B.Tech-II Year

(Compact Notes)

(DSTL-III SEM)

Q-1 Give an example of relation R which is-

(a) Symmetric but neither transitive nor reflexive.

(b) R is both symmetric and antisymmetric.

(c) R is transitive but $R^{-1} \cup R$ is not transitive.

Sol^n: Let $A = \{1, 2, 3\}$ be any set and a relation R is defined on A i.e. $R \subseteq A \times A$

(a) Define R as $R = \{(1,2)(2,1)(2,2)(3,3)\}$ clearly as $(1,1) \notin R$

so R is not reflexive.

Now again $(1,2) \in R$ & $(2,1) \in R$

$$\Rightarrow \left.\begin{array}{l}(x,y) \in R \\ (y,z) \in R\end{array}\right\} \Rightarrow (x,z) \in R$$

$$\Rightarrow (1,1) \notin R$$

so, R is not transitive.

(b) Define $R = \{(1,1)(2,2)\}$ which is both symmetric as well as antisymmetric

(c) Define $R = \{(1,2)\}$ then $R^{-1} = \{(2,1)\}$

Now $R \cup R^{-1} = \{(1,2)(2,1)\}$

as $(1,2) \in R \cup R^{-1}$

$(2,1) \in R \cup R^{-1}$

but $(1,1) \notin R \cup R^{-1}$

$\Rightarrow R \cup R^{-1}$ is not transitive.

Q-2 Let $U = \{1, 2, 3, \ldots, 9\}$ be the universal set and let,

$A = \{1, 2, 3, 4, 5\}$, $D = \{1, 3, 5, 7, 9\}$

$B = \{4, 5, 6, 7\}$ $E = \{2, 4, 6, 8\}$

$C = \{5, 6, 7, 8, 9\}$ $F = \{1, 5, 9\}$

then find -

(a) $A \cup B$ and $D \cup F$ (b) B^c, E^c, D^c

(c) A/B, D/E, E/D (d) $C \oplus D$, $E \oplus F$, $A \oplus B$

(e) $(A \cup C)/B$, $(B \oplus C)/A$

Solⁿ:- Here $U = \{1, 2, 3, \ldots, 9\}$ and

$A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7\}$, $C = \{5, 6, 7, 8, 9\}$

$D = \{1, 3, 5, 7, 9\}$, $E = \{2, 4, 6, 8\}$, $F = \{1, 5, 9\}$

(a) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$ and $D \cup F = \{1, 3, 5, 7, 9\}$

(b) $B^c = \{1, 2, 3, 8, 9\}$, $E^c = \{1, 3, 5, 7, 9\}$, $D^c = \{2, 4, 6, 8\}$

(c) $A/B = \{1, 2, 3, 4, 5\}/\{4, 5, 6, 7\}$, $D/E = \{1, 3, 5, 7, 9\}$

$= \{1, 2, 3\}$, $E/D = \{2, 4, 6, 8\}$

(d) $C \oplus D = C/D \cup D/C$, $E \oplus F = \{2, 4, 6, 8\} \cup \{1, 5, 9\}$

$= \{6, 8\} \cup \{1, 3\}$ $= \{1, 2, 4, 5, 6, 8, 9\}$

$= \{1, 3, 6, 8\}$

$A \oplus B = A/B \cup B/A$

$= \{1, 2, 3\} \cup \{6, 7\}$

$= \{1, 2, 3, 6, 7\}$

(a) $A \cup C / B$, $A \cup C = \{1,2,3,4,5,6,7,8,9\}$

$A \cup C / B = \{1,2,3,8,9\}$

$(B \oplus C)/A$, $B \oplus C = (B-C) \cup (C-B)$

$= \{4\} \cup \{8,9\} = \{4,8,9\}$

Q-3 Consider the relation $R = \{(1,3),(1,4),(3,2),(3,3),(3,4)\}$ on $A = \{1,2,3,4\}$.

(a) Find the matrix M_R of R (b) Find the domain and range of R

(c) Find R^{-1} (d) Draw the directed graph of R

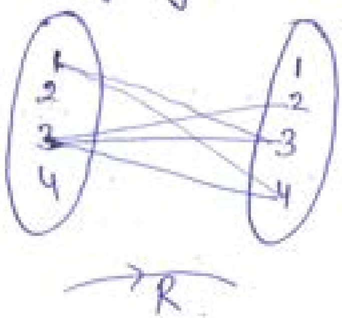
(e) Find the composition relation RoR (e) Find RoR^{-1} and $R^{-1}oR$

Solⁿ:- Here $R = \{(1,3),(1,4),(3,2),(3,3),(3,4)\}$ on $A = \{1,2,3,4\}$

(a) Find the matrix M_R of R [Note: if aRb then put 1 otherwise 0]

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 1 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

(b) Domain and Range of R



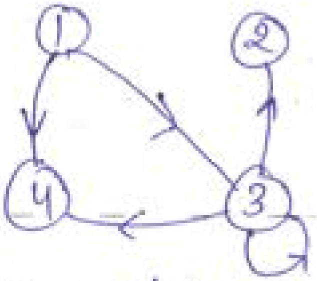
Domain of $R = \{1, 3\}$ and Range of $R = \{2, 3, 4\}$

(c) Find R^{-1}, $R = \{(1,3),(1,4),(3,2),(3,3)(3,4)\}$

$R^{-1} = \{(b,a) \mid (a,b) \in R\}$

$R^{-1} = \{(3,1),(4,1),(2,3),(3,3),(4,3)\}$

(d) Draw the directed graph of R



(e) Find the Composition relation RoR

[Note: for RoS, $(a,b) \in R$, $(b,c) \in S \Rightarrow (a,c) \in R \circ S$]

Here $R = \{(1,3),(1,4),(3,2),(3,3),(3,4)\}$

then $R \circ R = \{(1,2),(1,3),(1,4),(3,2),(3,3)(3,4)\}$

$(1,3) \in R$ & $(3,2) \in R$
$\Rightarrow (1,2) \in R \circ R$
$(1,3) \in R, (3,3) \in R$
$\Rightarrow (1,3) \in R \circ R$

(f) Find $R \circ R^{-1}$ & $R^{-1} \circ R$

$R = \{(1,3),(1,4),(3,2),(3,3),(3,4)\}$

$R^{-1} = \{(3,1),(4,1),(2,3),(3,3)\ (3,4)\}$

$R \circ R^{-1} = \{(1,1),(1,3),(1,4),(3,3),(3,1),(3,4)\}$

$R^{-1} \circ R = \{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,3)(4,4)\}$

Q-4: Prove that in the set of integers, the relation defined by the statement, $(x-y)$ divisible by 5 is an equivalence relation.

Proof: Here Set of integers, $\mathbb{Z} = \{0, \pm1, \pm2, \pm3, \ldots\}$

Now xRy is defined as $(x-y)$ divisible by 5

To Prove that R is equivalence relation, we need to show that R is (i) Reflexive (ii) Symmetric (iii) transitive

(i) R is reflexive → clearly xRx,

as $x-x=0$ and we know that 0 is divisible by 5

$\Rightarrow$ $(x-x)$ is divisible by 5

$\Rightarrow$ R is reflexive.

(ii) R is Symmetric → if $xRy \Rightarrow (x-y)$ is divisible by 5

i.e. $x-y = 5m$

$\Rightarrow y-x = -5m$

$\Rightarrow y-x = 5(-m)$, where $-m$ is also integer

$\Rightarrow (y-x)$ is divisible by 5

$\Rightarrow yRx$

$\Rightarrow$ so if $xRy \Rightarrow yRx$

$\Rightarrow$ R is symmetric.

(ii) Transitive → for transitive we have to show that xRy & $yRz \Rightarrow xRz$

Now $xRy \Rightarrow (x-y)$ is divisible by 5

$yRz \Rightarrow (y-z)$ is divisible by 5

Now $(x-y)$ & $(y-z)$ are divisible by 5

$\Rightarrow$ $(x-y)+(y-z)$ is also divisible by 5

$\Rightarrow$ $(x-z)$ is also divisible by 5

$\Rightarrow$ xRz, so R is transitive and thus R is equivalence

Q-5 Prove by mathematical induction for all positive integers that $3\cdot 5^{2n+1} + 2^{3n+1}$ is divisible by 17

Proof Principle of Mathematical Induction:- Let P be a proposition defined on the positive integers N: that is P(n) is either true or false for each $n \in N$. Suppose P has the following two properties

① P(1) is true

② P(k+1) is true whenever P(k) is true

Then P is true for every positive integer $n \in N$

Here $$P = 3\cdot 5^{2n+1} + 2^{3n+1}$$

(i) for n=1 $P(1) = 3\cdot 5^3 + 2^4 = 375+16 = 391$ is divisible by 17

(ii) Let for n=K P(K) is true

$\Rightarrow 3\cdot 5^{2K+1} + 2^{3K+1}$ is divisible by 17

$\Rightarrow 3\cdot 5^{2K+1} + 2^{3K+1} = 17m$ —— ① $\Rightarrow 3\cdot 5^{2K+1} = 17m - 2^{3K+1}$

(iii) We have to show that P is true for n=K+1

now $$P(K+1) = 3\cdot 5^{2(K+1)+1} + 2^{3(K+1)+1}$$

$$= 3\cdot 5^{2K+3} + 2^{3K+4}$$

$$= 5^2\cdot 3\cdot 5^{2K+1} + 2^3\cdot 2^{3K+1}$$

from eq^n ①

$$= 5^2(17m - 2^{3K+1}) + 2^3\cdot 2^{3K+1}$$

$$= 5^2 17m - 17\, 2^{3K+1}$$

$$= 17(5^2 m - 2^{3K+1})$$ which is divisible by 17

Hence by M.I $3\cdot 5^{2n+1} + 2^{3n+1}$ is divisible by 17 for all positive integers

Q.6 Prove that $\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+\frac{1}{9\cdot 13}+\cdots+\frac{1}{(4n-3)(4n+1)}=\frac{n}{4n+1}$

Proof: Now we are proving this by P.I.

Let is will be true for $n=1$

$$\Rightarrow L.H.S = \frac{1}{1\cdot 5} = \frac{1}{5}, \quad R.H.S = \frac{1}{4+1} = \frac{1}{5}$$

$$L.H.S = R.H.S.$$

Now let this will be true for n,

then $$\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+\frac{1}{9\cdot 13}+\cdots+\frac{1}{(4n-3)(4n+1)} = \frac{n}{(4n+1)} \quad \text{—(1)}$$

for $n = n+1$,

$$\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+\frac{1}{9\cdot 13}+\cdots+\frac{1}{[4(n+1)-3][4(n+1)+1]} = \frac{n+1}{4(n+1)+1}$$

$$= \frac{n+1}{(4n+5)}$$

L.H.S.

$$\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+\frac{1}{9\cdot 13}+\frac{1}{(4n+1)(4n+5)}$$

$$= \underbrace{\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+\frac{1}{9\cdot 13}+\cdots+\frac{1}{(4n-3)(4n+1)}}+\frac{1}{(4n+1)(4n+5)}$$

from eq^n (1)

$$= \frac{n}{4n+1}+\frac{1}{(4n+1)(4n+5)} = \frac{1}{4n+1}\left[n+\frac{1}{4n+5}\right]$$

$$= \frac{1}{4n+1}\left[\frac{4n^2+5n+1}{4n+5}\right] = \frac{(4n+1)(n+1)}{(4n+1)(4n+5)}$$

$$= \frac{n+1}{4n+5} = R.H.S. \Rightarrow L.H.S. = R.H.S.$$

thus this will be true for all n.

