P. U.T : Even Semester (2022-23)

(Solution)

Course | Branch : B.Tech - Ist Year

Semester : II

Sub. Name : Engineering Maths IInd.

Sub Code : BAS-203 (SET-A)

Section — A.

Solution. 1 :-

(A) :- given.
$$\frac{d^2y}{dx^2} + 4y = \sin 2x \quad \text{——} (1)$$

we write, $\frac{d}{dx} = D$.

now, eq^n (1) become

$$D^2y + 4y = \sin 2x$$

$$\Rightarrow (D^2+4)y = \sin 2x \Rightarrow f(D^2)y = \sin 2x$$

$$\because \quad P.I = \frac{\sin 2x}{f(D^2)}$$

$$= \frac{\sin 2x}{D^2+4} \quad (\because f(D^2) = D^2+4)$$

$$= \frac{\sin 2x}{-(2)^2+4} \quad (\because D^2 = -(2)^2)$$

$\because f(-2^2) = -4+4 = 0$

now, $P.I. = x \cdot \dfrac{\sin 2x}{f'(D^2)}$

$= x \cdot \dfrac{\sin 2x}{2D}$ $\left(\because \begin{array}{l} f(D^2) = D^2+4 \\ f'(D^2) = 2D \end{array}\right)$

$= \dfrac{x}{2} \cdot \dfrac{1}{D} \sin 2x$ $\left(\because \begin{array}{l} D = \frac{d}{dx} \text{ \&} \\ \frac{1}{D} = \int dx \end{array}\right)$

$= \dfrac{x}{2} \cdot \int \sin 2x \, dx$

$= \dfrac{x}{2} \left(-\dfrac{\cos 2x}{2}\right)$

$$\boxed{P.I = \frac{-x \cos 2x}{4}}$$

1/ 3.

given, $x^2 y'' + x y' - y = 0$ —— (1)

$\Rightarrow x^2 D^2 y + x D y - y = 0$ $\left(D = \frac{d}{dx}\right)$

$\because$ given eq^n is homogeneous linear differential eq^n.

$\therefore$ we will reduce this eq^n in linear diff. eq^n with constant coefficients by substituting.

$x = e^z \Rightarrow z = \log x$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{1}{x} \qquad \left(\because z = \log x \quad \therefore \frac{dz}{dx} = \frac{1}{x}\right)$$

$$\Rightarrow x\cdot\frac{dy}{dx} = \frac{dy}{dz} \Rightarrow x\frac{d}{dx} = \frac{d}{dz} = D \text{ (say).}$$

—— (A)

IIly.

$$x^2\frac{d^2}{dx^2} = D(D-1) \quad \text{—— (B)}$$

by using (A) & (B) we can write eq^n (1) as -

$$D(D-1)y + Dy - y = 0$$

$$\Rightarrow D^2y - \cancel{D}y + \cancel{D}y - y = 0$$

$$\Rightarrow (D^2-1)y = 0$$

Auxiliary eq^n is -

$$m^2 - 1 = 0 \Rightarrow (m+1)(m-1) = 0$$

$$\Rightarrow m = 1, -1$$

$$C.F = C_1e^z + C_2e^{-z}$$

$$C.F = C_1x + C_2\frac{1}{x} \qquad (\because x = e^z)$$

$$P.I = 0$$

Hence, complete solution is .

$$\boxed{y = C.F + P.I = C_1x + C_2\frac{1}{x}}$$

1(c) :- $\mathcal{L}\{t e^{-t} \cosh t\}$. —— (i)

Method - I :-

let $f(t) = e^{-t} \cosh t$

$$L\{f(t)\} = L\{e^{-t} \cosh t\}$$

$$\Rightarrow L\{e^{-t} \cosh t\} = \frac{p+1}{(p+1)^2 - 1^2}$$

$$= f(p). \quad \text{—— (ii)}$$

$\because$ for $p > |a|$, $L\{\cosh at\} = \frac{p}{p^2 - a^2}$

& $L\{e^{bt} \cosh at\} = \frac{(p-b)}{(p-b)^2 - a^2}$

$$\because L\{t \cdot f(t)\} = -\frac{d}{dp}[f(p)]$$

$$\Rightarrow L\{t \cdot \underbrace{e^{-t} \cosh t}_{f(t)}\} = -\frac{d}{dp}\left[\frac{p+1}{(p+1)^2 - 1}\right] \text{ (by (ii))}$$

$$= -\left\{\frac{(p+1)^2 - 1 - 2(p+1)^2}{((p+1)^2 - 1)^2}\right\}$$

$$= -\left\{\frac{p^2 + 2p - 2p^2 - 2 - 4p}{(p^2 + 2p)^2}\right\}$$

$$= -\left\{\frac{-p^2 - 2p - 2}{(p^2 + 2p)^2}\right\}$$

$$\boxed{\mathcal{L}\{t \cdot e^{-t} \cosh t\} = \frac{(p+1)^2 + 1}{p^2 (p+2)^2}}$$

Method. II.

Let $f(t) = t \cdot \cosh t$.

& $g(t) = \cosh t$

$$L.\{\cosh t\} = \frac{p}{p^2-1} = g(p)$$

$$L.\{t \cdot \cosh t\} = -\frac{d}{dp}\left\{\frac{p}{(p^2-1)}\right\} \qquad \left(\because L.\{t \cdot g(t)\} = -\frac{d}{dp} g(p)\right\}$$

$$= -\frac{(p^2-1) - 2p^2}{(p^2-1)^2}$$

$$L.\{t \cdot \cosh t\} = -\frac{(-p^2-1)}{(p^2-1)^2} = \frac{p^2+1}{(p^2-1)^2} = f(p).$$

Now,

$$L.\{e^{-t}\, t \cosh t\} = f(p+1) \qquad (\because L\{e^{at} f(t)\} = f(p-a)\}$$

$$= \frac{(p+1)^2+1}{((p+1)^2-1)^2}$$

$$\boxed{L.\{e^{-t}\, t \cosh t\} = \frac{(p+1)^2+1}{(p^2+2p)^2}}$$

1(d).

$$L^{-1}\left(\frac{e^{-\pi p}}{p^2}\right) \quad \text{——} (1)$$

$\because$ by second shifting property.

if $g(t) = \begin{cases} f(t-a) & , t>a \\ 0 & t<a \end{cases}$ then.

$$L.\{g(t)\} = e^{-ap} f(p).$$

$$\Rightarrow L^{-1}\{e^{-ap} f(p)\} = g(t).$$

$\because$ in eqⁿ ①

$$f(p) = \frac{1}{p^2}$$

$$f(t) = L^{-1}(f(p)) = L^{-1}\{\frac{1}{p^2}\} = t.$$

$$\Rightarrow f(t) = t$$

now,
$$g(t) = \begin{cases} t - a & t > a \\ 0 & t < a \end{cases}$$

$$\because L^{-1}\{e^{-ap} f(p)\} = g(t)$$

$$\Rightarrow L^{-1}\{e^{-\pi p} \frac{1}{p^2}\} = g(t)$$

i.e, $a = \pi$.

now,
$$L^{-1}\left[\frac{e^{-\pi p}}{p^2}\right] = \boxed{g(t) = \begin{cases} t-\pi & t > \pi \\ 0 & t < \pi \end{cases}}$$

$$= (t-\pi) \begin{cases} 1 & t > \pi \\ 0 & t < \pi \end{cases}$$

$$L\left[\frac{e^{\pi p}}{p^2}\right] = \underline{(t-\pi)} \; u(t-2).$$

when $u(t-2)$ is a unit-step function.

1/(E) :-

Bounded Sequence :- A sequence $\{a_n\}$ is said to be bounded when it is bounded both above and below.

