

## Association Analysis: Basic Concepts and Algorithms

Many business enterprises accumulate large quantities of data from their day-to-day operations. For example, huge amounts of customer purchase data are collected daily at the checkout counters of grocery stores. Table 6.1 illustrates an example of such data, commonly known as market basket transactions. Each row in this table corresponds to a transaction, which contains a unique identifier labeled *TID* and a set of items bought by a given customer. Retailers are interested in analyzing the data to learn about the purchasing behavior of their customers. Such valuable information can be used to support a variety of business-related applications such as marketing promotions, inventory management, and customer relationship management.

This chapter presents a methodology known as association analysis, which is useful for discovering interesting relationships hidden in large data sets. The uncovered relationships can be represented in the form of association

**Table 6.1.** An example of market basket transactions.

<i>TID</i>	Items
1	{Bread, Milk}
2	{Bread, Diapers, Beer, Eggs}
3	{Milk, Diapers, Beer, Cola}
4	{Bread, Milk, Diapers, Beer}
5	{Bread, Milk, Diapers, Cola}

rules or sets of frequent items. For example, the following rule can be extracted from the data set shown in Table 6.1:

$$\{\text{Diapers}\} \rightarrow \{\text{Beer}\}.$$

The rule suggests that a strong relationship exists between the sale of diapers and beer because many customers who buy diapers also buy beer. Retailers can use this type of rules to help them identify new opportunities for crossselling their products to the customers.

Besides market basket data, association analysis is also applicable to other application domains such as bioinformatics, medical diagnosis, Web mining, and scientific data analysis. In the analysis of Earth science data, for example, the association patterns may reveal interesting connections among the ocean, land, and atmospheric processes. Such information may help Earth scientists develop a better understanding of how the different elements of the Earth system interact with each other. Even though the techniques presented here are generally applicable to a wider variety of data sets, for illustrative purposes, our discussion will focus mainly on market basket data.

There are two key issues that need to be addressed when applying association analysis to market basket data. First, discovering patterns from a large transaction data set can be computationally expensive. Second, some of the discovered patterns are potentially spurious because they may happen simply by chance. The remainder of this chapter is organized around these two issues. The first part of the chapter is devoted to explaining the basic concepts of association analysis and the algorithms used to efficiently mine such patterns. The second part of the chapter deals with the issue of evaluating the discovered patterns in order to prevent the generation of spurious results.

### 6.1 Problem Definition

This section reviews the basic terminology used in association analysis and presents a formal description of the task.

Binary Representation Market basket data can be represented in a binary format as shown in Table 6.2, where each row corresponds to a transaction and each column corresponds to an item. An item can be treated as a binary variable whose value is one if the item is present in a transaction and zero otherwise. Because the presence of an item in a transaction is often considered more important than its absence, an item is an asymmetric binary variable.

**Table 6.2.** A binary 0/1 representation of market basket data.

TID	Bread	Milk	Diapers	Beer	Eggs	Cola
1	1	1	0	0	0	0
2	1	0	1	1	1	0
3	0	1	1	1	0	1
4	1	1	1	1	0	0
5	1	1	1	0	0	1

This representation is perhaps a very simplistic view of real market basket data because it ignores certain important aspects of the data such as the quantity of items sold or the price paid to purchase them. Methods for handling such non-binary data will be explained in Chapter 7.

**Itemset and Support Count** Let  $I = \{i_1, i_2, \dots, i_d\}$  be the set of all items in a market basket data and  $T = \{t_1, t_2, \dots, t_N\}$  be the set of all transactions. Each transaction  $t_i$  contains a subset of items chosen from  $I$ . In association analysis, a collection of zero or more items is termed an itemset. If an itemset contains  $k$  items, it is called a  $k$ -itemset. For instance, {Beer, Diapers, Milk} is an example of a 3-itemset. The null (or empty) set is an itemset that does not contain any items.

The transaction width is defined as the number of items present in a transaction. A transaction  $t_j$  is said to contain an itemset  $X$  if  $X$  is a subset of  $t_j$ . For example, the second transaction shown in Table 6.2 contains the item-set {Bread, Diapers} but not {Bread, Milk}. An important property of an itemset is its support count, which refers to the number of transactions that contain a particular itemset. Mathematically, the support count,  $\sigma(X)$ , for an itemset  $X$  can be stated as follows:

$$\sigma(X) = |\{t_i | X \subseteq t_i, t_i \in T\}|,$$

where the symbol  $|\cdot|$  denote the number of elements in a set. In the data set shown in Table 6.2, the support count for {Beer, Diapers, Milk} is equal to two because there are only two transactions that contain all three items.

**Association Rule** An association rule is an implication expression of the form  $X \rightarrow Y$ , where  $X$  and  $Y$  are disjoint itemsets, i.e.,  $X \cap Y = \emptyset$ . The strength of an association rule can be measured in terms of its support and confidence. Support determines how often a rule is applicable to a given

data set, while confidence determines how frequently items in  $Y$  appear in transactions that contain  $X$ . The formal definitions of these metrics are

$$\text{Support, } s(X \rightarrow Y) = \frac{\sigma(X \cup Y)}{N}; \quad (6.1)$$

$$\text{Confidence, } c(X \rightarrow Y) = \frac{\sigma(X \cup Y)}{\sigma(X)}. \quad (6.2)$$

**Example 6.1.** Consider the rule  $\{\text{Milk, Diapers}\} \rightarrow \{\text{Beer}\}$ . Since the support count for  $\{\text{Milk, Diapers, Beer}\}$  is 2 and the total number of transactions is 5, the rule's support is  $2/5 = 0.4$ . The rule's confidence is obtained by dividing the support count for  $\{\text{Milk, Diapers, Beer}\}$  by the support count for  $\{\text{Milk, Diapers}\}$ . Since there are 3 transactions that contain milk and diapers, the confidence for this rule is  $2/3 = 0.67$ . ■

**Why Use Support and Confidence?** Support is an important measure because a rule that has very low support may occur simply by chance. A low support rule is also likely to be uninteresting from a business perspective because it may not be profitable to promote items that customers seldom buy together (with the exception of the situation described in Section 6.8). For these reasons, support is often used to eliminate uninteresting rules. As will be shown in Section 6.2.1, support also has a desirable property that can be exploited for the efficient discovery of association rules.

Confidence, on the other hand, measures the reliability of the inference made by a rule. For a given rule  $X \rightarrow Y$ , the higher the confidence, the more likely it is for  $Y$  to be present in transactions that contain  $X$ . Confidence also provides an estimate of the conditional probability of  $Y$  given  $X$ .

Association analysis results should be interpreted with caution. The inference made by an association rule does not necessarily imply causality. Instead, it suggests a strong co-occurrence relationship between items in the antecedent and consequent of the rule. Causality, on the other hand, requires knowledge about the causal and effect attributes in the data and typically involves relationships occurring over time (e.g., ozone depletion leads to global warming).

**Formulation of Association Rule Mining Problem** The association rule mining problem can be formally stated as follows:

**Definition 6.1 (Association Rule Discovery).** Given a set of transactions  $T$ , find all the rules having support  $\geq \text{minsup}$  and confidence  $\geq \text{minconf}$ , where  $\text{minsup}$  and  $\text{minconf}$  are the corresponding support and confidence thresholds.

A brute-force approach for mining association rules is to compute the support and confidence for every possible rule. This approach is prohibitively expensive because there are exponentially many rules that can be extracted from a data set. More specifically, the total number of possible rules extracted from a data set that contains  $d$  items is

$$R = 3^d - 2^{d+1} + 1. \quad (6.3)$$

The proof for this equation is left as an exercise to the readers (see Exercise 5 on page 405). Even for the small data set shown in Table 6.1, this approach requires us to compute the support and confidence for  $3^6 - 2^7 + 1 = 602$  rules. More than 80% of the rules are discarded after applying  $minsup = 20\%$  and  $minconf = 50\%$ , thus making most of the computations become wasted. To avoid performing needless computations, it would be useful to prune the rules early without having to compute their support and confidence values.

An initial step toward improving the performance of association rule mining algorithms is to decouple the support and confidence requirements. From Equation 6.2, notice that the support of a rule  $X \rightarrow Y$  depends only on the support of its corresponding itemset,  $X \cup Y$ . For example, the following rules have identical support because they involve items from the same itemset,

{Beer, Diapers, Milk}:

{Beer, Diapers} $\rightarrow$ {Milk},	{Beer, Milk} $\rightarrow$ {Diapers},
{Diapers, Milk} $\rightarrow$ {Beer},	{Beer} $\rightarrow$ {Diapers, Milk},
{Milk} $\rightarrow$ {Beer, Diapers},	{Diapers} $\rightarrow$ {Beer, Milk}.

If the itemset is infrequent, then all six candidate rules can be pruned immediately without our having to compute their confidence values.

Therefore, a common strategy adopted by many association rule mining algorithms is to decompose the problem into two major subtasks:

Frequent Itemset Generation, whose objective is to find all the itemsets that satisfy the  $minsup$  threshold. These itemsets are called frequent itemsets.

Rule Generation, whose objective is to extract all the high-confidence rules from the frequent itemset found in the previous step. These rules are called strong rules.

The computational requirements for frequent itemset generation are generally more expensive than those of rule generation. Efficient techniques for generating frequent item sets and association rules are discussed in Sections 6.2 and 6.3, respectively.