Q-7 Consider the funⁿ $f: X \to Y$ and $g: Y \to Z$ be two funⁿ's such that $gof: X \to Z$ is bijective prove that

① f is injective

② g is surjective

Proof:- Let $f: X \to Y$ and $g: Y \to Z$ be two functions such that $gof: X \to Z$ is bijective

1. Let x_1 & x_2 be any elements in X such that

$$f(x_1) = f(x_2)$$

$$\Rightarrow g[f(x_1)] = g[f(x_2)] \quad [\because g \text{ is a fun}^n]$$

$$\Rightarrow gof(x_1) = gof(x_2)$$

$$\Rightarrow x_1 = x_2$$

$\because gof$ is bijective so gof is injective

thus $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Hence f is injective

2. Let z be any element in Z

Since gof is surjective, therefore $\exists x_1 \in X$ such that

$$(gof)x_1 = z$$

$$\Rightarrow g[\underbrace{f(x_1)}_{\in Y}] = z$$

thus for any $z \in Z$ there exists an element $f(x_1) \in Y$ such that $g[f(x_1)] = z$

Hence g is surjective.

Q-8 If $f : R \to R$, $g : R \to R$ and $h : R \to R$ defined by
$f(x) = 3x^2 + 2$, $g(x) = 7x - 5$ and $h(x) = \frac{1}{x}$

Compute the following composition fun^n's

(i) $(f \circ g \circ h)(x)$ (ii) $(g \circ g)(x)$

(iii) $(g \circ h)(x)$ (iv) $(h \circ g \circ f)(x)$

Sol^n: Here $f(x) = 3x^2 + 2$

$g(x) = 7x - 5$

$h(x) = \frac{1}{x}$

(i) $(f \circ g \circ h)(x) = (f \circ g)\left(\frac{1}{x}\right)$ $\because h(x) = \frac{1}{x}$

$= f\left(\frac{7}{x} - 5\right)$ $\because g(x) = 7x - 5$

$= 3\left(\frac{7}{x} - 5\right)^2 + 2$

$= 3\left(\frac{49}{x^2} + 25 - \frac{70}{x}\right) + 2 = \frac{147}{x^2} - \frac{210}{x} + 77$

$= \frac{77x^2 - 210x + 147}{x^2}$

(ii) $(g \circ g)(x) = g(7x - 5) = 7(7x - 5) - 5 = 49x - 40$

(iii) $(g \circ h)(x) = g\left(\frac{1}{x}\right) = \frac{7}{x} - 5 = \frac{7 - 5x}{x}$

(iv) $(h \circ g \circ f)(x) = (h \circ g)(3x^2 + 2) = h(7(3x^2 + 2) - 5)$

$= \frac{1}{21x^2 + 9}$

Q-9 Find the number between 100 to 1000 which is divisible by 3 or 5 or 7.

Solⁿ:- Let A = Set of all integer b/w 100 to 1000 divisible by 3

$A = \{102, 105, 108, ---, 999\}$

$|A| = 300$

[Note: 102, 105, 108, --- 999 in AP with $d=3$, $a=102$, $T_n = 999$

$T_n = a+(n-1)d$

$999 = 102+(n-1)3$

$n = \frac{999-99}{3} = 300$]

Let B = Set of all integers b/w 100 to 1000 divisible by 5

$B = \{105, 110, 115, ---, 995\}$

$|B| = 179$

[$T_n = a+(n-1)d$

$995 = 105+(n-1)5$

$n = \frac{995-100}{5} = \frac{895}{5} = 179$

Let C = Set of all integers b/w 100 to 1000 divisible by 7

$C = \{105, 112, 119, ---, 994\}$

$|C| = 128$

[$994 = 105+(n-1)7$

$n = \frac{994-98}{7} = 128$

Now $A \cap B$ = Set of all integers b/w 100 to 1000 divisible by 3 or 5

= Set of all integers b/w 100 to 1000 divisible by 15

$= \{105, 120, 135, ---, 990\}$

$|A \cap B| = 180$

Similarly $|A \cap C| = 43$

$|B \cap C| = 26$

$|A \cap B \cap C| = 9$

the set of numbers divisible by 3 or 5 or 7 $= A \cup B \cup C$

$$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$

$$= 300 + 179 + 128 - 180 - 43 - 26 + 9$$

$$= 367$$

Q-10 Suppose a list A Contains the 30 students in mathematics class and a list B Contains 35 students in an English class and suppose there are 20 names on both list. Find the number of students in

(a) Only on list A (b) Only on list B (c) on list A or B

(d) on exactly one list.

Solⁿ Inclusion-Exclusion Principle → $n(A\cup B) = n(A) + n(B) - n(A\cap B)$

(a) List A has 30 names, 20 names are on both lists

⇒ names Only on A = 30 − 20 = 10

(b) List B has 35 names, 20 names are on both lists

⇒ Names only on B = 35 − 20 = 15

(c) Names on list A or B = $n(A\cup B)$

$$n(A\cup B) = n(A) + n(B) - n(A\cap B)$$

$$= 30 + 35 - 20 = 45$$

(d) Names on exactly one list i.e. $n(A\oplus B)$

from (a) & (b) $n(A\oplus B) = 10 + 15 = 25$

Q-1 Consider the set Q of rational numbers and $*$ be the operation on Q defined by: $a*b = a+b-ab$

(a) (i) Find $3*4$ (ii) $2*(-5)$ (iii) $7*(\frac{1}{2})$

(b) Is $(Q,*)$ a semigroup? Is it Commutative?

(c) Find the identity element of $(Q,*)$.

(d) Do any of the element in $(Q,*)$ have an inverse? What is it?

Sol^n (a) Here $a*b = a+b-ab$, $a,b \in Q$

(i) $3*4 = 3+4-3\times4 = 7-12 = -5$

(ii) $2*(-5) = 2+(-5)-2(-5) = 2-5+10 = 7$

(iii) $7*(\frac{1}{2}) = 7+\frac{1}{2}-7\times\frac{1}{2} = 4$

(b) We have

$$a*(b*c) = a*[b+c-bc]$$
$$= a+(b+c-bc)-a(b+c-bc)$$
$$= a+b+c-bc-ab-ac+abc \quad \text{--- (1)}$$

$$(a*b)*c = (a+b-ab)*c$$
$$= (a+b-ab)+c-(a+b-ab)c$$
$$= a+b+c-ab-ac-bc+abc \quad \text{--- (2)}$$

from eq^n (1) & eq^n (2)

$$a*(b*c) = (a*b)*c$$

$\Rightarrow$ $(Q,*)$ is a semigroup.

Now $a*b = a+b-ab$

$b*a = b+a-ba = a+b-ab = a*b$

$\Rightarrow a*b = b*a$

$\Rightarrow$ $(Q,*)$ is Commutative

(c) Let e be an identity element of $*$ then $a*e = a$ for every $a \in Q$,

$$a*e = a+e-ae = a$$
$$\Rightarrow e(1-a) = 0$$
$$\Rightarrow e = 0$$, Hence 0 is the identity element of $(Q,*)$

(d) Let x be an inverse for $a \in Q$ then it satisfies $a * x = e$

$$\Rightarrow a * x = 0$$
$$\Rightarrow a + x - ax = 0$$
$$\Rightarrow a + x(1-a) = 0$$
$$x = \frac{a}{a-1}$$

thus if $a \neq 1$, then a has invers and it is $\frac{a}{(a-1)}$.

Q-2 Let G be a reduced residue system modulo 15, say $G = \{1, 2, 4, 7, 8, 11, 13, 14\}$. Then G is a group under the multiplication modulo 15

(a) find the multiplication table of G (b) find $2^{-1}, 7^{-1}, 11^{-1}$

(c) find the orders and subgroups generated by 2, 7 and 11

(d) Is G cyclic?

Solⁿ: (a) To find $a * b$ in G, find the remainder when the product ab is divided by 15. $G = \{1, 2, 4, 7, 8, 11, 13, 14\}$

$\times_{15}$	1	2	4	7	8	11	13	14
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	1	8	4
13	13	11	7	1	14	8	4	2
14	14	13	11	8	7	4	2	1

(b) find $2^{-1}, 7^{-1}, 11^{-1}$, for inverse $a * b = e$, here $e = 1$

$\Rightarrow a * b = 1$ so for inverse of 2, $2 * 2^{-1} = 1$

$2 \times_{15} 2^{-1} = 1$

$\Rightarrow 2^{-1} = 8$, $7^{-1} = 13$, $11^{-1} = 11$

(c) Here we have $2^2 = 2 \times_{15} 2 = 4$, $2^3 = 8$, $2^4 = 16 \times_{15} = 1$

Hence $O(2) = 4$

group generated by 2

$$\Rightarrow 2^1 = 2$$
$$2^2 = 4$$
$$2^3 = 8$$
$$2^4 = 1$$
$$2^5 = 2$$
$$2^6 = 4$$
$$2^7 = 8$$
$$2^8 = 1$$

group generated by $2 = \{1, 2, 4, 8\}$

Also $7^2 = 4$, $7^3 = 13$, $7^4 = 1$, hence $O(7) = 4$

group generated by $7 = \{1, 4, 7, 13\}$

Now $11^2 = 1$, Hence $O(11) = 2$

and group genrated by $11 = \{1, 11\}$

(d) Since G Has no element which generat the G so G is not Cyclic.

Q-3 Consider the symmetric group S_3,

(a) Find the order and the group generated by each element of S_3

(b) Find the number and all subgroups of S_3

(c) Let $A = \{\sigma_1, \sigma_2\}$ and $B = \{\phi_1, \phi_2\}$. Find AB, $\sigma_3 A$ and $A\sigma_3$

(d) Is S_3 cyclic?

