1 \le a_n$ $\forall\ n \in N$.

It is Unbounded sequence. (Unbounded above)

Q-3 (i) Test . Convergence of series

$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots$

Sol Convergence of series (If given Series is A.P OR G.P)

(1) Find S_n (Sum of first n terms)

(2) If $\lim_{n\to\infty} S_n = S$ (finite and Unique) then series convergent

(3) If $\lim_{n\to\infty} S_n = \infty / -\infty$ then Series is divergent

Note:-(1) G.P. is cgt if $|r|<1$, dgt if $r>1$

Here $S_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots$ n term.

$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$ $r<1$ | $\boxed{S_n = \frac{a(r^n-1)}{r-1}}$ $r>1$

Here $r = \frac{1}{3} < 1$

$S_n = \frac{1(1-\frac{1}{3^n})}{1-\frac{1}{3}} = \frac{3}{2}\left(1-\frac{1}{3^n}\right)$

Now $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{2}\left(1-\frac{1}{3^n}\right) = \frac{3}{2}$ (finite

$\Longrightarrow$ Series is cgt

p-Series

$\Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots + \frac{1}{n^p}$ is

cgt if $p>1$
dgt if $p \le 1$

Comparision Test If Σu_n and Σv_n be two p term series such that

$\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (finite, fixed, Non zero)

$\Rightarrow$ Test is applicable

Then both series will converges or diverge simultaneously.

If ΣV_n converge (by p-series) then $\Rightarrow \Sigma u_n$

If ΣV_n divergent (,) then $\Rightarrow \Sigma u_n$

ΣV_n is called Auxiliary series.

Q.3 (ii) Test the convergence of series

$$\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \cdots$$

Sol $\Sigma U_n = \frac{2n-1}{n(n+1)(n+2)}$ (Given series)

1, 3, 5 - - - -
N^{th} Term $= a+(n-1)d$
$= 1+(n-1)2 = 2n-1$

1, 2, 3 - - - -
N^{th} term $= 1+(n-1).1$
$= n$

2, 3, 4 - - - -
N^{th} term $= 2+(n-1)1$
$= n+1$

3. 4. 5 - - - -
N^{th} term $= 3+(n-1)$
$= n+2$

$\Sigma V_n = \frac{n}{n^3} = \frac{1}{n^2}$ [Aux. series]

$\left[\Sigma V_n = \Sigma \frac{1}{n^2} \text{ is cgt as } p=2>1\right]$

Comparision Test

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{2n-1}{n(n+1)(n+2)} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(2-\frac{1}{n}).\cancel{n^2}}{\cancel{n^3}(1+\frac{1}{n})(1+2/n)} = \frac{2}{1} \text{ (finite, fixed, Non zero)}$$

$\Rightarrow$ Test is applicable

By p-series, $\Sigma V_n = \Sigma \frac{1}{n^2}$ is cgt $(p>1) \Rightarrow \Sigma U_n$ is also convergent.

D'Alembert Ratio Test :-

If ΣU_n is positive term series such that

$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = l$ then
$\Rightarrow$ convergent if $l>1$
$\Rightarrow$ Divergent if $l<1$
$\Rightarrow$ Test fails if $l=1$

Q:-4 Test convergence of series $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots \frac{x^n}{n^2+1}$ by using D'Alembert Ratio Test.

Sol leaving first term,
$U_n = \frac{x^n}{n^2+1}$

$U_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$ $[n \to n+1]$

$$\frac{U_n}{U_{n+1}} = \frac{\cancel{x^n}}{n^2+1} \cdot \frac{(n+1)^2+1}{\cancel{x^{n+1}}} = \frac{\cancel{n^2}\left[(1+\frac{1}{n})^2 + \frac{1}{n^2}\right]}{\cancel{n^2}\left[1+\frac{1}{n^2}\right].x}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{\left[(1+\frac{1}{n})^2 + \frac{1}{n^2}\right]}{(1+\frac{1}{n^2})x} = \frac{1}{x}$$

If $\frac{1}{x} > 1$ then it is cgt i.e. $\boxed{x<1} \Rightarrow$ cgt

If $\frac{1}{x} < 1$ then it is dgt i.e $\boxed{x>1} \Rightarrow$ dgt

If $\frac{1}{x} = 1$ then Test fail i.e $(x=1) \Rightarrow$ Test fails

At $x=1$

$U_n = \frac{1}{n^2+1}$ $\quad V_n = \frac{1}{n^2}$ (Aux series).

Now By Comparision Test

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1}{n^2+1} \cdot n^2 = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}[1+\frac{1}{n^2}]} = 1$$

(fixed, finite, Non zero)

$\Rightarrow$ Test is applicable

By p-series, $\Sigma V_n = \Sigma \frac{1}{n^2}$ is cgt as $p=2>1$

$\Rightarrow \Sigma U_n$ is also cgt at $x=1$

finally, series is $\boxed{\text{cgt if } x \le 1}$, $\boxed{\text{dgt if } x>1}$

Raabe's Test

Raabe's Test is applied only when D'Alembert Ratio Test fails

i.e $\left(\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1 \text{ in D'Alembert Ratio Test} \right)$

If Σu_n is positive term series then

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = k.$$

$\Rightarrow$ series is convergent if $k > 1$

$\Rightarrow$ series is divergent if $k < 1$

$\Rightarrow$ Test fails if $k = 1$

Q→5 Test the convergence of series

$$1 + \frac{x}{2} + \frac{1\cdot3}{2\cdot4}x^2 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^3 + \cdots, \quad x>0$$

Sol Leaving first term.

$$u_n = \frac{x^n\,(1\cdot3\cdot5\cdots(2n-1))}{(2\cdot4\cdot6\cdots2n)}$$

$$u_{n+1} = \frac{1\cdot3\cdot5\cdots(2n-1)(2n+1)}{2\cdot4\cdot6\cdots2n\cdot(2n+2)}\,x^{n+1}.$$

1, 3, 5 - - -
$T_n = a+(n-1)d$
$= 1+(n-1)2$
$2n-1$

2, 4, 6 - - -
$T_n = 2+(n-1)2$
$= 2n$

Now $\frac{u_n}{u_{n+1}} = \frac{\cancel{(1\cdot3\cdot5\cdots2n-1)}}{\cancel{(2\cdot4\cdot6\cdots2n)}}\frac{x^{\cancel{n}}}{x^{\cancel{n}+1}}\frac{[\cancel{2\cdot4\cdot6\cdots2n}\,(2n+2)]}{[\cancel{1\cdot3\cdot5\cdots(2n-1)}\,(2n+1)]}$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)}{(2n+1)}\frac{1}{x} \quad \text{——— (A)}$$

Now By Ratio Test

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{2n+2}{2n+1}\frac{1}{x}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left[2+\frac{2}{n}\right]}{\cancel{n}\left[2+\frac{1}{n}\right]}\frac{1}{x} = \frac{2}{2}\frac{1}{x} = \frac{1}{x}$$

By Ratio Test Σu_n is

$\Rightarrow$ convergent if $\frac{1}{x} > 1$ i.e $x < 1$

$\Rightarrow$ Divergent if $\frac{1}{x} < 1$ i.e $x > 1$

$\Rightarrow$ Test fails if $\frac{1}{x} = 1$ i.e $\boxed{x=1}$

When $x = 1$

$$\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1} \quad [\text{from A}]$$

$$\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{2n+2}{2n+1} - 1 = \frac{\cancel{2n}+2-\cancel{2n}-1}{2n+1}$$

$$= \frac{1}{2n+1}$$

Now $n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n}{2n+1}$

$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \left(\frac{n}{2n+1}\right)$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(2+\frac{1}{n}\right)} = \frac{1}{2} < 1$$

By Raabe's Test,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{1}{2} < 1,$$ series is Divergent at $x = 1$.