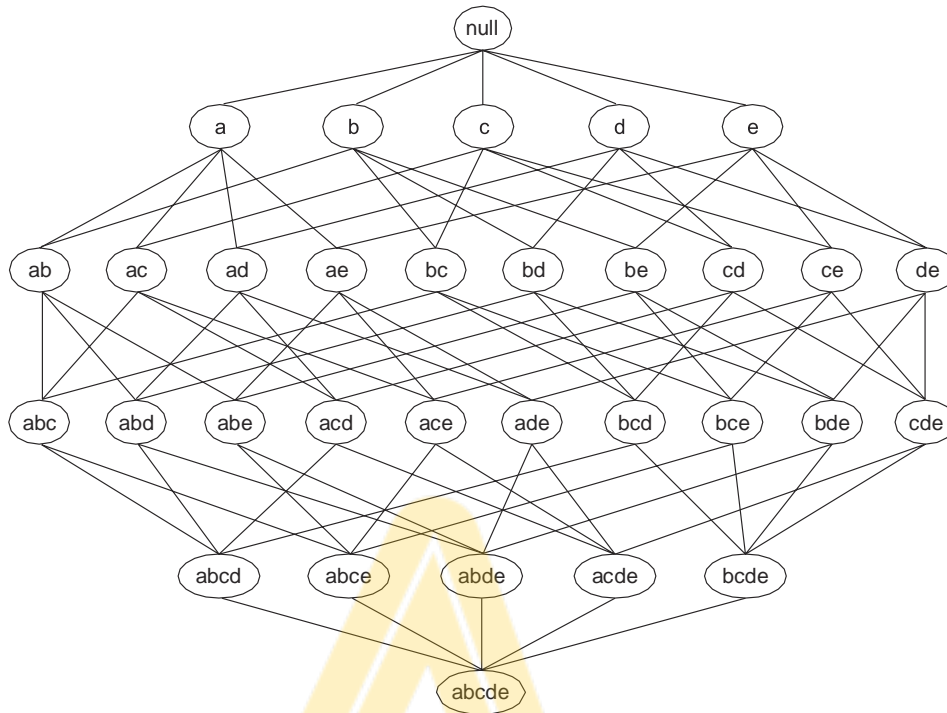


Figure 6.1. An itemset lattice.

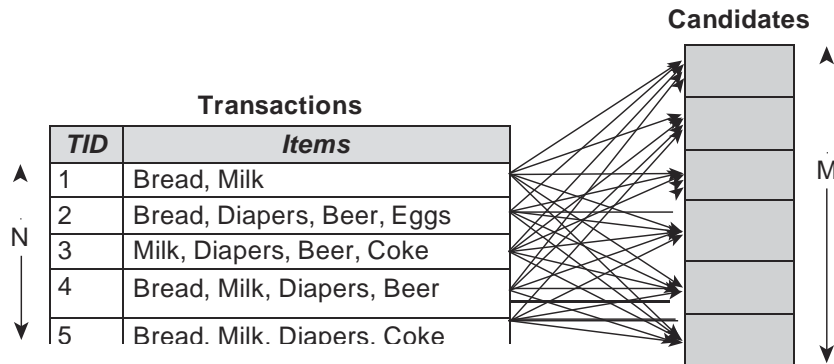
## 6.2 Frequent Itemset Generation

A lattice structure can be used to enumerate the list of all possible itemset. Figure 6.1 shows an itemset lattice for  $I = \{a, b, c, d, e\}$ . In general, a data set that contains  $k$  items can potentially generate up to  $2^k - 1$  frequent itemset, excluding the null set. Because  $k$  can be very large in many practical applications, the search space of item sets that need to be explored is exponentially large.

A brute-force approach for finding frequent itemset is to determine the support count for every candidate itemset in the lattice structure. To do this, we need to compare each candidate against every transaction, an operation that is shown in Figure 6.2. If the candidate is contained in a transaction,

its support count will be incremented. For example, the support for {Bread, Milk} is incremented three times because the itemset is contained in transactions 1, 4, and 5. Such an approach can be very expensive because it requires  $O(NMw)$  comparisons, where  $N$  is the number of transactions,  $M = 2^k - 1$  is the number of candidates itemset, and  $w$  is the maximum transaction width.





**Figure 6.2.** Counting the support of candidate itemsets.

There are several ways to reduce the computational complexity of frequent itemset generation.

Reduce the number of candidate itemset ( $M$ ). The *Apriori* principle, described in the next section, is an effective way to eliminate some of the candidate itemset without counting their support values.

Reduce the number of comparisons. Instead of matching each candidate itemset against every transaction, we can reduce the number of comparisons by using more advanced data structures, either to store the candidate itemsets or to compress the data set. We will discuss these strategies in Sections 6.2.4 and 6.6.

### 6.2.1 The *Apriori* Principle

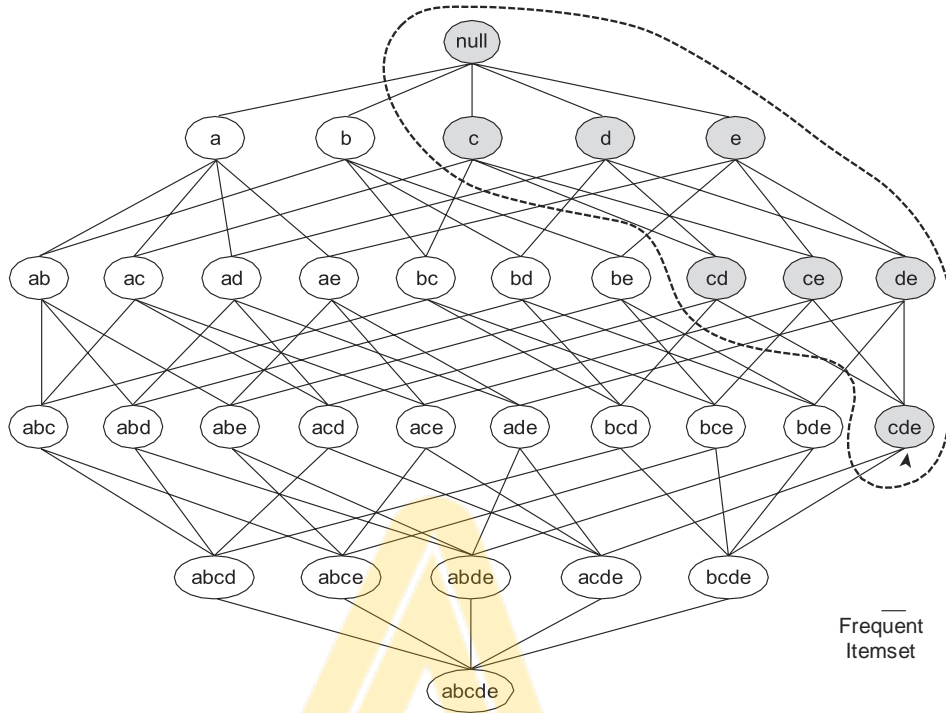
This section describes how the support measure helps to reduce the number of candidate itemset explored during frequent itemset generation. The use of support for pruning candidate itemset is guided by the following principle.

**Theorem 6.1 (*Apriori* Principle).** *If an itemset is frequent, then all of its subsets must also be frequent.*

To illustrate the idea behind the *Apriori* principle, consider the itemset lattice shown in Figure 6.3. Suppose  $\{c, d, e\}$  is a frequent itemset. Clearly, any transaction that contains  $\{c, d, e\}$  must also contain its subsets,  $\{c, d\}$ ,

$\{c, e\}$ ,  $\{d, e\}$ ,  $\{c\}$ ,  $\{d\}$ , and  $\{e\}$ . As a result, if  $\{c, d, e\}$  is frequent, then all subsets of  $\{c, d, e\}$  (i.e., the shaded itemset in this figure) must also be frequent.





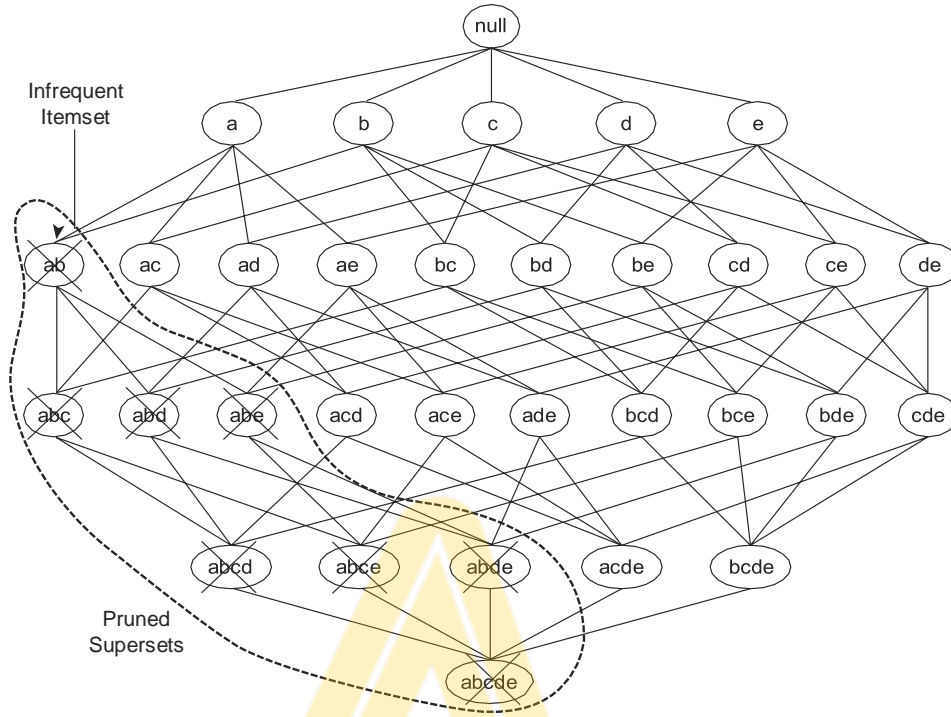
**Figure 6.3.** An illustration of the *Apriori* principle. If  $\{c, d, e\}$  is frequent, then all subsets of this itemset are frequent.

Conversely, if an itemset such as  $\{a, b\}$  is infrequent, then all of its supersets must be infrequent too. As illustrated in Figure 6.4, the entire subgraph containing the supersets of  $\{a, b\}$  can be pruned immediately once  $\{a, b\}$  is found to be infrequent. This strategy of trimming the exponential search space based on the support measure is known as support-based pruning. Such a pruning strategy is made possible by a key property of the support measure, namely, that the support for an itemset never exceeds the support for its subsets. This property is also known as the anti-monotone property of the support measure.

**Definition 6.2 (Monotonicity Property).** Let  $I$  be a set of items, and  $J = 2^I$  be the power set of  $I$ . A measure  $f$  is monotone (or upward closed) if

$$\forall X, Y \in J : (X \subseteq Y) \rightarrow f(X) \leq f(Y),$$





**Figure 6.4.** An illustration of support-based pruning. If  $\{a, b\}$  is infrequent, then all supersets of  $\{a, b\}$  are infrequent.