Solⁿ: The symmetric group S_3 $o(S_3) = 3! = 6$ has 6 elements

$$\varepsilon = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

the multiplication table of S_3

	ε	σ_1	σ_2	σ_3	ϕ_1	ϕ_2
ε	ε	σ_1	σ_2	σ_3	ϕ_1	ϕ_2
σ_1	σ_1	ε	ϕ_1	ϕ_2	σ_2	σ_3
σ_2	σ_2	ϕ_2	ε	ϕ_1	σ_3	σ_1
σ_3	σ_3	ϕ_1	ϕ_2	ε	σ_1	σ_2
ϕ_1	ϕ_1	σ_3	σ_1	σ_2	ϕ_2	ε
ϕ_2	ϕ_2	σ_2	σ_3	σ_1	ε	ϕ_1

Now for order of element $o(x) = x^n = \varepsilon$, $n \le o(S_3)$

(i) $\varepsilon^1 = \varepsilon$, $o(\varepsilon) = 1$ and group generated by $\varepsilon = \{\varepsilon\}$ → ε is identity element

(ii) $\sigma_1^2 = \varepsilon$, $o(\sigma_1) = 2$ and group generated by $\sigma_1 = \{\varepsilon, \sigma_1\}$

(iii) $\sigma_2^2 = \varepsilon$, $o(\sigma_2) = 2$ and group generated by $\sigma_2 = \{\varepsilon, \sigma_2\}$

(iv) $\sigma_3^2 = \varepsilon$, $o(\sigma_3) = 2$ and group generated by $\sigma_3 = \{\varepsilon, \sigma_3\}$

(v) $\phi_1^2 = \phi_2$, $\phi_1^3 = \varepsilon$, $o(\phi_1) = 3$ and group generated by $\phi_1 = \{\varepsilon, \phi_1, \phi_2\}$

(vi) $\phi_2^2 = \phi_1$, $\phi_2^3 = \varepsilon$, $o(\phi_2) = 3$ and group generated by $\phi_2 = \{\varepsilon, \phi_1, \phi_2\}$

(b) Here $o(S_3) = 6$ so S_3 has subgroup of order, 1, 2, 3, & 6

Subgroup of order $1 = \{\varepsilon\} = H_1$

Subgroup of order $6 = \{S_3\} = H_2$

Subgroup of order 2 $\Rightarrow H_3 = \{\epsilon, \sigma_1\}$, $H_4 = \{\epsilon, \sigma_2\}$, $H_5 = \{\epsilon, \sigma_3\}$

Subgroup of order 3 $\Rightarrow H_6 = \{\phi_1, \phi_2, \epsilon\}$

So, S_3 has 6 Subgroup of order, 1, 2, 3, & 6.

(c) Here $A = \{\sigma_1, \sigma_2\}$, $B = \{\phi_1, \phi_2\}$

Now multiply each element of A by each element of B

$$\sigma_1\phi_1 = \sigma_2 \qquad \sigma_2\phi_1 = \sigma_3$$

$$\sigma_1\phi_2 = \sigma_3 \qquad \sigma_2\phi_2 = \sigma_1$$

Hence $AB = \{\sigma_1, \sigma_2, \sigma_3\}$

Now for $\sigma_3 A$, $\sigma_3\sigma_1 = \phi_1$, $\sigma_3\sigma_2 = \phi_2$

hence $\sigma_3 A = \{\phi_1, \phi_2\}$

Now for $A\sigma_3$, $\sigma_1\sigma_3 = \phi_2$, $\sigma_2\sigma_3 = \phi_1$

hence $A\sigma_3 = \{\phi_1, \phi_2\}$

(d) Since S_3 has no element of order 6 thus S_3 has no generator therefore S_3 is not Cyclic.

Q-4 What do you mean by the Cosets of a subgroup? Consider the group $\mathbb{Z}$ of integers under the addition and the subgroup $H = \{---, -12, -6, 0, 6, 12, ---\}$ considering the multiple of 6

(a) Find the Cosets of H in G

(b) What is the index of H in G.

Soln Let $\phi \neq H$ subgroup of Group G and $a \in G$ then, the set

$$aH = \{a \cdot h \mid h \in H\}$$ is called Left coset of H in G

and the set

$$Ha = \{h \cdot a \mid h \in H\}$$ is called Right coset of H in G.

(a) Here $G = \{0 \pm 1, \pm 2, \pm 3 ---\} = \mathbb{Z}$

$H = 6\mathbb{Z} = \{---, -12, -6, 0, 6, 12, ---\}$

Such that $0 \in \mathbb{Z}$

then $0 + H = 0 + 6\mathbb{Z} = \{---, -12, -6, 0, 6, 12, ---\}$

$1 \in \mathbb{Z}$ $\quad 1 + H = 1 + 6\mathbb{Z} = \{---, -11, -5, 1, 7, 13, ---\}$

$2 \in \mathbb{Z}$, $\quad 2 + H = 2 + 6\mathbb{Z} = \{---, -10, -4, 2, 8, 14, ---\}$

$3 \in \mathbb{Z}$, $\quad 3 + H = 3 + 6\mathbb{Z} = \{---, -9, -3, 3, 9, 15, ---\}$

$4 \in \mathbb{Z}$, $\quad 4 + H = 4 + 6\mathbb{Z} = \{---, -8, -2, 4, 10, 16, ---\}$

$5 \in \mathbb{Z}$, $\quad 5 + H = 5 + 6\mathbb{Z} = \{---, -7, -1, 5, 11, 17, ---\}$

$6 \in \mathbb{Z}$, $\quad 6 + H = 6 + 6\mathbb{Z} = \{---, -6, 0, 6, 12, ---\} = 6\mathbb{Z}$

$0 + 6\mathbb{Z} = 6 + 6\mathbb{Z}$ $\quad 2 + 6\mathbb{Z} = 8 + 6\mathbb{Z}$ ---

$1 + 6\mathbb{Z} = 7 + 6\mathbb{Z}$, $\quad 3 + 6\mathbb{Z} = 9 + 6\mathbb{Z}$ ---

Hence H has only 6 distinct Cosets in $\mathbb{Z}$,

$0+H$, $1+H$, $2+H$

$3+H$, $4+H$, $5+H$

(5) Index of H in G = total number of distinct Cosets of H in G

Since, H is abelian, all left cosets are equal to right Cosets thus distinct Cosets of $H = 6$

Index of H in $G = 6$

Q-5: In a group G prove that

(i) $(a^{-1})^{-1} = a$

(ii) $(ab)^{-1} = b^{-1}a^{-1}$

Proof: (i) Let G be a group and $a \in G$ and a^{-1} is an inverse of a in G

$\Rightarrow \quad aa^{-1} = e = a^{-1}a$

and $a^{-1} \in G$ ----

$\Rightarrow$ Inverse of (a^{-1}) is 'a' by inverse property

$\Rightarrow \boxed{(a^{-1})^{-1} = a}$

Hence proved

(ii) Let G be a group and $a \in G$ & $b \in G$

$\Rightarrow ab \in G$ [$\because$ G is a group]

Now, we have to show that inverse of (ab) is $b^{-1}a^{-1}$, i.e.

$$(ab)(b^{-1}a^{-1}) = e = (b^{-1}a^{-1})(ab)$$

take left side

$ab(b^{-1}a^{-1}) = a(bb^{-1})(a^{-1})$ by associative ing

$= a(e)a^{-1}$ $\quad \because bb^{-1} = e$

$= aa^{-1}$ $\quad \because ae = a$

$= e$

IIly $b^{-1}a^{-1}(ab) = e$

$\Rightarrow \boxed{(ab)^{-1} = b^{-1}a^{-1}}$

Hence proved.

Q-6 (a): Let H be a subgroup of a finite group G. Prove that the order of H is a divisor of order of G. i.e. $o(H) | o(G)$

OR

State and prove Lagrange Theorem.

Proof: Let G be a finite group of order 'n' and H be a subgroup of G of order 'm' then we have to show that $o(H) | o(G)$ i.e. $m|n$.

Since H is a subgroup of G of order m then H has 'm' distinct elements, $h_1, h_2, h_3 --- h_m$

Now let $a \in G$, Claim that Ha has m-distinct elements i.e. $h_1 a, h_2 a, h_3 a, --- h_m a$

proof claim: let $h_i a = h_j a$ for $i \neq j$

$$h_i a a^{-1} = h_j a a^{-1} \quad [a \in G \text{ then } a^{-1} \in G]$$

$= h_i = h_j$ which is contradiction

$\Rightarrow Ha$ has 'm' distinct elements.

Now G is finite, then number of all cosets of H in G are finite. This implies that number of distinct right cosets of H in G are finite.

Suppose number of distinct right cosets of H in G is 'k'

Also we have,

$$G = \underbrace{Ha_1 \cup Ha_2 \cup Ha_3 \cup --- \cup Ha_k}_{k\text{-times}}$$

$$o(G) = o(Ha_1) + o(Ha_2) + o(Ha_3) + --- + o(Ha_k)$$

$$n = m + m + m + --- + m$$

$$n = km$$

$$\frac{n}{m} = k \Rightarrow o(H) | o(G)$$

Hence proved

Q.6(b) Prove that every group of prime order is cyclic.

Proof – Let G be a group of order p (say)

we have to show that G is cyclic.

as we know that order of every element of a group divides order of G then,

for $x \in G$, $o(x) \mid o(G)$

$\Rightarrow o(x) \mid p$ $\quad \because p$ is prime

$\Rightarrow o(x) = 1$ or p

we also know that identity is the only element in a group of order 1

so, number of element of order $p = p-1$

thus G has element of order p, therefore G is cyclic.

Hence proved

Q7 Let $G = \{1, -1, i, -i\}$ with the binary operation multiplication be an algebraic structure, where $i = \sqrt{-1}$, Determine whether G is an abelian or not.

Soln: First we construct the composition table for G

×	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

as by the table all element are belongs to G thus G is closed under the multiplication.

(1) Associative → By the composition table G holds associativity.

(2) Identity → 1 is the identity element for G as $1 \times a = 1$ for every $a \in G$

(3) Inverse → from the composition table inverse of as for inverse $a * a^{-1} = 1$

$1^{-1} = 1$
$(-1)^{-1} = -1$
$(i)^{-1} = -i$
$(-i)^{-1} = i$

(4) Commutative :- As by the composition table each rows and columns of corresponding element are identical. Therefore the given binary operation is commutative on G thus $(G, \times)$ is commutative and G is abelian.

Q.8 The subgroup H of a group G is normal subgroup iff $g^{-1}hg \in H$ for every $h \in H$ & $g \in G$.

Proof Let H be normal subgroup of G then

$$Hg = gH \quad \text{for } \forall\, g \in G$$

where $Hg = \{hg \mid h \in H\}$ & $gH = \{gh \mid h \in H\}$

$$\Rightarrow hg = gh$$

Now $g \in G,\ g^{-1} \in G \Rightarrow gg^{-1} = e$

$$h \in H \Rightarrow eh \in H$$
$$\Rightarrow g^{-1}gh \in H$$
$$\Rightarrow g^{-1}hg \in H \qquad \because gh = hg$$

Conversely, Let $g^{-1}hg \in H$ $\forall\, g \in G$ and $h \in H$ then we have to show that H is normal i.e.

$$Hg = gH$$

Let $x \in Hg \Rightarrow x = hg$
$$\Rightarrow x = ehg$$
$$\Rightarrow x = g\underline{g^{-1}hg}$$
$$\Rightarrow x \in gH$$
$$\Rightarrow Hg \subseteq gH \quad \text{— (1)}$$

IIIly, we can show that $gH \subseteq Hg$ — (2)

from eq^n (1) & eq^n (2)

$$\boxed{Hg = gH}$$

Q-9 (a) Prove that $\mathbb{Z}_6$ is Cyclic.

(b) Obtain all distinct left cosets of $H = \{0, 3\}$ in the group $(\mathbb{Z}_6, +_6)$ and find their Union.