Finally Σu_n is

$\Rightarrow$ Convergent when $x < 1$

$\Rightarrow$ Divergent when $x \geq 1$

FOURIER SERIES

Question 6 a) Define Dirichlet's conditions for a Fourier Series

b) Explain odd and Even function with Example

Solution a) Any function $f(x)$ can be expressed as a fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$

where a_0, a_n, b_n are constants, provided

i) $f(x)$ is periodic, single valued function.

ii) $f(x)$ has a finite number of finite discontinuities in any one period.

iii) $f(x)$ has a finite number of Maxima and Minima.

When these conditions are satisfied, the fourier series converges to $f(x)$ at every point of continuity. At a point of discontinuity the sum of series is equal to mean of limits on right and left i.e.

$$\frac{1}{2}\left[f(x+0) + f(x-0)\right]$$

b) **Even Function** A function $f(x)$ is said to be Even or symmetric function if, $f(-x) = f(x)$

for Example → $\cos x$, x^2 are Even functions.

Note For Even function $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

Odd function A function $f(x)$ is called odd or skew symmetric function if $f(-x) = -f(x)$

for Example → $\sin x$, x^3 are odd functions.

Note For odd function $f(x)$

$$\int_{-a}^{a} f(x)\,dx = 0$$

CHAIN RULE

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$$

where u and v are functions of x and dashes denote differentiation and suffixes denotes integration w.r. to x

#

$\sin 0 = 0$,	$\cos 0 = 1$
$\sin n\pi = 0$,	$\cos n\pi = (-1)^n$
$\sin 2n\pi = 0$,	$\cos 2n\pi = 1$

Question 7 → Find Fourier Series for the function $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 < x < 2\pi$

Solution → Fourier series for $f(x)$ is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n}$$

where,

$$\boxed{a_0 = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 dx}$$

$$a_0 = \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{3}(-1)\right]_0^{2\pi}$$

$$a_0 = \frac{-1}{12\pi}\left[-\pi^3 - \pi^3\right]$$

$$\boxed{a_0 = \frac{\pi^2}{6}}$$

$$\boxed{a_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^2 \cos nx\,dx}$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - 2(\pi-x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2(1)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\cdot\left[-\frac{2}{n^2}\left\{-\pi\cos 2n\pi - \pi\cos 0\right\}\right]$$

$$\left[\because \sin 2n\pi = 0,\quad \sin 0 = 0\right]$$

$$= \frac{1}{4\pi}\left[-\frac{2}{n^2}\left\{-2\pi\right\}\right]$$

$$\left[\because \cos 2n\pi = 1,\ \cos 0 = 1\right]$$

$$\boxed{a_n = \frac{1}{n^2}}$$

$$\boxed{b_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^2 \sin nx\,dx}$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left\{-\frac{\cos nx}{n}\right\} - 2(\pi-x)(-1)\left\{\frac{-\sin nx}{n^2}\right\} + 2(-1)(-1)\left\{-\frac{\cos nx}{n^3}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[-\frac{\pi^2}{n}\cos 2n\pi + \frac{\pi^2}{n}\cos 0 - 0 - \frac{2}{n^3}\left\{\cos 2n\pi - \cos 0\right\}\right]$$

$$\left[\because \cos 0 = \cos 2n\pi = 1\right]$$

$$\boxed{b_n = 0}$$

$\therefore$ Fourier Series is

$$\boxed{f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}}$$

Question 8 Expand the function $f(x) = x\sin x$ as a fourier Series in the interval $-\pi \le x \le \pi$. Hence deduce that $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots = \frac{\pi-2}{4}$

Solution

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\,dx$$

$$\boxed{a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx}$$

$$\left[\because x\sin x \text{ is an Even function}\right]$$

$$= \frac{2}{\pi}\left[x(-\cos x) - 1(-\sin x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}(\pi)$$

$$\boxed{a_0 = 2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos nx\,dx$$

$$\boxed{a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx}$$

$$\left[\because x\sin x\cos nx \text{ is an Even function}\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left\{\sin(n+1)x - \sin(n-1)x\right\}dx$$

$$\left[\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx - \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1}\right\}_0^{\pi} - (1)\left\{-\frac{\sin(n+1)x}{(n+1)^2}\right\}\right]_0^{\pi}$$

$$-\frac{1}{\pi}\left[x\left\{-\frac{\cos(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\sin(n+1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\pi\,\frac{(-1)^{n+1}}{n+1} + 0\right] - \frac{1}{\pi}\left[-\frac{\pi(-1)^{n-1}}{n-1} - 0\right]$$

$$= \frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}$$

$$= (-1)^{n+1}\left[-\frac{1}{n+1} + \frac{1}{n-1}\right] = \boxed{\frac{2(-1)^{n+1}}{n^2-1}}, \quad n \neq 1$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos 2x}{2}\right\} - (1)\left\{-\frac{\sin 2x}{4}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2}\right]$$

$$\boxed{a_1 = -\frac{1}{2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x \sin nx \, dx$$

$$b_n = 0 \qquad [\because x \sin x \sin nx \text{ is an odd function}]$$

$$\therefore \quad f(x) = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}\cos nx$$

or

$$x \sin x = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{(n-1)(n+1)}\cos nx$$

or

$$x \sin x = 1 + 2\left[-\frac{1}{4}\cos x - \frac{1}{1\cdot 3}\cos 2x + \frac{1}{2\cdot 4}\cos 3x - \frac{1}{3\cdot 5}\cos 4x \ldots\right] \quad (1)$$

putting $x = \pi/2$ in Equation (1), we get

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

$$\Rightarrow \quad \frac{\pi}{4} - \frac{1}{2} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

$$\Rightarrow \quad \frac{\pi - 2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

Change of Interval

Fourier series $f(x)$ in the interval $c < x < c+2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where,

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Question 9 Find fourier series for

(i) $f(x) = \begin{cases} 2, & -2 \le x \le 0 \\ x, & 0 < x < 2 \end{cases}$

In interval $(-2,2)$

<u>Solution</u>

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\,dx + \int_0^2 x\,dx\right]$$

$$= \frac{1}{2}\left[\{2x\}_{-2}^{0} + \left\{\frac{x^2}{2}\right\}_0^2\right]$$

$$\boxed{a_0 = 3}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\cos\frac{n\pi x}{2}\,dx + \int_0^2 x\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{4}{n\pi}\left\{\sin\frac{n\pi x}{2}\right\}_{-2}^{0} + \left\{x\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right\}_0^2\right]$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\right]$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \frac{-4}{n^2\pi^2}, \quad \text{when } n \text{ is odd}$$

$$= 0, \quad \text{when } n \text{ is Even}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \int_0^2 x\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[2\left\{-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right\}\right]_{-2}^{0}$$

$$+ \frac{1}{2}\left[x\left\{\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right\} + (1)\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2$$

$$b_n = \frac{1}{2}\left[-\frac{4}{n\pi} + \frac{4}{n\pi}\cos n\pi\right] + \frac{1}{2}\left[\frac{-4}{n\pi}\cos n\pi + \frac{4}{n^2\pi^2}\sin n\pi\right]$$

$$b_n = \frac{1}{2}\left[\frac{-4}{n\pi}\right] = \boxed{-\frac{2}{n\pi}}$$

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{l} + a_2\cos\frac{2\pi x}{l} + a_3\cos\frac{3\pi x}{l} + \cdots + b_1\sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l} + b_3\sin\frac{3\pi x}{l} + \cdots$$

$$f(x) = \frac{3}{2} - \frac{4}{\pi^2}\left\{\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \cdots\right\} - \frac{2}{\pi}\left\{\frac{1}{1}\sin\frac{\pi x}{2} + \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \cdots\right\}$$

