Extra Important Questions

Engg. Mathematics-II – EIQ Solutions

2022-23

B.Tech I Year

Q1. **Unit- I**

Solve $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Sol. A.E. is $(m+2)(m-1)^2 = 0 \Rightarrow m = -2, 1, 1$

$$C.F. = c_1 e^{-2x} + (c_2 + c_3 x) e^{x}$$

$$\left[\because \sinh x = \frac{e^x - e^{-x}}{2}\right]$$

$$P.I. = \frac{1}{(D+2)(D-1)^2}(e^{-2x} + 2\sinh x)$$

$$= \frac{1}{(D+2)(D-1)^2} e^{-2x} + \frac{1}{(D+2)(D-1)^2} e^{x} - \frac{1}{(D+2)(D-1)^2} e^{-x}$$

$$= \frac{1}{D+2}\left[\frac{1}{(D-1)^2} e^{-2x}\right] + \frac{1}{(D-1)^2}\left[\frac{1}{D+2} e^{x}\right] - \frac{1}{(-1+2)(-1-1)^2} e^{-x}$$

$$= \frac{1}{D+2}\left[\frac{1}{(-2-1)^2} e^{-2x}\right] + \frac{1}{(D-1)^2}\left[\frac{1}{1+2} e^{x}\right] - \frac{1}{1.4} e^{-x}$$

$$= \frac{1}{9} \cdot \frac{1}{D+2} e^{-2x} + \frac{1}{3} \cdot x \cdot \frac{1}{2(D-1)} e^{x} - \frac{1}{4} e^{-x}$$

$$= \frac{1}{9} \cdot x \cdot \frac{1}{1} e^{-2x} + \frac{1}{3} \cdot x^2 \cdot \frac{1}{2 \cdot 1} e^{x} - \frac{1}{4} e^{-x}$$

$$= \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^{x} - \frac{1}{4} e^{-x}$$

Hence the complete solution is

$y = C.F. + P.I.$

$$y = c_1 e^{-2x} + (c_2 + c_3 x) e^{x} + \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^{x} - \frac{1}{4} e^{-x}$$

Ans.

Q2. Solve the differential equation

$$(D^3-1)y = 3x^4 - 2x^3$$

Sol. A.E. is $m^3-1=0$

$(m-1)(m^2+m+1)=0$

$\Rightarrow m-1=0$ or $m^2+m+1=0$

$\therefore m=1$, $m = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm\sqrt{-3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$C.F. = c_1e^{x} + e^{-\frac{x}{2}}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$$

$$P.I. = \frac{1}{D^3-1}(3x^4-2x^3)$$

$$= \frac{1}{-(1-D^3)}(3x^4-2x^3)$$

$$= -(1-D^3)^{-1}(3x^4-2x^3)$$

$$= -(1+D^3)(3x^4-2x^3)$$

$$= -[(3x^4-2x^3) + 3D^3x^4 - 2D^3x^3]$$

$$= -[3x^4-2x^3+3.4.3.2x-2.3.2.1]$$

$$= -[3x^4-2x^3+72x-12]$$

Hence C.S. is

$$y = C.F. + P.I.$$

$$y = c_1e^{x} + e^{-\frac{x}{2}}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right] - [3x^4-2x^3+72x-12]$$

Ans.

Q3. Solve $(D^2-2D+1)y = xe^x \sin x$

Sol. A.E. is $m^2-2m+1=0$

$$(m-1)^2=0 \Rightarrow m=1,1$$

$$C.F. = (c_1+c_2x)e^x$$

$$P.I. = \frac{1}{D^2-2D+1} xe^x \sin x$$

$$= \frac{1}{(D-1)^2} e^x . x\sin x$$

$$= e^x \frac{1}{(D+1-1)^2} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x$$

$$= e^x \frac{1}{D}\left[\int x \sin x \, dx\right]$$

$$= e^x \frac{1}{D}\left[x(-\cos x) - 1.(-\sin x)\right]$$

$$= e^x \left[-\int x\cos x \, dx + \int \sin x \, dx\right]$$

$$= e^x\left[-(x\sin x - 1.(-\cos x)) + (-\cos x)\right]$$

$$= e^x\left[-x\sin x - \cos x - \cos x\right]$$

$$= -e^x(x\sin x + 2\cos x)$$

Hence C.S. is

$$y = C.F. + P.I.$$

$$y = (c_1+c_2x)e^x - e^x(x\sin x + 2\cos x)$$

Ans.

Q4. Solve $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

Sol. Given diff. eq. is of the form of Homogeneous linear diff. eq. (Euler-Cauchy eq.)

Put $x = e^z$ so that $z = \log x$ and let $D \equiv \frac{d}{dz}$

then given diff. eq. reduces to

$[D(D-1)(D-2) + 2D(D-1) + 2]y = 10\left(e^z + \frac{1}{e^z}\right)$

$\Rightarrow [D^3 - 3D^2 + 2D + 2D^2 - 2D + 2]y = 10(e^z + e^{-z})$

or $(D^3 - D^2 + 2)y = 10(e^z + e^{-z})$

A.E. is $m^3 - m^2 + 2 = 0 \Rightarrow (m+1)(m^2 - 2m + 2) = 0$

$\therefore m = -1, \frac{2 \pm \sqrt{4-8}}{2} = -1, 1 \pm i$

$\therefore$ C.F. $= c_1 e^{-z} + e^z(c_2 \cos z + c_3 \sin z)$

$= \frac{c_1}{x} + x[c_2 \cos(\log x) + c_3 \sin(\log x)]$

P.I. $= 10\left[\frac{1}{D^3 - D^2 + 2} e^z + \frac{1}{D^3 - D^2 + 2} e^{-z}\right]$

$= 10\left[\frac{1}{1 - 1 + 2} e^z + z \frac{1}{3D^2 - 2D} e^{-z}\right]$

$= 10\left[\frac{1}{2} e^z + z \frac{1}{3(-1)^2 - 2(-1)} e^{-z}\right]$

$= 10\left[\frac{1}{2} e^z + z \frac{1}{3+2} e^{-z}\right] = 5e^z + 2z e^{-z}$

$= 5x + \frac{2}{x} \log x$

Hence C.S. is

$y =$ C.F. + P.I.

$y = \frac{c_1}{x} + x[c_2 \cos(\log x) + c_3 \sin(\log x)] + 5x + \frac{2}{x} \log x$

Ans

Q5. Solve $\frac{d^2x}{dt^2}+\frac{dy}{dt}+3x=e^{-t}$

$\frac{d^2y}{dt^2}-4\frac{dx}{dt}+3y=\sin 2t$

Sol. Let $D\equiv\frac{d}{dt}$ then we have

$(D^2+3)x+Dy=e^{-t}$ — ①

$-4Dx+(D^2+3)y=\sin 2t$ — ②

Operating ① by (D^2+3) and ② by D then subtracting, we get

$[(D^2+3)^2+4D^2]x=4e^{-t}-2\cos 2t$

$(D^4+10D^2+9)x=4e^{-t}-2\cos 2t$

A.E. is $m^4+10m^2+9=0 \Rightarrow m=\pm i, \pm 3i$

C.F. $=c_1\cos t+c_2\sin t+c_3\cos 3t+c_4\sin 3t$

P.I. $=\frac{1}{D^4+10D^2+9}4e^{-t}-\frac{1}{D^4+10D^2+9}2\cos 2t$

$=\frac{1}{1+10+9}4e^{-t}-\frac{1}{16-40+9}2\cos 2t=\frac{1}{5}e^{-t}+\frac{2}{15}\cos 2t$

$\therefore x=c_1\cos t+c_2\sin t+c_3\cos 3t+c_4\sin 3t+\frac{1}{5}e^{-t}+\frac{2}{15}\cos 2t$ — ③

Again operating ① by $4D$ and ② by (D^2+3) then adding we get

$[(D^2+3)^2+4D^2]y=-4e^{-t}-\sin 2t$

or $(D^4+10D^2+9)y=-4e^{-t}-\sin 2t$

C.F. $=c_5\cos t+c_6\sin t+c_7\cos 3t+c_8\sin 3t$

P.I. $=\frac{1}{D^4+10D^2+9}(-4e^{-t})-\frac{1}{D^4+10D^2+9}\sin 2t$

$=-\frac{1}{5}e^{-t}+\frac{1}{15}\sin 2t$

$\therefore y=c_5\cos t+c_6\sin t+c_7\cos 3t+c_8\sin 3t-\frac{1}{5}e^{-t}+\frac{1}{15}\sin 2t$ — ④

Equations ③ and ④, when taken together, give the complete solution.

Ans.

Q6. By changing the independent variable, solve the diff. eq. $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2 y = x^4$

Sol. Compare given eq. with standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R, \quad \text{we get}$$

$$P = -\frac{1}{x}, \quad Q = 4x^2, \quad R = x^4$$

Choose z such that $\left(\frac{dz}{dx}\right)^2 = 4x^2 \Rightarrow \frac{dz}{dx} = 2x$

on integration, $z = x^2$; on diff. $\frac{d^2z}{dx^2} = 2$

$$\therefore P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 - \frac{1}{x}(2x)}{(2x)^2} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{4x^2} = 1, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{4x^2} = \frac{x^2}{4}$$

Reduced eq. is $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$

$$\frac{d^2y}{dz^2} + y = \frac{z}{4} \qquad (\because z = x^2)$$

A.E. is $m^2 + 1 = 0 \Rightarrow m = \pm i$

C.F. $= c_1 \cos z + c_2 \sin z$

$$\text{P.I.} = \frac{1}{D^2+1}\cdot\frac{z}{4} = (1+D^2)^{-1}\frac{z}{4}$$

$$= (1 - D^2 - -)\frac{z}{4} = \left(\frac{z}{4}\right) = \frac{z}{4}$$

Hence C.S. is $y =$ C.F. + P.I.

$$y = c_1 \cos z + c_2 \sin z + \frac{z}{4}$$

or $y = c_1 \cos x^2 + c_2 \sin x^2 + \frac{x^2}{4}$ Ans.