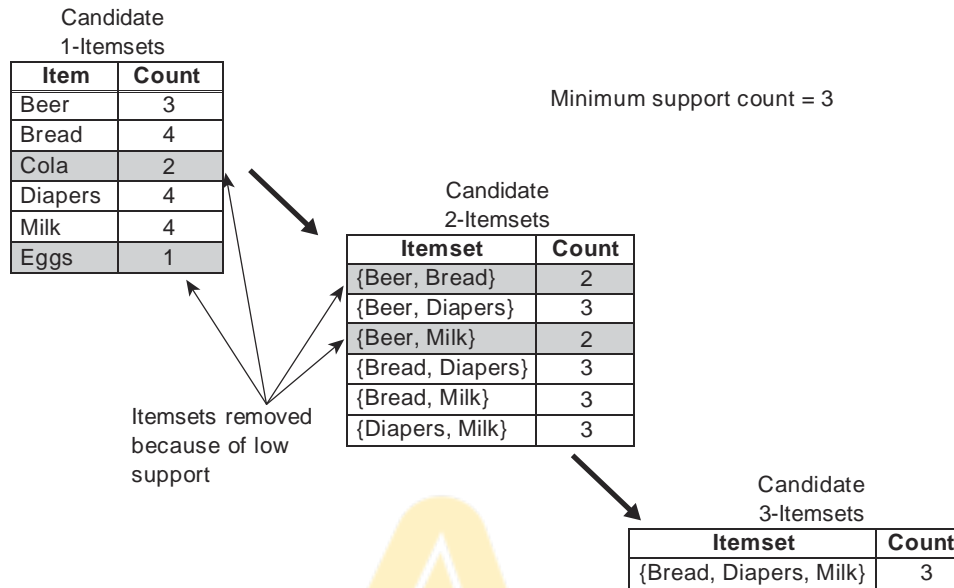
which means that if  $X$  is a subset of  $Y$ , then  $f(X)$  must not exceed  $f(Y)$ . On the other hand,  $f$  is anti-monotone (or downward closed) if

$$\forall X, Y \in J: (X \subseteq Y) \rightarrow f(Y) \leq f(X),$$

which means that if  $X$  is a subset of  $Y$ , then  $f(Y)$  must not exceed  $f(X)$ . Any measure that possesses an anti-monotone property can be incorporated directly into the mining algorithm to effectively prune the exponential search space of candidate itemset, as will be shown in the next section.

### 6.2.2 Frequent Itemset Generation in the *Apriori* Algorithm

*Apriori* is the first association rule mining algorithm that pioneered the use of support-based pruning to systematically control the exponential growth of candidate itemset. Figure 6.5 provides a high-level illustration of the frequent itemset generation part of the *Apriori* algorithm for the transactions shown in



**Figure 6.5.** Illustration of frequent itemset generation using the *Apriori* algorithm.

Table 6.1. We assume that the support threshold is 60%, which is equivalent to a minimum support count equal to 3. Initially, every item is considered as a candidate 1-itemset. After counting their supports, the candidate itemset {Cola} and {Eggs} are discarded because they appear in fewer than three transactions. In the next iteration, candidate 2-itemsets are generated using only the frequent 1-itemsets because the *Apriori* principle ensures that all supersets of the infrequent 1-itemsets must be infrequent. Because there are only four frequent 1-itemset, the number of candidate 2-itemsets generated by the algorithm is  $\binom{4}{2} = 6$ . Two of these six candidates, {Beer, Bread} and {Beer, Milk}, are subsequently found to be infrequent after computing their support values. The remaining four candidates are frequent, and thus will be used to generate candidate 3-itemsets. Without support-based pruning, there are  $\binom{6}{3} = 20$  candidate 3-itemsets that can be formed using the six items given in this example. With the *Apriori* principle, we only need to keep candidate 3-itemsets whose subsets are frequent. The only candidate that has this property is {Bread, Diapers, Milk}.

The effectiveness of the *Apriori* pruning strategy can be shown by counting the number of candidates itemset generated. A brute-force strategy of

enumerating all itemset (up to size 3) as candidates will produce

$$\binom{6}{1} + \binom{6}{2} + \binom{6}{3} = 6 + 15 + 20 = 41$$

candidates. With the *Apriori* principle, this number decreases to

$$\binom{6}{1} + \binom{4}{2} + 1 = 6 + 6 + 1 = 13$$

candidates, which represents a 68% reduction in the number of candidates itemset even in this simple example.

The pseudocode for the frequent itemset generation part of the *Apriori* algorithm is shown in Algorithm 6.1. Let  $C_k$  denote the set of candidate  $k$ -itemsets and  $F_k$  denote the set of frequent  $k$ -itemset:

The algorithm initially makes a single pass over the data set to determine the support of each item. Upon completion of this step, the set of all frequent 1-itemsets,  $F_1$ , will be known (steps 1 and 2).

Next, the algorithm will iteratively generate new candidate  $k$ -itemset using the frequent  $(k - 1)$ -itemset found in the previous iteration (step 5). Candidate generation is implemented using a function called *apriori-gen*, which is described in Section 6.2.3.

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**Algorithm 6.1** Frequent itemset generation of the *Apriori* algorithm.

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1:  $k = 1$ .
2:  $F_k = \{ i \mid i \in I \wedge \sigma(\{i\}) \geq N \times \text{minsup} \}$ .    {Find all frequent 1-itemsets}
3: repeat
4:    $k = k + 1$ .
5:    $C_k = \text{apriori-gen}(F_{k-1})$ .    {Generate candidate itemsets}
6:   for each transaction  $t \in T$  do
7:      $C_t = \text{subset}(C_k, t)$ .    {Identify all candidates that belong to  $t$ }
8:     for each candidate itemset  $c \in C_t$  do
9:        $\sigma(c) = \sigma(c) + 1$ .    {Increment support count}
10:    end for
11:  end for
12:   $F_k = \{ c \mid c \in C_k \wedge \sigma(c) \geq N \times \text{minsup} \}$ .    {Extract the frequent  $k$ -itemsets}
13: until  $F_k = \emptyset$ 
14: Result =  $F_k$ .
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To count the support of the candidates, the algorithm needs to make an additional pass over the data set (steps 6–10). The subset function is used to determine all the candidate itemset in  $C_k$  that are contained in each transaction  $t$ . The implementation of this function is described in Section 6.2.4.

After counting their supports, the algorithm eliminates all candidate itemset whose support counts are less than *minsup* (step 12).

The algorithm terminates when there are no new frequent itemset generated, i.e.,  $F_k = \emptyset$  (step 13).

The frequent itemset generation part of the *Apriori* algorithm has two important characteristics. First, it is a level-wise algorithm; i.e., it traverses the itemset lattice one level at a time, from frequent 1-itemsets to the maximum size of frequent itemset. Second, it employs a generate-and-test strategy for finding frequent itemset. At each iteration, new candidate itemset are generated from the frequent itemset found in the previous iteration. The support for each candidate is then counted and tested against the *minsup* threshold. The total number of iterations needed by the algorithm is  $k_{\max} + 1$ , where  $k_{\max}$  is the maximum size of the frequent itemset.

### Candidate Generation and Pruning

The *apriori-gen* function shown in Step 5 of Algorithm 6.1 generates candidate itemset by performing the following two operations:

**Candidate Generation.** This operation generates new candidate  $k$ -itemset based on the frequent  $(k - 1)$ -itemset found in the previous iteration.

**Candidate Pruning.** This operation eliminates some of the candidate  $k$ -itemset using the support-based pruning strategy.

To illustrate the candidate pruning operation, consider a candidate  $k$ -itemset,  $X = \{i_1, i_2, \dots, i_k\}$ . The algorithm must determine whether all of its proper subsets  $X - \{i_j\}$  ( $\forall j = 1, 2, \dots, k$ ), are frequent. If one of them is infrequent, then  $X$  is immediately pruned. This approach can effectively reduce the number of candidates itemset considered during support counting. The complexity of this operation is  $O(k)$  for each candidate  $k$ -itemset. However, as will be shown later, we do not have to examine all  $k$  subsets of a given candidate itemset. If  $m$  of the  $k$  subsets were used to generate a candidate, we only need to check the remaining  $k - m$  subsets during candidate pruning.

In principle, there are many ways to generate candidate itemset. The following is a list of requirements for an effective candidate generation procedure:

It should avoid generating too many unnecessary candidates. A candidate itemset is unnecessary if at least one of its subsets is infrequent. Such a candidate is guaranteed to be infrequent according to the anti-monotone property of support.

It must ensure that the candidate set is complete, i.e., no frequent itemset are left out by the candidate generation procedure. To ensure completeness, the set of candidate itemset must subsume the set of all frequent itemset, i.e.,  $\forall k : F_k \subseteq C_k$ .

It should not generate the same candidate itemset more than once. For example, the candidate itemset  $\{a, b, c, d\}$  can be generated in many ways—by merging  $\{a, b, c\}$  with  $\{d\}$ ,  $\{b, d\}$  with  $\{a, c\}$ ,  $\{c\}$  with  $\{a, b, d\}$ , etc. Generation of duplicate candidates leads to wasted computations and thus should be avoided for efficiency reasons.

Next, we will briefly describe several candidate generation procedures, including the one used by the *apriori-gen* function.

**Brute-Force Method** The brute-force method considers every  $k$ -itemset as a potential candidate and then applies the candidate pruning step to remove any unnecessary candidates (see Figure 6.6). The number of candidate itemset generated at level  $k$  is equal to  $d^k$ , where

$d$  is the total number of items.

Although candidate generation is rather trivial, candidate pruning becomes extremely expensive because a large number of itemset must be examined.

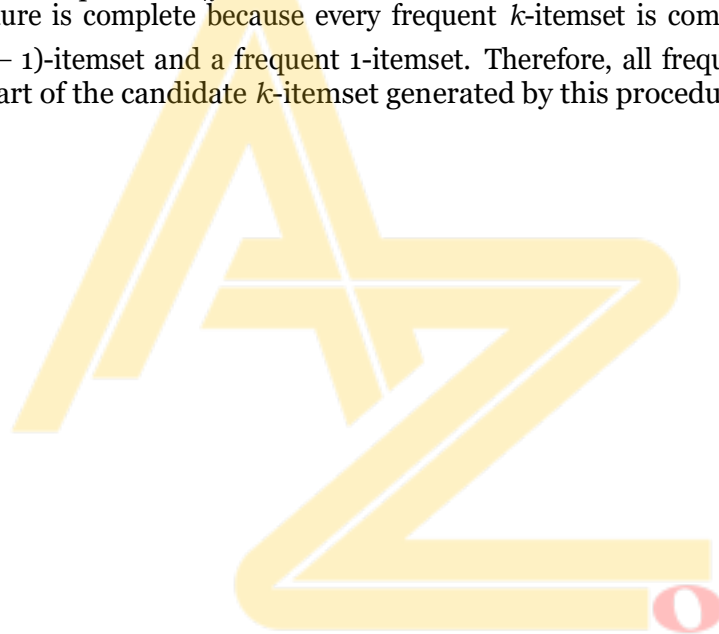
Given that the amount of computations needed for each candidate is  $O(k)$ , the overall complexity of this method is  $O\left(\sum_{k=1}^d k \times \binom{d}{k}\right) = O(d \cdot 2^{d-1})$ .