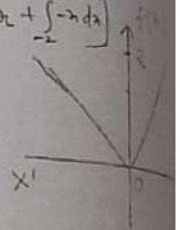
Question 9(ii) A periodic function of period 4 is defined as

$f(x) = |x|$, $-2 < x < 2$

Find its Fourier Series Exp

Solution $f(x) = |x|$, $-2 < x < 2$

$$\Rightarrow f(x) = \begin{cases} x, & 0 < x < 2 \\ -x, & -2 < x < 0 \end{cases}$$



$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{1}{2}\left[\int_0^2 x\,dx + \int_{-2}^0 -x\,dx\right]$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 + \frac{1}{2}\left[-\frac{x^2}{2}\right]_{-2}^0$$

$$= \frac{1}{4}(4-0) + \frac{1}{4}(0+4)$$

$$\boxed{a_0 = 2}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2}\left[\int_0^2 x\cos\frac{n\pi x}{2}\,dx + \int_{-2}^0 (-x)\cos\frac{n\pi x}{2}\right.$$

$$= \frac{1}{2}\left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\cos\right.\right.$$

$$+ \frac{1}{2}\left[(-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (-1)\left(\frac{-4}{n^2\pi^2}\right.\right.$$

$$= \frac{1}{2}\left[0 + \frac{4}{n^2\pi^2}(-1)^n - \frac{4}{n^2\pi^2}\right) + \frac{1}{2}\left[0 - \frac{4}{n^2\pi^2} + \frac{4}{n^2\pi^2}(-1)^n\right)$$

$$= \frac{1}{2}\cdot\frac{4}{n^2\pi^2}\left[(-1)^n - 1 - 1 + (-1)^n\right)$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right)$$

$$= \frac{-8}{n^2\pi^2}, \quad (\text{If } n \text{ is odd})$$

$$= 0, \quad (\text{If } n \text{ is Even})$$

$b_n = 0$ $\left[\because f(x) \text{ is Even function}\right]$

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{l} + a_2 \cos\frac{2\pi x}{l} + \dots + \dots b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} \dots$$

$$= 1 - \frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \dots\right]$$

Half Range Series

A function $f(x)$ defined over the interval $0 < x < l$ is capable of two distinct half range series

The half range cosine series is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}}$$

where, $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$;

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

The half range Sine Series is

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}}$$

where, $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

Question 10 Find half range Fourier Sine Series of $f(x)$ defined over the range $0 < x < 4$ as

$$f(x) = \begin{cases} x, & 0 < x < 2 \\ 4-x, & 2 < x < 4 \end{cases}$$

Solution Here, we have

$$f(x) = \begin{cases} x, & 0 < x < 2 \\ 4-x, & 2 < x < 4 \end{cases}$$

Now, $$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$\therefore \quad b_n = \frac{2}{4}\left[\int_0^2 x \sin\frac{n\pi x}{4}\,dx + \int_2^4 (4-x)\sin\frac{n\pi x}{4}\,dx\right]$$

$$[\because c = 4]$$

$$\Rightarrow \quad b_n = \frac{1}{2}\left[x\cdot\frac{4}{n\pi}\left\{-\cos\frac{n\pi x}{4}\right\} - (1)\frac{16}{n^2\pi^2}\left\{-\sin\frac{n\pi x}{4}\right\}\right]_0^2$$

$$+\frac{1}{2}\left[(4-x)\cdot\frac{4}{n\pi}\left\{-\cos\frac{n\pi x}{4}\right\} - (-1)\frac{16}{n^2\pi^2}\left\{-\sin\frac{n\pi x}{4}\right\}\right]_2^4$$

$$\Rightarrow \quad b_n = \frac{1}{2}\left[(2)\cdot\frac{4}{n\pi}\left\{-\cos\frac{2n\pi}{4}\right\} - \frac{16}{n^2\pi^2}\left\{-\sin\frac{2n\pi}{4}\right\}\right]$$

$$+\frac{1}{2}\left[0 - \frac{16}{n^2\pi^2}\sin\frac{4n\pi}{4} - (4-2)\frac{4}{n\pi}\left\{-\cos\frac{n\pi}{2}\right\} + \frac{16}{n^2\pi^2}\sin\frac{2n\pi}{4}\right]$$

$$\Rightarrow \quad b_n = -\frac{4}{n\pi}\cos\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{16}{2n^2\pi^2}n\pi + \frac{4}{n\pi}\cos\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\Rightarrow \quad b_n = \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\Rightarrow \quad b_1 = \frac{16}{\pi^2}, \quad b_2 = 0, \quad b_3 = -\frac{16}{3^2\pi^2},$$

$$b_4 = 0, \quad b_5 = \frac{-16}{5^2\pi^2} \cdots\cdots$$

$$\therefore \quad f(x) = b_1\sin\frac{\pi x}{4} + b_2\sin\frac{2\pi x}{4} + b_3\sin\frac{3\pi x}{4} + b_4\sin\frac{4\pi x}{4} + \cdots$$

$$\therefore \quad f(x) = \frac{16}{\pi^2}\left[\sin\frac{\pi x}{4} - \frac{1}{3^2}\sin\frac{3\pi x}{4} + \frac{1}{5^2}\sin\frac{5\pi x}{4} - \cdots\right]$$

UNIT-4

Unit-4

Ques1:- Q. Prove that $f(z) = \sinh z$ is analytic and find its derivative.

Soln:- Formula used:- function and all its partial derivatives are continuous and C-R eqns are satisfied

$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$, $\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

Here $f(z) = u + iv = \sinh z = \sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y$

$u = \sinh x \cos y$ and $v = \cosh x \sin y$

$\frac{\partial u}{\partial x} = \cosh x \cos y$, $\frac{\partial u}{\partial y} = -\sinh x \sin y$

$\frac{\partial v}{\partial x} = \sinh x \sin y$, $\frac{\partial v}{\partial y} = \cosh x \cos y$

$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ and $\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$, Thus C-R eqns are satisfied.

Since $\sinh x$, $\cosh x$, $\sin x$ and $\cos y$ are continuous functions, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are also continuous functions satisfying C-R eqns.

Hence $f(z)$ is an analytic functn.

Now, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \cosh x \cos y + i \sinh x \sin y$

$= \cosh(x+iy) = \cosh z$

$\Rightarrow \boxed{f'(z) = \cosh z}$

6. Show that $f(z) = \log z$ is analytic everywhere in the complex plane except at the origin and that its derivative is $\left(\frac{1}{z}\right)$.

Soln:- Here $f(z) = u + iv = \log z = \log(x+iy)$

Let $x = r\cos\theta$, $y = r\sin\theta$ so that $x+iy = r(\cos\theta + i\sin\theta)$

$\Rightarrow x+iy = re^{i\theta}$, where $\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$

$\log(x+iy) = \log(re^{i\theta}) = \log r + i\theta = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$

$\Rightarrow u = \frac{1}{2}\log(x^2+y^2)$, $v = \tan^{-1}(y/x)$

Now, $\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$, $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$

$\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}$, $\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$

$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ and $\boxed{\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}}$ except when $x^2+y^2 = 0$

i.e. when $x = 0$ and $y = 0$. Also, derivatives are continuous at the origin.

Hence $f(z)$ is analytic everywhere in the complex plane except at the origin.