Bounded above :- A sequence $\{a_n\}$ is said to be bounded above if $\exists$ a real number K such that -

$$a_n \le K \quad \forall\ n \in \mathbb{N}.$$

Bounded below :- A sequence $\{a_n\}$ is said to be bdd below if $\exists$ a real number k such that.

$$a_n \ge k \quad \forall\ n \in \mathbb{N}.$$

ie, A seq^n $\{a_n\}$ is bdd if if $\exists$ two real numbers k & K $(k \le K)$ such that -

$$k \le a_n \le K \quad \forall\ n \in \mathbb{N}$$

Choosing $M = \max\{|k|, |K|\}$.

we can also define a bdd seq^n as.

$$|a_n| \le M \quad \forall\ n \in \mathbb{N}.$$

Eg. let we have a seq^n $\{a_n\} = \{\frac{1}{n}\}$

$$a_n = \frac{1}{n} \Rightarrow \{\tfrac{1}{n}\} = \{1, \tfrac{1}{2}, \tfrac{1}{3}, ---\}$$

all the terms of $\{a_n\}$ lie b/w 0 & 1

i.e,
$$0 < a_n \le 1 \quad \forall\ n \in \mathbb{N}.$$

$\therefore$ $\{a_n\}$ is bounded.

Unbounded Sequence :- If sequence $\{a_n\}$ is not bounded above or not bdd below, or neither bdd above nor below then $\{a_n\}$ is said to be unbounded.

i.e,

If $\exists$ a real number M such that $|a_n| \leq M \; \forall n \in \mathbb{N}$,

then the seq^n $\{a_n\}$ is said to be unbounded.

Eg :-

(i) $\{a_n\} = \{2^{n-1}\}$. $n \in \mathbb{N}$.

$= \{1, 2, 2^2, 2^3 - - \}$.

$\forall n \in \mathbb{N}$, $\exists$ no real number K such that $a_n \leq K$

i.e, $\{a_n\}$ is unbdd.

we can also say. as $n \to \infty$ $a_n \to \infty$

i.e, $\lim_{n \to \infty} a_n = \infty$ i.e, $\{a_n\}$ is not bdd above.

i.e, unbounded.

(ii) $\{a_n\} = \{-2^n\}$. $= \{-2, -4, -8, -16 - - \}$

$0 < a_n$, but as $n \to \infty$ $a_n \to -\infty$

$\lim_{n \to \infty} a_n = -\infty$ i.e, it is not bdd below

hence. unbounded.

(iii)

1(F):- Dirichlet's Condition :- The sufficient conditions for the uniform convergence of a Fourier series is called Dirichlet's condition

Any function $f(x)$ can be expressed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

where a_n, b_n, c_n are constants provided -

(i) $f(x)$ is periodic, single valued and finite.

(ii) $f(x)$ has a finite number of finite-discontinuities in any one period.

(iii) $f(x)$ has a finite number of maxima and minima.

When these conditions satisfied fourier series converge to $f(x)$ at every pt. of continuity

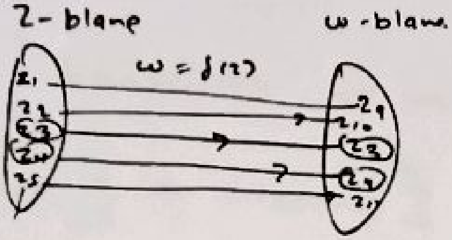
At a point of discontinuity, the sum of the series is equal to the mean of the limits on the left & right.

i.e, $\frac{1}{2}[f(x+0) + f(x-0)]$.

limit on the right

limit on the left.

1/(G) :- **Fixed points:-** The points which coincide with their transformation are called fixed point or invariant point of the transformation.



z_3 & z_4 are coincides with $w = f(z)$

i.e, z_3 & z_4 are fixed points.

$\because \quad w = f(z)$

$(w)_{z_3} = f(z_3) = z_3. \quad (\because \ z_3 \xrightarrow{} z_3)$ (z-plane → w-plane)

Ily. $(w)_{z_4} = f(z_4) = z_4 \quad (\because \ z_4 \xrightarrow{} z_4)$ (z-plane → w-plane)

$\Rightarrow$ i.e, Fixed point of a transformation $w = f(z)$ are obtained by the eqn $\underline{z = f(z)}$

Ex. $w = \dfrac{2z-5}{z+4}$ — (1) fixed points ??

$\because$ we know that for fixed points we will replace w by z

i.e, $z = \dfrac{2z-5}{z+4} \Rightarrow z^2 + 4z = 2z - 5$

$\Rightarrow z^2 + 2z + 5 = 0 \Rightarrow z = \dfrac{-2 \pm \sqrt{4-20}}{2}$

$\Rightarrow z = -1 \pm 2i$

$\Rightarrow z = -1+2i, \ -1-2i$ are the fixed points.

1/(a) :- if $u = 3x^2y - y^3$ find the analytic func $f(z) = u + iv$. (1)

$\because \quad u = 3x^2y - y^3$ —— (i)

on. differentiating partially w.r.t. x, we get-

$$\frac{\partial u}{\partial x} = 6xy \quad \text{—— (ii)}$$

again diff. (ii) w.r.t. 'x' partially -

$$\frac{\partial^2 u}{\partial x^2} = 6y. \quad \text{—— (iii)}$$

diff. (i) w.r.t y partially. we get-

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 \quad \text{—— (iv)}$$

agaoing diff. (iv) w.r.t. 'y' partially, we get-

$$\frac{\partial^2 u}{\partial y^2} = -6y$$

Since, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y - 6y = 0$

$\therefore$ u is a harmonic func.

Now, $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$= \left(-\frac{\partial u}{\partial y}\right) dx + \frac{\partial u}{\partial x} dy$ (By C-R eq^n)

$= (3y^2 - 3x^2) dx + 6xy\, dy$ By (ii) & (iv)

$= 3y^2 dx + 6xy\, dy - 3x^2 dx$

$dv = 3\, d(y^2 x) - 3x^2 dx$

on integrating, we get -

$$v = 3y^2x - x^3 + c_1$$

$\therefore$ $f(z) = u + iv + ic_1$

$$\boxed{f(z) = 3x^2y - y^3 + i(3y^2x - x^3) + c_2}$$

1 /(I) :-

$$f(z) = \frac{z^2+2z}{(z-2)(z-1)}$$

Poles of $f(z)$ are given by -

$$(z-2)(z-1) = 0$$

$\Rightarrow$ $z = 1, 2$ both are simple poles.

Residue at $z=1$ is -

$$R_1 = \lim_{z\to 1} \cancel{(z-1)}\left(\frac{z^2+2z}{(z-2)\cancel{(z-1)}}\right)$$

[$\because$ ~~for~~ simple pole at $z=a$ is - $\lim_{z\to a}(z-a)f(z)$

$$= \lim_{z\to 1} \frac{z^2+2z}{(z-2)} = -3$$

Residue at $z=2$, is -

$$R_2 = \lim_{z\to 2} \cancel{(z-2)}.\left(\frac{z^2+2z}{\cancel{(z-2)}(z-1)}\right)$$

$$R_2 = \lim_{z\to 2} \frac{z^2+2z}{(z-1)} = 8$$

1(J). If a function $f(z)$ is analytic in a domain D, then at any point $z=a$ of D, $f(z)$ has derivatives of all orders, all of which are again analytic functions in D, their values are given by

$$f^{n}(a) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} dz$$

where C is any closed contour in D. surrounding the point $z=a$.

1(H). II^{nd} Method. :-

$$u = 3x^2y - y^3 \quad \text{(given)}$$

$$\Rightarrow \frac{\partial u}{\partial x} = 6xy \quad , \frac{\partial u}{\partial y} = 3x^2 - 3y^2$$

let $\phi_1(x,y) = 6xy$ & $\phi_2(x,y) = 3x^2 - 3y^2$

now, $\phi_1(z,0) = 0$ & $\phi_2(z,0) = 3z^2$

now,

$$f(z) = \int \{ \phi_1(z,0) - i\phi_2(z,0) \, dz + c$$

$$= \int (0 - i3z^2) dz + c$$

$$f(z) = -iz^3 + c_1$$

$$= -(x+iy)^3 i + c_1$$

$$= -(x^3 + (iy)^3 + 3x^2 iy + 3(iy)^2 x) i + c_1$$

$$\Rightarrow (-x^3 + iy^3 - 3x^2 iy + 3xy^2) i + c$$

$$\boxed{f(z) = (3x^2y - y^3) + i(3xy^2 - x^3) + c_1}$$

SECTION-B

Solⁿ 2: Given: $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = y$ and $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 25x + 16e^t$ —— (A)