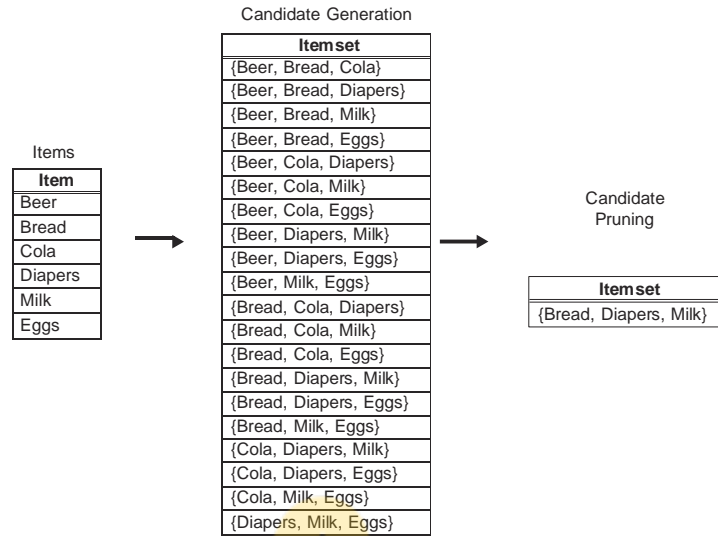
**$F_{k-1} \times F_1$  Method** An alternative method for candidate generation is to extend each frequent  $(k-1)$ -itemset with other frequent items. Figure 6.7 illustrates how a frequent 2-itemset such as {Beer, Diapers} can be augmented with a frequent item such as Bread to produce a candidate 3-itemset

{Beer, Diapers, Bread}. This method will produce  $O(|F_{k-1}| \times |F_1|)$  candidate  $k$ -itemset, where  $|F_j|$  is the number of frequent  $j$ -itemset. The overall

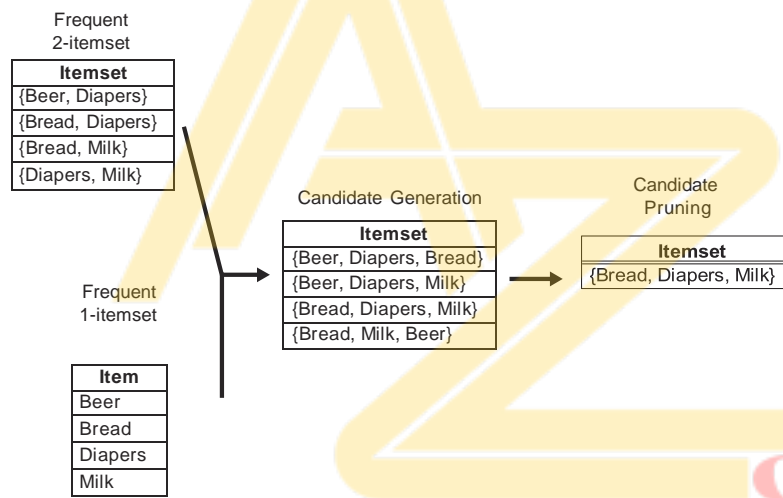
complexity of this step is  $O\left(\sum_k k |F_{k-1}| |F_1|\right)$ .

The procedure is complete because every frequent  $k$ -itemset is composed of a frequent  $(k-1)$ -itemset and a frequent 1-itemset. Therefore, all frequent  $k$ -itemsets are part of the candidate  $k$ -itemset generated by this procedure.





**Figure 6.6.** A brute-force method for generating candidate 3-itemsets.



**Figure 6.7.** Generating and pruning candidate  $k$ -itemset by merging a frequent  $(k - 1)$ -itemset with a frequent item. Note that some of the candidates are unnecessary because their subsets are infrequent.

This approach, however, does not prevent the same candidate itemset from being generated more than once. For instance, {Bread, Diapers, Milk} can be generated by merging {Bread, Diapers} with {Milk}, {Bread, Milk} with {Diapers}, or {Diapers, Milk} with {Bread}. One way to avoid generating

duplicate candidates is by ensuring that the items in each frequent itemset are kept sorted in their lexicographic order. Each frequent  $(k-1)$ -itemset  $X$  is then extended with frequent items that are lexicographically larger than the items in  $X$ . For example, the itemset {Bread, Diapers} can be augmented with {Milk} since Milk is lexicographically larger than Bread and Diapers. However, we should not augment {Diapers, Milk} with {Bread} nor {Bread, Milk} with {Diapers} because they violate the lexicographic ordering condition.

While this procedure is a substantial improvement over the brute-force method, it can still produce a large number of unnecessary candidates. For example, the candidate itemset obtained by merging {Beer, Diapers} with

{Milk} is unnecessary because one of its subsets, {Beer, Milk}, is infrequent.

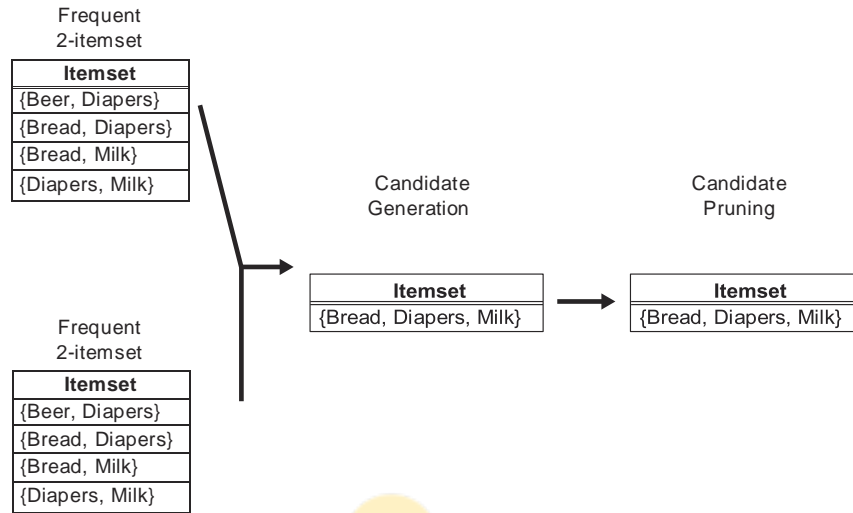
There are several heuristics available to reduce the number of unnecessary candidates. For example, note that, for every candidate  $k$ -itemset that survives the pruning step, every item in the candidate must be contained in at least  $k-1$  of the frequent  $(k-1)$ -itemsets. Otherwise, the candidate is guaranteed to be infrequent. For example, {Beer, Diapers, Milk} is a viable candidate 3-itemset only if every item in the candidate, including Beer, is contained in at least two frequent 2-itemsets. Since there is only one frequent 2-itemset containing Beer, all candidate itemsets involving Beer must be infrequent.

$F_{k-1} \times F_{k-1}$  Method The candidate generation procedure in the apriori-gen function merges a pair of frequent  $(k-1)$ -itemset only if their first  $k-2$  items are identical. Let  $A = \{a_1, a_2, \dots, a_{k-1}\}$  and  $B = \{b_1, b_2, \dots, b_{k-1}\}$  be a pair of frequent  $(k-1)$ -itemset.  $A$  and  $B$  are merged if they satisfy the following conditions:

$$a_i = b_i \text{ (for } i = 1, 2, \dots, k-2 \text{) and } a_{k-1} \neq b_{k-1}.$$

In Figure 6.8, the frequent itemset {Bread, Diapers} and {Bread, Milk} are merged to form a candidate 3-itemset {Bread, Diapers, Milk}. The algorithm does not have to merge {Beer, Diapers} with {Diapers, Milk} because the first item in both itemset is different. Indeed, if {Beer, Diapers, Milk} is a viable candidate, it would have been obtained by merging {Beer, Diapers} with {Beer, Milk} instead. This example illustrates both the completeness of the candidate generation procedure and the advantages of using lexicographic ordering to prevent duplicate candidates. However, because each candidate is obtained by merging a pair of frequent  $(k-1)$ -itemset, an additional candidate pruning step is needed to ensure that the remaining  $k-2$  subsets of the candidate are frequent.





**Figure 6.8.** Generating and pruning candidate  $k$ -itemsets by merging pairs of frequent  $(k-1)$ -itemset.

### Support Counting

Support counting is the process of determining the frequency of occurrence for every candidate itemset that survives the candidate pruning step of the apriori-gen function. Support counting is implemented in steps 6 through 11 of Algorithm 6.1. One approach for doing this is to compare each transaction against every candidate itemset (see Figure 6.2) and to update the support counts of candidates contained in the transaction. This approach is computationally expensive, especially when the numbers of transactions and candidate itemset are large.

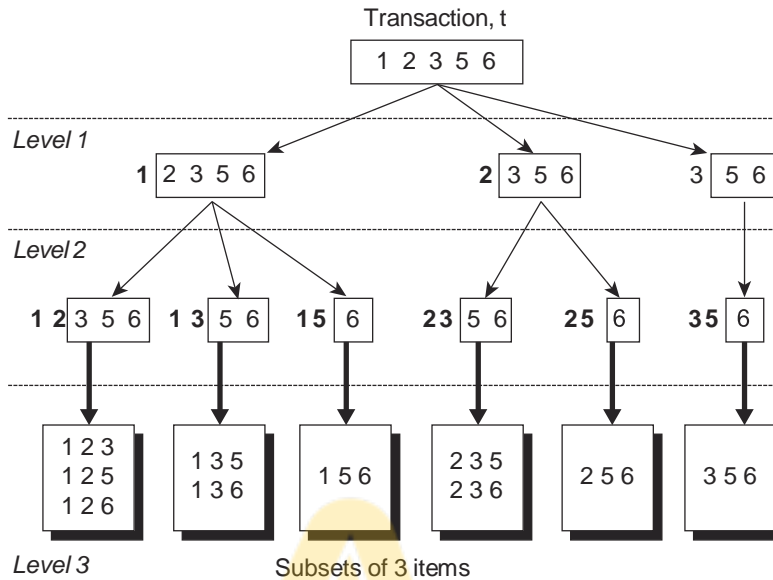
An alternative approach is to enumerate the itemset contained in each transaction and use them to update the support counts of their respective candidate itemset. To illustrate, consider a transaction  $t$  that contains five items,

{1, 2, 3, 5, 6}. There are  $\binom{5}{3} = 10$  itemset of size 3 contained in this transaction. Some of the itemset may correspond to the candidate 3-itemsets under investigation, in which case, their support counts are incremented. Other subsets of  $t$  that do not correspond to any candidates can be ignored.