Also, $z = x+iy \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f(z) = u + iv$

$\Rightarrow f'(z) = \frac{x - iy}{x^2+y^2} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z}$

$\Rightarrow \boxed{f'(z) = \frac{1}{z}}$

Ques 2:- Prove that the funct^n $f(z)$ defined by
$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$ and $f(0) = 0$ is continuous and the C-R eq^{n}s are satisfied at the origin yet $f'(0)$ does not exist.

Soln:- Formula used:- At origin

$\frac{\partial u}{\partial x} = \lim_{h\to 0}\left\{\frac{u(h,0) - u(0,0)}{h}\right\}$ | $\frac{\partial v}{\partial x} = \lim_{h\to 0}\left\{\frac{v(h,0) - v(0,0)}{h}\right\}$

$\frac{\partial u}{\partial y} = \lim_{k\to 0}\left\{\frac{u(0,k) - u(0,0)}{k}\right\}$ | $\frac{\partial v}{\partial y} = \lim_{k\to 0}\left\{\frac{v(0,k) - v(0,0)}{k}\right\}$

$f'(0) = \lim_{z\to 0}\left\{\frac{f(z) - f(0)}{z}\right\}$ — (1), $f'(0)$ exists if limit (1) exists uniquely along any path.

$\Rightarrow$ Here, $f(z) = \frac{(x^3 - y^3) + i(x^3+y^3)}{(x^2+y^2)}$, $z \neq 0$ and $f(0) = 0$

Let $f(z) = u + iv = \frac{x^3 - y^3}{x^2+y^2} + i\frac{(x^3+y^3)}{x^2+y^2}$,

$u = \frac{x^3 - y^3}{x^2+y^2}$, $v = \frac{x^3+y^3}{x^2+y^2}$; $\because z \neq 0 \Rightarrow x \neq 0, y \neq 0$

$\therefore$ u and v are rational funct^{n}s of x and y with non-zero denominators. Thus u, v and hence $f(z)$ is continuous when $\boxed{z \neq 0}$. On changing u, v to polar co-ordinates by putting $x = r\cos\theta$, $y = r\sin\theta$, we get $u = r(\cos^3\theta - \sin^3\theta)$ and $v = r(\cos^3\theta + \sin^3\theta)$
when $z \to 0$, $r \to 0$ $\therefore$ limit $\lim_{z\to 0} u = \lim_{r\to 0} r(\cos^3\theta - \sin^3\theta)$
Similarly, $\lim_{z\to 0} v = 0$, $\therefore$ $\lim_{z\to 0} f(z) = 0 = f(0)$ Thus $f(z)$ is continuous

At the origin (0, 0), we have

$\frac{\partial u}{\partial x} = \lim_{h\to 0}\left(\frac{h-0}{h}\right) = 1$, $\frac{\partial u}{\partial y} = \lim_{k\to 0}\left(\frac{-k-0}{k}\right) = -1$

$\frac{\partial v}{\partial x} = \lim_{h\to 0}\left(\frac{h-0}{h}\right) = 1$, $\frac{\partial v}{\partial y} = \lim_{k\to 0}\left(\frac{k-0}{k}\right) = 1$

$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ and $\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

Hence C-R eqn's are satisfied at the origin.

Now, $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0}\left\{\frac{(x^3-y^3)+i(x^3+y^3)-0}{(x^2+y^2)(x+iy)}\right\}$

Let $z\to 0$ along the line $y=x$, then

$$f'(0) = \lim_{x\to 0}\left\{\frac{0+2ix^3}{2x^3(1+i)}\right\} = \frac{i}{1+i} = \frac{i(1-i)}{2} = \frac{1+i}{2} \quad (1)$$

Also, let $z\to 0$ along the x-axis (i.e. $y=0$), then

$$f'(0) = \lim_{x\to 0} \frac{x^3+ix^3}{x^3} = 1+i \quad (2).$$

Since the limits by (1) and (2) are different, thus $f'(0)$ does not exist.

Hence Proved

Ques 3:- Prove that $u = x^2-y^2$ and $v = \frac{y}{x^2+y^2}$ are harmonic funct^n of x, y, but are not harmonic conjugates.

Soln:- Formula used:- A funct^n possesses continuous partial derivatives of the first and second orders and satisfies Laplace eq^n. $\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}}$

Here $u = x^2-y^2$ and $v = \frac{y}{x^2+y^2}$

$\Rightarrow \frac{\partial u}{\partial x} = 2x \Rightarrow \frac{\partial^2 u}{\partial x^2} = 2$ | $\frac{\partial v}{\partial x} = \frac{-2xy}{(x^2+y^2)^2} \Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{6x^2y-2y^3}{(x^2+y^2)^3}$

$\frac{\partial u}{\partial y} = -2y \Rightarrow \frac{\partial^2 u}{\partial y^2} = -2$ | $\frac{\partial v}{\partial y} = \frac{(x^2-y^2)}{(x^2+y^2)^2} \Rightarrow \frac{\partial^2 v}{\partial y^2} = \frac{-6x^2y+2y^3}{(x^2+y^2)^3}$

Now, $\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$ and $\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$

$\Rightarrow$ u and v have continuous partial derivatives of the first and second orders respectively and satisfies Laplace's eq^n.

Thus, both u and v both are harmonic functions.

But $\boxed{\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}}$ and $\boxed{\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}}$.

Hence, u and v are not harmonic conjugates.
(As, they both are not analytic).

Ques 4:- Determine an analytic funct^n $f(z)$ in terms of z whose real part is $e^{-x}(x\sin y - y\cos y)$.

Soln:- Formula used:- when only u is given:-

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz + c$$

where $\phi_1(x,y) = \frac{\partial u}{\partial x}$ and $\phi_2(x,y) = \frac{\partial u}{\partial y}$, $\phi_1(z,0) = [\phi_1(x,y)]_{\substack{at\ x=z \\ and\ y=0}}$

$\phi_2(z,0) = [\phi_2(x,y)]_{\substack{at\ x=z \\ and\ y=0}}$

we have, $u = e^{-x}(x\sin y - y\cos y)$

$$\frac{\partial u}{\partial x} = e^{-x}\sin y - e^{-x}(x\sin y - y\cos y) = \phi_1(x,y) \text{ say}$$

$$\frac{\partial u}{\partial y} = e^{-x}(x\cos y - \cos y + y\sin y) = \phi_2(x,y) \text{ say}$$

$\therefore$ $\phi_1(z,0) = 0$ and $\phi_2(z,0) = e^{-z}(z-1)$

By Milne's Thomson method,

$$f(z) = \int \{\phi_1(z,0) - i\phi_2(z,0)\}\,dz + c$$

$$= -i\int e^{-z}(z-1)\,dz + c$$

$$= -i\left[(z-1)(-e^{-z}) - \int (-e^{-z})\,dz\right] + c$$

$$= -i\left[(1-z)e^{-z} - e^{-z}\right] + c$$

$\boxed{f(z) = ize^{-z} + c}$, where c is constant.

Ques 5:- Find an analytic funct^n whose imaginary part is $e^{-x}(x\cos y + y\sin y)$.