Q7. Solve by changing the independent variable

$$\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$$

Sol. Write it in standard form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

we get $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} - 2\cos^2 x \cdot y = 2\cos^4 x$

Here $P = \tan x$, $Q = -2\cos^2 x$, $R = 2\cos^4 x$

Choose z such that $\left(\frac{dz}{dx}\right)^2 = \cos^2 x \Rightarrow \frac{dz}{dx} = \cos x$

$\therefore$ $z = \sin x$ and $\frac{d^2z}{dx^2} = -\sin x$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-\sin x + \tan x \cdot \cos x}{\cos^2 x} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-2\cos^2 x}{\cos^2 x} = -2, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{2\cos^4 x}{\cos^2 x}$$

$$R_1 = 2\cos^2 x = 2(1-z^2)$$

Reduced eq. is $\frac{d^2y}{dz^2} - 2y = 2(1-z^2)$

A.E. is $m^2 - 2 = 0 \Rightarrow m = \pm\sqrt{2}$

C.F. $= c_1 \cosh\sqrt{2}z + c_2 \sinh\sqrt{2}z$

$$P.I. = \frac{1}{D^2-2} 2(1-z^2) = \frac{1}{-2\left(1-\frac{D^2}{2}\right)} 2(1-z^2)$$

$$= -\left[1-\frac{D^2}{2}\right]^{-1}(1-z^2) = -\left[1+\frac{D^2}{2}\right](1-z^2)$$

$$= -\left[1-z^2-1\right] = z^2$$

C.S. is $y = $ C.F. $+$ P.I.

$$y = c_1\cosh\sqrt{2}z + c_2\sinh\sqrt{2}z + z^2$$

or $y = c_1\cosh(\sqrt{2}\sin x) + c_2\sinh(\sqrt{2}\sin x) + \sin^2 x$

Ans

Question No. 8 :– Solve by the method of variation of parameters:

$$\frac{d^2y}{dx^2} + a^2y = \sec ax$$

Solution :– Here, $u = \cos ax$, $v = \sin ax$ are two parts of C.F.

Also, $R = \sec ax$

Let the complete solution be

$$y = A\cos ax + B\sin ax \qquad \text{——} (1)$$

where A and B are suitable functions of x.

To determine the values of A and B, we have

$$A = \int \frac{-Rv}{uv_1 - u_1v}\,dx + c_1$$

$$\Rightarrow A = \int \frac{-\sec ax \cdot \sin ax}{\{\cos ax \cdot a\cos ax - (-a\sin ax)\sin ax\}}\,dx + c_1$$

$$\Rightarrow A = -\int \frac{\tan ax}{a}\,dx + c_1 = \frac{1}{a^2}\log\cos ax + c_1$$

where c_1 is an arbitrary constant of integration.

$$B = \int \frac{Ru}{uv_1 - u_1v}\,dx + c_2$$

$$\Rightarrow B = \int \frac{\sec ax \cdot \cos ax}{\{\cos ax \cdot a\cos ax - (-a\sin ax)\sin ax\}}\,dx + c_2$$

$$\Rightarrow B = \frac{1}{a}\int dx + c_2 = \frac{x}{a} + c_2$$

where c_2 is an arbitrary constant of integration.

$$y = A\cos ax + B\sin ax$$

$$\Rightarrow \boxed{y = \left(\frac{\log\cos ax}{a^2} + c_1\right)\cos ax + \left(\frac{x}{a} + c_2\right)\sin ax}$$

Q: Using variation of parameters method, Solve $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-12y=x^3\log x$

Ans: Let $x=e^z \Rightarrow z=\ln x$

$x\frac{d}{dx}\equiv\frac{d}{dz}\equiv D$ and $x^2\frac{d^2}{dx^2}\equiv D(D-1)$

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-12y=x^3\log x$$

$$D(D-1)y+2Dy-12y=e^{3z}z$$

$$[D^2+D-12]y=ze^{3z}$$

A.E. $m^2+m-12=0 \Rightarrow m=3,-4$

C.F. $c_1e^{3z}+c_2e^{-4z}=c_1x^3+c_2x^{-4}$

Part of C.F. $u=x^3$ and $v=x^{-4}$, $R=x\log x$

Let general solⁿ is $y=Au+Bv$

$$A=-\int\frac{Rv\,dx}{uv_1-u_1v}+c_1=-\int\frac{x\log x\,x^{-4}\,dx}{x^3(-4x^{-5})-3x^2(x^{-4})}+c_1$$

$$=\frac{1}{7}\int\frac{\log x}{x}dx+c_1=\frac{1}{14}(\log x)^2+c_1$$

$$B=\int\frac{Ru\,dx}{uv_1-u_1v}+c_2=\int\frac{x\log x\cdot x^3}{-7x^{-2}}+c_2=\frac{-1}{7}\int x^6\log x\,dx+c_2$$

$$=\frac{-1}{7}\left[\log x\cdot\frac{x^7}{7}-\frac{1}{7}\left(\frac{x^7}{7}\right)\right]+c_2$$

$$=\frac{x^7}{49}\left[\frac{1}{7}-\log x\right]+c_2$$

Hence the complete solⁿ $y=Ax^3+Bx^{-4}$

$$y=c_1x^3+c_2x^{-4}+\frac{x^3}{14}(\log x)^2+\frac{x^3}{49}\left(\frac{1}{7}-\log x\right)$$

Q10 Solve

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 48x^5$$

Consider the eqn $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 0$ for finding parts of C.F.

Put $x = e^z$ so that $z = \log x$ & let $D \equiv \frac{d}{dz}$ then the given eqn be

$$[D(D-1) + D - 9]y = 0$$

$$(D^2 - 9)y = 0$$

Auxiliary eqn is $m^2 - 9 = 0 \Rightarrow m = \pm 3$

$\therefore$ C.F $= c_1 e^{3z} + c_2 e^{-3z} = c_1 x^3 + c_2 x^{-3}$

Hence parts of C.F are x^3 and x^{-3}.

Let $u = x^3$ and $v = x^{-3}$ Also $R = 48x^3$

Let $y = Au + Bv$ be the complete solution &

$$A = -\int \frac{Rv}{uv_1 - u_1 v}dx + c_1 = -48\int \frac{x^3 \cdot x^{-3}}{x^3(-3x^{-4}) - 3x^2(x^{-3})} + c_1$$

$$= \frac{48}{6}\int \frac{1}{x^{-1}} \cdot dx + c_1 = 8\int x\,dx + c_1$$

$$\boxed{A = (4x^2 + c_1)}$$

$$B = \int \frac{Ru}{uv_1 - u_1 v}dx + c_2 = \frac{48}{-6}\int \frac{x^3 \cdot x^3}{x^{-1}}dx + c_2$$

$$= -8\int x^7 dx + c_2$$

$$= -\frac{\cancel{8} \cdot x^8}{\cancel{8}} + c_2 = -x^8 + c_2$$

$$\boxed{B = -x^8 + c_2}$$

Hence the complete solution is

$$y = Au + Bv$$

$$y = (4x^2 + c_1)x^3 + (-x^8 + c_2)x^{-3}$$

ques:-1

(a) find the Laplace transform of $\frac{1-\cos 2t}{t}$.

Soln:- We have $L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$.

using $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s)\,ds$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \frac{1}{2}\left[\log s^2 - \log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{s^2}{s^2+4}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{s^2}{s^2\left[1+\frac{4}{s^2}\right]}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log \frac{s^2}{s^2+4}\right]$$

$$= \frac{1}{2}\log \frac{s^2+4}{s^2}.$$

(b) Evaluate $\int_0^\infty \frac{e^{-2t} - e^{-4t}}{t}\,dt$.

Soln:- $\int_0^\infty e^{-2t}\frac{(1-e^{-2t})}{t}\,dt$ — ①

We know $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ — ②

on comparing eqn ① & ②

$f(t) = \frac{1-e^{-2t}}{t}$ & $s = 2$

now $L\{f(t)\} = L\left\{\frac{1-e^{-2t}}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+2}\right) ds$

$$= \left[\log s - \log(s+2)\right]_s^\infty$$

$$= \left[\log \frac{s}{s+2}\right]_s^\infty = \left[\log \frac{s}{s[1+2/s]}\right]_s^\infty$$

$$= \left[\log 1 - \log \frac{s}{s+2}\right]$$

$$= 0 + \log \frac{s+2}{s}$$

putting $s = 2$

$$L\{f(t)\} = \log \frac{4}{2} = \log 2.$$

(c) Evaluate $\int_0^\infty \frac{e^{-t}\sin^2 t}{t}\,dt.$

<u>Soln</u>:- $L(\sin^2 t) = \frac{1}{2} L[1 - \cos 2t] = \frac{1}{2}\left(\frac{1}{p} - \frac{p}{p^2+4}\right)$

$$L\left(\frac{\sin^2 t}{t}\right) = \frac{1}{2}\int_p^\infty \left(\frac{1}{p} - \frac{p}{p^2+4}\right) dp = \frac{1}{4}\log\left(\frac{p^2+4}{p^2}\right)$$

by definition, $\int_0^\infty e^{-pt}\frac{\sin^2 t}{t}\,dt = \frac{1}{4}\log\left(\frac{p^2+4}{p^2}\right).$

put $p = 1$

$$\int_0^\infty e^{-t}\frac{\sin^2 t}{t}\,dt = \frac{1}{4}\log 5.$$

Ques:-2 Laplace transform of the square wave function of period a given by

$$f(t) = \begin{cases} 1, & 0 \le t \le a/2 \\ -1, & a/2 < t < a \end{cases}.$$

Soln:- We know that $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)\,dt$.