Figure 6.9 shows a systematic way for enumerating the 3-itemsets contained in  $t$ . Assuming that each itemset keeps its items in increasing lexicographic order, an itemset can be enumerated by specifying the smallest item first, followed by the larger items. For instance, given  $t =$

{1, 2, 3, 5, 6}, all the 3-

itemsets contained in  $t$  must begin with item 1, 2, or 3. It is not possible to construct a 3-itemset that begins with items 5 or 6 because there are only two



**Figure 6.9.** Enumerating subsets of three items from a transaction  $t$ .

items in  $t$  whose labels are greater than or equal to 5. The number of ways to specify the first item of a 3-itemset contained in  $t$  is illustrated by the Level

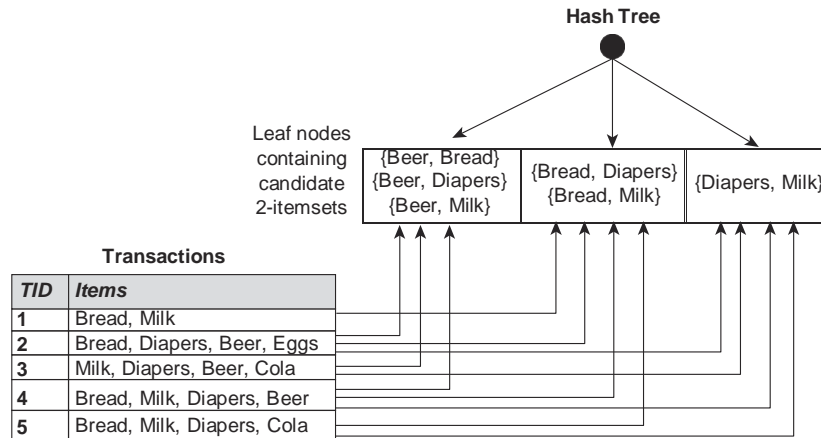
1 prefix structures depicted in Figure 6.9. For instance,  $1 \boxed{2 \ 3 \ 5 \ 6}$  represents a 3-itemset that begins with item 1, followed by two more items chosen from the set  $\{2, 3, 5, 6\}$ .

After fixing the first item, the prefix structures at Level 2 represent the

number of ways to select the second item. For example,  $1 \ 2 \boxed{3 \ 5 \ 6}$  corresponds to itemset that begin with prefix  $\{1 \ 2\}$  and are followed by items 3, 5, or 6. Finally, the prefix structures at Level 3 represent the complete set of 3-itemsets contained in  $t$ . For example, the 3-itemsets that begin with prefix  $\{1 \ 2\}$  are

$\{1, 2, 3\}$ ,  $\{1, 2, 5\}$ , and  $\{1, 2, 6\}$ , while those that begin with prefix  $\{2 \ 3\}$  are  $\{2, 3, 5\}$  and  $\{2, 3, 6\}$ .

The prefix structures shown in Figure 6.9 demonstrate how itemset contained in a transaction can be systematically enumerated, i.e., by specifying their items one by one, from the leftmost item to the rightmost item. We still have to determine whether each enumerated 3-itemset corresponds to an existing candidate itemset. If it matches one of the candidates, then the support count of the corresponding candidate is incremented. In the next section, we illustrate how this matching operation can be performed efficiently using a hash tree structure.

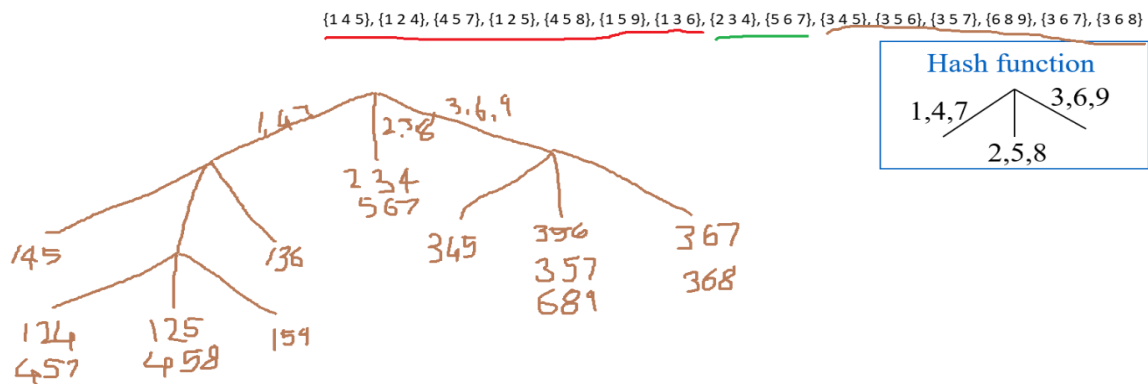


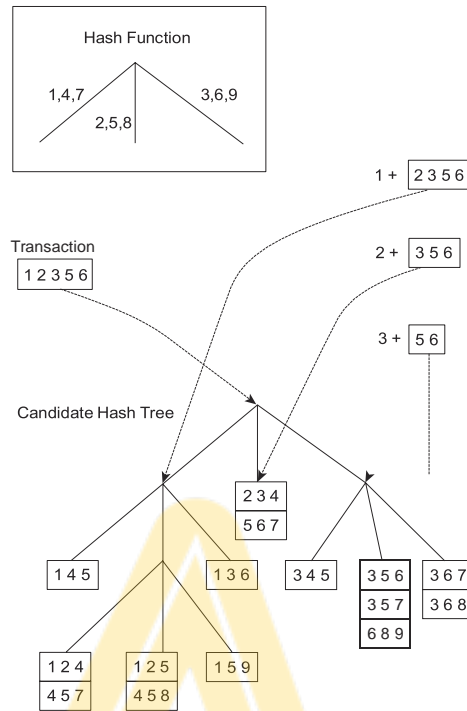
### Support Counting Using a Hash Tree

In the *Apriori* algorithm, candidate itemset are partitioned into different buckets and stored in a hash tree. During support counting, itemset contained in each transaction are also hashed into their appropriate buckets. That way, instead of comparing each itemset in the transaction with every candidate itemset, it is matched only against candidate itemset that belong to the same bucket, as shown in Figure 6.10.

Figure 6.11 shows an example of a hash tree structure. Each internal node of the tree uses the following hash function,  $h(p) = p \bmod 3$ , to determine which branch of the current node should be followed next. For example, items 1, 4, and 7 are hashed to the same branch (i.e., the leftmost branch) because they have the same remainder after dividing the number by 3. All candidate itemsets are stored at the leaf nodes of the hash tree. The hash tree shown in Figure 6.11 contains 15 candidate 3-itemsets, distributed across 9 leaf nodes.

Consider a transaction,  $t = \{1, 2, 3, 5, 6\}$ . To update the support counts of the candidate itemset, the hash tree must be traversed in such a way that all the leaf nodes containing candidate 3-itemsets belonging to  $t$  must be visited at least once. Recall that the 3-itemsets contained in  $t$  must begin with items 1, 2, or 3, as indicated by the Level 1 prefix structures shown in Figure 6.9. Therefore, at the root node of the hash tree, the items 1, 2, and 3 of the transaction are hashed separately. Item 1 is hashed to the left child of the root node, item 2 is hashed to the middle child, and item 3 is hashed to the right child. At the next level of the tree, the transaction is hashed on the second





**Figure 6.11.** Hashing a transaction at the root node of a hash tree.

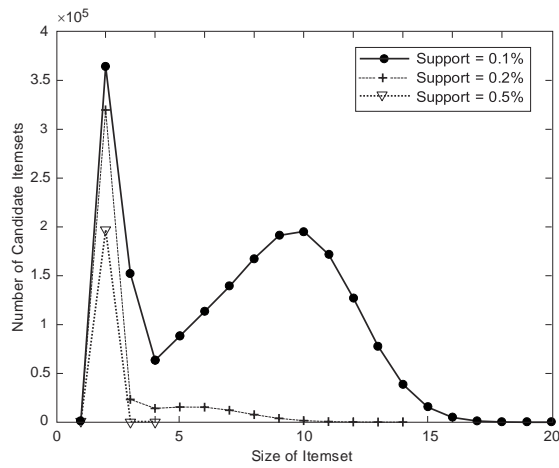
item listed in the Level 2 structures shown in Figure 6.9. For example, after hashing on item 1 at the root node, items 2, 3, and 5 of the transaction are hashed. Items 2 and 5 are hashed to the middle child, while item 3 is hashed to the right child, as shown in Figure 6.12. This process continues until the leaf nodes of the hash tree are reached. The candidate itemset stored at the visited leaf nodes are compared against the transaction. If a candidate is a subset of the transaction, its support count is incremented. In this example, 5 out of the 9 leaf nodes are visited and 9 out of the 15 itemset are compared against the transaction.

### Computational Complexity

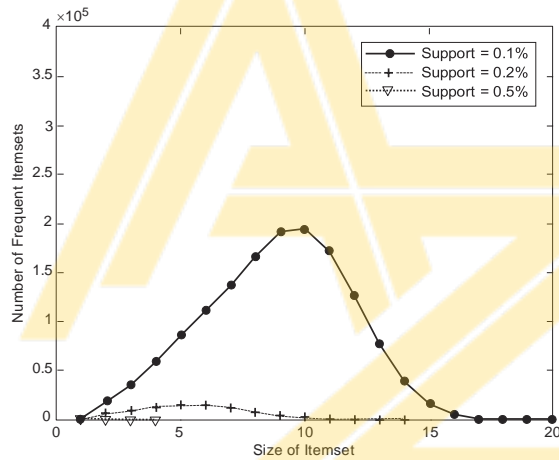
The computational complexity of the *Apriori* algorithm can be affected by the following factors.