Soln:- Formula used:- When only v is given:-

$$f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)]\,dz + c, \text{ where}$$

$\psi_1(x,y) = \frac{\partial v}{\partial y}$ and $\psi_2(x,y) = \frac{\partial v}{\partial x}$,

$\psi_1(z,0) = [\psi_1(x,y)]_{\substack{at\ x=z \\ and\ y=0}}$ and $\psi_2(z,0) = [\psi_2(x,y)]_{\substack{at\ x=z \\ and\ y=0}}$

$\Rightarrow$ Here, $v = e^{-x}(x\cos y + y\sin y)$

$$\frac{\partial v}{\partial y} = e^{-x}(-x\sin y + y\cos y + \sin y) = \psi_1(x,y) \text{ say}$$

$$\frac{\partial v}{\partial x} = e^{-x}\cos y - e^{-x}(x\cos y + y\sin y) = \psi_2(x,y) \text{ say}$$

$\therefore$ $\psi_1(z,0) = 0$ and $\psi_2(z,0) = e^{-z} - e^{-z}z = (1-z)e^{-z}$

By Milne's Thomson Method

$$f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)]\,dz + c$$

$$= i\int (1-z)e^{-z}\,dz + c$$

$$= i\left[(1-z)(-e^{-z}) - \int (-1)(-e^{-z})\,dz\right] + c$$

$$= i\left[(z-1)e^{-z} + e^{-z}\right] + c$$

$\boxed{f(z) = ize^{-z} + c}$

Case III When $u-v$ is given

1. $f(z) = u+iv$; $i f(z) = iu - v$

2. adding both $\underbrace{(1+i) f(z)}_{F(z)} = \underbrace{u-v}_{U} + i \underbrace{(u+v)}_{V}$

   $F(z) = U + iV$

3. Find $\frac{\partial}{\partial x}(u-v) = \frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$

4. Put $\frac{\partial U}{\partial x} = \phi_1(x,y)$, $\frac{\partial V}{\partial y} = \phi_2(x,y)$

5. Find $\phi_1(z,0)$ and $\phi_2(z,0)$ by putting $x=z, y=0$

4. Find $F(z) = \int [\phi_1(z,0) - i\phi_2(z,0)] dz + c$

5. Find $f(z) = \frac{F(z)}{1+i}$ in terms of z.

Ques 6 If $u-v = (x-y)(x^2+4xy+y^2)$ and $f(z) = u+iv$ is an analytic function of $z = x+iy$, find $f(z)$ in terms of z.

Soln 6: We have $F(z) = U+iV$ where

$F(z) = (1+i) f(z)$, $U = u-v$, $V = u+v$

$U = u-v$ is given

$U = (x-y)(x^2+4xy+y^2)$

$$\frac{\partial U}{\partial x} = 1\cdot(x^2+4xy+y^2) + (x-y)(2x+4y)$$

$$= 3x^2+6xy-3y^2 = \phi_1(x,y)$$

$$\frac{\partial V}{\partial y} = -(x^2+4xy+y^2) + (x-y)(4x+2y)$$

$$= 3x^2 - 6xy - 3y^2 = \phi_2(x,y)$$

$\phi_1(z,0) = 3z^2$, $\phi_2(z,0) = 3z^2$

B. Tech I Year [Subject Name: Engineering Mathematics-II]

By Milne's Thomson method.

$$F(z) = \int [\phi_1(z,0) - i\phi_2(z,0)] dz + c$$

$$= \int [3z^2 - i(3z^2)] dz + c$$

$F(z) = (1-i) z^3 + c$

$(1+i) f(z) = (1-i) z^3 + c$

$f(z) = \left(\frac{1-i}{1+i}\right) z^3 + \frac{c}{1+i} = \left(\frac{-2i}{2}\right) z^3 + c_1$

$f(z) = -iz^3 + c_1$ $\left(\text{where } c_1 = \frac{c}{1+i}\right)$

Ques 7. If $u+v = \frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$ and $f(z) = u+iv$ is an analytic function of $z = x+iy$, find $f(z)$ in terms of z.

Soln 7 $F(z) = U+iV$ where $F(z) = (1+i) f(z)$

$U = u-v$, $V = u+v$

$V = u+v = \frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$ is given $= \frac{\sin 2x}{\cosh 2y - \cos 2x}$

as $e^{2y}+e^{-2y} = 2\cosh 2y$

$$\frac{\partial V}{\partial y} = \frac{-2\sin 2x \sinh y}{\cosh 2y - \cos 2x} = \psi_1(x,y)$$

$$\frac{\partial V}{\partial x} = \frac{2\cos 2x\cdot(\cosh 2y - \cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial V}{\partial x} = \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} = \psi_2(x,y)$$

$\psi_1(z,0) = 0$, $\psi_2(z,0) = \frac{2(\cos 2z - 1)}{(1-\cos 2z)^2} = \frac{-2}{1-\cos 2z} =$

$$= \frac{-2}{1-1+2\sin^2 x} = -\operatorname{cosec}^2 z$$

By Michael's Thomson method we have

$$F(z) = \int \phi_1(z,0) + i\,\phi_2(z,0)\,dz + c$$

$$= \int -i\,\operatorname{cosec}^2 z\,dz + c = i\cot z + c$$

Replacing $F(z)$ by $(1+i)\,f(z)$

$$(1+i)\,f(z) = i\cot z + c$$

$$f(z) = \frac{i}{1+i}\cot z + \frac{c}{1+i}$$

$$f(z) = \frac{1}{2}(1+i)\cot z + c_1 \qquad c_1 = \frac{c}{1+i}$$

Case IV when $u+v$ is given

1) $f(z) = u+iv$, $if(z) = ui - v$

2) $\underbrace{(1+i)f(z)}_{F(z)} = \underbrace{u-v}_{U} + i\underbrace{(u+v)}_{V}$

3) Find $\frac{\partial V}{\partial y}$, $\frac{\partial V}{\partial x}$

4) Put $\frac{\partial V}{\partial y} = \phi_1(x,y)$

$\frac{\partial V}{\partial x} = \phi_2(x,y)$

5) Find $\phi_1(z,0)$ & $\phi_2(z,0)$ by replacing $x \Rightarrow z$, $y = 0$

6) Find $F(z) = \int [\phi_1(z,0) + i\,\phi_2(z,0)]\,dz + c$

7) Find $f(z) = \frac{F(z)}{1+i}$ in terms of z

Conformal Mapping

A transformation which preserves angles both in magnitude and sense between every pair of curves through a point is said to be conformal at the p

Some transformation

1) Translation $\quad w = z + \beta \qquad (\beta = a+ib)$

2) Rotation $\quad w = z e^{i\gamma} \qquad (\gamma = \text{real})$

3) Magnification. $\quad w = cz \qquad (c \text{ real})$

4) Inversion (Reflection) $\quad w = \frac{1}{z}$

To solve put $w = u+iv$, $z = x+iy$ in given transform then seperate u and v to get transformation relation in u & v and x & y. Then draw figures to find image.

Ques 8 Find image of the infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $w = \frac{1}{z}$. Also show the regions graphically.

Soln $w = \frac{1}{z} \Rightarrow z = \frac{1}{w} \qquad \begin{pmatrix} z = x+iy \\ w = u+iv \end{pmatrix}$

$$x+iy = \frac{1}{u+iv}$$

$$x+iy = \frac{u-iv}{u^2+v^2} \qquad \text{(Rationalizing)}$$

seperating Real & imaginary part.

$$\Rightarrow x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

z plane	transformation	w plane
Now $y < 1/2$ $y > 1/4$	$x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$	$\frac{-v}{u^2+v^2} < \frac{1}{2}$ $\frac{-v}{u^2+v^2} > \frac{1}{4}$

$\frac{-v}{u^2+v^2} < \frac{1}{2}$

$\Rightarrow -2v < u^2+v^2 \Rightarrow u^2+v^2+2v+1 \geq 1$

$u^2+(v+1)^2 \geq 1$.