Soln: (a) Here $\mathbb{Z}_6$ is a group with the operation addition modulo 6

$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

Now $o(0) = 1$ $\quad o(2) = 3$ $\quad o(4) = 3$

$o(1) = 6$ $\quad o(3) = 2$ $\quad o(5) = 6$

as $\mathbb{Z}_6$ has two element of order 6 are, 1 & 5

$\langle 1 \rangle = \{n \cdot 1 \mid n \in \mathbb{Z}\}$

$1 \cdot 1 = 1$	$1 \cdot 5 = 5$
$2 \cdot 1 = 2$	$2 \cdot 5 = 4$
$3 \cdot 1 = 3$	$3 \cdot 5 = 3$
$4 \cdot 1 = 4$	$4 \cdot 5 = 2$
$5 \cdot 1 = 5$	$5 \cdot 5 = 1$

So $\langle 1 \rangle$ & $\langle 5 \rangle$ are generator of $\mathbb{Z}_6$

thus $(\mathbb{Z}_6, +_6)$ is Cyclic.

(b) Here $H = \{0, 3\}$

Left Cosets of H are

$0 \in \mathbb{Z}_6$, $0 + H = \{0, 3\} = H_1$

$1 \in \mathbb{Z}_6$, $1 + H = \{1, 4\} = H_2$

$2 \in \mathbb{Z}_6$, $2 + H = \{2, 5\} = H_3$

$3 \in \mathbb{Z}_6$, $3 + H = \{3, 0\} = H_4$

$4 \in \mathbb{Z}_6$, $4 + H = \{4, 1\} = H_5$

$5 \in \mathbb{Z}_6$, $5 + H = \{5, 2\} = H_6$

, as $H_1 = H_4$

$H_2 = H_5$

$H_3 = H_6$

thus $H = \{0, 3\}$ has 3 distinct left Cosets in $\mathbb{Z}_6$

$H_1 \cup H_2 \cup H_3 = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

Q-10 (a) Define Ring? Give the example of both Commutative and non-Commutative rings.

Sol^n: Ring :- Let R be a non-empty set with two binary operations + & •, (R,+,•) is said to be a ring if

① (R,+) is Commutative group

② (R,•) is Semigroup

③ Left & right distributive law holds

① (R,+) is Commutative group

(i) $a+(b+c) = (a+b)+c$, $\forall a,b,c \in R$

(ii) $a+0 = a = 0+a$, $\forall a \in R$

(iii) $a+b = 0 = b+a$ for each $a \in R$

(iv) $a+b = b+a$ $\forall a,b \in R$

② (R,•) is Semigroup

$\forall a,b,c \in R$, $a\cdot(b\cdot c) = (a\cdot b)\cdot c$.

③ Left and Right distributive law hold

(i) $a\cdot(b+c) = (a\cdot b)+(a\cdot c)$ left distributive law

(ii) $(a+b)\cdot c = (a\cdot c)+(b\cdot c)$ right " "

} $\forall a,b,c \in R$

Ex:- $(\mathbb{Z},+,\cdot)$ is a ring.

Commutative Ring :- A ring which satisfies $a\cdot b = b\cdot a$ $\forall a,b \in R$ is said to be Commutative.

Ex:- $(\mathbb{Z},+,\cdot)$

Example of non-Commutative ring. $M_{2\times2}(\mathbb{R})$ as take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$AB \neq BA$ ---

Q10(b) Find the zero divisor of $\mathbb{Z}_6$ & $\mathbb{Z}_4$.

Solⁿ: A Ring is said to be a ring with zero divisor if a & b are two non-zero element of R such that $a \cdot b = 0$ then a & b are zero divisor.

$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

here $2 \times 3 = 6 = 0$, $+_6$

$3 \times 4 = 12 = 0$, $+_6$

thus $2, 3,$ & 4 are zero divisor of $\mathbb{Z}_6$

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$

here $2 \cdot 2 = 4 = 0$, $+_4$

thus 2 is only zero divisor of $\mathbb{Z}_4$

Q.1 Consider the poset $P = \{1, 3, 4, 12, 24, 48, 72\}$ with respect to the relation "divides". Is P a lattice?

Solⁿ Here $P = \{1, 3, 4, 12, 24, 48, 72\}$ with relation of divisibility. i.e. in (x, y), x/y

Now

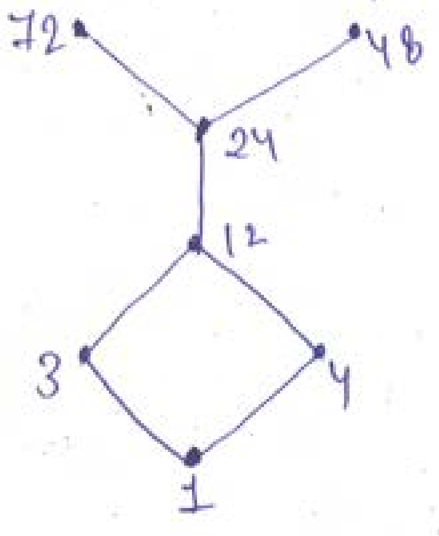
let x, y be any two elements in P, then $\sup\{x, y\}$ under divisibility relation is least positive integer 'z' such that x/z & y/z

i.e. $\sup(x,y) = \text{lcm of } (x, y)$

IIIly $\inf\{x,y\}$ is the greatest positive integer 'w' such that w/x & w/y i.e.

$\inf\{x,y\} = \gcd(x,y)$

Now, Draw the Hasse Diagram for P



Clearly from the diagram $\sup\{48, 72\}$ does not belongs to P, therefore P is not a lattice under divisibility.

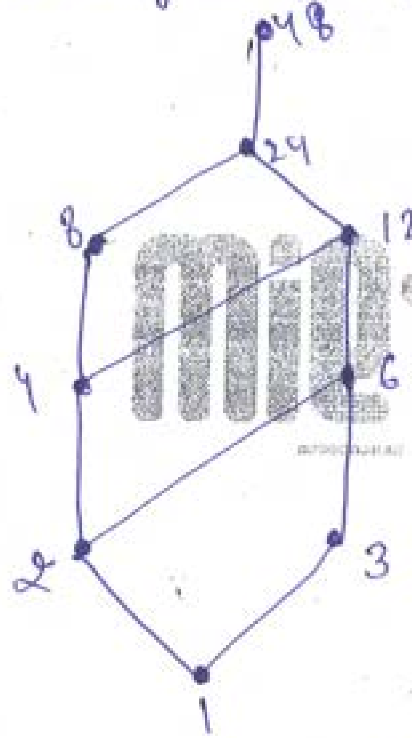
Q-2 (i) Is D_{48} a lattice? Explain your answer.
(ii) Is D_{48} a complmented lattice? Justify your answer
(iii) Show that D_{42} is a complmented lattice

Solⁿ:

D_{48} = set of all positive divisor of 48

$D_{48} = \{1, 2, 3, 4, 6, 8, 12, 24, 48\}$ is a poset under the relation of divisibility

Now, The Hasse diagram of D_{48} is given by

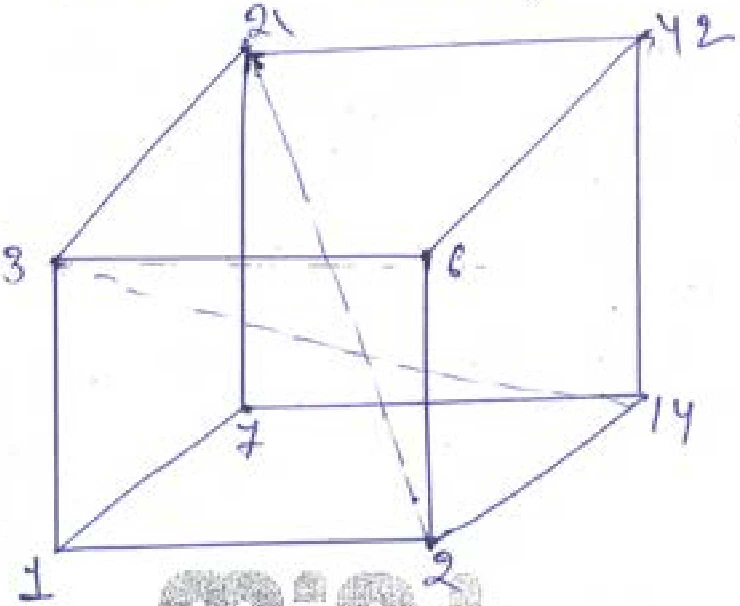


Clearly from the diagram D_{48} is a lattice as for any pair $a, b \in D_{48}$ we have

$a \vee b = LCM\{a, b\}$ & $a \wedge b = GCD\{a, b\}$ exist in D_{48}

(ii) Here from diagram $I = 48$ & $\overline{48} = 1$ but the Complimant of 2, 4, 8, 3 does not exist, So D_{48} is not a Complimented lattice. [$\because$ in D_n, $a, b \in D_n$ $a \vee b = n$, $a \wedge b = 1$] then a & b are complements]

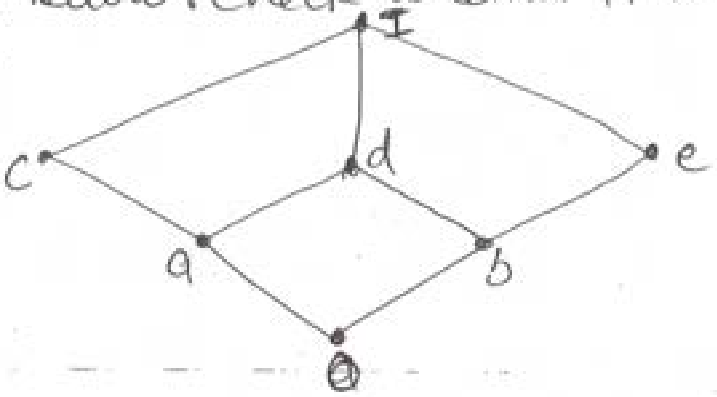
(iii) $D_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$ is a lattice under the relation division. Now the Hasse diagram of D_{42} is given as a Cube.



In any such lattice, the diagonally opposit elements are Complements of each other.
For example $\overline{3} = 14$ & $\overline{14} = 3$, $\overline{2} = 21$, $\overline{21} = 2$ --- and so on.

Hence D_{42} is a Complemented lattice as each element has complement.

Q-3 What do you mean by distributed lattice and Complimented lattice? Consider the bounded lattice L given below. Check whether it is distributive or not.