**Support Threshold** Lowering the support threshold often results in more itemset being declared as frequent. This has an adverse effect on the





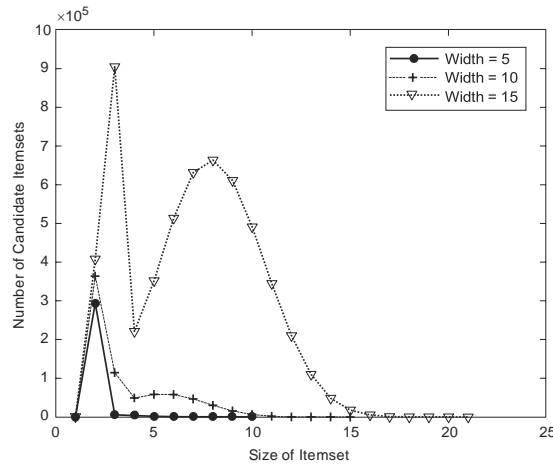
(a) Number of candidate itemsets.



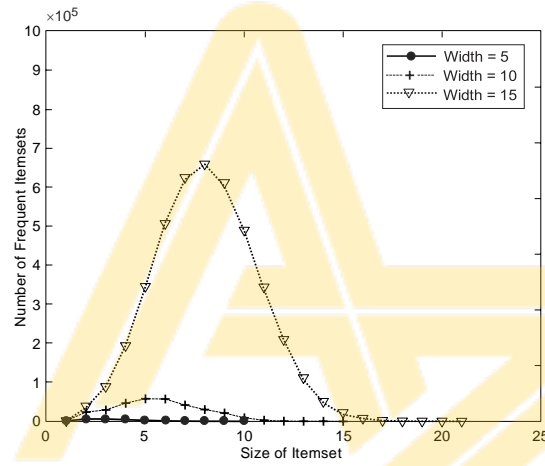
(b) Number of frequent itemsets.

**Figure 6.13.** Effect of support threshold on the number of candidate and frequent itemsets.

average transaction width increases. As a result, more candidate itemset must be examined during candidate generation and support counting, as illustrated in Figure 6.14. Second, as the transaction width increases, more itemset



(a) Number of candidate itemsets.



(b) Number of Frequent Itemsets.

**Figure 6.14.** Effect of average transaction width on the number of candidate and frequent itemsets.

are contained in the transaction. This will increase the number of hash tree traversals performed during support counting. A detailed analysis of the time complexity for the *Apriori* algorithm is presented next.



Generation of frequent 1-itemsets for each transaction, we need to update the support count for every item present in the transaction. Assuming that  $w$  is the average transaction width, this operation requires  $O(Nw)$  time, where  $N$  is the total number of transactions.

**Candidate generation** To generate candidate  $k$ -itemset, pairs of frequent  $(k-1)$ -itemset are merged to determine whether they have at least  $k-2$  items in common. Each merging operation requires at most  $k-2$  equality comparisons. In the best-case scenario, every merging step produces a viable candidate  $k$ -itemset. In the worst-case scenario, the algorithm must merge every pair of frequent  $(k-1)$ -itemset found in the previous iteration. Therefore, the overall cost of merging frequent itemset is

$$\sum_{k=2}^{\infty} (k-2)|C_k| \leq \text{Cost of merging} \leq \sum_{k=2}^{\infty} (k-2)|F_{k-1}|^2.$$

A hash tree is also constructed during candidate generation to store the candidate itemset. Because the maximum depth of the tree is  $k$ , the cost for

populating the hash tree with candidate itemsets is  $O \sum_{k=2}^w k|C_k|$ . During

candidate pruning, we need to verify that the  $k-2$  subsets of every candidate  $k$ -itemset are frequent. Since the cost for looking up a candidate in a hash tree is  $O(k)$ , the candidate pruning step requires  $O \sum_{k=2}^w k(k-2)|C_k|$  time.

**Support counting** Each transaction of length  $|t|$  produces  $\binom{|t|}{k}$  itemset of size  $k$ . This is also the effective number of hash tree traversals performed for each transaction. The cost for support counting is  $O \sum_{k=2}^w N \binom{|t|}{k} a_k$  where  $w$  is the maximum transaction width and  $a_k$  is the cost for updating the support count of a candidate  $k$ -itemset in the hash tree.

## Rule Generation

This section describes how to extract association rules efficiently from a given frequent itemset. Each frequent  $k$ -itemset,  $Y$ , can produce up to  $2^k - 2$  association rules, ignoring rules that have empty antecedents or consequents ( $\emptyset \rightarrow Y$  or  $Y \rightarrow \emptyset$ ). An association rule can be extracted by partitioning the itemset  $Y$  into two non-empty subsets,  $X$  and  $Y - X$ , such that  $X \rightarrow Y - X$  satisfies the confidence threshold. Note that all such rules must have already met the support threshold because they are generated from a frequent itemset.

Example 6.2. Let  $X = \{1, 2, 3\}$  be a frequent itemset. There are six candidate association rules that can be generated from  $X$ :  $\{1, 2\} \rightarrow \{3\}$ ,  $\{1, 3\} \rightarrow \{2\}$ ,  $\{2, 3\} \rightarrow \{1\}$ ,  $\{1\} \rightarrow \{2, 3\}$ ,  $\{2\} \rightarrow \{1, 3\}$ , and  $\{3\} \rightarrow \{1, 2\}$ . As each of their support is identical to the support for  $X$ , the rules must satisfy the support threshold. ■

Computing the confidence of an association rule does not require additional scans of the transaction data set. Consider the rule  $\{1, 2\} \rightarrow \{3\}$ , which is generated from the frequent itemset  $X = \{1, 2, 3\}$ . The confidence for this rule is  $\sigma(\{1, 2, 3\})/\sigma(\{1, 2\})$ . Because  $\{1, 2, 3\}$  is frequent, the anti-monotone property of support ensures that  $\{1, 2\}$  must be frequent, too. Since the support counts for both itemset were already found during frequent itemset generation, there is no need to read the entire data set again.

### Confidence-Based Pruning

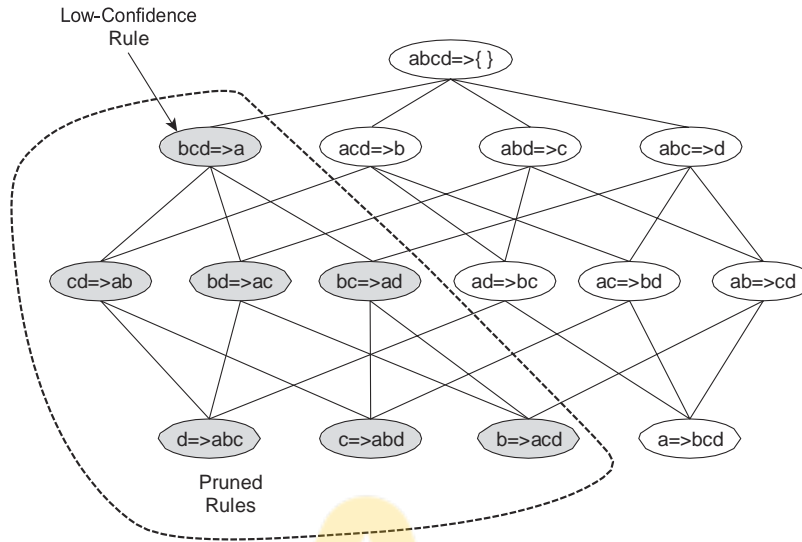
Unlike the support measure, confidence does not have any monotone property. For example, the confidence for  $X \rightarrow Y$  can be larger, smaller, or equal to the confidence for another rule  $\tilde{X} \rightarrow \tilde{Y}$ , where  $\tilde{X} \subseteq X$  and  $\tilde{Y} \subseteq Y$  (see Exercise 3 on page 405). Nevertheless, if we compare rules generated from the same frequent itemset  $Y$ , the following theorem holds for the confidence measure.

*Theorem 6.2. If a rule  $X \rightarrow Y - X$  does not satisfy the confidence threshold, then any rule  $\tilde{X} \rightarrow Y - \tilde{X}$ , where  $\tilde{X}$  is a subset of  $X$ , must not satisfy the confidence threshold as well.*

To prove this theorem, consider the following two rules:  $\tilde{X} \rightarrow Y - \tilde{X}$  and  $X \rightarrow Y - X$ , where  $\tilde{X} \subset X$ . The confidence of the rules are  $\sigma(Y)/\sigma(\tilde{X})$  and  $\sigma(Y)/\sigma(X)$ , respectively. Since  $\tilde{X}$  is a subset of  $X$ ,  $\sigma(\tilde{X}) \geq \sigma(X)$ . Therefore, the former rule cannot have a higher confidence than the latter rule.

### Rule Generation in Apriori Algorithm

The *Apriori* algorithm uses a level-wise approach for generating association rules, where each level corresponds to the number of items that belong to the rule consequent. Initially, all the high-confidence rules that have only one item in the rule consequent are extracted. These rules are then used to generate new candidate rules. For example, if  $\{acd\} \rightarrow \{b\}$  and  $\{abd\} \rightarrow \{c\}$  are high-confidence rules, then the candidate rule  $\{ad\} \rightarrow \{bc\}$  is generated by merging the consequents of both rules. Figure 6.15 shows a lattice structure for the association rules generated from the frequent itemset  $\{a, b, c, d\}$ . If any node in the lattice has low confidence, then according to Theorem 6.2, the



**Figure 6.15.** Pruning of association rules using the confidence measure.

entire subgraph spanned by the node can be pruned immediately. Suppose the confidence for  $\{bcd\} \rightarrow \{a\}$  is low. All the rules containing item  $a$  in its consequent, including  $\{cd\} \rightarrow \{ab\}$ ,  $\{bd\} \rightarrow \{ac\}$ ,  $\{bc\} \rightarrow \{ad\}$ , and  $\{d\} \rightarrow \{abc\}$  can be discarded.