$\frac{-v}{u^2+v^2} > \frac{1}{4}$

$-4v > u^2+v^2 \Rightarrow u^2+v^2+4v+4 < 4$

Finally $\frac{1}{4} < y < \frac{1}{2} \Rightarrow u^2+(v+2)^2 < 4$ and

$u^2+(v+1)^2 \geq 1$

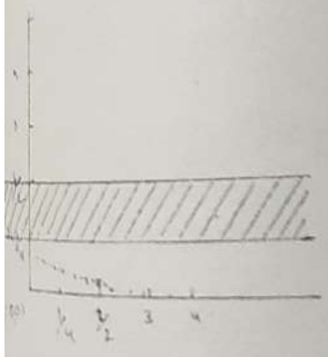
This shows that Region R' in w plane is bounded by two circles

$u^2+(v+2)^2 = 4$ Centre $(0,-2)$ Radius $= 2$

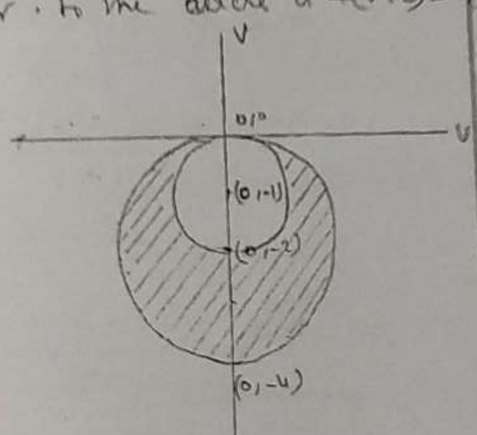
$[(x-h)^2+(y-k)^2 = r^2]$

$u^2+(v+1)^2 = 1$ centre $(0,-1)$ Radius $= 1$

such that the region is exterior to circle $u^2+(v+1)^2 = 1$ and interior to the circle $u^2+(v+2)^2 = 4$



z plane



w plane

Bilinear Transformation:
A transformation of the form $w = \frac{az+b}{cz+d}$ where a, b, c, d are complex constants and $ad - bc \neq 0$ is called linear transformation.

Fixed points: Points which coincide with their transformation are called fixed points i.e. $z = f(z)$

Method to find fixed points
we have $w = \frac{az+b}{cz+d}$

Put $\boxed{w = z}$ $z = \frac{az+b}{cz+d}$ —(1)

Solve 1 to get values of z.

Cross ratio:
If there are four points w, w_1, w_2, w_3 in w plane and z, z_1, z_2, z_3 in z plane ~~then~~ maps by some Bilinear transformation then cross ratio is given by.

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Ques 9 Find the bilinear transformation which maps the points $z = 1, -i, -1$ to the points $w = i, 0, -i$ respectively. ~~show also that transformation maps the region outside the circle $|z| = 1$ into the half-plane $R(w) \geq 0$.~~

Soln 9. The bilinear transformation mapping $z = 1, -i, -1$ into $w = i, 0, -i$ resp.

cross ratio is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-i)(0+i)}{(w+1)(-1)} = -\frac{(z-1)(1-i)}{(z+1)(i+1)}$$

$$\frac{w-i}{w+i} = \frac{(z-1)(1-i)}{(z+1)(i+1)} = \frac{z-zi-1+i}{zi+z+i+1}$$

$$\frac{w-i}{w+i} = \frac{(i-1)z+i(1-i)}{(i+1)z+(1+i)}$$

Applying componendo & Dividendo Rule $\frac{a}{b}=\frac{c}{d} \Rightarrow$

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

$$\frac{w-\cancel{i}+w+\cancel{i}}{w-i-w-i} = \frac{z-\cancel{zi}-\cancel{1}+i+\cancel{zi}+z+i+\cancel{1}}{\cancel{z}-zi-1+\cancel{i}-zi-\cancel{z}-\cancel{i}-1}$$

$$\frac{\cancel{2}w}{-\cancel{2}i} = \frac{\cancel{2}(z+i)}{-\cancel{2}(zi+1)}$$

$$\frac{w}{i} = \frac{z+i}{zi+1} \Rightarrow w = i\left(\frac{z+i}{zi+1}\right) = \frac{iz-1}{iz+1}$$

which is required transformation $\boxed{w = \frac{iz-1}{iz+1}}$

Ques 10 (a) Find the fixed points of the Bilinear transformation $w = \frac{3z-4}{z-1}$

(b) Find the bilinear transformation which maps the points $z = 0, -1, i$ onto $w = i, 0, \infty$.

Soln 10 (a) For fixed pt $w = z \Rightarrow z = \frac{3z-4}{z-1}$

$z^2 - z = 3z-4 \Rightarrow z^2 - z - 3z + 4 = 0$.

$z^2 - 4z + 4 = 0 \Rightarrow (z-2)^2 = 0$

$z = 2$ only one fixed pt This transformation is parabolic

(b) cross ratio is

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{(w_1-w_2)\left(1-\frac{w}{w_3}\right)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \quad (\because w_3 = \infty)$$

$$\frac{(w-i)(-1)}{(i+0)(i-0)} = \frac{(z-0)(-1-i)}{(0+1)(i-z)}$$

$$\frac{(w-i)(-1)}{i} = -\frac{(z)(1+i)}{i-z}$$

$$w-i = \frac{z(1+i)i}{i-z}$$

$$w-i = \frac{zi+zi^2}{i-z} + i$$

$$w = \frac{\cancel{zi}-z+i^2-\cancel{zi}}{i-z} = \frac{-z-1}{i-z}$$

$$\Rightarrow w = \frac{-(z+1)}{i-z}$$

$$\boxed{w = \frac{z+1}{z-i}}$$

which is required bilinear transformation

UNIT-5

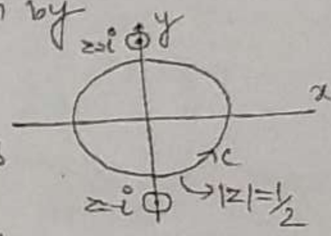
UNIT-5

Ques.(1) ⇒ State Cauchy's integral theorem & hence evaluate $\int_{|z|=\frac{1}{2}} \frac{e^z}{z^2+1}\,dz$

Sol^n. Cauchy's integral theorem ⇒ If $f(z)$ is an analytic function & $f'(z)$ is continuous at each point within & on a simple closed curve C then $\boxed{\oint_C f(z)\,dz = 0}$

Now given integral is $I = \int_{|z|=\frac{1}{2}} \frac{e^z}{z^2+1}\,dz$

singular points (poles) are given by



$z^2+1=0 \Rightarrow z=\pm i$

$z=i \Rightarrow (0,1)$ } both are
$z=-i \Rightarrow (0,-1)$ } simple poles

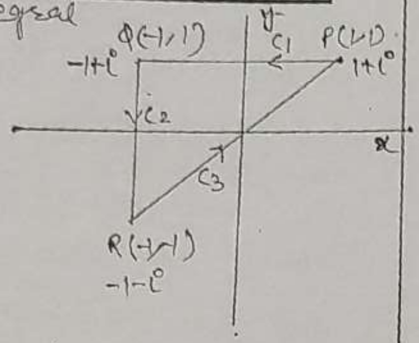
$|z|=\frac{1}{2}$ represents a circle with centre $(0,0)$ & radius $\frac{1}{2}$

clearly both the singular points $z=\pm i$ lie outside the circle $|z|=\frac{1}{2}$, Hence function is analytic everywhere inside c so by the Cauchy's integral theorem

$$\oint_{|z|=\frac{1}{2}} \frac{e^z}{z^2+1}\,dz = 0$$

Ques 2 ⇒ Verify Cauchy's theorem by integrating $f(z)=e^{iz}$ along the boundary of the triangle with vertices at the points $1+i, -1+i, -1-i$