Let $\frac{d}{dt} = D$, then the given system of eqⁿ becomes:-

$$(D^2-4D+4)x - y = 0 \quad \text{—— ①}$$

$$-25x + (D^2+4D+4)y = 16e^t \quad \text{—— ②}$$

On eqⁿ ① × (D^2+4D+4) and adding with ②, we get -

$$(D^2-4D+4)(D^2+4D+4)x - 25x = 16e^t$$

$$\Rightarrow (D^4 - 8D^2 - 9)x = 16e^t$$

∴ Auxiliary Eqⁿ is: $m^4 - 8m^2 - 9 = 0$

$$\Rightarrow (m^2-9)(m^2+1) = 0 \qquad \Rightarrow m = \pm 3, \pm i$$

$$\therefore \quad C.F = C_1 e^{3t} + C_2 e^{-3t} + C_3 \cos t + C_4 \sin t$$

and $P.I = \frac{1}{f(D)} e^{ax} = \frac{1}{D^4 - 8D^2 - 9} 16e^t = 16 \cdot \frac{1}{1-8-9} e^t = 16\frac{e^t}{-16}$

$$\Rightarrow \boxed{P.I = -e^t}$$

$$\therefore \boxed{x = C_1 e^{3t} + C_2 e^{-3t} + C_3 \cos t + C_4 \sin t - e^t} \quad \text{—— ③}$$

On diff. eqⁿ ③ w.r.t. 't', we get.

$$\frac{dx}{dt} = 3C_1 e^{3t} - 3C_2 e^{-3t} - C_3 \sin t + C_4 \cos t - e^t$$

$$\frac{d^2x}{dt^2} = 9C_1 e^{3t} + 9C_2 e^{-3t} - C_3 \cos t - C_4 \sin t - e^t$$

From (A), $y = \frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x$.

$$\therefore y = 9C_1 e^{3t} + 9C_2 e^{-3t} - C_3 \cos t - C_4 \sin t - e^t - 4[3C_1 e^{3t} - 3C_2 e^{-3t} - C_3 \sin t + C_4 \cos t - e^t] + 4[C_1 e^{3t} + C_2 e^{-3t} + C_3 \cos t + C_4 \sin t - e^t]$$

$$\therefore \boxed{y = C_1 e^{3t} + 7C_2 e^{-3t} - C_3 \sin t + C_4 \cos t - e^t}$$

OR

Apply method of Variation of Parameters to find the general solution of: $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 3x = \frac{e^t}{1+e^t}$. ②

Solⁿ: Given: $(D^2 - 4D + 3)x = \frac{e^t}{1+e^t}$ $\quad \left[\because \frac{d}{dt} = D\right]$

Auxiliary eqⁿ is $m^2 - 4m + 3 = 0$

$m^2 - 3m - m + 3 = 0$

$m(m-3) - 1(m-3) = 0$

$(m-1)(m-3) = 0 \Rightarrow m = 1, 3.$

$\therefore C.F = Ae^t + Be^{3t}$

Here, $u = e^t$ and $v = e^{3t}$ and $R = \frac{e^t}{1+e^t}$

$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^t & e^{3t} \\ e^t & 3e^{3t} \end{vmatrix} = 3e^{4t} - e^{4t} = 2e^{4t}$

$\therefore A = -\int \frac{Rv}{W} dt + C_1$

$A = -\int \frac{e^t \cdot e^{3t}}{(1+e^t)2e^{4t}} dt + C_1$

$= -\frac{1}{2}\int \frac{dt}{1+e^t} + C_1$

$= -\frac{1}{2}\int \frac{e^{-t}}{e^{-t}+1} dt + C_1$

$A = +\frac{1}{2}\int \frac{dp}{p} + C_1$ $\quad$ Let $1+e^{-t} = p$, $-e^{-t}dt = dp$.

$A = \frac{1}{2}\log p + C_1$

$\boxed{A = \log\sqrt{p} + C_1}$

$\boxed{A = \frac{1}{2}\log[1+e^{-t}] + C_1}$

$B = \int \frac{Ru}{W} dt + C_2$

$B = \int \frac{e^t \, e^t \, dt}{(1+e^t)2e^{4t}} + C_2$

$B = \frac{1}{2}\int \frac{dt}{e^{2t}(1+e^t)} + C_2$

$B = \frac{1}{2}\int \frac{e^{-2t}}{(1+e^t)} dt + C_2$

$= \frac{1}{2}\int \frac{e^{-3t}}{e^{-t}+1} dt + C_2$

$B = \frac{1}{2}\int \frac{e^{-t} \cdot e^{-2t} dt}{1+e^{-t}} + C_2$ $\quad$ Let $1+e^{-t} = r$, $-e^{-t}dt = dr$, $\boxed{e^{-t}dt = -dr}$

$B = -\frac{1}{2}\int \frac{(r-1)^2}{r} dr + C_2$ and $e^{-t} = r-1$, $e^{-2t} = (r-1)^2$

$B = -\frac{1}{2}\int \left[r + \frac{1}{r} - 2\right] dr + C_2$

$B = -\frac{1}{2}\left[\frac{r^2}{2} + \log r - 2r\right] + C_2$

$\boxed{B = \left[-\frac{1}{4}(1+e^{-t})^2 - \frac{1}{2}\log(1+e^{-t}) + (1+e^{-t})\right] + C_2}$

$\therefore$ Complete Solution is $x = Ae^{t} + Be^{3t}$ (3)

$$\therefore x = \left[\frac{1}{2}\log(1+e^{-t}) + c_1\right]e^{t} + \left[-\frac{1}{4}(1+e^{-t})^2 - \frac{1}{2}\log(1+e^{-t}) + (1+e^{-t}) + c_2\right]e^{3t}$$

Ans:

Q.3. Using Laplace Transform, solve simultaneous diffⁿ eqⁿs.

(a) $\frac{d^2x}{dt^2} + 5\frac{dy}{dt} - x = t$, $2\frac{dx}{dt} - \frac{d^2y}{dt^2} + 4y = 2$, given that $t=0, x=0, y=0, \frac{dx}{dt}=0, \frac{dy}{dt}=0$.

Solⁿ: Let $L\{x(t)\} = \bar{x}(p)$ and $L\{y(t)\} = \bar{y}(p)$

Now, Taking Laplace Transform of given eqⁿs., we get

$$\{p^2\bar{x} - px(0) - x'(0)\} + 5\{p\bar{y} - y(0)\} - \bar{x} = \frac{1}{p^2} \quad \text{and}$$

$$2\{p\bar{x} - x(0)\} - \{p^2\bar{y} - py(0) - y'(0)\} + 4\bar{y} = \frac{2}{p}$$

Using the given initial conditions, these equations reduces to

$$(p^2-1)\bar{x} + 5p\bar{y} = \frac{1}{p^2} \quad (1) \quad \text{and} \quad 2p\bar{x} - (p^2-4)\bar{y} = \frac{2}{p} \quad (2)$$

Eliminating $\bar{y}$ between (1) & (2), we get:

$$\{(p^2-1)(p^2-4) + 10p^2\}\bar{x} = \frac{p^2-4}{p^2} + 10$$

$$\bar{x} = \frac{11p^2-4}{p^2(p^2+1)(p^2+4)} = \frac{-1}{p^2} + \frac{5}{p^2+1} - \frac{4}{p^2+4}$$

Taking Inverse Laplace Transform, we get:

$$\boxed{x = -t + 5\sin t - 2\sin 2t} \quad (3)$$

Again eliminating $\bar{x}$ between (1) & (2), we get:

$$\{10p^2 + (p^2-1)(p^2-4)\}\bar{y} = \frac{2}{p} - \frac{2(p^2-1)}{p}$$

$$\bar{y} = \frac{4-2p^2}{p(p^2+1)(p^2+4)} = \frac{1}{p} - \frac{2p}{p^2+1} + \frac{p}{p^2+4}$$

Taking Inverse Laplace Transform, we have —

$$\boxed{y = 1 - 2\cos t + \cos 2t} \quad (4)$$

Eqⁿ (3) & (4) is required solution.