A pseudocode for the rule generation step is shown in Algorithms 6.2 and 6.3. Note the similarity between the apriori gen rules procedure given in Algorithm 6.3 and the frequent itemset generation procedure given in Algorithm 6.1. The only difference is that, in rule generation, we do not have to make additional passes over the data set to compute the confidence of the candidate rules. Instead, we determine the confidence of each rule by using the support counts computed during frequent itemset generation.

Algorithm 6.2 Rule generation of the *Apriori* algorithm.

- 
- 
- ```

1: for each frequent  $k$ -itemset  $f_k$ ,  $k \geq 2$  do
2:    $H_1 = \{i \mid i \in f_k\}$       {1-item consequents of the rule.}
3:   call ap-genrules( $f_k$ ,  $H_1$ .)
4: end for

```
-

---

**Algorithm 6.3 Procedure ap-genrules( $f_k, H_m$ ).**

---

```
1:  $k = |f_k|$  {size of frequent itemset.}
2:  $m = |H_m|$  {size of rule consequent. 3:}
if  $k > m + 1$  then
4:  $H_{m+1} = \text{apriori-gen}(H_m)$ .
5: for each  $h_{m+1} \in H_{m+1}$  do
6:    $\text{conf} = \sigma(f_k) / \sigma(f_k - h_{m+1})$ .
7:   if  $\text{conf} \geq \text{minconf}$  then
8:     output the rule  $(f_k - h_{m+1}) \rightarrow h_{m+1}$ .
9:   else
10:    delete  $h_{m+1}$  from  $H_{m+1}$ .
11:   end if
12: end for
13: call ap-genrules( $f_k, H_{m+1}$ .)
14: end if
```

---

### An Example: Congressional Voting Records

This section demonstrates the results of applying association analysis to the voting records of members of the United States House of Representatives. The data is obtained from the 1984 Congressional Voting Records Database, which is available at the UCI machine learning data repository. Each transaction contains information about the party affiliation for a representative along with his or her voting record on 16 key issues. There are 435 transactions and 34 items in the data set. The set of items are listed in Table 6.3.

The Apriori algorithm is then applied to the data set with  $\text{minsup} = 30\%$  and  $\text{minconf} = 90\%$ . Some of the high-confidence rules extracted by the algorithm are shown in Table 6.4. The first two rules suggest that most of the members who voted yes for aid to El Salvador and no for budget resolution and MX missile are Republicans; while those who voted no for aid to El Salvador and yes for budget resolution and MX missile are Democrats. These high-confidence rules show the key issues that divide members from both political parties. If  $\text{minconf}$  is reduced, we may find rules that contain issues that cut across the party lines. For example, with  $\text{minconf} = 40\%$ , the rules suggest that corporation cutbacks is an issue that receives almost equal number of votes from both parties—52.3% of the members who voted no are Republicans, while the remaining 47.7% of them who voted no are Democrats.

**Table 6.3.** List of binary attributes from the 1984 United States Congressional Voting Records. Source: The UCI machine learning repository.

|                                       |                                       |
|---------------------------------------|---------------------------------------|
| 1. Republican                         | 18. aid to Nicaragua = no             |
| 2. Democrat                           | 19. MX-missile = yes                  |
| 3. handicapped-infants = yes          | 20. MX-missile = no                   |
| 4. handicapped-infants = no           | 21. immigration = yes                 |
| 5. water project cost sharing = yes   | 22. immigration = no                  |
| 6. water project cost sharing = no    | 23. synfuel corporation cutback = yes |
| 7. budget-resolution = yes            | 24. synfuel corporation cutback = no  |
| 8. budget-resolution = no             | 25. education spending = yes          |
| 9. physician fee freeze = yes         | 26. education spending = no           |
| 10. physician fee freeze = no         | 27. right-to-sue = yes                |
| 11. aid to El Salvador = yes          | 28. right-to-sue = no                 |
| 12. aid to El Salvador = no           | 29. crime = yes                       |
| 13. religious groups in schools = yes | 30. crime = no                        |
| 14. religious groups in schools = no  | 31. duty-free-exports = yes           |
| 15. anti-satellite test ban = yes     | 32. duty-free-exports = no            |
| 16. anti-satellite test ban = no      | 33. export administration act = yes   |
| 17. aid to Nicaragua = yes            | 34. export administration act = no    |

**Table 6.4.** Association rules extracted from the 1984 United States Congressional Voting Records.

| Association Rule                                                                     | Confidence |
|--------------------------------------------------------------------------------------|------------|
| {budget resolution = no, MX-missile=no, aid to El Salvador = yes }<br>→ {Republican} | 91.0%      |
| {budget resolution = yes, MX-missile=yes, aid to El Salvador = no }<br>→ {Democrat}  | 97.5%      |
| {crime = yes, right-to-sue = yes, physician fee freeze = yes}<br>→ {Republican}      | 93.5%      |
| {crime = no, right-to-sue = no, physician fee freeze = no}<br>→ {Democrat}           | 100%       |

## Alternative Methods for Generating Frequent Itemsets

*Apriori* is one of the earliest algorithms to have successfully addressed the combinatorial explosion of frequent itemset generation. It achieves this by applying the *Apriori* principle to prune the exponential search space. Despite its significant performance improvement, the algorithm still incurs considerable I/O overhead since it requires making several passes over the transaction data set. In addition, as noted in Section 6.2.5, the performance of the *Apriori* algorithm may degrade significantly for dense data sets because of the increasing width of transactions. Several alternative methods have been developed to overcome these limitations and improve upon the efficiency of the *Apriori*

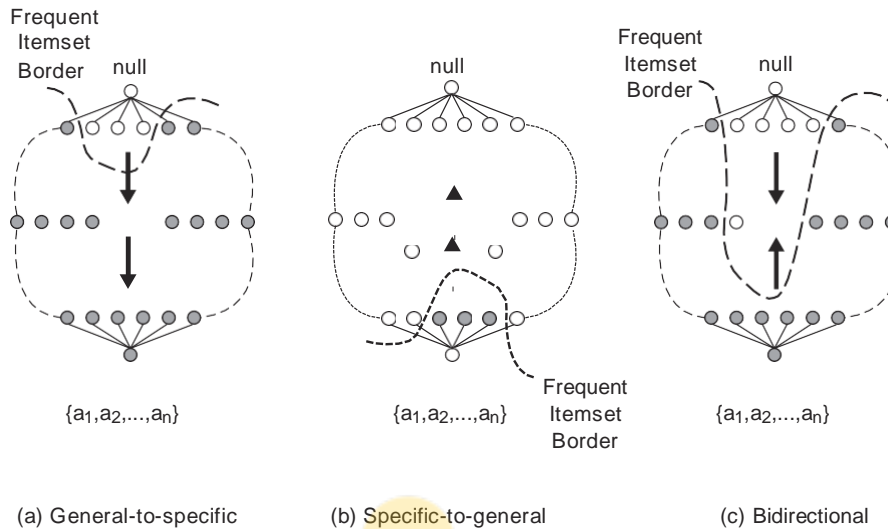
algorithm. The following is a high-level description of these methods.

**Traversal of Itemset Lattice** A search for frequent itemsets can be conceptually viewed as a traversal on the itemset lattice shown in Figure 6.1. The search strategy employed by an algorithm dictates how the lattice structure is traversed during the frequent itemset generation process. Some search strategies are better than others, depending on the configuration of frequent itemset in the lattice. An overview of these strategies is presented next.

**General-to-Specific versus Specific-to-General:** The *Apriori* algorithm uses a general-to-specific search strategy, where pairs of frequent( $k - 1$ )-itemsets are merged to obtain candidate  $k$ -itemsets. This general-to-specific search strategy is effective, provided the maximum length of a frequent itemset is not too long. The configuration of frequent itemsets that works best with this strategy is shown in Figure 6.19(a), where the darker nodes represent infrequent itemsets. Alternatively, a specific-to-general search strategy looks for more specific frequent

itemsets first, before finding the more general frequent itemsets. This strategy is useful to discover maximal frequent itemsets in dense transactions, where the frequent itemset border is located near the bottom of the lattice, as shown in Figure 6.19(b). The *Apriori* principle can be applied to prune all subsets of maximal frequent itemsets. Specifically, if a candidate  $k$ -itemset is maximal frequent, we do not have to examine any of its subsets of size  $k - 1$ . However, if the candidate  $k$ -itemset is infrequent, we need to check all of its  $k - 1$  subsets in the next iteration. Another approach is to combine both general-to-specific and specific-to-general search strategies. This bidirectional approach requires more space to





**Figure 6.19.** General-to-specific, specific-to-general, and bidirectional search.

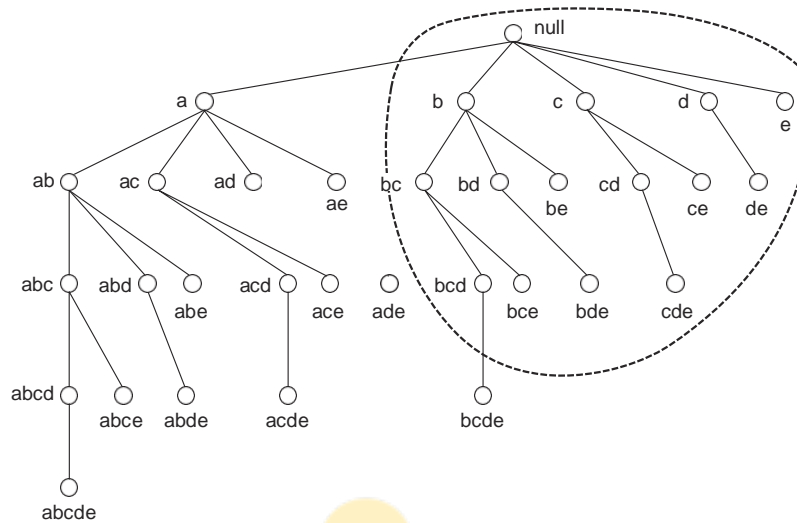
store the candidate itemsets, but it can help to rapidly identify the frequent itemset border, given the configuration shown in Figure 6.19(c).