$\boxed{T = a}$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} 1 \cdot e^{-st}\,dt + \int_{a/2}^{a} (-1)\, e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} + \left(\frac{e^{-st}}{s}\right)_{a/2}^{a}\right]$$

$$= \frac{1}{(1-e^{-as})s}\left[-e^{-as/2} + 1 + e^{-as} - e^{-as/2}\right]$$

$$= \frac{1}{(1-e^{-as})s}\left[1 - 2e^{-as/2} + e^{-as}\right]$$

$$= \frac{1}{(1-e^{-as})s}\left[(1-e^{-as/2})^2\right]$$

$$= \frac{1}{s}\,\frac{(1-e^{-as/2})^{\cancel{2}}}{\cancel{(1-e^{-as/2})}(1+e^{-as/2})} = \frac{(1-e^{-as/2})}{s(1+e^{-as/2})}$$

$$= \frac{1}{s}\tanh\frac{as}{2}.$$

Ques:-3 (a) If $L^{-1}\left(\frac{e^{-1/s}}{\sqrt{s}}\right) = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$. find $L^{-1}\left(\frac{e^{-a/s}}{\sqrt{s}}\right)$.

Soln:- Let $F(s) = \frac{e^{-1/s}}{\sqrt{s}}$ and $L^{-1}\{F(s)\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$ (given) $= f(t)$.

$$L^{-1}[F(s/a)] = a f(at) \Rightarrow L^{-1}\left\{\frac{e^{-a/s}}{\sqrt{s/a}}\right\} = a \cdot \frac{\cos 2\sqrt{at}}{\sqrt{\pi a t}}$$

$$L^{-1}\left\{\sqrt{a}\cdot\frac{e^{-a/s}}{\sqrt{s}}\right\} = a\cdot\frac{\cos 2\sqrt{at}}{\sqrt{\pi a t}}$$

Hence, $$L^{-1}\left\{\frac{e^{-a/s}}{\sqrt{s}}\right\} = \frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}.$$

(b) Find Inverse Laplace Transform of $\frac{1}{(s+2)^2}$.

$$L^{-1}\left\{\frac{1}{(s+2)^2}\right\} = e^{-2t}\,t$$

Ques:-4 Find the Laplace Transform of "Saw-tooth wave" function f(t). which is periodic with period 1. and defined as, $f(t) = kt$ in $0 < t < 1$.

Soln:- given function is "saw-tooth" function with period $T = 1$ so,

$$L\{f(t)\} = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)\, dt$$

$$= \frac{1}{1-e^{-p}} \int_0^1 e^{-pt}\, kt\, dt$$

$$= \frac{k}{1-e^{-p}} \int_0^1 e^{-pt}\, t\, dt$$

$$= \frac{k}{1-e^{-p}} \left[-t\, e^{-pt} - \frac{1}{p^2} e^{-pt} \right]_0^1$$

$$= \frac{k}{1-e^{-p}} \left[-\frac{1}{p} e^{-p} - \frac{1}{p^2} e^{-p} + 0 + \frac{1}{p^2} \right]$$

$$= \frac{k}{1-e^{-p}} \left[\frac{1}{p^2}(1-e^{-p}) - \frac{e^{-p}}{p} \right] = \frac{k}{p^2} - \frac{k e^{-p}}{p(1-e^{-p})}$$

ques:-5

(a) find the Laplace Transform of $e^{-2t} u(t-2)$.

Soln:- we know that $L\{u(t-a)\} = \frac{e^{-as}}{s}$

using first shifting property —

$$L\{e^{-2t} u(t-2)\} = \frac{e^{-2(s+2)}}{s+2}.$$

(b) find the Laplace transform of $t^2 u(t-3)$.

Soln:- $\Rightarrow L\{t^2 u(t-3)\} = L[\{(t-3)^2 + 6(t-3) + 9\} u(t-3)]$

$$= L[\{(t-3)^2 u(t-3)\}] + 6L[(t-3)u(t-3)] + 9L\{u(t-3)\}.$$

$$= 2\frac{e^{-3s}}{s^3} + 6\frac{e^{-3s}}{s^2} + 9\frac{e^{-3s}}{s}$$

$$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right].$$

(c) find the Laplace Transform of $\int_0^t \int_0^t \cos t \, dt \, dt$.

$$L\left\{\int_0^t \int_0^t \cos t \, dt \, dt\right\} = \frac{1}{s^2} \cdot \frac{s}{s^2+1}$$

$$= \frac{1}{s(s^2+1)}$$

ques:- 6 use convolution theorem to find

$$L^{-1}\left[\frac{1}{(p^2+4)(p+2)}\right].$$

Soln:- If $L^{-1}\{f(p)\} = F(t)$

$L^{-1}\{g(p)\} = G(t)$

then $L^{-1}\{f(p)\cdot g(p)\} = F*G = \int_0^t F(u)\, G(t-u)\, du.$

now let, $f(p) = \frac{1}{p^2+4}$, $g(p) = \frac{1}{p+2}$

$L^{-1}\{F(p)\} = \frac{1}{2}\sin 2t$; $L^{-1}\{g(p)\} = e^{-2t}$

$F(u) = \frac{1}{2}\sin 2u$; $G(t-u) = e^{-2(t-u)}$

$\therefore L^{-1}\{f(p)\cdot g(p)\} = \int_0^t \frac{1}{2}\sin 2u \; e^{-2(t-u)}\, du$

$$= \frac{e^{-2t}}{2}\int_0^t e^{2u}\sin 2u\, du.$$

$$= \frac{e^{-2t}}{2}\left[\frac{e^{2u}}{4+4}\{2\sin 2u - 2\cos 2u\}\right]_0^t$$

$$= \frac{e^{-2t}}{2}\left[\frac{e^{2t}}{8}(2\sin 2t - 2\cos 2t) + 2\right]$$

$$= \frac{1}{8}\left[\sin 2t - \cos 2t + e^{-2t}\right].$$

ques:-7 use convolution theorem to find

$$L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\}$$

soln:- $f(s) = \frac{1}{(s+2)^2}$ $\qquad G(s) = \frac{1}{s-2}$

$f(t) = e^{-2t}.t$ $\qquad G(t) = e^{2t}.$

$f(u) = e^{-2u}.u$ $\qquad G(u) = e^{2u}.$

$$L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\} = \int_0^t e^{-2u}.u.e^{2t}.e^{-2u}\,du$$

$$= e^{2t}\int_0^t e^{-4u}\,u\,du$$

$$= e^{2t}\left[u.\frac{e^{-4u}}{-4} - (1)\left(\frac{e^{-4u}}{16}\right)\right]_0^t$$

$$= e^{2t}\left[t\frac{e^{-4t}}{-4} - \frac{e^{-4t}}{16} + \frac{1}{16}\right]$$

$$= e^{2t}\left[\frac{t\,e^{-4t}}{-4} - \frac{(e^{-4t}-1)}{16}\right].$$

ques:- 8 solve the Differential Equation by Laplace transform

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0 \quad \text{where } y=1, \frac{dy}{dt}=2,$$

$$\frac{d^2y}{dt^2} = 2 \text{ at } t=0.$$

Soln:- Taking Laplace transform on both sides, we get,

$$= [p^3\bar{y} - p^2y(0) - py'(0) - y''(0)] + 2[p^2\bar{y} - py(0) - y'(0)]$$
$$- [p\bar{y} - y(0)] - 2\bar{y} = 0 \quad \text{--- (1)}$$

using given conditions $y(0)=1$, $y'(0)=2$, $y''(0)=2$ in eqn (1)

$$(p^3+2p^2-p-2)\bar{y} = p^2+4p+5$$

$$\bar{y} = \frac{p^2+4p+5}{p^3+2p^2-p-2} = \frac{p^2+4p+5}{(p-1)(p+1)(p+2)}$$

$$\bar{y} = \frac{5}{3(p-1)} - \frac{1}{p+1} + \frac{1}{3(p+2)}$$

taking the Inverse Laplace transforms of both sides, we get,

$$y = \frac{5}{3}e^{t} - e^{-t} + \frac{1}{3}e^{-2t}.$$

Ques:-9 Solve the following DE's by Laplace Transform
$y''' - 2y'' + 5y' = 0$; $y(0) = 0$, $y'(0) = 1$ & $y = 1$ at $t = \frac{\pi}{8}$.