OR

Q.3(b) Use Convolution Theorem to evaluate $L^{-1}\left\{\frac{p^2}{(p^2+a^2)(p^2+b^2)}\right\}$. (4)

Sol^n: Given: $\frac{p^2}{(p^2+a^2)(p^2+b^2)} = \frac{p}{p^2+a^2} \cdot \frac{p}{p^2+b^2}$

Let $f(p) = \frac{p}{p^2+a^2}$ and $g(p) = \frac{p}{p^2+b^2}$

$\therefore$ $F(t) = L^{-1}\{f(p)\} = L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

$G(t) = L^{-1}\{g(p)\} = L^{-1}\left\{\frac{p}{p^2+b^2}\right\} = \cos bt$

Now, $F(u) = \cos au$ and $G(t-u) = \cos b(t-u)$

$\therefore$ By Convolution Theorem, we have–

$$L^{-1}\left\{\frac{p^2}{(p^2+a^2)(p^2+b^2)}\right\} = \int_0^t \cos au . \cos b(t-u)\, du$$

$$= \frac{1}{2}\int_0^t [\cos\{(a-b)u+bt\} + \cos\{(a+b)u-bt\}]\, du$$

$$= \frac{1}{2}\left[\frac{\sin\{(a-b)u+bt\}}{(a-b)} + \frac{\sin\{(a+b)u-bt\}}{(a+b)}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right]$$

$$\boxed{L^{-1}\left\{\frac{p^2}{(p^2+a^2)(p^2+b^2)}\right\} = \frac{a\sin at - b\sin bt}{a^2-b^2}}$$ Ans

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Q.4.(a) Find Fourier Series for function $f(x) = \begin{cases} x, & 0<x<1 \\ 1-x, & 1<x<2 \end{cases}$.

Sol^n. Here, the interval is $(0,2)$. So, $c=0$, $l=1$.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$ —— ①

Now, $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \int_0^2 f(x)\, dx$

$\therefore a_0 = \left[\int_0^1 x\,dx + \int_1^2 (1-x)\,dx\right] = \left[\frac{x^2}{2}\right]_0^1 + \left[x - \frac{x^2}{2}\right]_0^1$

$\Rightarrow a_0 = \frac{1}{2} + \left\{\left(2 - \frac{4}{2}\right) - \left(1 - \frac{1}{2}\right)\right\} \quad \Rightarrow a_0 = \frac{1}{2} + 0 - \frac{1}{2}$

$\Rightarrow \boxed{a_0 = 0}$

$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx = \int_0^2 f(x)\cos n\pi x\,dx$

$a_n = \int_0^1 x\cos n\pi x\,dx + \int_1^2 (1-x)\cos n\pi x\,dx$

$a_n = \left[x\frac{\sin n\pi x}{n\pi} - \left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[(1-x)\frac{\sin n\pi x}{n\pi} - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$

$a_n = \left[\frac{x\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]_0^1 + \left[(1-x)\frac{\sin n\pi x}{n\pi} - \frac{\cos n\pi x}{n^2\pi^2}\right]_1^2$

$a_n = \left[\frac{\sin n\pi}{n\pi} + \frac{\cos n\pi}{n^2\pi^2} - 0 - \frac{1}{n^2\pi^2}\right] + \left[-\frac{\sin 2n\pi}{n\pi} - \frac{\cos 2n\pi}{n^2\pi^2} - 0 + \frac{1}{n^2\pi^2}\right]$

$a_n = \left[0 + \frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} + 0 - \frac{1}{n^2\pi^2} - 0 + \frac{(-1)^n}{n^2\pi^2}\right]$

$\therefore a_n = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right] \Rightarrow a_n = \begin{cases} 0 &, \text{if } n \text{ is even} \\ \frac{-4}{n^2\pi^2} &, \text{if } n \text{ is odd.} \end{cases}$

and, $b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$

$b_n = \int_0^2 f(x)\sin n\pi x\,dx = \int_0^1 x\sin n\pi x\,dx + \int_1^2 (1-x)\sin n\pi x\,dx$

$b_n = \left[x\left(-\frac{\cos n\pi x}{n\pi}\right) - \left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[(1-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$

$b_n = \left[-\frac{x\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 + \left[(x-1)\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right]_1^2$

$$b_n = \left[-\frac{\cos n\pi}{n\pi} + 0 + 0 - 0\right] + \left[\frac{\cos 2n\pi}{n\pi} - 0 - 0 + \frac{\sin n\pi}{n^2\pi^2}\right]$$

$$b_n = -\frac{(-1)^n}{n\pi} + \frac{1}{n\pi} \Rightarrow b_n = \frac{1}{n\pi}[1-(-1)^n]$$

$$\Rightarrow \quad b_n = \begin{cases} 0 &, \text{ if } n \text{ is even} \\ \frac{2}{n\pi} &, \text{ if } n \text{ is odd.} \end{cases}$$

Putting the values of a_0, a_n and b_n in ①, we get:

$$\boxed{f(x) = -\frac{4}{\pi^2}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right] + \frac{2}{\pi}\left[\frac{\sin \pi x}{1} + \frac{\sin 3\pi x}{3} + \cdots\right]}$$ Ans.

OR

(b) Find the half-Range Cosine Series of $f(x) = (lx - x^2)$ in the interval $(0, l)$.

Sol.ⁿ Given: $f(x) = (lx - x^2)$ in the interval $(0, l)$.

Half Range Cosine Series is given by-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$ and $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$.

Now, $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l (lx - x^2)\,dx$

$$\Rightarrow a_0 = \frac{2}{l}\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_0^l = \frac{2}{l}\left[\frac{l^3}{2} - \frac{l^3}{3}\right]_0^l \Rightarrow \boxed{a_0 = \frac{l^2}{3}}$$

And, $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$

$$= \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$a_n = \frac{2}{l}\left[(lx-x^2)\sin\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right) - (l-2x)\left(-\cos\frac{n\pi x}{l}\right)\left(\frac{l^2}{n^2\pi^2}\right) + \frac{(-2)l^3}{n^3\pi^3}\left(-\sin\frac{n\pi x}{l}\right)\right]_0^l$$ 7

$$= \frac{2}{l}\left[(lx-x^2)\frac{l}{n\pi}\sin\frac{n\pi x}{l} + (l-2x)\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} + \frac{2l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2}{l}\left[\left(0 + (l-2l)\frac{l^2}{n^2\pi^2}\cos n\pi + \frac{2l^3}{n^3\pi^3}\sin n\pi\right) - \left(0 + \frac{l\cdot l^2}{n^2\pi^2}\cos 0 + \frac{2l^3}{n^3\pi^3}\sin 0\right)\right]$$

$$= \frac{2}{l}\left[\frac{-l^3}{n^2\pi^2}(-1)^n - \frac{l^3}{n^2\pi^2}\right] = -\frac{2l^3}{l n^2\pi^2}\left[(-1)^n + 1\right]$$

$$\Rightarrow a_n = -\frac{2l^2}{n^2\pi^2}\left[1+(-1)^n\right] \Rightarrow a_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ -\dfrac{4l^2}{n^2\pi^2}, & \text{if } n \text{ is even} \end{cases}$$

$\therefore$ Required Half-Range Cosine Series is-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$\boxed{f(x) = \frac{l^2}{6} + \sum_{n=1}^{\infty}\left[-\frac{2l^2}{n^2\pi^2}\left(1+(-1)^n\right)\right]\cos\frac{n\pi x}{l}}$$ Ans.

OR

$$\boxed{f(x) = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\left[\frac{\cos\frac{2\pi x}{l}}{2^2} + \frac{\cos\frac{4\pi x}{l}}{4^2} + \cdots\right]}$$ Ans.

Q.5(a) Given: Sol^n:
$$f(z) = \begin{cases} \dfrac{2xy(x+iy)}{x^2+y^2}; & z \neq 0 \\ 0; & z = 0. \end{cases}$$

$$\Rightarrow f(z) = \frac{2x^2y + i2xy^2}{x^2+y^2}$$

$$\Rightarrow u = \frac{2x^2y}{x^2+y^2} \quad \text{and} \quad v = \frac{2xy^2}{x^2+y^2}$$

At Origin,

$$\frac{\partial u}{\partial x} = \lim_{h\to 0}\frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0,k) - u(0,0)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(h,0) - v(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0,k) - v(0,0)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

Here, $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ and $\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

$\Rightarrow$ C-R eq^ns. are satisfied at Origin.