Solⁿ ⇒ The boundary of the integral C, consists of three lines C_1, C_2 & C_3 so



$$\oint_C e^{iz}dz = \int_{C_1} e^{iz}dz + \int_{C_2} e^{iz}dz + \int_{C_3} e^{iz}dz \quad \text{—(1)}$$

Now $\int_{C_1} e^{iz}dz = \int_{1+i}^{-1+i} e^{iz}dz$

$$= \left(\frac{e^{iz}}{i}\right)_{1+i}^{-1+i} = \frac{1}{i}\left[e^{i(-1+i)} - e^{i(1+i)}\right]$$

$$= \frac{1}{i}\left[e^{-i-1} - e^{i-1}\right] \quad \text{—(2)}$$

$$\int_{C_2} e^{iz}dz = \int_{-1+i}^{-1-i} e^{iz}dz = \left(\frac{e^{iz}}{i}\right)_{-1+i}^{-1-i}$$

$$= \frac{1}{i}\left[e^{i(-1-i)} - e^{i(-1+i)}\right] = \frac{1}{i}\left[e^{-i+1} - e^{-i-1}\right] \quad \text{—(3)}$$

$$\int_{C_3} e^{iz}dz = \int_{-1-i}^{1+i} e^{iz}dz = \left(\frac{e^{iz}}{i}\right)_{-1-i}^{1+i}$$

$$= \frac{1}{i}\left[e^{i(1+i)} - e^{i(-1-i)}\right] = \frac{1}{i}\left[e^{i-1} - e^{-i+1}\right] \quad \text{—(4)}$$

putting values of (2), (3), (4) in (1)

$$\oint_C f(z)dz = \oint_C e^{iz}dz = 0 \quad \text{—(5)}$$

Now e^{iz} is analytic inside the triangle C

⇒ By Cauchy's integral thᵐ

$$\oint_C e^{iz}dz = 0 \quad \text{—(6)}$$

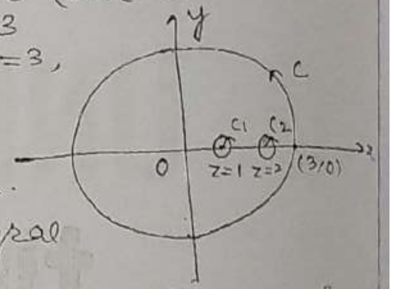
(5) & (6) gives the verification of Cauchy's Integral theorem.

Ques(3) ⇒ State Cauchy's integral formula. Hence evaluate $\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}$, where c is $|z|=3$

Solⁿ ⇒ Cauchy's integral formula ⇒ If $f(z)$ is analytic with in & on a simple closed curve c & a is any point with in c, then $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)}dz$

Given integral is, $I = \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}dz$

where c is the circle $|z|=3$ with centre $(0,0)$ & radius $=3$,



Poles are given by

$(z-1)(z-2)=0$

⇒ $z = 1$ & 2 are simple poles.

Now By Cauchy's integral formula

$$\oint_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz = \oint_{C_1} \frac{\left[\frac{\sin\pi z^2+\cos\pi z^2}{(z-2)}\right]}{(z-1)}dz + \oint_{C_2} \frac{\left[\frac{\sin\pi z^2+\cos\pi z^2}{z-1}\right]}{(z-2)}dz$$

$$= 2\pi i\left[\frac{\sin\pi z^2+\cos\pi z^2}{z-2}\right]_{at\, z=1} + 2\pi i\left[\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)}\right]_{z=2}$$

$$= 2\pi i\left[\frac{\sin\pi + \cos\pi}{1-2}\right] + 2\pi i\left[\frac{\sin 4\pi + \cos 4\pi}{2-1}\right]$$

$$= \frac{2\pi i}{(-1)}[0-1] + \frac{2\pi i}{1}[0+1]$$

$$= +2\pi i + 2\pi i = 4\pi i \quad \text{Ans.}$$

Ques ④ ⇒ Evaluate by Cauchy's integral formula

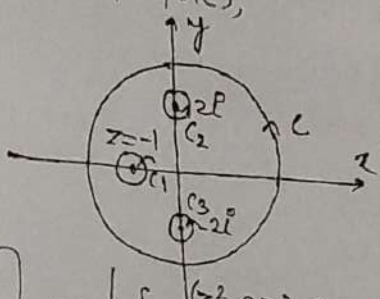
$\oint_c \frac{z^2-2z}{(z+1)^2(z^2+4)} dz$, where C is the circle $|z|=3$.

Sol^n $|z|=3$ ⇒ circle with centre (0,0) & Radius = 3

Poles are $(z+1)^2(z^2+4)=0$ ⇒ $z=\pm 2i$ (simple pole),

$z=-1$ (Pole of order 2)

All poles lie inside C.



$$\oint_c \frac{z^2-2z}{(z+1)^2(z^2+4)} dz = \oint_c \frac{(z^2-2z)}{(z+1)^2(z+2i)(z-2i)} dz$$

$$= \oint_{C_1} \frac{\left(\frac{z^2-2z}{z^2+4}\right)}{(z+1)^2} dz + \oint_{C_2} \left[\frac{\frac{z^2-2z}{(z+1)^2(z+2i)}}{(z-2i)}\right] dz + \oint_{C_3} \left[\frac{\frac{(z^2-2z)}{(z+1)^2(z-2i)}}{(z+2i)}\right] dz$$

Cauchy integral formula for derivatives:→

If $z=a$ is a pole of order $(n+1)$

$$\oint_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{(n)!}\left[\frac{d^n}{dz^n} f(z)\right]_{z=a}$$

for Pole of order (2)

$$\oint_c \frac{f(z)}{(z-a)^2} dz = \frac{2\pi i}{(1)!}\left[\frac{d}{dz} f(z)\right]_{z=a}$$

$$= \frac{2\pi i}{(1)!}\left[\frac{d}{dz}\left(\frac{z^2-2z}{z^2+4}\right)\right]_{z=-1} + 2\pi i\left[\frac{z^2-2z}{(z+1)^2(z+2i)}\right]_{z=2i}$$

$$+ 2\pi i\left[\frac{z^2-2z}{(z+1)^2(z-2i)}\right]_{z=-2i}$$

$$= 2\pi i\left[\frac{(z^2+4)(2z-2)-(z^2-2z)2z}{(z^2+4)^2}\right]_{z=-1} + \frac{2\pi i(-4)(1+i)}{(-4+1+4i)4i}$$

$$+ \frac{2\pi i(-4)(1-i)}{(-4+1-4i)(-4i)}$$

$$= 2\pi i\left(\frac{-14}{25}\right) - \frac{2\pi i(1+i)}{(-3i-4)} + \frac{2\pi i(1-i)}{(-3i+4)}$$

$$= \frac{-28}{25}\pi i + \frac{2\pi i(1+i)}{(-4+3i)} + \frac{2\pi i(1-i)}{(4-3i)}$$

$$= 2\pi i\left[-\frac{14}{25} + \frac{1+i}{4+3i} + \frac{1-i}{4-3i}\right] = 2\pi i\left[-\frac{14}{25} + \frac{14}{25}\right]$$

$= 0$ Ans

Ques ⑤ ⇒ Expand $\frac{1}{(z-1)(z-2)}$ in the regions

ⓐ $|z|<1$ ⓑ $1<|z|<2$ ⓒ $|z|>2$ ⓓ $0<|z-1|<1$

Sol^n ⇒ To expand $f(z)$ we will use Binomial theorem

$$\left.\begin{aligned}(1+z)^{-1} &= 1-z+z^2-z^3+z^4-\dots\infty = \sum_{n=0}^{\infty}(-1)^n z^n \\ (1-z)^{-1} &= 1+z+z^2+z^3+z^4\dots\infty = \sum_{n=0}^{\infty} z^n\end{aligned}\right\} \text{where } |z|<1$$

(1) If all the power are +ive in expansion then there is Taylor's series.