Soln:- Taking Laplace transform on both sides, we get,

$$L(y''') - 2L(y'') + 5L(y') = 0$$

$$\Rightarrow p^3\bar{y} - p^2y(0) - py'(0) - py''(0) - 2[p^2\bar{y} - py(0) - y'(0)] + 5[p\bar{y} - y(0)] = 0$$

$$\Rightarrow (p^3 - 2p^2 + 5p)\bar{y} - p - A + 2 = 0 \qquad [\text{Let } y''(0) = A]$$

$$\Rightarrow \bar{y} = \frac{(A-2) + p}{p(p^2 - 2p + 5)}$$

$$= \frac{A-2}{5}\left\{\frac{1}{p} - \frac{p-2}{p^2-2p+5}\right\} + \frac{1}{p^2-2p+5}$$

$$= \left(\frac{A-2}{5}\right)\frac{1}{p} - \left(\frac{A-2}{5}\right)\left\{\frac{p-1}{(p-1)^2+4}\right\} + \left(\frac{A+3}{10}\right)\left\{\frac{2}{(p-1)^2+4}\right\}$$

Taking Inverse Laplace transform on both sides, we get,

$$y = \left(\frac{A-2}{5}\right) - \left(\frac{A-2}{5}\right)\{e^t \cos 2t\} + \left(\frac{A+3}{10}\right)e^t \sin 2t.$$

$$y(\pi/8) = 1.$$

$$1 = \left(\frac{A-2}{5}\right) - \left(\frac{A-2}{5}\right)e^{\pi/8}\cdot\frac{1}{\sqrt{2}} + \left(\frac{A+3}{10}\right)e^{\pi/8}\cdot\frac{1}{\sqrt{2}}$$

$$\Rightarrow A = 7$$

Hence required solution is

$$y = 1 + e^t(\sin 2t - \cos 2t).$$

ques:-10 Solve the following DE's by Laplace Transform

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = e^t; \quad y(0) = 0 \quad y'(0) = 1$$

Soln:- taking Laplace transform both sides we get,

$$\{s^2\bar{y} - s\,y(0) - y'(0)\} - 2\{s\bar{y} - y(0)\} + \bar{y} = \frac{1}{s-1}$$

$$(s^2\bar{y} - 1) - 2\{s\bar{y}\} + \bar{y} = \frac{1}{s-1}$$

$$s^2\bar{y} - 1 - 2s\bar{y} + \bar{y} = \frac{1}{s-1}$$

$$(s^2 - 2s + 1)\bar{y} = \frac{1}{s-1} + 1$$

$$\bar{y} = \frac{1}{(s-1)(s-1)^2} + \frac{1}{(s-1)^2}$$

taking Inverse laplace transform both sides,
we get,

$$y = L^{-1}\left\{\frac{1}{(s-1)^3}\right\} + L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= e^t.\frac{t^2}{2} + e^t.t.$$

Ques ① Show that $\frac{l}{2} - x = \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{l}\right)$

when $0 < x < l$

Solution: Let $\frac{l}{2} - x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ — ①

where $b_n = \frac{2}{l}\int_0^l \left(\frac{l}{2} - x\right)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{2}{l}\left[\left(\frac{l}{2} - x\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(-\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left(\frac{-l}{2}\right)\left(\frac{-\cos n\pi}{n\pi/l}\right) + 0 + \frac{l}{2}\,\frac{1}{(n\pi/l)} - 0\right]$$

$$= \frac{l}{n\pi}\left[1 + (-1)^n\right]$$

$$\therefore \frac{l}{2} - x = \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{[1+(-1)^n]}{n}\sin\frac{n\pi x}{l}$$

$$= \frac{l}{\pi}\left[\frac{2}{2}\sin\frac{2\pi x}{l} + \frac{2}{4}\sin\frac{4\pi x}{l} + \cdots\right]$$

$$= \frac{l}{\pi}\left[\frac{1}{1}\sin\frac{2\pi x}{l} + \frac{1}{2}\sin\frac{4\pi x}{l} + \cdots\right]$$

$$\therefore \frac{l}{2} - x = \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}$$

Ques ② Obtain the fourier series expansion of $f(x) = \left(\frac{\pi - x}{2}\right)$ for $0 < x < 2$

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Here $l = 1$

$\therefore \frac{\pi - x}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$ —— ①

Here, $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \int_0^2 \left(\frac{\pi - x}{2}\right)dx = \frac{1}{2}\left(\pi x - \frac{x^2}{2}\right)_0^2 = \frac{1}{2}(2\pi - 2) = \pi - 1$

$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos n\pi x\,dx = \int_0^2 \left(\frac{\pi - x}{2}\right)\cos n\pi x\,dx$

$= \frac{1}{2}\left[\left\{(\pi - x)\frac{\sin n\pi x}{n\pi}\right\}_0^2 - \int_0^2 (-1)\frac{\sin n\pi x}{n\pi}\,dx\right] = \frac{1}{2n\pi}\left(\frac{-\cos n\pi x}{n\pi}\right)_0^2$

$= 0$

$\Rightarrow \boxed{a_n = 0}$

$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin n\pi x\,dx = \int_0^2 \left(\frac{\pi - x}{2}\right)\sin n\pi x\,dx$

$= \frac{1}{2}\left[\left\{(\pi - x)\left(\frac{-\cos n\pi x}{n\pi}\right)\right\}_0^2 - \int_0^2 (-1)\left(-\frac{\cos n\pi x}{n\pi}\right)dx\right]$

$= -\frac{1}{2n\pi}\left[(\pi - 2) - \pi\right] = \frac{1}{n\pi}$

$\Rightarrow \boxed{b_n = \frac{1}{n\pi}}$

Hence from ①,

$$\frac{\pi - x}{2} = \frac{(\pi - 1)}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin n\pi x$$

Ques 3 Obtain Fourier series for

$$f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$$

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

then

$$a_0 = \int_0^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \pi\left(\frac{1}{2}\right) + \pi\left[(4-2) - \left(2 - \frac{1}{2}\right)\right]$$

$$\boxed{a_0 = \pi}$$

$$a_n = \int_0^2 f(x)\cos n\pi x\,dx$$

$$= \int_0^1 \pi x \cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \left[\pi x \cdot \frac{\sin n\pi x}{n\pi} - \pi\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\cdot\frac{\sin n\pi x}{n\pi} - (-\pi)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi}\right] + \left[\frac{-\cos 2n\pi}{n^2\pi} + \frac{\cos n\pi}{n^2\pi}\right]$$

$$= \frac{2}{n^2\pi}(\cos n\pi - 1) = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$= 0 \quad \text{or} \quad -\frac{4}{n^2\pi}$$ according as n is even or odd

$$b_n = \int_0^2 f(x)\sin n\pi x\,dx$$

$$= \int_0^1 \pi x \sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \left[\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 +$$

$$\left[\pi(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[-\frac{\cos n\pi}{n}\right] + \left[\frac{\cos n\pi}{n}\right] = 0$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)$$

Ques 4 Expand the function $f(x) = x \sin x$ as a fourier series in the interval $-\pi \le x \le \pi$.

Deduce that $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \dots = \frac{\pi - 2}{4}$

Solution: Since $x \sin x$ is an even function of x,

$b_n = 0$

Let $f(x) = x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x \, dx = \frac{2}{\pi}\left[x(-\cos x) - 1\cdot(-\sin x)\right]_0^{\pi}$

$$= \frac{2}{\pi}(-\pi\cos\pi) = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x (2\cos nx \sin x)\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\, dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1\cdot\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\}\right]$$

$$= \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}, \quad n \ne 1$$

When n is odd, $n \ne 1$, $n-1$ and $n+1$ are even

$$\therefore a_n = \frac{1}{n-1} - \frac{1}{n+1} = \frac{2}{n^2-1}$$

When n is even, $n-1$ and $n+1$ are odd

$$\therefore \quad a_n = \frac{-1}{n-1} + \frac{1}{n+1} = \frac{-2}{n^2-1}$$

When $n=1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{-\pi \cos 2\pi}{2}\right]$$

$$= -\frac{1}{2}$$

$$\therefore x \sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \frac{\cos 5x}{5^2-1} + \cdots\right)$$

Putting $x = \frac{\pi}{2}$, we get

$$\frac{\pi}{2} = 1 - 2\left(\frac{-1}{2^2-1} + \frac{1}{4^2-1} - \frac{1}{6^2-1} + \cdots\right)$$

$$\Rightarrow \frac{\pi}{2} - 1 = 2\left(\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \cdots\right)$$

$$\Rightarrow \frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots$$

Q5 Expand the function $f(x) = x$ as fourier series in the interval $0 < x < 2\pi$.

Ans
$$f(x) = x \quad ;(0, 2\pi)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{2\pi} = \frac{1}{2\pi}(4\pi^2)$$

$$\boxed{a_0 = 2\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\left\{\frac{\sin nx}{n}\right\} - (1)\left\{\frac{-\cos nx}{n^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\right]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left\{\frac{-\cos nx}{n}\right\} - (1)\left\{\frac{-\sin nx}{n^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi\cos 2n\pi}{n}\right] = \frac{1}{\pi}\left[\frac{-2\pi}{n}\right]$$

$$\boxed{b_n = \frac{-2}{n}}$$

So $f(x) = \frac{2\pi}{2} + \sum_{n=1}^{\infty} 0.\cos nx + \sum_{n=1}^{\infty}\left(\frac{-2}{n}\right)\sin nx$

$$\boxed{x = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}} \text{ Ans}$$

Q6 Test the nature of the series

$$1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots$$

Ans $u_n = \frac{1}{5^{n-1}}$ and $u_{n+1} = \frac{1}{5^n}$

Now $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1/5^{n-1}}{1/5^n} = \lim_{n\to\infty}\frac{5^n}{5^{n-1}} = 5$

$\because \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = 5 > 1$

Hence By D'Alembert ratio test.

$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\frac{1}{5^{n-1}}$ is convergent.

Q⑦ Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$ using D'Alembert test.