Now, $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{(x,y) \to (0,0)} \frac{\frac{2xy(x+iy)}{x^2+y^2} - 0}{(x+iy)}$$

$$f'(0) = \lim_{(x,y) \to (0,0)} \frac{2xy}{x^2+y^2}$$

Case I: Taking $z \to 0$ along x-axis i.e. $y=0$,

$$f'(0) = \lim_{x \to 0} \frac{2 \cdot x \cdot (0)}{x^2+0} = 0$$

Case II: Taking $z \to 0$ along a line $y = mx$, then

$$f'(0) = \lim_{x \to 0} \frac{2x \cdot mx}{x^2+m^2x^2} = \lim_{x \to 0} \frac{2mx^2}{x^2(1+m^2)} = \frac{2m}{(1+m^2)}$$

Since, we are getting different values of the limit for diff. value of m.

$\Rightarrow f(z)$ is not differentiable at $(0,0)$.

$\Rightarrow f'(0)$ does not exist.

$\Rightarrow f'(z)$ does not exist at Origin.

Hence, the C-R eq^ns. are satisfied at origin but derivative of $f(z)$ at origin does not exist.

OR

Find the image of the Circle $|z-1|=1$ in the Complex plane under the mapping $w=\frac{1}{z}$. ⑨

Solⁿ: Given: $w=\frac{1}{z} \Rightarrow u+iv=\frac{1}{x+iy}$.

$$\Rightarrow x+iy=\frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$\Rightarrow x=\frac{u}{u^2+v^2} \quad \underline{\text{and}} \quad y=\frac{-v}{u^2+v^2}$$

The given circle is—

$$|z-1|=1$$

$$|x+iy-1|=1$$

$$|(x-1)+iy|=1$$

$$(x-1)^2+y^2=1$$

$$\Rightarrow \left[\frac{u}{u^2+v^2}-1\right]^2+\left(\frac{-v}{u^2+v^2}\right)^2=1$$

$$\Rightarrow \frac{u^2}{(u^2+v^2)^2}+1-\frac{2u}{u^2+v^2}+\frac{v^2}{(u^2+v^2)^2}=1$$

$$\Rightarrow 1-2u=0$$

$\Rightarrow \boxed{u=\frac{1}{2}}$ which is a line in w-plane.

Q.6. Find the Laurent's expansion of $f(z)=\frac{1}{z^2-3z+2}$ in the region given by:

(a) $|z|<1$ (b) $1<|z|<2$ (c) $|z|>2$ (d) $0<|z-1|<1$

Solⁿ: Here $f(z)=\frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)}-\frac{1}{(z-1)}$ {by Partial Fractions}

(a) when $|z|<1$

$$\therefore f(z)=\frac{1}{(z-2)}-\frac{1}{(z-1)} = \frac{1}{-2\left(1-\frac{z}{2}\right)}+\frac{1}{(1-z)}$$

$$\Rightarrow f(z)=-\frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}+(1-z)^{-1}$$

$\boxed{f(z)=-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n+\sum_{n=0}^{\infty}z^n}$ OR $\boxed{f(z)=-\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\cdots\right]+\left[1+z+z^2+\cdots\right]}$

(b) When $1<|z|<2$

$$\therefore f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$\Rightarrow \boxed{f(z) = -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)}$$ OR

$$\boxed{f(z) = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n}$$

(c) When $|z|>2$

$$\therefore f(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$\boxed{f(z) = \frac{1}{z}\left[1+\frac{2}{z}+\frac{4}{z^2}+\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]}$$ OR

$$\boxed{f(z) = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n}$$

(d) When $0<|z-1|<1$

$$\therefore f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{(z-1)-1} - \frac{1}{z-1}$$

$$= \frac{-1}{[1-(z-1)]} + \frac{1}{1-z}$$

$$= -[1-(z-1)]^{-1} + [1-z]^{-1}$$

$$= -[1+(z-1)+(z-1)^2+\cdots] + [1+z+z^2+\cdots]$$

$$\boxed{f(z) = -\sum_{n=0}^{\infty}(z-1)^n - \frac{1}{z-1}}$$ Ans.

Q.6.(b) OR

State Cauchy Integral Formula. Also, evaluate $\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz$, where C is $|z|=3$.

Solⁿ: CAUCHY INTEGRAL FORMULA →

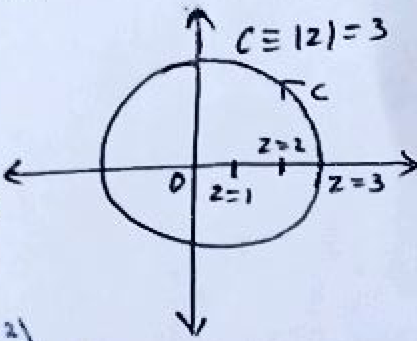
If $f(z)$ is analytic within and on a closed curve C and a is any point within C, then

$$\boxed{f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz}$$

Now, Given to evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

The integrand has singularities given by $(z-1)(z-2)=0$
$\Rightarrow z = 1, 2.$

Given circle $|z|=3$ with centre $z=0$ and radius $= 3$ encloses both the singularities.



Then, by Cauchy Integral Formula,

$$\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz = \int_{C_1} \frac{\left(\frac{\sin\pi z^2 + \cos\pi z^2}{z-2}\right)}{(z-1)}\,dz + \int_{C_2} \frac{\left(\frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)}\right)}{(z-2)}\,dz$$

$$= 2\pi i\left[\frac{\sin\pi z^2 + \cos\pi z^2}{z-2}\right]_{z=1} + 2\pi i\left[\frac{\sin\pi z^2 + \cos\pi z^2}{z-1}\right]_{z=2}$$

$$= 2\pi i\left[\frac{0-1}{-1}\right] + 2\pi i\left[\frac{0+1}{1}\right]$$

$$= 2\pi i + 2\pi i = 4\pi i$$

$$\therefore \boxed{\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz = 4\pi i}$$ **Ans.**

Q.7 (a) Solve the differential equation: **SECTION - C**

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$$

Sol^n

Homogeneous Linear Differential Equation :→

An equation of the form

$$x^n \frac{d^ny}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} +$$

$$--- + a_{n-1} x\frac{dy}{dx} + a_n y = Q.$$

where a_i's are Constant and Q is a function of x, is Called Cauchy's homogeneous linear equation.

put $x = e^z$ so that $z = \log x$

let $x\frac{d}{dx} = \frac{d}{dz} = D$

$$x^2\frac{d^2}{dx^2} = D(D-1)$$

Then Differential Equation Reduce to

$$\{D(D-1) + 4D + 2\}y = e^{e^z}$$

$$\Rightarrow (D^2 + 3D + 2)y = e^{e^z}$$

A.E is put $D = m$, $y = 1$, $e^{e^z} = 0$

$$m^2 + 3m + 2 = 0$$

$$m = -1, -2$$

$$C.F = c_1 e^{-z} + c_2 e^{-2z}$$

$$P.I = \frac{1}{D^2+3D+2}\left(e^{e^z}\right) = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^z}$$

$$= \frac{1}{D+1} e^{e^z} - \frac{1}{D+2} e^{e^z} = \quad \left(\because \frac{1}{D+\alpha} Q = e^{-\alpha x} \int e^{\alpha x} Q . dx\right.$$

$$= e^{-z} \int e^{z} . e^{e^z} dz - e^{-2z} \int e^{2z} . e^{e^z} . dZ$$

$$= e^{-z} I_1 - I_2 . e^{-2z} \text{ (say).}$$

$$I_1 = \int e^{z} . e^{e^z} . dz.$$

put $e^z = t$.

$$e^z . dz = dt$$

$$I_1 = \int e^t . dt = e^t = e^{e^z}$$

$$I_2 = \int e^{2z} . e^{e^z} . dz$$

put $e^z = t$

$$e^z . dz = dt$$

$$I_2 = \int t . e^t . dt = t . e^t - e^t = e^t (t-1)$$

$$I_2 = e^{e^z} (e^z - 1)$$

$$P.I = e^{-z} e^{e^z} - e^{e^z} (e^z - 1) . e^{-2z}$$

$$= e^{-z} e^{e^z} - e^{e^z} (e^{-z} - e^{-2z})$$

$$= e^{-z} . e^{e^z} - e^{e^z} e^{-z} + e^{-2z} e^{e^z} = e^{-2z} e^{e^z}$$

Hence complete Solution is

$$y = C.F + P.I = C_1 e^{-z} + C_2 e^{-2z} + e^{-2z} \cdot e^{e^z}$$

$$\boxed{y = \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{1}{x^2} e^x}$$ $(\because \text{put } z = \log x)$

Ans.