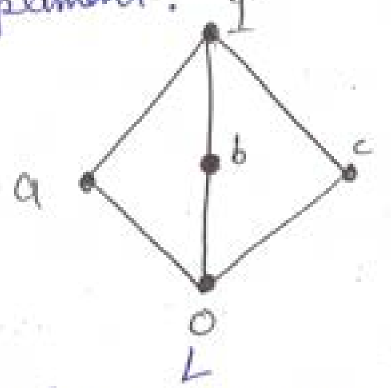


Sol^n: Distributive Lattice:- A lattice L is called distributive lattice if for any $a, b, c \in L$, if it satisfies the distributive property →

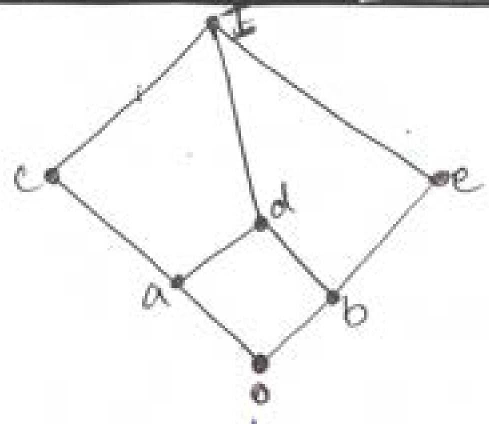
① $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

② $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Complimented Lattice:- A lattice L is called a complemented lattice if L is bounded and every element in L has a Compliment.



the given lattice L is Complemented as all elements has Compliment. $\bar{0} = I$, $\bar{I} = 0$, $\bar{a} = c$, $\bar{c} = a$, $\bar{b} = a$



from given lattice L c, d, e ∈ L such that

$$c \wedge (d \vee e) = c \wedge I$$
$$= c$$

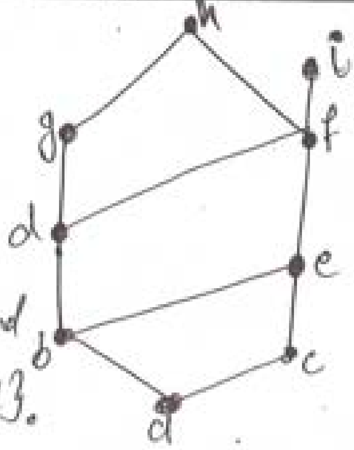
but

$$(c \wedge d) \vee (c \wedge e) = a \vee 0$$
$$= a$$

i.e. $c \wedge (d \vee e) \neq (c \wedge d) \vee (c \wedge e)$

thus given lattice is not distributive.

Q-4 Find the lower and upper bounds of the subsets {a, b, c}, {i, h} and {a, c, d, f} in the poset with the given Hasse diagram. Also find the greatest lower bound and least upper bound of {b, d, g}.



Soln:- The lower bound of {a, b, c} = a

lower bounds of {i, h} = a, b, c, d, e, f

lower bound of {a, c, d, f} = a

Upper bound of {a, b, c} = e, f, h, i

Upper bound of {i, h} doesn't exist.

Upper bound of {a, c, d, f} = f, h, i

glb {b, d, g} = b

lub {b, d, g} = g.

Q-5 Let $(L, \leq)$ be a lattice then for any $a, b \in L$, then

① $a \leq b \Leftrightarrow a \wedge b = a$

② $a \leq b \Leftrightarrow a \vee b = b$

Proof ① Suppose $a \wedge b = a$

Since $a \wedge b = \inf\{a, b\}$, therefore $a \wedge b \leq b$

$\Rightarrow a \leq b$ [$\because a \wedge b = a$]

Conversely suppose that $a \leq b$

Since $\leq$ is reflexive, we have

$a \leq a$

$a \leq b$ & $a \leq a \Rightarrow a$ is lower bound of $\{a, b\}$

$\Rightarrow a \leq \inf\{a, b\}$

$\Rightarrow a \leq a \wedge b$

Since $a \wedge b$ is infimum of $\{a, b\}$, $a \wedge b \leq a$

$\Rightarrow \boxed{a \wedge b = a}$

② Suppose $a \vee b = b$

Since $a \leq a \vee b$ and $a \vee b = b$

$\boxed{a \leq b}$

Conversely, suppose that $a \leq b$

we know for reflexivity $b \leq b$, therefore $\sup\{a, b\} \leq b$

$\Rightarrow a \vee b \leq b$ — ①

by the definition $a \vee b = \sup\{a, b\}$

$b \leq a \vee b$ — ②

from eq^n ① & eq^n ②

$\boxed{a \vee b = b}$

Q.6 In a lattice, $a, b, c \in L$ prove that the following properties

① $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

② $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Proof ① we know that

$a \wedge b \leq a$ $\because [a \wedge b = \inf\{a, b\}]$

and $a \wedge b \leq b \leq b \vee c$

$\Rightarrow a \wedge b$ is lower bound of $\{a, b \vee c\}$

$\Rightarrow a \wedge b \leq a \wedge (b \vee c)$ — ①

Now again $a \wedge c \leq a$ $[\because a \wedge c = \inf\{a, c\}]$

and $a \wedge c \leq c \leq b \vee c$

$\Rightarrow a \wedge c \leq a \wedge (b \vee c)$ — ②

from eq① & eq②, $a \wedge (b \vee c)$ is upper bound of $\{a \wedge b, a \wedge c\}$

$\Rightarrow (a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$

Hence proved

(ii) This inequality can be proved in similar manner or using the principle of duality then we get

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

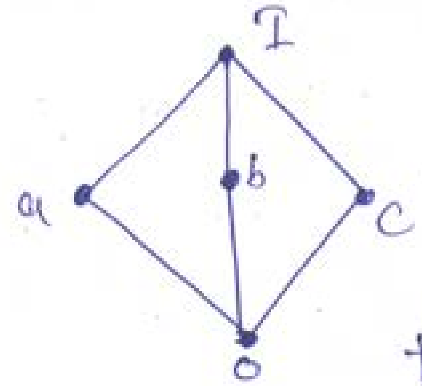
Q-7 (1) Prove that every finite lattice is bounded.
(2) Prove that every distributive lattice is modular.
(3) Give an example of lattice which is modular but not distributive.

Sol^n (1) Let L be a finite lattice
$$L = \{a_1, a_2, a_3 --- a_n\}$$

Now L is finite then
greatest element of $L = a_1 \vee a_2 \vee a_3 \vee --- \vee a_n$
and least element of $L = a_1 \wedge a_2 \wedge a_3 \wedge --- \wedge a_n$
and both are exist in L
$\Rightarrow L$ is a bounded lattice.

(2) Let L be a distributive lattice and $a, b, c \in L$
then we have to show that
$$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$$
Since L is distributive, the L hold
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad \text{—} (1)$$
and also $a \leq c \Rightarrow a \vee c = c$ — (2)
from (1) & (2)
$$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$$
thus L is a modular lattice.

(3)



as $a \wedge (b \vee c) = a \wedge I = a$
$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$

$a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$
thus the given lattice is modular but NOT distributive.

Q-8 Simplify the following Boolean expression using K-map

① $Y = [(AB)' + A' + AB]'$

② $A'B'C'D' + A'B'C'D + A'B'CD + A'B'CD' = A'B'$

Solⁿ:

① $Y = [(AB)' + A' + AB]'$

$= ((AB)')' (A' + AB)'$

$= AB (A')' (AB)'$

$= AB (A (A' + B'))$

$= AB (AA' + AB')$

$= ABAB' = 0$

$\because (A+B)' = A'B'$

$(AB)' = A' + B'$

$(A')' = A$

$BB' = 0$

$A + A' = 1$

② Here $A'B'C'D' + A'B'C'D + A'B'CD + A'B'CD'$

By K-map

CD \ AB	AB	AB'	A'B'	A'B
CD	0	0	1	0
CD'	0	0	1	0
C'D'	0	0	1	0
C'D			1	

thus we get $A'B'$ = R.H.S.

Q-9 Find the Sum-of-Products and Product-of-Sum expasion of the Boolean funn $f(x,y,z) = (x+y)z'$

Ans: $f(x,y,z) = (x+y)z'$

x	y	z	$x+y$	z'	$(x+y)z'$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	0	1	0

for minterm = 1
maxterm = 0

SOP → $f(x,y,z) = xyz' + xy'z' + x'yz'$

POS → $f(x,y,z) = (x+y+z)(x+y'+z)(x'+y+z)(x'+y'+z)(x'+y'+z')$

Q-10 (a) Use Karnaugh map to simplify the Boolean fun^n given by

$$f(a,b,c,d) = ab'c + b'c'd' + bcd + acd' + a'b'c + a'bc'd$$

Sol^n: Here Boolean fun^n

$$f(a,b,c,d) = ab'c + b'c'd' + bcd + acd' + a'b'c + a'bc'd$$

cd \ ab	ab	ab'	a'b'	a'b
cd	1	1	1	1
cd'	1	1	1	
c'd'		1	1	
c'd				1

Hence $f(a,b,c,d) = cd + b'd' + abc + a'bd$

(b) Simplify the Boolean fun^n

$$F(A,B,C,D) = \Sigma(0,1,2,3,4,5,6,7,8,9,11)$$

Here the fun^n is of 4 variables

AB \ CD	CD	CD'	C'D'	C'D
AB	1 $_{0}$	1 $_{1}$	1 $_{3}$	1 $_{2}$
AB'	1 $_{4}$	1 $_{5}$	1 $_{7}$	1 $_{6}$
A'B'	$_{12}$	$_{13}$	$_{15}$	$_{14}$
A'B	1 $_{8}$	1 $_{9}$	1 $_{11}$	0 $_{10}$

Hence $f(A,B,C,D) = AB + AB' + A'BC + A'BD'$

$= AB + AB' + A'B(C + D')$

Q-1 (i) Show that the propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

Sol^n: Construct the truth tables for $\neg(p \wedge q)$ & $\neg p \vee \neg q$

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Since from the truth tables the truth values of $\neg(p \wedge q)$ & $(\neg p \vee \neg q)$ are same in all possible cases, the proposition $\neg(p \wedge q)$ & $(\neg p \vee \neg q)$ are logically equivalent.

(ii) Use truth table to show that the two propositions $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent.

Sol^n:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	M
T	T	T	F	F	F	T
T	F	F	F	T	F	F
F	T	F	T	F	F	F
F	F	F	T	T	T	T

So from the truth table the truth values for both proposition are same for all possible cases, thus they are logically equivalent.

Q.2 Show that $\neg(p \vee q) \vee (\neg p \wedge q) \equiv \neg p$ without using truth table.

Proof

Statements	Reason
① $\neg(p \vee q) \vee (\neg p \wedge q)$	
② $\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge q)$	By DeMorgan's Law
③ $\equiv \neg p \wedge (\neg q \vee q)$	By Distributive Law
④ $\equiv \neg p \wedge T$	By Complement Law
⑤ $\equiv \neg p$	By Identity Law

thus $\boxed{\neg(p \vee q) \vee (\neg p \wedge q) \equiv \neg p}$

Q3 Define tautologies, contradications, and Contingency. Check whether the $(p \vee q) \vee (\neg p \vee r) \rightarrow (q \vee r)$ is tautology, contradication or contingency.

Solⁿ. Tautologies :- A statement which is true for all possible truth values of its propositional variables is called a tautology.

Contradication :- A statement which is always false for all possible truth values is called a Contradiction.

Contingency :- A statement which is true as well as false for some possible truth values is called a Contingency.

Here give $(p \vee q) \vee (\neg p \vee r) \rightarrow (q \vee r) = M$

p	q	r	$p \vee q$	$\neg p$	$\neg p \vee r$	$q \vee r$	$(p \vee q) \vee (\neg p \vee r)$	M
T	T	T	T	F	T	T	T	T
T	T	F	T	F	F	T	T	T
T	F	T	T	F	T	T	T	T
T	F	F	T	F	F	F	T	F
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	F	T	F

Thus from the truth table the given proposition is a Contingency

Q-4 Define Converse, inverse and Contrapositive of $p \to q$. write the Converse, inverse and Contrapositive of given "If he has Courage, then he will win."

Soln Let $p \to q$ be any Conditional statement then

(a) Converse:- Converse of $p \to q$ is $q \to p$

(b) Inverse: Inverse of $p \to q$ is $\neg p \to \neg q$

(c) Contrapositive: Contrapositive of $p \to q$ is $\neg q \to \neg p$

The given statement is "If he has Courage then he will win."