Ans $u_n = \frac{2^n+5}{3^n}$ and $u_{n+1} = \frac{2^{n+1}+5}{3^{n+1}}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\frac{2^n+5}{3^n}}{\frac{2^{n+1}+5}{3^{n+1}}} = \lim_{n\to\infty}\left(\frac{2^n+5}{2^{n+1}+5}\right)\left(\frac{3^{n+1}}{3^n}\right)$$

$$= \lim_{n\to\infty} \frac{2^n\left[1+\frac{5}{2^n}\right]}{2^{n+1}\left[1+\frac{5}{2^{n+1}}\right]}\cdot 3$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(\frac{1+5/2^n}{1+5/2^{n+1}}\right)\cdot 3$$

$$= \frac{3}{2}\lim_{n\to\infty}\left(\frac{1+5/2^n}{1+5/2^{n+1}}\right)$$

$$= \frac{3}{2}\left(\frac{1+0}{1+0}\right) = \frac{3}{2} > 1$$

$\because \lim_{n\to\infty} \frac{u_n}{u_{n+1}} > 1$ Hence By D'Alembert Ratio test $\sum u_n = \sum \frac{2^n+5}{3^n}$ convergent.

Q⑧ Find the Fourier half range cosine series for the function $f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ 0 & \pi/2 < x < \pi \end{cases}$

Ans $f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ 0 & \pi/2 < x < \pi \end{cases}$

Fourier cosine series for the interval $(0,\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} 1\,dx + \int_{\pi/2}^{\pi} 0\,dx\right] = \frac{2}{\pi}\left[(x)_0^{\pi/2}\right]$$

$$= \frac{2}{\pi}\left(\frac{\pi}{2}\right) = 1$$

$$\boxed{a_0 = 1}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} f(x)\cdot\cos nx\,dx + \int_{\pi/2}^{\pi} f(x)\cdot\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} 1\cdot\cos nx\,dx + \int_{\pi/2}^{\pi} 0\cdot\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{\sin nx}{n}\right)_0^{\pi/2}\right] = \frac{2}{\pi}\left[\frac{\sin\left(\frac{n\pi}{2}\right)}{n} - 0\right]$$

$$a_n = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) \Rightarrow a_n = \frac{2}{n\pi}\sin(n\pi/2)$$

so $$\boxed{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right)\cos nx}$$

Q9 Test the convergence of the series

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots$$

Ans $u_n = \dfrac{1}{\sqrt{n}+\sqrt{n+1}}$ and $v_n = \dfrac{1}{\sqrt{n}}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} = \lim_{n\to\infty}\frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\left[1+\sqrt{1+\frac{1}{n}}\right]}$$

$$= \frac{1}{1+\sqrt{1+0}} = 1 \quad \text{(finite \& Non-zero)}$$

$\Rightarrow$ Comparison test is applicable.

Now since $\sum v_n = \sum \frac{1}{\sqrt{n}}$ is divergent by p-series test as $n = \frac{1}{2} < 1$

Hence by Comparison test

$\because \sum v_n$ is divergent $\Rightarrow \sum u_n$ also divergent.

So $\sum \dfrac{1}{\sqrt{n}+\sqrt{n+1}}$ also divergent.

Q1 (a) Write the Cauchy-Riemann equations in Cartesian form.

The Cauchy-Riemann equations for $f(z) = u(x,y) + i\,v(x,y)$ to be analytic are

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

or

$$u_x = v_y \quad \& \quad u_y = -v_x$$

Q1 (b) Find the values of c_1 & c_2 such that the function $f(z) = x^2 + c_1 y^2 - 2xy + i(c_2 x^2 - y^2 + 2xy)$ is analytic

Here $f(z) = x^2 + c_1 y^2 - 2xy + i(c_2 x^2 - y^2 + 2xy)$ — (1)

Comparing (1) with $f(z) = u + iv$ we get

$u = x^2 + c_1 y^2 - 2xy$ — (2)

$v = c_2 x^2 - y^2 + 2xy$ — (3)

For the function $f(z)$ to be analytic, it should satisfy CR Equations

Now from (2) $\frac{\partial u}{\partial x} = 2x - 2y$ & $\frac{\partial u}{\partial y} = 2c_1 y - 2x$

also from (3) $\frac{\partial v}{\partial x} = 2c_2 x + 2y$ & $\frac{\partial v}{\partial y} = -2y + 2x$

C-R Equations are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$2x - 2y = -2y + 2x$$

& $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $\quad 2c_1 y - 2x = -2c_2 x - 2y$ — (4)

Comparing the coefficients of x & y in eq^n(4) we get

$2c_1 = -2 \quad \Rightarrow \quad c_1 = -1$

& $-2 = -2c_2 \quad \Rightarrow \quad c_2 = 1$

Hence $c_1 = -1$ and $c_2 = 1$ $\quad$ A

Q2: if $f(z) = \begin{cases} \dfrac{x^3 y(y-ix)}{x^6+y^2}, & z \neq 0 \\ 0 & z = 0 \end{cases}$, Prove that $\dfrac{f(z)-f(0)}{z} \to 0$

as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner and also that $f(z)$ is not analytic at $z=0$

$$\frac{f(z)-f(0)}{z} = \left[\frac{x^3 y(y-ix)}{x^6+y^2} - 0\right] \cdot \frac{1}{x+iy} = \frac{-i\,x^3 y(x+iy)}{(x^6+y^2)} \cdot \frac{1}{x+iy}$$

$$= -i\,\frac{x^3 y}{x^6+y^2}$$

Let $z \to 0$ along radius vector $y = mx$ then

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{-i\,x^3(mx)}{x^6+(mx)^2} = \lim_{x\to 0} \frac{-i\,m x^4}{x^6+m^2x^2}$$

$$= \lim_{x\to 0} \frac{x^2(-i\,m x^2)}{x^2(x^4+m^2)} = 0$$

Hence $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius Vector.

Now let $z \to 0$ along a curve $y = x^3$ then

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{-i\,x^3 \cdot x^3}{x^6+x^6} = \frac{-i}{2}$$

Hence $f(z)-f(0)$ does not tend to zero as $z \to 0$ along the curve $y = x^3$.

we observe that $f'(0)$ does not exist hence $f(z)$ is not analytic at $z=0$.

Q3: Show that $u(x,y) = x^2 - y^2 - y$ is harmonic. Also determine the analytic function $f(z)$ in terms of z whose real part is $u(x,y) = x^2 - y^2 - y$.

$$u = x^2 - y^2 - y$$

$$\frac{\partial u}{\partial x} = 2x \Rightarrow \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y - 1 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -2$$

Since $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 2 - 2 = 0 \therefore$ u is harmonic function.

By Milne's Thomson method:

if u is given then $f(z) = \int \{\phi_1(z,0) - i\,\phi_2(z,0)\} dz + c$

Let say $\phi_1(x,y) = \dfrac{\partial u}{\partial x} = 2x$

& $\phi_2(x,y) = \dfrac{\partial u}{\partial y} = -2y - 1$

$\therefore \phi_1(z,0) = 2z$

$\phi_2(z,0) = -1$

Then Milne's Thomson method:

$$f(z) = \int \{2z - i(-1)\} dz + c$$

$$= \int (2z + i)\, dz + c$$

$$= \frac{2z^2}{2} + iz + c$$

$$f(z) = z^2 + iz + c$$

where c is a constant.

Question No.4 :→ If $u+v = \dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$ and $f(z)=u+iv$ is an analytic function of $z=x+iy$, find $f(z)$ in terms of z.

Solution :→ let $f(z)=u+iv$ —— (1)

Multiplying both sides by i

$if(z)=iu-v$ —— (2)

Adding (1) & (2), we get

$(1+i)f(z)=(u-v)+i(u+v)$ —— (3)

$\Rightarrow F(z)=U+iV$ —— (4)

where $F(z)=(1+i)f(z)$ —— (5)

$U=u-v$ & $V=u+v$ —— (6)

It means that we have been given

$$V=\frac{\sin 2x}{\cosh 2y-\cos 2x} \quad \text{—— (7)} \quad \Big|\ \because e^{2y}+e^{-2y}=2\cosh 2y$$

Now $\dfrac{\partial V}{\partial y} = \dfrac{-2\sin 2x \sinh 2y}{(\cosh 2y-\cos 2x)^2} = \psi_1(x,y)$ | say

$$\frac{\partial V}{\partial x} = \frac{2\cos 2x(\cosh 2y-\cos 2x)-2\sin^2 2x}{(\cosh 2y-\cos 2x)^2}$$

$\Rightarrow \dfrac{\partial V}{\partial x} = \dfrac{2\cos 2x\cosh 2y-2}{(\cosh 2y-\cos 2x)^2} = \psi_2(x,y)$ | say

$\therefore \psi_1(z,0)=0$

$$\psi_2(z,0)=\frac{2(\cos 2z-1)}{(1-\cos 2z)^2} = \frac{-2}{1-\cos 2z} = \frac{-2}{1-1+2\sin^2 z} = -\operatorname{cosec}^2 z$$

By Milne's Thomson method, we have

$$F(z)=\int\{\psi_1(z,0)+i\psi_2(z,0)\}dz+c$$

$$=\int -i\operatorname{cosec}^2 z\,dz+c = i\cot z+c$$

Replacing $F(z)$ by $(1+i)f(z)$, from eq^n (5), we get

$(1+i)f(z)=i\cot z+c$

$$\Rightarrow \boxed{f(z)=\frac{i}{(1+i)}\cot z+\frac{c}{1+i} = \frac{1}{2}(1+i)\cot z+c_1}$$

where $c_1=\dfrac{c}{1+i}$

Question No.5: → If $u-v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$ and $f(z) = u+iv$ is an analytic function of $z = x+iy$. Find $f(z)$ in terms of z.