Q.7.b Solve the following differential Equation by Changing the independent Variable

$$x \frac{d^2y}{dx^2} + (4x^2 - 1) \frac{dy}{dx} + 4x^3 y = 2x^3$$

Sol^n Find the complete solution of $y'' + Py' + Qy = R$ by changing the independent variable. :→

Step-I :- Make the Coefficient of $\frac{d^2y}{dx^2}$ as 1 if it is not so.

Step-2 :- find P, Q, R

Step-3 :→ Choose z such that $\left(\frac{dz}{dx}\right)^2 = Q$.

Step-4 :→ Find $\frac{dz}{dx}$ and hence obtain z and $\frac{d^2z}{dx^2}$

Step-5 :→ Find P_1, Q_1, R_1 using formula

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Step-6 :- Reduced Equation is $\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$ which we solve for y in terms of z.

Step-7 :→ We write Complete Solution as y in terms of x by replacing the value of z in terms of x

Given Equation can be written as

Step-I $\frac{d^2y}{dx^2} + \left(\frac{4x^2-1}{x}\right)\frac{dy}{dx} + 4x^2y = 2x^2$

on Comparing with $y'' + Py' + Qy = R$

Step-2 $P = \frac{4x^2-1}{x}$, $Q = 4x^2$, $R = 2x^2$

Step-3 :- Now Choose z such that $\left(\frac{dz}{dx}\right)^2 = 4x^2$

$$\Rightarrow \frac{dz}{dx} = 2x$$

$$\Rightarrow z = x^2.$$

Step-4 :- $$\Rightarrow \frac{d^2z}{dx^2} = 2$$

Step-5 :→

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 + \left(\frac{4x^2-1}{x}\right)2x}{4x^2} = \boxed{2. = P_1}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{4x^2} = 1 \Rightarrow \boxed{Q_1 = 1}$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{2x^2}{4x^2} = \boxed{\frac{1}{2} = R_1}$$

Step-6 :-> $$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

$$\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + y = \frac{1}{2}.$$

A.E $\quad m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$

C.F $= (C_1 + z C_2) e^{-z}$

P.I $= \frac{1}{D^2+2D+1}\left(\frac{1}{2}\right) = \frac{1}{D^2+2D+1}\frac{1}{2}e^{0z}$

$= \frac{1}{2}\frac{1}{0+2(0)+1}e^{0z}$ (Put $D=0$)

$= \frac{1}{2}$

Hence $y = C.F + P.I = (C_1 + z C_2)e^{-z} + \frac{1}{2}.$

Step-7 :- put $z = x^2$

$$\boxed{y = (C_1 + x^2 C_2)e^{-x^2} + \frac{1}{2}}$$ Ans.

Q.7.C An inductance L of 5.0H and a resistance R of 25 Ω are connected in series with e.m.f of E Volt. If the current I is zero when $t=0$, find the current I at the end of 1 second if $E = 100V$

Sol

Solve the following differential equation:

$$L\frac{di}{dt} + Ri = E$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \quad \text{---} (1)$$

~~I.F =~~

(use $i = I$)

$$I.f = e^{\frac{R}{L}\int dt} = e^{R/L \cdot dt}$$

Solution of Equation (1)

$$i e^{\frac{R}{L}t} = \frac{E}{L}\int e^{Rt/L}\,dt = \frac{E}{R}e^{Rt/L} + C_1 \quad \text{---} (2)$$

If $t=0$, $i=0$ from (2) $C_1 = \frac{-E}{R}$

Put in (2)

$$i e^{Rt/L} = \frac{E}{R}\left(-1 + e^{Rt/L}\right)$$

$$i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$$

Putting the value of $E = 100v$, $R = 25\Omega$, $L = 5.0H$, $t = 1$

$$i = \frac{\cancel{100}^{\,4}}{\cancel{25}}\left(1 - e^{-25/5}\right) = 4\left(1 - e^{-5}\right) \text{ ampere.}$$

Q.8.a Find $L\{f(t)\}$, where $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$

Solⁿ Firstly we convert $f(t)$ in unit-step function

$$f(t) = \sin t\,[u(t) - u(t-\pi)] + 0.\,[u(t-\pi) - u(t-2\pi)]$$

$$= \sin t.\, u(t) - \sin t.\, u(t-\pi)$$

$$= \sin t.\, u(t) - \sin((t-\pi)+\pi).\, u(t-\pi)$$

$$f(t) = \sin t.\, u(t) + \sin(t-\pi).\, u(t-\pi) \quad \left\{ \because \sin(\pi+\theta) = -\sin\theta \right.$$

Now apply Laplace on both side, we have

$$L\{f(t)\} = L\{\sin t.\, u(t)\} + L\{\sin(t-\pi).\, u(t-\pi)\}$$

$$L\{f(t)\} = \frac{1}{p^2+1} + \frac{e^{-\pi p}}{p^2+1} \quad \left\{ \text{using second shifting property} \right.$$

$$\left\{ \because L(\sin t) = \frac{1}{p^2+1} \right.$$

Q.8.b Prove that $\int_0^\infty \int_0^t e^{-t} \frac{\sin u}{u} du\, dt = \frac{\pi}{4}$ (5)

Sol^n

$L\{\sin u\} = \frac{1}{p^2+1}$

$L\left\{\frac{F(u)}{u}\right\} = \int_p^\infty f(p)\, dp$ where $L\{F(u)\} = f(p)$

$L\{F(t)\} = f(p) = \int_0^\infty e^{-pt} F(t)\, dt$

$L\{\int_0^t F(t)dt\} = \frac{1}{p} f(p)$ where $L\{F(t)\} = f(p)$

$L\{\sin u\} = \frac{1}{p^2+1}$

$L\left\{\frac{\sin u}{u}\right\} = \int_p^\infty \frac{1}{p^2+1} dp = \tan^{-1}\left(\frac{1}{p}\right)$

$L\left\{\int_0^t \frac{\sin u}{u} du\right\} = \frac{1}{p} \tan^{-1}\left(\frac{1}{p}\right)$

$\Rightarrow \int_0^\infty e^{-pt}\left(\int_0^t \frac{\sin u}{u} du\right) dt = \frac{1}{p} \tan^{-1} \frac{1}{p}$

Puting $p = 1$ we get

$\int_0^\infty \int_0^t e^{-t} \frac{\sin u}{u} du. dt = \tan^{-1}(1) = \frac{\pi}{4}$

Proved.

$\boxed{Q.8.C}$ Find the inverse Laplace Transformation of

$\log\left(1+\frac{1}{s^2}\right)$

Soln

① If $L^{-1}\{f(p)\} = F(t)$, then

$L^{-1}\{f^n(p)\} = L^{-1}\left[\frac{d^n}{dp^n} f(p)\right] = (-1)^n t^n F(t)$

② $L^{-1}\left(\frac{1}{p}\right) = 1$

③ $L^{-1}\left(\frac{p}{p^2+a^2}\right) = \cos at$

$$L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\} = F(t)$$

$$L^{-1}\left[\frac{d}{ds}\left\{\log\left(1+\frac{1}{s^2}\right)\right\}\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{1}{1+\frac{1}{s^2}}\left(\frac{-2}{s^3}\right)\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{-2s^2}{(s^2+1)s^3}\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{-2}{s(s^2+1)}\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] = \frac{t}{2}F(t) \quad (\because \text{using partial fraction})$$

(6)

$$\Rightarrow L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{s}{s^2+1}\right\} = \frac{t}{2} F(t)$$

$$\Rightarrow 1 - \cos t = \frac{t}{2} F(t)$$

$$\Rightarrow \boxed{F(t) = \frac{2(1-\cos t)}{t}}$$

Ans.