Here $p \to$ He has Courage
$q \to$ He will win
$p \to q$

Converse:- $q \to p$
"If he will win then he has Courage."

Inverse:- $\neg p \to \neg q$
"If he has no Courage then he will not win."

Contrapositive: $\neg q \to \neg p$
"If he will not win then he has no Courage."

Q:-5 Show that $[(p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))] \vee [(\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)]$ is a tautology by using laws of logic.

Sol^n^ Here we have

$[(p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))] \vee [(\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)]$

$\equiv [(p \vee q) \wedge \neg(\neg p \wedge \neg(q \wedge r))] \vee (\neg(p \vee q) \vee \neg(p \vee r))]$ By DeMorgan's Law

$\equiv [(p \vee q) \wedge (p \vee (q \wedge r))] \vee \neg[(p \vee q) \wedge (p \vee r)]$ By DeMorgan's Law

$\equiv [(p \vee q) \wedge (p \vee q) \wedge (p \vee r)] \vee \neg[(p \vee q) \wedge (p \vee r)]$ By Distributive Law

$\equiv [(p \vee q) \wedge (p \vee r)] \vee \neg[(p \vee q) \wedge (p \vee r)]$

this can be written as

$\equiv x \vee \neg x$ where $x = (p \vee q) \wedge (p \vee r)$

$\equiv T$ By Complement law

thus the given proposition is a tautology.

Q6 (a) Let $A = \{1,2,3,4,5\}$. Determine the truth value of each of the following statements.

(a) $(\exists x \in A)(x+3=10)$ (b) $(\forall x \in A)(x+3<10)$

(c) $(\exists x \in A)(x+3<5)$ (d) $(\forall x \in A)(x+3 \leq 7)$

Solⁿ Here $A = \{1,2,3,4,5\}$

(a) False as for no number in A is a solution of $x+3=10$

(b) True, for every number in A satisfies $x+3<10$

(c) True for $x=1$, $1+3<5$ i.e. has only one solution 1

(d) False as $x=5$, $5+3 \nleq 7$

(b) Determine the truth value of each of the following statements where $U=\{1,2,3\}$ is the universal set:

(i) $\exists x \forall y, x^2 < y+1$ (ii) $\forall x \exists y, x^2+y^2<12$ (iii) $\forall x \forall y\ x^2+y^2<12$

Solⁿ (i) $\exists x \forall y, x^2<y+1$

True for $x=1$, $y=1,2,3$

(ii) True for $x=1,2,3$, $y=1$ $\forall x \exists y, x^2+y^2<12$

(iii) $\forall x \forall y, x^2+y^2<12$

false as $x=2$, $y=3$

$2^2+3^2 \nless 12$

Q 7 Explain various Rules of Inference for propositional logic.

Sol^n. Rules of Inference :- In this method, we reduce the given argument to a series of arguments each of which is known to be valid.

There are various rules of Inference are given below.

① Addition :- If p is a premise, we can use addition rule to derive $p \vee q$

$$\frac{p}{\therefore p \vee q}$$

② Conjunction :- If p & q are two ~~premi~~ premises then we can use Conjuction to derive $p \wedge q$

$$\frac{\begin{matrix} p \\ q \end{matrix}}{\therefore p \wedge q}$$

③ Simplification :- If $p \wedge q$ is a premise then we can use simplification to derive p

$$\frac{p \wedge q}{\therefore p}$$

④ Law of Detachment (Modus Pones) :- In this if $p \to q$ & p are two premises and q is the conclusion, then the conclusion of this implication is true.

$$\frac{\begin{matrix} p \\ p \to q \end{matrix}}{q}$$

(5) Law of Contrapositive (Modus Tollens) :- If $p \to q$ & $\neg q$ are two premises and $\neg p$ is Conclusion then Conclusion of this Implication is true.

$$\begin{array}{c} p \to q \\ \neg q \\ \hline \neg p \end{array}$$

(6) Disjunctive Syllogism :- If $p \vee q$ & $\neg p$ are two premises then we can use Disjunctive Syllogism to desire q

$$\begin{array}{c} \neg p \\ p \vee q \\ \hline q \end{array}$$

(7) Law of Hypothetical Syllogism :- It is one of the most useful rule of Inference and also known as Chain Rule.

If $p \to q$ & $q \to r$ are two premises then we can use Hypothetical Syllogism to desire $p \to r$.

$$\begin{array}{c} p \to q \\ q \to r \\ \hline p \to r \end{array}$$

Q'-8 Check the validity of the following argument
"If I get the job and work hard, then I will get promoted. If I get promoted, then I will be happy. I will not be happy. Therefore, either I will not get the job or I will not work hard."

Sol^n' Let p: I get the job
q: I work hard
r: I get promoted
s: I will be happy

then the above argument can be written in symbolic form

$$(p \wedge q) \rightarrow r$$
$$r \rightarrow s$$
$$\neg s$$

Now
① $(p \wedge q) \rightarrow r$ premise (given)
② $r \rightarrow s$ premise (given)
③ $(p \wedge q) \rightarrow s$ Hypothetical Syllogism using 1 & 2
④ $\neg s$ premise (given)
⑤ $\neg(p \wedge q)$ Modus tollens using 3 & 4
⑥ $\neg p \vee \neg q$ By DeMorgan's Law in ⑤
Conclusion

Hence the given argument is valid.

Q9 Translate the following sentences in quantified expressions of predicate logic.

(i) All students need financial aid.

(ii) Some cows are not white.

(iii) Suresh will get division if and only if he gets first division.

(iv) If water is hot then shyam will swim in pool.

(v) All integers are either even or odd integers.

Soln (i) All students need finacial aid.

M → Set of students

$P(x)$ → x need finacial aid

$\forall x \in M \; P(x)$

(ii) Some cows are not white.

M → Set of cows

P → x is a white cow

$\exists x \in M \; \neg P(x)$

(iii) Suresh will get division if and only if he gets first division.

p → Suresh get division

q → Suresh get first division

$p \leftrightarrow q$

(iv) If water is hot then Shyam will swim in pool.

$p \rightarrow$ Water is hot.

$q \rightarrow$ Shyam will swim in pool.

$p \rightarrow q$

(v) All integers are either even or odd integers.

$\mathbb{Z} \rightarrow$ Set of integers

$p \rightarrow$ x is even integer

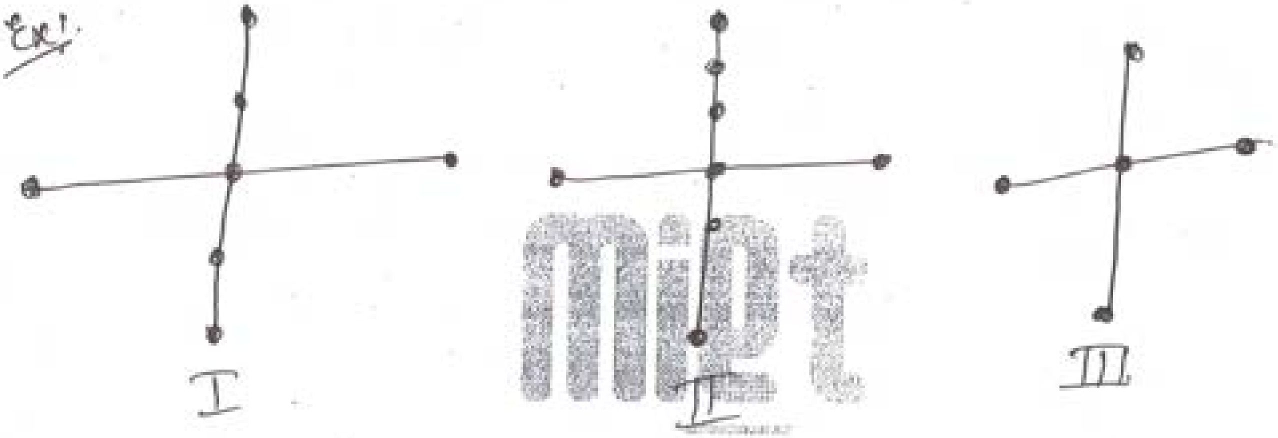
$q \rightarrow$ x is odd integer

$\forall x \in \mathbb{Z} \, (p(x) \vee q(x))$

Q-1 Explain the following terms with example
(i) Homomorphism and Isomorphism graph
(ii) Euler graph and Hamiltonian graph
(iii) Planar and Complete bipartite graph.

Sol^n: (i) Homomorphic Graphs:- Two graph $G_1(V_1, E_1)$ & $G_2(V_2, E_2)$ are said to be Homomorphic if they can be obtained from the same graph.

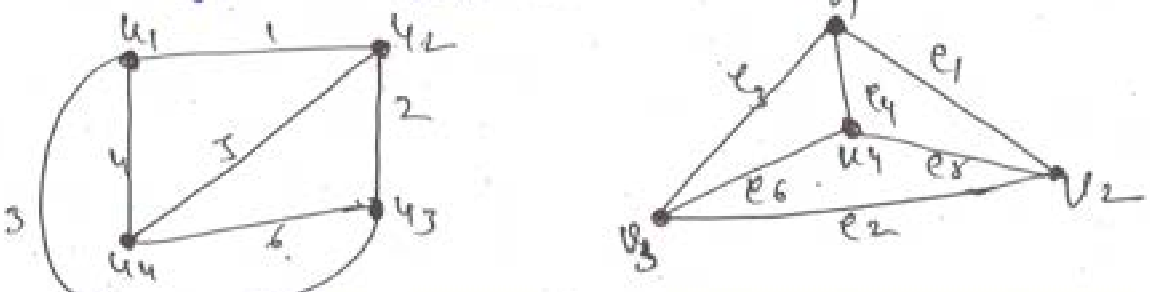
Ex:



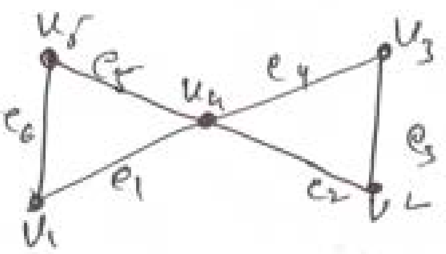
Graph I & III are homomorphic because both are obtained from II

Isomorphic Graph:- Two Graphs $G_1(V_1, E_1)$ & $G_2(V_2, E_2)$ are said to be isomorphic if there exists one-to-one correspondence b/w their vertex sets (V_1 & V_2) and one-to-one correspondence b/w their edges sets (E_1 & E_2)

Ex:

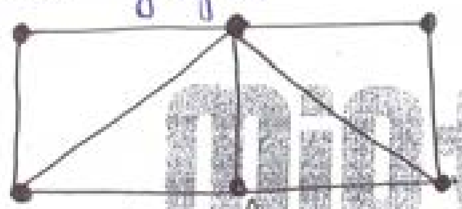


(ii) Euler Graph:- A graph which contain an Eulerian circuit is called an Eulerain graph.



Hamiltonian Graph :- An Hamiltonian circuit is a closed path that visit every vertex of G exactly once. If a graph G has a Hamiltonian circuit is called a Hamiltonian graph.