Solution:- Let $f(z) = u+iv$

$if(z) = iu - v$

then $(1+i)f(z) = (u-v) + i(u+v)$

$\Rightarrow \quad F(z) = U + iV$

where $F(z) = (1+i)f(z)$

$U = u-v$ & $V = u+v$

$$\therefore U = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$$

$$\frac{\partial U}{\partial x} = \frac{(\cosh y - \cos x)\cos x - (\sinh y + \sin x)\sin x}{(\cosh y - \cos x)^2} = \phi_1(x,y) \text{ (say)}$$

$$\therefore \phi_1(z,0) = \frac{1}{\cos z - 1}$$

$$\frac{\partial U}{\partial y} = \frac{(\cos x - \cosh y)\cosh y - (\sin x + \sinh y)\sinh y}{(\cosh y - \cos x)^2} = \phi_2(x,y) \text{ (say)}$$

$$\therefore \phi_2(z,0) = \frac{1}{\cos z - 1}$$

By Milne's Thomson method

$$F(z) = \int \{\phi_1(z,0) - i\phi_2(z,0)\} dz + c$$

$$F(z) = (1-i)\int \frac{1}{(\cos z - 1)} dz + c = -(1-i)\int \frac{1}{1-\cos z} dz + c$$

$$\Rightarrow F(z) = (1-i)\cot\frac{z}{2} + c$$

$$\Rightarrow f(z) = \frac{(1-i)}{(1+i)}\cot\frac{z}{2} + \frac{c}{1+i}$$

$$\Rightarrow \boxed{f(z) = \frac{(1-i)}{(1+i)}\cot\frac{z}{2} + c_1}, \text{ where } c_1 = \frac{c}{1+i}$$

Question No 6 :→ Find the image of the region bounded by $(0,0), (1,0), (1,2), (0,2)$ by the transformation $w = (1+i)z + 2 - i$.

Solution :– The given region is a rectangle R bounded by the lines $x=0$, $x=1$, $y=0$ & $y=2$ in z-plane.

The given transformation is

$$w = (1+i)z + 2 - i = (1+i)(x+iy) + 2 - i$$

$$\Rightarrow w = u + iv = (x - y + 2) + i(x + y - 1)$$

$$\therefore u = x - y + 2 \quad \& \quad v = x + y - 1$$

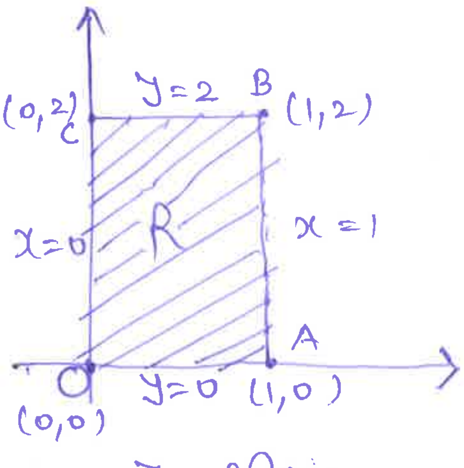
Now $x = 0 \Rightarrow \left.\begin{array}{l} u = -y+2 \\ v = y-1 \end{array}\right\} \Rightarrow u + v = 1$

$x = 1 \Rightarrow \left.\begin{array}{l} u = -y+3 \\ v = y \end{array}\right\} \Rightarrow u + v = 3$

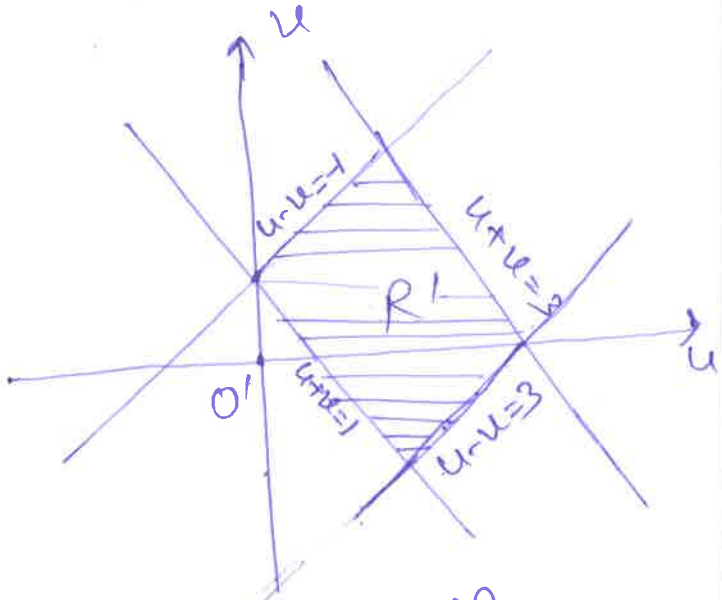
$y = 2 \Rightarrow \left.\begin{array}{l} u = x \\ v = x+1 \end{array}\right\} \Rightarrow u - v = -1$

$y = 0 \Rightarrow \left.\begin{array}{l} u = x+2 \\ v = x-1 \end{array}\right\} \Rightarrow u - v = 3$

The required image R' in w-plane is the region bounded by the lines $u+v=1$, $u+v=3$, $u-v=3$ and $u-v=-1$



z- plane



w-plane

Q7 Find the bilinear transformation which maps the points $z = 0, -1, i$ onto $w = i, 0, \infty$.

The bilinear transformation mapping $z = 0, -1, i$ into $w = i, 0, \infty$ is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[Here given
$z_1 = 0$, $z_2 = -1$, $z_3 = i$
$w_1 = i$, $w_2 = 0$, $w_3 = \infty$]

$$\Rightarrow \frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[Since w_3 is ∞ $\therefore$ formula cannot be applied directly]

$$\Rightarrow \frac{(w-i)(-1)}{(-1)(0-i)} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\frac{(w-i)}{(-i)} = \frac{z(1+i)}{(z-i)}$$

$$(w-i) = \frac{(-i+1)z}{(z-i)}$$

$$w = \frac{(1-i)z}{z-i} + i$$

$$\boxed{w = \frac{z+1}{z-i}}$$

which is the required bilinear transformation.

Question No. 8:- Find the bilinear transform which maps the points $z = 1, i, -1$ into the points $w = i, 0, -i$. Hence find the image of $|z| < 1$.

Solution:- we have

$$\frac{(w-i)\,i}{(w+i)(-i)} = \frac{(z-1)(1+i)}{(z+1)(i-1)}$$

$$\Rightarrow \frac{w-i}{w+i} = i\left(\frac{z-1}{z+1}\right)$$

$$\Rightarrow \frac{2w}{-2i} = \frac{iz-i+z+1}{iz-i-z-1} = \frac{(i+1)z-(i-1)}{(i-1)z-(i+1)} = -i\left(\frac{z-i}{z+i}\right)$$

(Applying C & D formula)

$$\Rightarrow \boxed{w = \frac{i-z}{i+z}} \quad \text{--- (1)}$$

which is required bilinear transformation

(1) can be rewritten as

$$z = i\left(\frac{1-w}{1+w}\right)$$

$\therefore |z| < 1$ is mapped into the region

$$\left|i\left(\frac{1-w}{1+w}\right)\right| < 1$$

$$\Rightarrow \frac{|i|\,|1-w|}{|1+w|} < 1$$

$$\Rightarrow |1-w| < |1+w| \qquad \because |i| = 1$$

$$\Rightarrow |1-u-iv| < |1+u+iv|$$

$$\Rightarrow (1-u)^2+v^2 < (1+u)^2+v^2$$

$$\Rightarrow 1+u^2+v^2-2u < 1+u^2+v^2+2u$$

$$\Rightarrow u > 0$$

Hence the interior of the circle $|z| = 1$ in z-plane is mapped into the entire half of the w-plane to the right of the imaginary axis.

UNIT-5

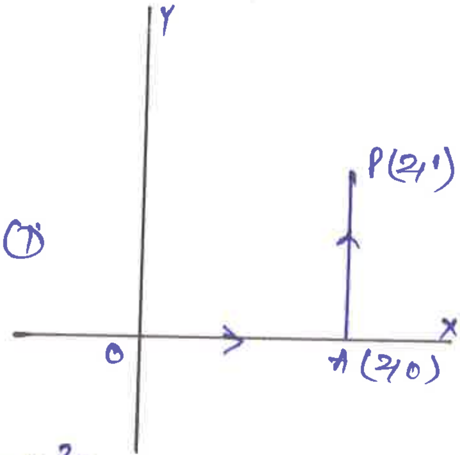
Ques 1| Evaluate $\int_0^{2+i} (\bar{z})^2 dz$, along the real axis from $z=0$ to $z=2$ & then along a line parallel to y-axis from $z=2$ to $z=2+i$

Sol^n ⇒ $(\bar{z})^2 = (x-iy)^2 = (x^2-y^2) - 2ixy$

$$\Rightarrow \int_0^{2+i} (\bar{z})^2 dz = \int_0^{2+i} [(x^2-y^2)-2ixy](dx+i\,dy)$$

According to the question

$$\int_0^{2+i} (\bar{z})^2 dz = \int_{OA} (x^2-y^2-2ixy)\,dz + \int_{AP} (x^2-y^2-2ixy)\,dz \quad \text{—(1)}$$



Along OA ⇒ $y=0 \Rightarrow dy=0$ & x varies from 0 to 2

$$\int_{OA} (x^2-y^2-2ixy)\,dz = \int_{x=0}^{2} x^2\,dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3}$$

Along AP ⇒ $x=2 \Rightarrow dx=0$ & y varies from 0 to 1.

$$\int_{AP} (x^2-y^2-2ixy)\,dz = \int_0^1 (4-y^2-4iy)\,dy\cdot i$$

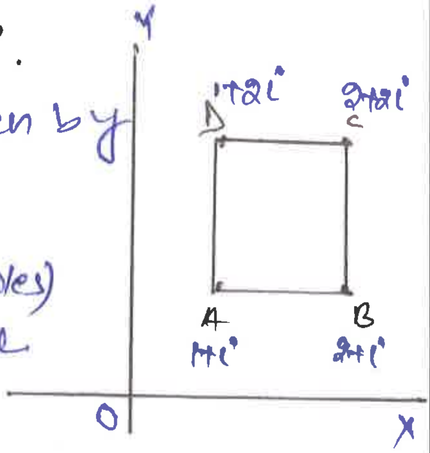
$$= \left(4iy - i\frac{y^3}{3} + 2y^2\right)_0^1 = 4i - \frac{1}{3}i + 2 = 2 + \frac{11}{3}i$$

Hence from (1) $\int_0^{2+i} (\bar{z})^2 dz = \frac{8}{3} + 2 + \frac{11}{3}i = \frac{14}{3} + \frac{11}{3}i$

Ques 2| Evaluate $\oint_C \frac{2z^2+5}{(z+2)^3(z^2+4)}\,dz$, where C is the square with vertices at $1+i,\ 2+i,\ 2+2i,\ 1+2i$.