Que-9. (a) Test the series;

$$\frac{x}{1} + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \cdots$$

Solⁿ.- Ratio test - Let $\sum U_n$ be a series of +ve terms

$$\lim_{n \to \infty} \frac{U_n}{U_{n+1}} = l$$

$\sum U_n$ is convergent if $l > 1$

$\sum U_n$ is divergent if $l < 1$

test fail if $l = 1$

Raabe's test - $\lim_{n \to \infty} n \left(\frac{U_n}{U_{n+1}} - 1 \right) = l$

$\sum U_n$ is convergent if $l > 1$

$\sum U_n$ is divergent if $l < 1$

Test fail if $l = 1$

Now ⇒ $\frac{x}{1} + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \cdots$

First of all we find the n^{th} term of the given series

$$U_n = \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots (2n)} \cdot \frac{x^{2n-1}}{(2n-1)}$$

$$U_{n+1} = \frac{1.3.5 \ldots (2n-1)(2n+1)}{2.4.6 \ldots (2n)(2n+2)} \cdot \frac{x^{2n+1}}{(2n+1)}$$

$$\frac{U_n}{U_{n+1}} = \frac{\cancel{1.3.5 \ldots (2n-1)}}{\cancel{2.4.6 \ldots (2n)}} \cdot \frac{x^{2n-1}}{(2n-1)} \times \frac{\cancel{2.4.6 \ldots (2n)}(2n+2)\cancel{(2n+1)}}{\cancel{1.3.5 \ldots (2n-1)}\cancel{(2n+1)}\, x^{2n+1}}$$

$$\frac{U_n}{U_{n+1}} = \frac{(2n+2)}{(2n-1)} \cdot \frac{x^{2n-1}}{x^{2n+1}}$$

$$\frac{U_n}{U_{n+1}} = \frac{\cancel{n}\left(2+2/n\right)}{\cancel{n}\left(2-1/n\right)} \cdot \frac{1}{x^2}$$

Now, $$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{(2+2/n)}{(2-1/n)} \cdot \frac{1}{x^2}$$

$$= \frac{1}{x^2}$$

Now, $\sum U_n$ is cgt if $\frac{1}{x^2} > 1 \Rightarrow x^2 < 1$

$\sum U_n$ is dgt if $\frac{1}{x^2} < 1 \Rightarrow x^2 > 1$

test fail if $\frac{1}{x^2} = 1 \Rightarrow x^2 = 1$

Now check for $x^2 = 1$, $\Rightarrow x = 1$

$$U_n = \frac{1.3.5...(2n-1)}{2.4.6...(2n)} \cdot \frac{x^{2n-1}}{2n-1} = \frac{1.3.5...(2n-1)}{2.4.6...(2n)} \cdot \frac{1}{(2n-1)}$$

$$U_{n+1} = \frac{1.3.5...(2n-1)(2n+1)}{2.4.6...(2n)(2n+2)} \cdot \frac{1}{(2n+1)}$$

$$\Rightarrow \frac{U_n}{U_{n+1}} = \cancel{\frac{1.3.5...(2n-1)}{2.4.6...(2n)}} \cdot \frac{1}{(2n-1)} \times \frac{2.4.6...\cancel{(2n)}(2n+2).\cancel{(2n+1)}}{1.3.5...\cancel{(2n-1)}\cancel{(2n+1)}}$$

$$\frac{U_n}{U_{n+1}} = \frac{2n+2}{2n-1}$$

$$\frac{U_n}{U_{n+1}} - 1 = \frac{2n+2}{2n-1} - 1 = \frac{\cancel{2n}+2-\cancel{2n}+1}{2n-1}$$

$$n\left(\frac{U_n}{U_{n+1}} - 1\right) = \frac{n.3}{2n-1} = \left(\frac{3}{2-1/n}\right)$$

$$\lim_{n\to\infty} n.\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n\to\infty}\left(\frac{3}{2-1/n}\right) = \frac{3}{2} > 1 \Rightarrow \underline{\text{cgt}}$$

Que-9. (b) Find the fourier series expansion of the following function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Hence find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

Solution- Fourier series of a function $f(x)$ in $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad \text{---} (1)$$

where,
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Now,
$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx\right]$$

$$= \frac{1}{\pi}\left[0 + \int_0^{\pi} x\, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$a_0 = \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

$$\boxed{a_0 = \frac{\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{\cos n\pi}{n^2} - 0 - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$a_n = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

when n is odd $\Rightarrow$ $a_n = \frac{-2}{\pi n^2}$

when n is even $\Rightarrow$ $a_n = 0$

Now, $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\cancel{\pi}}\left[-\cancel{\pi}\frac{\cos n\pi}{n} + 0 - 0 - 0\right]$$

$$b_n = \frac{-(-1)^n}{n}$$

Hence the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\boxed{f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin nx}$$

"when n is odd"

Now $\Rightarrow$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

$$- \left[-\frac{\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} - \cdots\right] \quad \text{——— (2)}$$

Now, we find $f(x)$ at $x = 0$

$$f(0) = \frac{1}{2}[\,LHL + RHL\,]$$

$$f(0) = \frac{1}{2}[\,f(0-0) + f(0+0)\,]$$

$$f(0) = 0$$

From eqn. (2), we have

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] - 0$$

$$\cancel{-}\frac{\pi}{4} = \cancel{-}\frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\boxed{\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}}$$

Que-10 (a) Find Mobius transformation that maps points $z = i, -i, 1$ into the points $w = 0, 1, \infty$ respectively.

Sol^n. We know that the Mobius transformation mapping $z = z_1, z_2, z_3$ into $w = w_1, w_2, w_3$ respectively is,

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \quad \text{—(1)}$$

Here, $w_1 = 0, \ w_2 = 1, \ w_3 = \infty$ &

$z_1 = i, \ z_2 = -i, \ z_3 = 1$

From (1),
$$\frac{(w-w_1)\left[\cancel{w_3}\left(\frac{w_2}{w_3}-1\right)\right]}{(w_1-w_2)\left[\cancel{w_3}\left(1-\frac{w}{w_3}\right)\right]} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)\left(\frac{1}{\infty}-1\right)}{(0-1)\left(1-\frac{w}{\infty}\right)} = \frac{(z-i)(-i-1)}{(i-(-i))(1-z)}$$

$$w = \frac{-(z-i)(1+i)}{2i(1-z)} = \frac{(z-i)(1+i)}{2i(z-1)} \times \frac{i}{i}$$

$$= \frac{(z-i)(i-1)}{-2(z-1)} = \frac{zi - z + 1 + i}{-2z+2}$$

$$w = \frac{(1-z) + i(1+z)}{-2(z-1)} \quad \text{or}$$

$$\boxed{w = \frac{(i-1)z + (1+i)}{-2z+2}}$$

Que-10 (b) If $f(z) = u + iv$ is analytic function and $u - v = e^x(\cos y - \sin y)$. Find $f(z)$ in terms of z

Solⁿ- We know, $f(z) = u + iv$ —— (1)

$$if(z) = iu - v \quad —— (2)$$

Now, add eqⁿ (1) and (2), we have

$$(1+i)f(z) = (u-v) + i(u+v)$$

$$F(z) = U + iV$$

where $F(z) = (1+i)f(z)$

$U = u - v$ & $V = u + v$

$$\Rightarrow \boxed{f(z) = \frac{F(z)}{1+i}}$$

If we have given $\left(\frac{u-v}{U}\right)$ then First find

$$U_x = \frac{\partial U}{\partial x} = \phi_1(x, y)$$

then put $x = z$ and $y = 0$, we get

$$\Rightarrow \underline{\phi_1(z, 0)}$$

Now find $U_y = \frac{\partial U}{\partial y} = \phi_2(x, y)$

then put $x = z$ and $y = 0$, we get

$$\Rightarrow \underline{\phi_2(z, 0)}$$

then $F(z) = \int[\phi_1(z, 0) - i\phi_2(z, 0)]dz + c$

$$\boxed{f(z) = \frac{F(z)}{1+i}}$$

$(u-v) = U = e^x(\cos y - \sin y)$

$U_x = e^x(\cos y - \sin y) = \phi_1(x,y)$

$\phi_1(z,0) = e^z(1-0)$

$\phi_1(z,0) = e^z$

Now, $U_y = e^x(-\sin y - \cos y) = \phi_2(x,y)$

$\phi_2(z,0) = e^z(-0-1)$

$\phi_2(z,0) = -e^z$

Now, $F(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + c$

$= \int (e^z - i(-e^z))\,dz + c$

$= \int (e^z + ie^z)\,dz + c$

$F(z) = (1+i)e^z + c$

Now, $f(z) = \dfrac{F(z)}{1+i}$

$= e^z + \dfrac{c}{1+i}$

$$\boxed{f(z) = e^z + c_1}$$

where, $\boxed{c_1 = \dfrac{c}{1+i}}$

Que- 10. (c) Show that the function $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic. Find the harmonic conjugate of u.