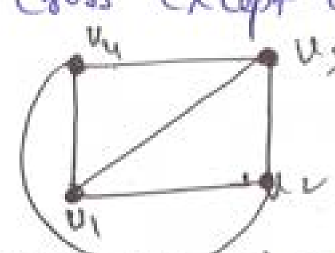
Ex:-



Hamiltonian graph.

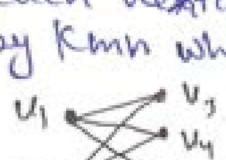
(iii) Planar Graph:- A graph G is said to be planar if the graph can be drawn in a plane so that no edges cross except at vertices.

Ex:-

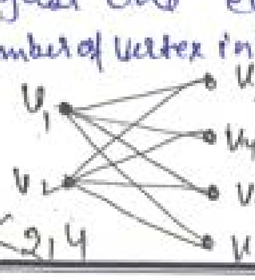


Complete Bipartite graph:- The bipartite graph G is called complete bipartite if each vertex in V_1 is joined to each vertex in V_2 by just one edge. This graph is denoted by K_{mn} where m is the number of vertex in V_1 & n is the number of vertex in V_2.

Ex:-



$K_{2,3}$



$K_{2,4}$

Q-2 Show that the maximum number of edges in a simple graph with n-vertices is $\frac{n(n-1)}{2}$

Proof Let G be a graph with 'n' vertices and 'e' edges

Now By Hand Shaking Lemma

$$\sum_{i=1}^{n} d(v_i) = 2e$$

$$\Rightarrow d(v_1) + d(v_2) + d(v_3) + \cdots + d(v_n) = 2e \quad \text{--- (1)}$$

Since the maximum degree of each vertex in the graph G can be $(n-1)$ then from eq^n (1)

$$\underbrace{(n-1) + (n-1) + (n-1) + \cdots + (n-1)}_{n\text{-times}} = 2e$$

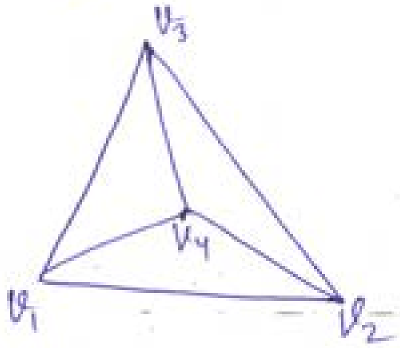
$$\Rightarrow n(n-1) = 2e$$

$$\Rightarrow \boxed{e = \frac{n(n-1)}{2}}$$

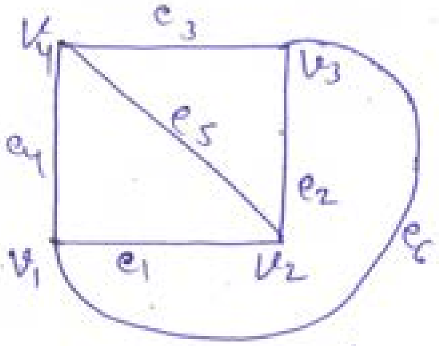
Hence the maximum number of edges in any simple graph with n-vertices is $\frac{n(n-1)}{2}$.

Q3 ⓐ Prove that Complete graph K_4 is planar

Proof The Complete graph K_4 has 4 vertices and 6 edges as



or



As we know that for a Connected planar graph

$$3v - e \geq 6$$

Now $v = 4$, $e = 6$ $\Rightarrow$ $3 \times 4 - 6 \geq 6$

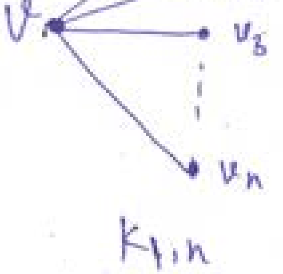
$12 - 6 \geq 6$

which satisfies the property

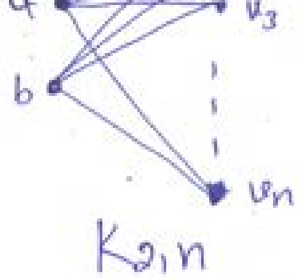
Thus K_4 is a planar graph

ⓑ Show that a Complete bipartite graph $K_{m,n}$ is planar if m or n is less than or equal to 2. $m \leq 2$ or $n \leq 2$

Proof We shall show that both $K_{1,n}$, $K_{2,n}$ Can be drawn on a plane without Crossing edges



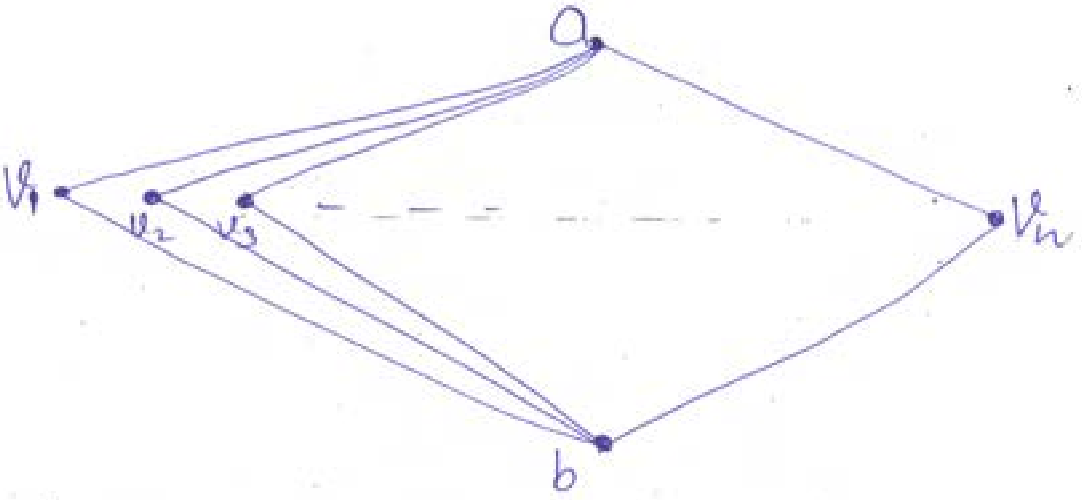
$K_{1,n}$



$K_{2,n}$

Clearly the graph $K_{1,n}$ is without crossing of edges and therefore it is planar.

Now, redraw $K_{2,n}$



thus $K_{2,n}$ is a planar graph.

Hence proved

Q-4 If a connected planar graph G has n-vertices and 'e' edges and 'r' region then

$$\boxed{n-e+r=2}$$

OR

Euler's Formula

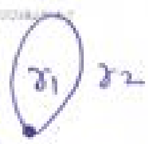
Solⁿ: Let G be a connected planar graph. We shall prove the result by induction on the number of edges of G.

If $e=0$, then G must have just one vertex i.e. $n=1$ and one infinite region i.e. $r=1$

$$\Rightarrow n-e+r = 1-0+1 = 2$$

Now if $e=1$ then $n=1$ or 2 as

$e=1, n=2$



$e=1, n=1$

When $e=1$, $n=2$ then $r=1$, then we have

$$n-e+r = 2-1+1 = 2$$

and if $e=1$ and $n=1$ i.e. G has self loop then $r=2$

$$n-e+r = 1-1+2 = 2$$

thus the result is true for $e=1$

Now, we assume that result is true for all graphs with at most $e-1$ edges

Now, let G be a connected graph with edges and 'r' regions. If G is a tree then $e=n-1$ and has only one infinite region, then

$$n-e+r = n-n+1+1 = 2$$

Hence result is true in this case.

Now if G is not a tree then it has some circuit.

Now, let 'a' be an edges in some circuit.

Removal of edge 'a' from the plane representation of G will merge the two region into one region

thus G-a is a connected graph with 'n' vertices and e-1 edges and (r-1) regions

Now

$$n-(e-1)+(r-1)$$

$$n-e+r-2$$

$$\Rightarrow \boxed{n-e+r=2}$$

Hence proved

Q-5 What do you mean generating function? Solve the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$, $n \geq 2$ given $a_0 = 3$, $a_1 = -2$ using generating function

Sol^n Generating fun^n → The generating function for the sequence $a_0, a_1, a_2, \ldots a_k \ldots$ of real numbers is infinite series

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_k x^k + \cdots$$

$$= \sum_{k=0}^{\infty} a_k x^k$$

Here given recurrence relation is

$$a_n = 2a_{n-1} - a_{n-2} \quad \text{①} \quad n \geq 2$$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

multiplying eq^n ① by x^n and adding from 2 to ∞ we get,

$$\sum_{n=2}^{\infty} a_n x^n = 2\sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$G(x) - a_0 - a_1 x = 2x(G(x) - a_0) - x^2 G(x)$$

$a_0 = 3$, $a_1 = -2$

$$G(x) - 3 + 2x = 2x(G(x) - 3) - x^2 G(x)$$

$$G(x) - 3 + 2x = 2x\,G(x) - 6x - x^2 G(x)$$

$$G(x)[1 - 2x + x^2] = 3 - 8x$$

$$G(x) = \frac{3 - 8x}{(1-x)^2} = \frac{3}{(1-x)^2} - \frac{8x}{(1-x)^2}$$

$$a_n = 3(n+1) - 8n = 3 - 5n$$

Generating fun^n

$\because \frac{1}{(1-x)^2} = n+1$

$\frac{x}{(1-x)^2} = n$

Q-6 Solve the recurrence relation $a_{n+2} - 5a_{n+1} + 6a_n = 5^n$ subject to condition $a_0 = 0$, $a_1 = 2$

It is LNHRR

$$a_{n+2} - 5a_{n+1} + 6a_n = 5^n \quad \text{—(1)}$$

Solⁿ The associated homogeneous recurrence relation is

$$a_{n+2} - 5a_{n+1} + 6a_n = 0 \quad \text{—(2)}$$

Let $a_n = r^n$ be a solⁿ of eqⁿ (1)

put $a_n = r^n$ in eqⁿ (1)

$$r^{n+2} - 5r^{n+1} + 6r^n = 0 \quad \text{—(2)}$$

divide eqⁿ (2) by r^n

$$r^2 - 5r + 6 = 0 \qquad \text{[Char eqⁿ]}$$

$$r = 2, 3$$

So the solⁿ of eqⁿ (2) is

$$a_n^{(h)} = c_1 2^n + c_2 3^n$$

to find the particular solⁿ of eqⁿ (1)

$$a_n^{(p)} = c_3 5^n \qquad \because 5 \text{ is not a root of Char. eqⁿ.}$$

put in eqⁿ (1) we get

$$c_3 5^{n+2} - 5c_3 5^{n+1} + 6c_3 5^n = 5^n$$

$$5^2 c_3 - 5^2 c_3 + 6c_3 = 1$$

$$6c_3 = 1$$

$$c_3 = \frac{1}{6}$$

$$a_n^{(p)} = \frac{1}{6} 5^n$$

then the general solⁿ of eqⁿ (1) is

$$a_n = a_n^{(h)} + a_n^{(p)} = c_1 2^n + c_2 3^n + \frac{1}{6} 5^n \quad \text{—(3)}$$