Sol^n $f(z) = \frac{2z^2+5}{(z+2)^3(z^2+4)}$, Poles are given by

$(z+2)^3(z^2+4) = 0 \Rightarrow$ $z=-2$ (order 3), $z = \pm 2i$ (simple poles)



Since all poles are outside the square ABCD, Hence by Cauchy's Integral th^m

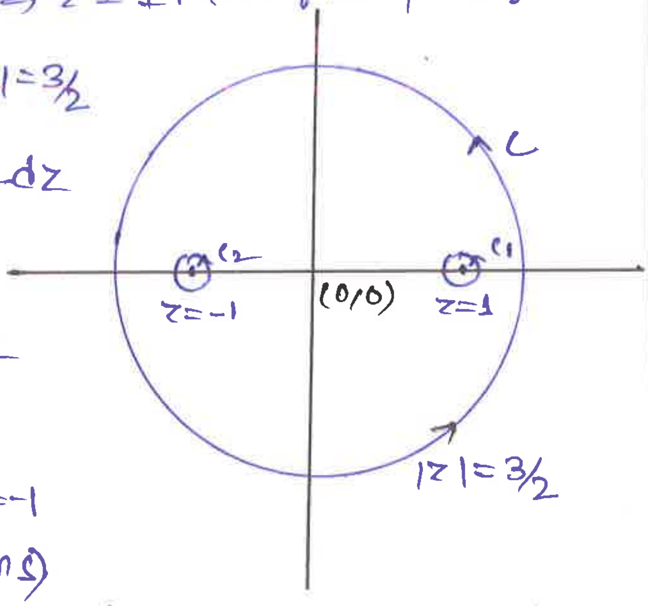
$$\oint_C \frac{2z^2+5}{(z+2)^3(z^2+4)}\,dz = 0$$

Ques3| Evaluate $\int_C \frac{z^2+1}{z^2-1}\,dz$, where C is a circle

a) $|z| = \frac{3}{2}$, (b) $|z-1| = 1$ (c) $|z| = \frac{1}{2}$

Sol^n| Poles are given by $z^2-1=0 \Rightarrow z = \pm 1$ (simple poles).

(a) $|z| = \frac{3}{2}$, $z = \pm 1$ lies inside $|z| = 3/2$



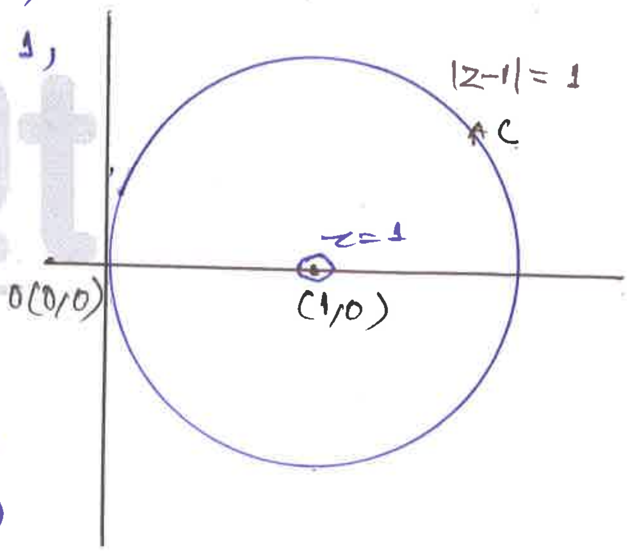
so $\int_C \frac{z^2+1}{z^2-1}\,dz = \oint_{C_1} \frac{z^2+1}{z^2-1}\,dz + \oint_{C_2} \frac{z^2+1}{z^2-1}\,dz$

$$= \oint_{C_1} \frac{\left(\frac{z^2+1}{z+1}\right)}{z-1}\,dz + \oint_{C_2} \frac{\left(\frac{z^2+1}{z-1}\right)}{(z+1)}\,dz$$

$$= 2\pi i\left(\frac{z^2+1}{z+1}\right)_{z=1} + 2\pi i\left(\frac{z^2+1}{z-1}\right)_{z=-1}$$

$$= 2\pi i(1) + 2\pi i(-1) = 0 \text{ (Ans)}$$

(b) $|z-1| = 1$, centre $(1,0)$, radius 1, only $z=1$ lies inside the circle $|z-1|=1$ so

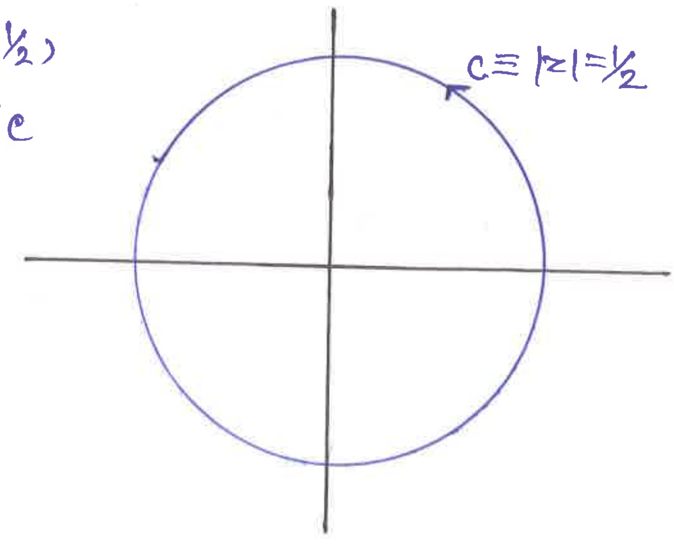


$$\int_C \frac{z^2+1}{z^2-1}\,dz = \int_{C_1} \frac{\left(\frac{z^2+1}{z+1}\right)}{(z-1)}\,dz$$

$$= 2\pi i\left(\frac{z^2+1}{z+1}\right)_{z=1} = 2\pi(1)i = 2\pi i$$

(Ans.)

(c) $|z| = \frac{1}{2}$, centre $(0,0)$ & Radius $1/2$, Any pole doesnot lie inside the circle, hence by cauchy's integral th^m

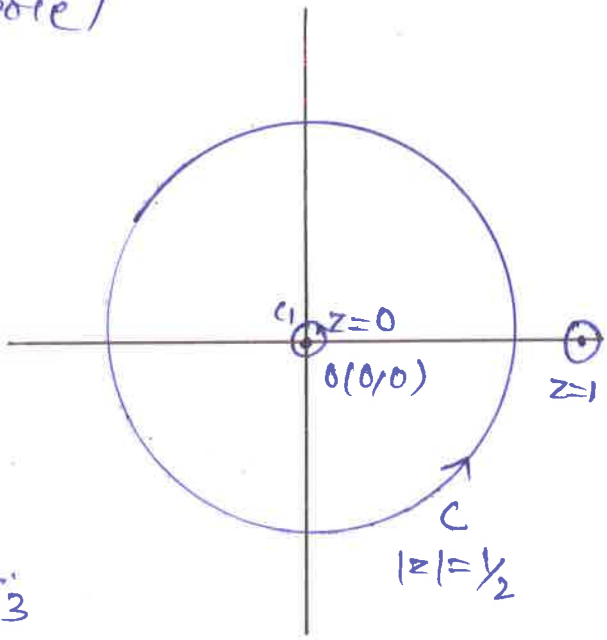


$$\int_C \frac{z^2+1}{z^2-1}\,dz = 0 \quad \text{(Ans)}.$$

Ques 4] Evaluate $\oint_C \frac{e^z}{z(1-z)^3}dz$ where C is $|z|=\frac{1}{2}$

Sol^n Poles are given by $z(1-z)^3=0 \Rightarrow z=0$ & $z=1$
(Simple Pole) (order 3)

$|z|=\frac{1}{2} \rightarrow$ centre $(0,0)$
& Radius $1/2$



$\Rightarrow$ only $z=0$ lies inside c.

$$\oint_C \frac{e^z}{z(1-z)^3}dz$$

$$= \oint_{C_1} \left[\frac{\frac{e^z}{(1-z)^3}}{z}\right]dz$$

$$= \left[\frac{e^z}{(1-z)^3}\right]_{z=0} \cdot 2\pi i = \frac{2\pi i\, e^0}{(1-0)^3}$$

$$= 2\pi i \quad \text{(Ans)}.$$

Ques 5] Find the residue of $f(z)=\frac{z^3}{z^2-1}$ at $z=\infty$

Sol^n $\Rightarrow$ Res. of $f(z)$ at $z=\infty = \lim_{z\to\infty}[-z f(z)]$ (If limit exist)

OR

$= -$[coefficient of $\frac{1}{z}$ in the expansion of $f(z)$)

So

$$\text{Res}(z=\infty) = \lim_{z\to\infty}\left[-z\left(\frac{z^3}{z^2-1}\right)\right] = \text{limit does not exist so}$$

$$f(z) = \frac{z^3}{z^2-1} = \frac{z^3}{z^2(1-\frac{1}{z^2})} = z\left(1-\frac{1}{z^2}\right)^{-1}$$

$$= z\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\cdots\right) = z+\frac{1}{z}+\frac{1}{z^3}+\cdots$$

$= -$ coeff^n of $\frac{1}{z}$

$= -(1)$

$= -1$.