Soln.- Harmonic function - A function $f(x,y)$ is said to be harmonic if it satisfy laplace equation i.e $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

Now, we have $u = \frac{1}{2}\log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{\cancel{2}}\frac{1}{x^2+y^2} \times \cancel{2}x = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\cancel{2}}\frac{1}{x^2+y^2} \times \cancel{2}y = \frac{y}{x^2+y^2}$$

Now, $$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\cdot(1) - x(2x)}{(x^2+y^2)} = \frac{y^2-x^2}{(x^2+y^2)}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\cdot(1) - y(2y)}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

Now, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{x^2+y^2} + \frac{x^2-y^2}{x^2+y^2}$$

$$= \frac{\cancel{y^2} - \cancel{x^2} + \cancel{x^2} - \cancel{y^2}}{x^2+y^2} = 0$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Hence, u is harmonic function

To find harmonic conjugate,

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy \quad \text{--- (1)}$$

since u is harmonic $\Rightarrow$ it satisfy C-R eqn.

i.e $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

From (1), we have

$$dv = \left(-\frac{\partial u}{\partial y}\right) dx + \frac{\partial u}{\partial x} \cdot dy$$

$$dv = -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

$$dv = \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$dv = d\left(\tan^{-1}\frac{y}{x}\right)$$

On Integration, we have

$$\boxed{v = \tan^{-1}\left(\frac{y}{x}\right) + c}$$

Que-11. (a) Determine the poles and residues at each poles of the function $f(z) = \dfrac{z}{z^2-3z+2}$ and hence evaluate $\oint_c f(z)\,dz$, where c is the circle $|z-2| = \frac{1}{2}$.

Solution - To find Poles we put denominator = 0

And to find residue of simple pole at $z \to a$

$$R = \lim_{z \to a} (z-a) \cdot f(z)$$

We have, $f(z) = \dfrac{z}{(z^2-3z+2)}$

To find poles

$$z^2-3z+2=0$$
$$(z-2)(z-1) = 0$$
$$z = 1, 2$$

Here, we have a simple pole at $z=1$
and a simple pole at $z=2$

Residue At $z=1$, Residue $(R_1) = \lim_{z \to 1} \cancel{(z-1)} \cdot \dfrac{z}{\cancel{(z-1)}(z-2)}$

$$R_1 = \frac{1}{-1} = -1$$

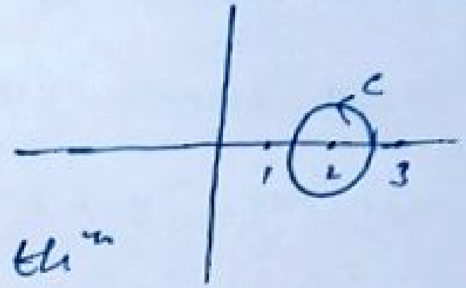
$$\boxed{R_1 = -1}$$

Residue at $z=2$, $R_2 = \lim_{z \to 2} \cancel{(z-2)} \cdot \dfrac{z}{(z-1)\cancel{(z-2)}} = 2$

$$\boxed{R_2 = 2}$$

Now, we find $\oint_c f(z)\, dz$, where $c \equiv |z-2| = \frac{1}{2}$

First we draw the circle.



Here, only $z=2$ lie inside the circle, and $\underline{z=1}$ lie outside the circle so by cauchy integral thm

$\int f(z)\, dz = 0$ at $z=1$

Now, we find $\oint_c f(z)\, dz$, $c \equiv |z-2| = \frac{1}{2}$, at $z=2$

we have Residue at $z=2$

$$R = 2$$

$$\oint_c f(z)\, dz = 2\pi i \times R$$

$$= 2\pi i \times 2$$

$$\oint_c f(z)\, dz = 4\pi i \qquad c : |z-2| = \frac{1}{2}$$

Que-11 (b) State Cauchy-Integral theorem for an analytic function. Verify the theorem by integrating the function $f(z) = z^3 + iz$ along the boundary of the rectangle with vertices $1, -1, i, -i$.

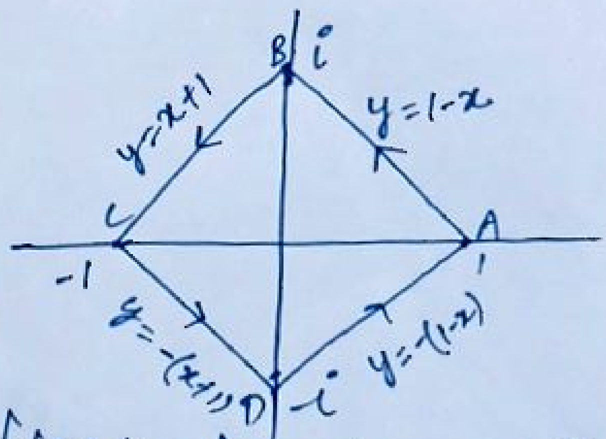
Sol^n: Cauchy-Integral theorem-

If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a simple closed curve C, then

$$\boxed{\oint_C f(z)\, dz = 0}$$

First we draw rectangle with vertices $1, -1, i, -i$

R: 1, 1, i, i



$$\oint_R f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz \quad \text{---} (1)$$

$$f(z) = z^3 + iz = (x+iy)^3 + i(x+iy)$$
$$f(z) = x^3 - iy^3 + 3x^2 iy - 3xy^2 + ix - y$$

Now, $\displaystyle\int_{AB} f(z)\,dz = \int_{AB} (z^3 + iz)\, dz$

$$\int_{AB} f(z)\,dz = \int_{AB} [x^3 - iy^3 + 3x^2 iy - 3xy^2 + ix - y][dx + idy]$$

put $y = 1 - x \Rightarrow dy = -dx$, $1 < x < 0$

$$= \int_1^0 [x^3 - i(1-x)^3 + 3x^2 i(1-x) - 3x(1-x)^2 + ix - (1-x)] [dx - idx]$$

$$= \int_1^0 (-2x^3 - 2x^3 i + 6x^2 + 4ix - 2x - i - 1)\,dx\,(1-i)$$

$$= (1-i)\left[-\frac{2x^4}{4} - \frac{2x^4}{4}i + \frac{6x^3}{3} + \frac{4ix^2}{2} - \frac{2x^2}{2} - ix - x\right]_1^0$$

$$= (1-i)\left[\frac{1}{2} + \frac{i}{2} - \cancel{2} - 2i + \cancel{1} + i + \cancel{1}\right]$$

$$= (1-i)\left(\frac{1}{2} - \frac{i}{2}\right)$$

$$\boxed{\int_{AB} f(z)\,dz = -i}$$

Now $\int_{BC} f(z)\,dz$, BC: $y = x + 1$

$dy = dx$

$0 < x < -1$

$$\Rightarrow \boxed{\int_{BC} f(z)\,dz} = \int_{BC} [x^3 - iy^3 + 3x^2 iy - 3xy^2 + ix - y][dx + idy]$$

$$= \int_0^{-1} [x^3 - i(1+x)^3 + 3x^2 i(1+x) - 3x(1+x)^2 + ix - (1+x)][dx + idx]$$

$$= (1+i)\int_0^{-1} (-2x^3 + 2x^3 i - 6x^2 - 4x - 2ix - 1 - i)\,dx$$

$$\int_{BC} f(z)\,dz = (1+i)\left[-2\frac{x^4}{4}+\frac{2x^4}{4}i-\frac{6x^3}{3}-\frac{4x^2}{2}-\frac{2ix^2}{2}-x-ix\right]_0$$

$$= (1+i)\left[\frac{1}{2}+\frac{i}{2}\right]$$

$$\boxed{\int_{BC} f(z)\,dz = i}$$

similarly $\boxed{\int_{CD} f(z)\,dz = -i}$

similarly $\boxed{\int_{DA} f(z)\,dz = i}$

Now, from ① we have

$$\int_{R} f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz$$

$$= \cancel{-i} + \cancel{i} \cancel{-i} \cancel{+i}$$

$$\boxed{\int_{R} f(z)\,dz = 0}$$

=