(2) If atleast one negative power of z is present in expansion then there is Laurent's series

(Basic)

Here $f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z^2-3z+2} = \frac{1}{z-2} - \frac{1}{z-1}$ (By partial fractions)

ⓐ When $|z|<1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{-2\left(1-\frac{z}{2}\right)} + \frac{1}{1-z} = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n$$

All the powers of z are +ive so it is the expansion of $f(z)$ in Taylor's series.

(b) when $1<|z|<2$ i.e $|z|>1$ & $|z|<2$

$$f(z)=\frac{1}{z-2}-\frac{1}{z-1} = \frac{-1}{2}\frac{1}{(1-\frac{z}{2})}-\frac{1}{z(1-\frac{1}{z})}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

This is a series in +ive & -ive powers of z so it is Laurent's series.

(c) when $|z|>2$

$$f(z)=\frac{1}{z-2}-\frac{1}{z-1} = \frac{1}{z(1-\frac{2}{z})}-\frac{1}{z(1-\frac{1}{z})}$$

$$= \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}-\frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

This series contains negative power terms of z ⇒ series is Laurent's series.

(d) when $0<|z-1|<1$

$$f(z)=\frac{1}{z-2}-\frac{1}{z-1} = \frac{1}{(z-1)-1}-\frac{1}{(z-1)}$$

$$= -\frac{1}{1-(z-1)}-\frac{1}{(z-1)} = -\left[1-(z-1)\right]^{-1}-\frac{1}{z-1}$$

$$= -\sum_{n=0}^{\infty}(z-1)^n-\frac{1}{z-1}$$

This series is Laurent's series since it contains -ive power terms of z.

Q.6 Define singular point of an analytic function. Find nature and location of the singularity of

$$f(z) = \frac{z-\sin z}{z^2}$$

Sol. Singularity:- A singularity of a function $f(z)$ is a point at which function ceases to be analytic. i.e. those points where $f(z)$ is not analytic.

Types of Singularity:- In Laurent's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

second term is called the Principal Part (P.P.) of $f(z)$.

Three types of singularities are there,

(i) If no terms in P.P. then removable singularity

(ii) If finite number of terms in P.P. then 'Pole'

(iii) If infinite number of terms in P.P. then 'essential singularity'.

Now, $$f(z)= \frac{z-\sin z}{z^2}$$

$$= \frac{1}{z^2}\left[z-\left\{z-\frac{z^3}{\lfloor 3}+\frac{z^5}{\lfloor 5}-\frac{z^7}{\lfloor 7}+\cdots\right\}\right]$$

$$= \frac{1}{z^2}\left[\frac{z^3}{\lfloor 3}-\frac{z^5}{\lfloor 5}+\frac{z^7}{\lfloor 7}-\cdots\right]$$

$$= \frac{z}{\lfloor 3}-\frac{z^3}{\lfloor 5}+\frac{z^5}{\lfloor 7}-\cdots$$

Since, there is no term in the principal part of $f(z)$. Hence $f(z)$ has removable singularity at $z=0$.

Ans

Find the poles (with its order) and residue at each pole of the following function $\frac{1-2z}{z(z-1)(z-2)^2}$

Sol Poles of $f(z) = \frac{1-2z}{z(z-1)(z-2)^2}$ are given by

$z(z-1)(z-2)^2 = 0$

$z = 0, 1$ (Simple Pole)

and $z = 2$ Pole of order 2.

Residue of $f(z)$ at $z=0$ is,

$R_1 = \lim_{z\to 0} z\, f(z) = \lim_{z\to 0} \frac{1-2z}{(z-1)(z-2)^2} = \frac{1}{(-1)(-2)^2} = \frac{-1}{4}$

Residue of $f(z)$ at $z=1$ is,

$R_2 = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \frac{1-2z}{z(z-2)^2}$

$= \frac{1-2}{1(1-2)^2} = -1$

Residue of $f(z)$ at $z=2$ is.

$R_3 = \lim_{z\to 2} \frac{d}{dz}[(z-2)^2 f(z)]$

$= \lim_{z\to 2} \frac{d}{dz}\left[\frac{1-2z}{z(z-1)}\right] = \lim_{z\to 2} \frac{d}{dz}\left[\frac{(1-z)-z}{z(z-1)}\right]$

$= \lim_{z\to 2} \frac{d}{dz}\left[-\frac{1}{z} - \frac{1}{z-1}\right] = \lim_{z\to 2}\left[\frac{1}{z^2} + \frac{1}{(z-1)^2}\right]$

$= \frac{1}{2^2} + \frac{1}{(2-1)^2} = \frac{1}{4} + 1 = \frac{5}{4}.$ Ans

<u>Formula to find Residue</u>

① If $f(z)$ has simple pole at $z=a$ then Residue $= \lim_{z\to a}(z-a)f(z)$

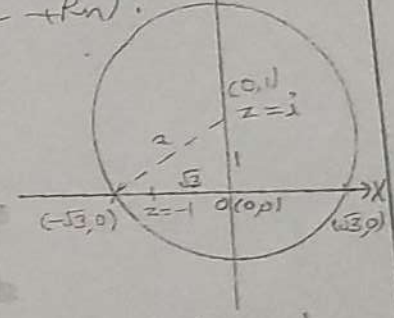
② If $f(z)$ has a pole of order m at $z=a$ then

Residue $= \frac{1}{\lfloor m-1} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]$

Q2. State Cauchy's Residue theorem. Determine the poles and residues at each pole for $\frac{z-1}{(z+1)^2(z-2)}$ and hence evaluate $\oint_C f(z)\,dz$ where C is the circle $|z-i| = 2$.

Sol Statement – Let $f(z)$ be a single valued and analytic function within and on a closed contour C except at a finite number of poles $z_1, z_2, \ldots, z_n$ and let $R_1, R_2, \ldots, R_n$ be respectively the residues of $f(z)$ at these poles then

$\oint_C f(z)\,dz = 2\pi i\,(R_1 + R_2 + \cdots + R_n).$



Now Poles of $f(z) = \frac{z-1}{(z+1)^2(z-2)}$ are

$(z+1)^2(z-2) = 0$

$\Rightarrow z = 2$ (Simple Pole)

& $z = -1$ (Double Pole)

Residue of $f(z)$ at $z=2$ is

$R_1 = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2} \frac{(z-1)}{(z+1)^2} = \frac{2-1}{(3)^2} = \frac{1}{9}$

Residue of $f(z)$ at $z=-1$ is

$R_2 = \frac{1}{\lfloor 1} \lim_{z\to -1}\left[\frac{d}{dz}\left\{\frac{(z+1)^2 (z-1)}{(z+1)^2(z-2)}\right\}\right] = \lim_{z\to -1}\left[\frac{d}{dz}\left(\frac{z-1}{z-2}\right)\right]$

$= \lim_{z\to -1} \frac{d}{dz}\left[\frac{z-2+1}{z-2}\right] = \lim_{z\to -1} \frac{d}{dz}\left[1 + \frac{1}{z-2}\right]$

$= \lim_{z\to -1}\left[0 + \frac{-1}{(z-2)^2}\right] = \frac{-1}{(-3)^2} = -\frac{1}{9}$

Now $|z-i| = 2$ is the circle with centre at $z=i$ and radius 2. clearly only $z=-1$ lies inside C. Hence by Cauchy's Residue theorem

$\oint_C f(z)\,dz = 2\pi i\left(-\frac{1}{9}\right) = -\frac{2\pi i}{9}$ Ans