Ques-6 Find the poles (with its order) and residue at each pole of the function $f(z) = \dfrac{1-2z}{(z-1)(z-2)^2}$

Soln Here $f(z) = \dfrac{1-2z}{(z-1)(z-2)^2}$

Poles are given by $(z-1)(z-2)^2 = 0$

$z = 1 \quad , \quad z = 2$ (Order 2) $\quad \boxed{z = 2, 2}$

$z=1$ is a simple Pole and $z = 2$ is a pole of Order 2

Residue of $f(z)$ at Simple Pole $z = 1$

$$R_1 = \lim_{z \to 1} (z-1) \cdot f(z)$$

$$= \lim_{z \to 1} \cancel{(z-1)} \times \frac{(1-2z)}{\cancel{(z-1)}(z-2)^2}$$

(Put Limit)

$$R_1 = \frac{(-1)}{1} = -1 \qquad \boxed{R_1 = -1}$$

Residue of $f(z)$ at $z = 2$ (double Pole)

$$R_2 = \frac{1}{\lfloor 2-1} \left[\frac{d}{dz} \left\{ (z-2)^2 \cdot f(z) \right\} \right]_{z=2}$$

Formula for Order m

$$\therefore \operatorname{Res} f(z) = \frac{1}{\lfloor m-1} \left[\frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\} \right]_{z=a}$$

$$R_2 = \frac{1}{\lfloor 1} \left[\frac{d}{dz} \left\{ (z-2)^2 \times \frac{(1-2z)}{(z-1)(z-2)^2} \right\} \right]_{z=2}$$

$$R_2 = \left[\frac{d}{dz} \left\{ \left(\frac{1-2z}{z-1} \right) \right\} \right]_{z=2}$$

$\therefore$ ~~diff~~ ~~by~~

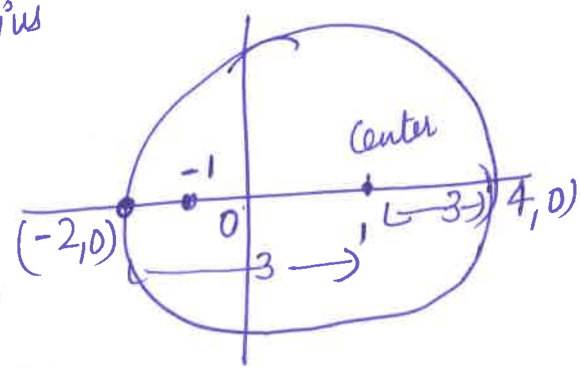
$$R_2 = \left[\frac{(z-1)(-2) - (1-2z) \times 1}{(z-1)^2} \right]_{z=2}$$

$$R_2 = \frac{1}{1} = 1 \quad \text{Ans}$$

Ques. 7 Evaluate $\oint_C \frac{e^z}{(z+1)^2}\,dz$, where C is the Circle $|z-1|=3$

Soln Here $f(z) = \frac{e^z}{(z+1)^2}$ has one singular point $z=-1$ which is a Pole or order 2 and it lies inside the Circle $|z-1|=3$

[Circle with Center (1,0) radius 3]



Residue of $f(z)$ at $z=-1$

$$R_1 = \lim_{z\to -1} \left\{ \frac{d}{dz} \left\{ [z-(-1)]^2 f(z) \right] \right\}_{z=-1}$$

$$R_1 = \lim_{z\to(-1)} \left[\frac{d}{dz} \left\{ (z+1)^2 \times \frac{e^z}{(z+1)^2} \right] \right]_{z=-1}$$

$$R_1 = \lim_{z\to(-1)} \left[\frac{d}{dz} (e^z) \right]_{z=-1}$$

$$R_1 = e^{-1}$$

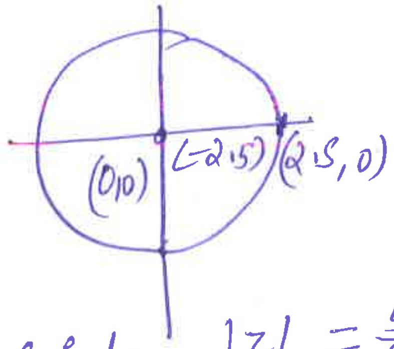
By Residue Theorem $\oint_C \frac{e^z}{(z+1)^2}\,dz = 2\pi i (R_1)$

$$= 2\pi i \times \frac{1}{e}$$

$$= \frac{2\pi i}{e} \quad \text{Ans}$$

Ques-8 Find the residue of $f(z) = \frac{z^3}{(z-1)^4 (z-2)(z-3)}$ at its Pole and hence evaluate $\int_C f(z)\,dz$, where C is Circle $|z| = \frac{5}{2}$

Ans-8 Here $f(z) = \dfrac{z^3}{(z-1)^4 (z-2)(z-3)}$ C is the Circle of radius $= 2.5$ (Center at $(0,0)$)



Poles are given by

$z = 1$ (order 4), $z=2$, $z=3$

Only $z=1$ and $z=2$ lies inside Circle $|z| = \frac{5}{2}$

Residue at $z=1$ (R_1) (order 4) $= \dfrac{1}{\lfloor 4-1} \left[\dfrac{d^3}{dz^3} \left\{ \cancel{(z-1)^4} \cdot \dfrac{z^3}{\cancel{(z-1)^4}(z-2)(z-3)} \right\} \right]_{z=1}$

$$= \frac{1}{\lfloor 3} \left[\frac{d^3}{dz^3} \left\{ \frac{z^3}{(z-2)(z-3)} \right\} \right]_{z=1}$$

$$= \frac{1}{6} \left[\frac{d^3}{dz^3} \left\{ z+5 + \frac{19z-30}{z^2-5z+6} \right\} \right]_{z=1}$$

$$= \frac{1}{6} \left[\frac{d^3}{dz^3} \left\{ (z+5) + \frac{27}{z-3} - \frac{8}{z-2} \right\} \right]_{z=1}$$

$$= \frac{1}{6} \left[\frac{d^2}{dz^2} \left\{ 1 - \frac{27}{(z-3)^2} + \frac{8}{(z-2)^2} \right\} \right]_{z=1}$$

$$= \frac{1}{6} \left[\frac{d}{dz} \left\{ \frac{54}{(z-3)^3} - \frac{16}{(z-2)^3} \right\} \right]_{z=1}$$

$$= \frac{1}{6} \left[\frac{-162}{(z-3)^4} + \frac{48}{(z-2)^4} \right]_{z=1}$$

$$= \frac{1}{6} \left[\frac{-162}{16} + 48 \right] = 8 - \frac{27}{16}$$

$$= \frac{101}{16}$$

Residue at $z=2$

$$R_2 = \lim_{z\to 2} (z-2) \times \frac{z^3}{(z-1)^4 (z-2)(z-3)}$$

$$R_2 = \lim_{z\to 2} \left[\frac{z^3}{(z-1)^4 (z-3)}\right] = \frac{8}{(-1)} = -8$$

Hence By Cauchy Residue Theorem

$$\int_C f(z)\,dz = 2\pi i (R_1 + R_2) = 2\pi i \left(\frac{101}{16} - 8\right)$$

$$= \frac{-27\pi i}{8} \quad \underline{\text{Ans}}$$

Ques-9 Find the Taylor's and laurent Series which represent the function

$$f(z) = \frac{z^2-1}{(z+2)(z+3)}$$ when (i) $|z| < 2$ II $2 < |z| < 3$ III $|z| > 3$

Solⁿ $$f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

(i) $|z| < 2$

$$f(z) = 1 + \frac{3}{2\left(\frac{z}{2}+1\right)} - \frac{8}{3\left(\frac{z}{3}+1\right)}$$ or

$$f(z) = 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$f(z) = 1 + \frac{3}{2}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n$$

This is a series in positive powers of z so it is Taylor's series within a circle $|z| = 2$

Ans- 9 II $\quad 2<|z|<3$

Here $2<|z|$ $\quad |z|<3$

$\frac{2}{|z|}<1$, $\frac{|z|}{3}<1$

$$f(z) = 1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$f(z) = 1+\frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n-\frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

This is Laurent Series within annulas $2<|z|<3$

[. +ive , -ive Powers of z]

Ans 9 III $\quad |z|>3$

$\frac{3}{z}<1$

$$f(z) = 1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1+\frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n-\frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$$

This is Laurent Series in annulas $3<|z|<R$.

Ques-10 Discuss the Singularity of $\frac{1}{\sin z-\cos z}$ at $z=\frac{\pi}{4}$

Solⁿ at $z=\frac{\pi}{4}$, Simple Pole

$\because$ if $\lim_{z\to a} f(z)=\infty$, then $f(z)$ has a pole at $z=a$

Here $f(z)=\frac{1}{\sin z-\cos z}$ at $z=\frac{\pi}{4}$

$$\lim_{z\to\frac{\pi}{4}} f(z)=\infty$$

Simple Pole