Now, given $a_1 = 2$, $a_0 = 0$

from eq^n ③

put $n=0$

$$a_0 = c_1 + c_2 + \frac{1}{6}$$

$$c_1 + c_2 = -\frac{1}{6} \quad \text{— (a)}$$

put $n=1$

$$a_1 = 2c_1 + 3c_2 + \frac{5}{6}$$

$$2c_1 + 3c_2 = 2 - \frac{5}{6} = \frac{7}{6} \quad \text{— (b)}$$

from (a) & (b)

$$c_1 = \frac{-5}{3}, \quad c_2 = \frac{3}{2}$$

put c_1 & c_2 in eq^n ⑤

$$\boxed{a_n = \frac{1}{6}5^n + \frac{3}{2}3^n - \frac{5}{3}2^n}$$

Q. 7 Solve the recurrence relation

$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$

with $a_0 = 3$, $a_1 = 3$

Solⁿ: We have

$$a_n - 7a_{n-1} + 10a_{n-2} = 0 \quad \text{---(1)}$$

Let $a_n = r^n$ is a solⁿ of eqⁿ (1)

then put $a_n = r^n$

$$r^n - 7r^{n-1} + 10r^{n-2} = 0$$

divid by r^{n-2}

$$r^2 - 7r^1 + 10 = 0 \quad \text{[Char. Eqⁿ]}$$

$$r^2 - (5+2)r + 10 = 0$$

$$(r-5)(r-2) = 0$$

$$r = 2, 5$$

both are distinct root

thus the general solⁿ is

$$a_n = b_1(2)^n + b_2(5)^n \quad \text{---(2)}$$

from initial condition $a_0 = 3$

put $n=0$ in eqⁿ (2)

$$a_0 = b_1 + b_2$$

$$b_1 + b_2 = 3$$

put $n = 1$ $\quad a_1 = 2b_1 + 5b_2$

$$3 = 2b_1 + 5b_2$$

then $b_1 = 4 \quad b_2 = -1$

put in eqⁿ (2)

$$\boxed{a_n = 4 \cdot 2^n - 5^n}$$

Q-8 A simple graph with 'n' vertices and 'K' Components Can have at most $\frac{(n-K)(n-K+1)}{2}$ edges.

Solⁿ. Let G be a graph with n vertices and K Component. Let $n_1, n_2, n_3, \ldots, n_K$ be the number of vertices in each of K Component of G then,

$$n_1 + n_2 + n_3 + \cdots + n_K = n \quad \text{and } n_i \geq 1$$

Now $\sum (n_i - 1) = n_1 + n_2 + n_3 + \cdots + n_K - K = n - K$

squaring on both side we get

$$\left(\sum_{i=1}^{K} (n_i - 1)\right)^2 = n^2 + K^2 - 2nK$$

$$\Rightarrow \sum_{i=1}^{K} (n_i - 1)^2 + \text{sum of the terms type } 2(n_i - 1)(n_j - 1) = n^2 + K^2 - 2nK$$

$$\Rightarrow \sum_{i=1}^{K} (n_i^2 - 2n_i) + K \leq n^2 + K^2 - 2nK \quad [\because (n_i - 1) \geq 0 \ \forall i]$$

$$\Rightarrow \sum_{i=1}^{K} n_i^2 - 2\sum_{i=1}^{K} n_i \leq n^2 + K^2 - 2nK - K$$

$$\Rightarrow \sum_{i=1}^{K} n_i^2 \leq n^2 + K^2 - 2nK - K + 2n \quad \because \sum n_i = n$$

$$\Rightarrow \sum_{i=1}^{n} n_i^2 \leq n^2 - [(K-1)(2n-K)]$$

Since we know that maximum number of edges in a simple graph with 'n' vertices is $\frac{n(n-1)}{2}$

So the maximum number of edges in i^{th} Component is

$$\frac{n_i(n_i - 1)}{2}$$

thus the maximum number of edges in G

$$\sum_{i=1}^{k} \frac{1}{2} n_i (n_i - 1) = \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right]$$

$$= \frac{1}{2}\left[n^2 - (k-1)(2n-k) - n\right]$$

$$= \frac{1}{2}(n-k)(n-k+1)$$

thus it can has at most $\frac{(n-k)(n-k+1)}{2}$ edges

Q9 Define traversing of binary tree. A binary tree has 11 nodes. It's inorder and postorder traversals node sequence are

Inorder – DBFE AGCLJHK

Postorder – DFE BGLJKHCA

Draw the tree.

Soln There are three standard ways of traversing a binary tree T with root R. They are follows

① Preorder :- ① Process the R

② Traverse the left subtree of R in preorder

③ Traverse the right subtree of R in preorder.

② Inorder :- (i) Traverse the left subtree of R in inorder

(ii) Process the root R

(iii) Traverse the right subtree of R in inorder

③ Postorder :- (i) Traverse the left subtree of R in postorder

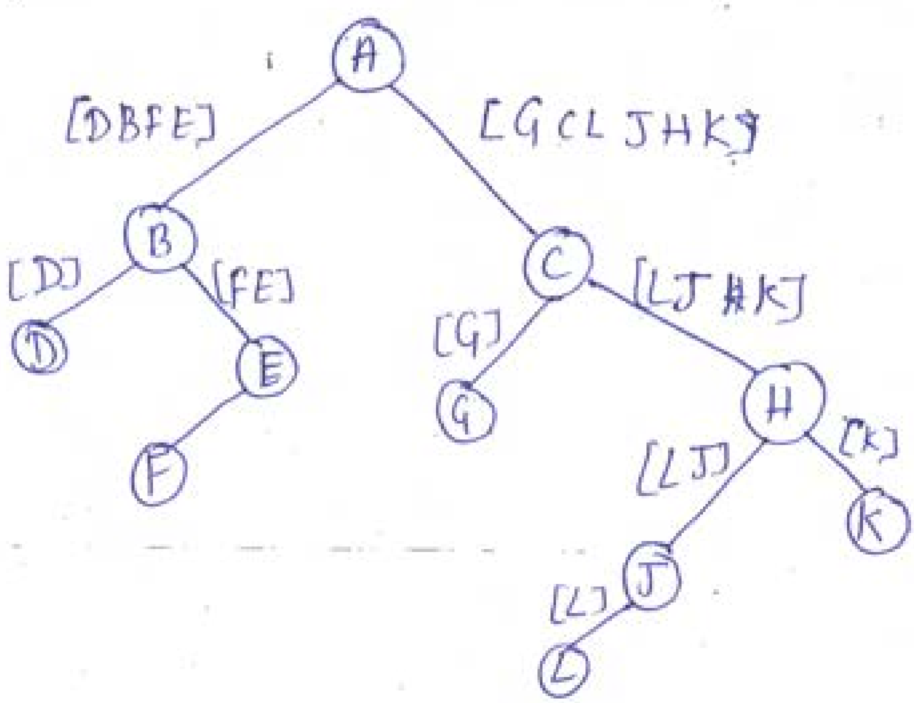
(ii) Traverse the right subtree of R in postorder

(iii) Process the root R

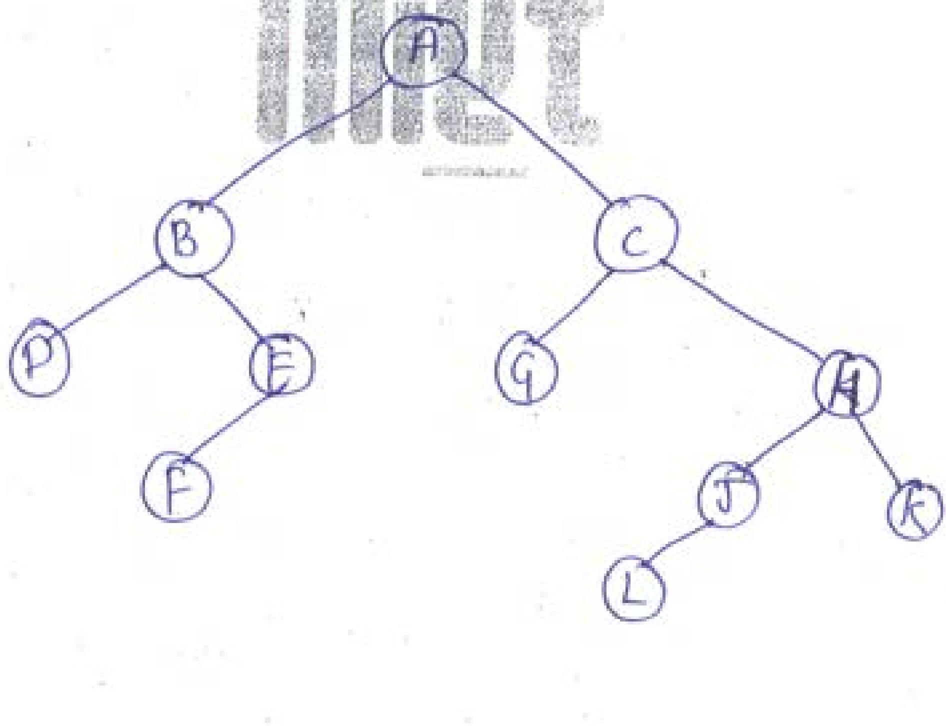
The given sequence of postorder and inorder are

Inorder → $\underbrace{\text{DBFE}}_{\text{Left}}$ A $\underbrace{\text{GCLJHK}}_{\text{Right}}$ (Left, Root right)

Postorder → DFE BGLJKHC A (Left, Right, root)



thus the final tree is



Q-10 Solve the recurrence relation $a_{n+2} - 5a_{n+1} + 6a_n = (n+1)^2$

Sol^n: Here given LNHRR is

$$a_{n+2} - 5a_{n+1} + 6a_n = (n+1)^2 \quad \text{--- (1)}$$

the associated relation is

$$a_{n+2} - 5a_{n+1} + 6a_n = 0 \quad \text{--- (2)}$$

Put $a_n = r^n$ in eq^n (2)

$$r^{n+2} - 5r^{n+1} + 6r^n = 0$$

divide by r^n

$$r^2 - 5r + 6 = 0$$

$$(r-3)(r-2) = 0$$

$$r = 2, 3$$

then associated homogeneous recurrence relation is

$$a_n^{(h)} = C_1(2)^n + C_2(3)^n$$

Now let the particular Sol^n of eq^n (1) is

$$a_n^{(p)} = A_0 + A_1 n + A_2 n^2$$

[∵ f(n) is a polynomial of degree 2]

put this in eq^n (1)

$$[A_0 + A_1(n+2) + A_2(n+2)^2] - 5[A_0 + A_1(n+1) + A_2(n+1)^2] + 6[A_0 + A_1 n + A_2 n^2] = (n+1)^2$$

$$(2A_0 - 3A_1 - A_2) + n(2A_1 - 6A_2) + 2A_2 n^2 = n^2 + 2n + 1$$

on Comparing

$$2A_0 - 3A_1 - A_2 = 1$$

$$2A_1 - 6A_2 = 2$$

$$2A_2 = 1$$

$$A_2 = \frac{1}{2} \qquad A_1 = \frac{5}{2}, \quad A_0 = \frac{9}{2}$$

$$a_n^{(p)} = \frac{9}{2} + \frac{5}{2}n + \frac{n^2}{2}$$

$a_n = C_1 2^n + C_2 3^n + \frac{9}{2} + \frac{5}{2}n + \frac{n^2}{2}$ is the final Solution.