**Equivalence Classes:** Another way to envision the traversal is to first partition the lattice into disjoint groups of nodes (or equivalence classes). A frequent itemset generation algorithm search for frequent itemset within a particular equivalence class first before moving to another equivalence class. As an example, the level-wise strategy used in the *Apriori* algorithm can be considered to be partitioning the lattice on the basis of itemset sizes; i.e., the algorithm discovers all frequent 1-itemsets first before proceeding to larger-sized itemset. Equivalence classes can also be defined according to the prefix or suffix labels of an itemset. In this case, two itemset belong to the same equivalence class if they share a common prefix or suffix of length  $k$ . In the prefix-based approach, the algorithm can search for frequent itemset starting with the prefix  $a$  before looking for those starting with prefixes  $b$ ,  $c$ , and so on. Both prefix-based and suffix-based equivalence classes can be demonstrated using the tree-like structure shown in Figure 6.20.

**Breadth-First versus Depth-First:** The *Apriori* algorithm traverses the lattice in a breadth-first manner, as shown in Figure 6.21(a). It first discovers all the frequent 1-itemsets, followed by the frequent 2-itemsets, and so on, until no new frequent itemset are generated. The itemset







**Figure 6.22.** Generating candidate itemsets using the depth-first approach.

performed on its subsets. For example, if the node *bcde* shown in Figure 6.22 is maximal frequent, then the algorithm does not have to visit the subtrees rooted at *bd*, *be*, *c*, *d*, and *e* because they will not contain any maximal frequent itemset. However, if *abc* is maximal frequent, only the nodes such as *ac* and *bc* are not maximal frequent (but the subtrees of *ac* and *bc* may still contain maximal frequent itemset). The depth-first approach also allows a different kind of pruning based on the support of itemsets. For example, suppose the support for  $\{a, b, c\}$  is identical to the support for  $\{a, b\}$ . The subtrees rooted at *abd* and *abe* can be skipped because they are guaranteed not to have any maximal frequent itemset. The proof of this is left as an exercise to the readers.

**Representation of Transaction Data Set** There are many ways to represent a transaction data set. The choice of representation can affect the I/O costs incurred when computing the support of candidate itemset. Figure 6.23 shows two different ways of representing market basket transactions. The representation on the left is called a horizontal data layout, which is adopted by many association rule mining algorithms, including *Apriori*. Another possibility is to store the list of transaction identifiers (TID-list) associated with each item. Such a representation is known as the vertical data layout. The support for each candidate itemset is obtained by intersecting the TID-lists of its subset items. The length of the TID-lists shrinks as we progress to larger



| Horizontal Data Layout |         | Vertical Data Layout |    |   |   |   |
|------------------------|---------|----------------------|----|---|---|---|
| TID                    | Items   | a                    | b  | c | d | e |
| 1                      | a,b,e   | 1                    | 1  | 2 | 2 | 1 |
| 2                      | b,c,d   | 4                    | 2  | 3 | 4 | 3 |
| 3                      | c,e     | 5                    | 5  | 4 | 5 | 6 |
| 4                      | a,c,d   | 6                    | 7  | 8 | 9 |   |
| 5                      | a,b,c,d | 7                    | 8  | 9 |   |   |
| 6                      | a,e     | 8                    | 10 |   |   |   |
| 7                      | a,b     | 9                    |    |   |   |   |
| 8                      | a,b,c   |                      |    |   |   |   |
| 9                      | a,c,d   |                      |    |   |   |   |
| 10                     | b       |                      |    |   |   |   |

**Figure 6.23.** Horizontal and vertical data format.

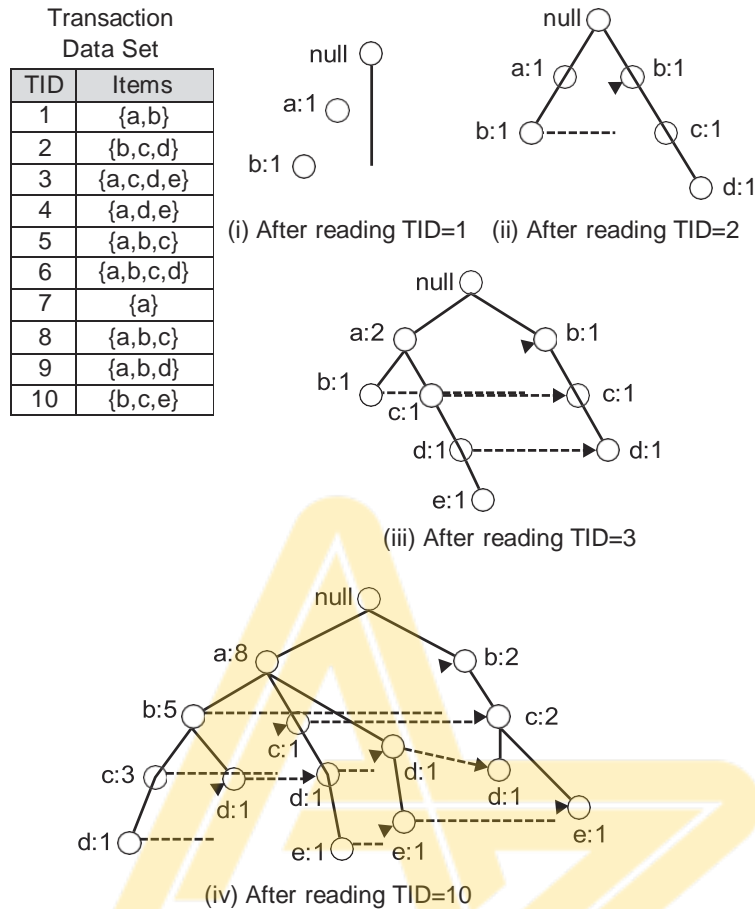
sized itemsets. However, one problem with this approach is that the initial set of TID-lists may be too large to fit into main memory, thus requiring more sophisticated techniques to compress the TID-lists. We describe another effective approach to represent the data in the next section.

### FP-Growth Algorithm

This section presents an alternative algorithm called FP-growth that takes a radically different approach to discovering frequent itemsets. The algorithm does not subscribe to the generate-and-test paradigm of *Apriori*. Instead, it encodes the data set using a compact data structure called an FP-tree and extracts frequent itemsets directly from this structure. The details of this approach are presented next.

#### FP-Tree Representation

An FP-tree is a compressed representation of the input data. It is constructed by reading the data set one transaction at a time and mapping each transaction onto a path in the FP-tree. As different transactions can have several items in common, their paths may overlap. The more the paths overlap with one another, the more compression we can achieve using the FP-tree structure. If the size of the FP-tree is small enough to fit into main memory, this will allow us to extract frequent itemsets directly from the structure in memory instead of making repeated passes over the data stored on disk.



**Figure 6.24.** Construction of an FP-tree.

Figure 6.24 shows a data set that contains ten transactions and five items. The structures of the FP-tree after reading the first three transactions are also depicted in the diagram. Each node in the tree contains the label of an item along with a counter that shows the number of transactions mapped onto the given path. Initially, the FP-tree contains only the root node represented by the null symbol. The FP-tree is subsequently extended in the following way: The data set is scanned once to determine the support count of each item. Infrequent items are discarded, while the frequent items are sorted in decreasing support counts. For the data set shown in Figure 6.24, *a* is the most frequent item, followed by *b*, *c*, *d*, and *e*.

The algorithm makes a second pass over the data to construct the FP- tree. After reading the first transaction,  $\{a, b\}$ , the nodes labeled as  $a$  and  $b$  are created. A path is then formed from  $\text{null} \rightarrow a \rightarrow b$  to encode the transaction. Every node along the path has a frequency count of 1.

After reading the second transaction,  $\{b, c, d\}$ , a new set of nodes is created for items  $b$ ,  $c$ , and  $d$ . A path is then formed to represent the transaction by connecting the nodes  $\text{null} \rightarrow b \rightarrow c \rightarrow d$ . Every node along this path also has a frequency count equal to one. Although the first two transactions have an item in common, which is  $b$ , their paths are disjoint because the transactions do not share a common prefix.

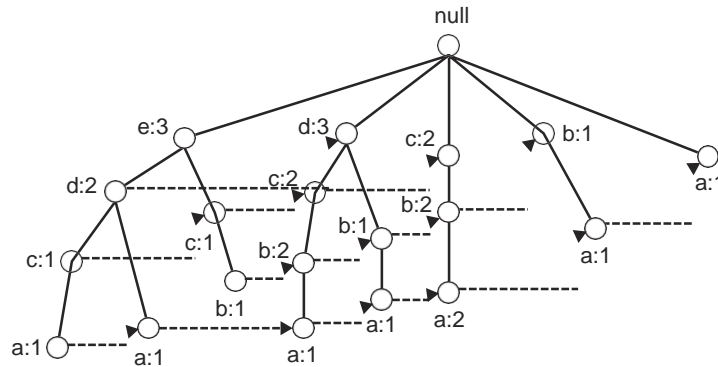
The third transaction,  $\{a, c, d, e\}$ , shares a common prefix item (which is  $a$ ) with the first transaction. As a result, the path for the third transaction,  $\text{null} \rightarrow a \rightarrow c \rightarrow d \rightarrow e$ , overlaps with the path for the first transaction,  $\text{null} \rightarrow a \rightarrow b$ . Because of their overlapping path, the frequency count for node  $a$  is incremented to two, while the frequency counts for the newly created nodes,  $c$ ,  $d$ , and  $e$ , are equal to one.

This process continues until every transaction has been mapped onto one of the paths given in the FP-tree. The resulting FP-tree after reading all the transactions is shown at the bottom of Figure 6.24.

The size of an FP-tree is typically smaller than the size of the uncompressed data because many transactions in market basket data often share a few items in common. In the best-case scenario, where all the transactions have the same set of items, the FP-tree contains only a single branch of nodes. The worst-case scenario happens when every transaction has a unique set of items. As none of the transactions have any items in common, the size of the FP-tree is effectively the same as the size of the original data. However, the physical storage requirement for the FP-tree is higher because it requires additional space to store pointers between nodes and counters for each item.

The size of an FP-tree also depends on how the items are ordered. If the ordering scheme in the preceding example is reversed, i.e., from lowest to highest support item, the resulting FP-tree is shown in Figure 6.25. The tree appears to be denser because the branching factor at the root node has increased from 2 to 5 and the number of nodes containing the high support items such as  $a$  and  $b$  has increased from 3 to 12. Nevertheless, ordering by decreasing support counts does not always lead to the smallest tree. For example, suppose we augment the data set given in Figure 6.24 with 100

transactions that contain  $\{e\}$ , 80 transactions that contain  $\{d\}$ , 60 transactions



**Figure 6.25.** An FP-tree representation for the data set shown in Figure 6.24 with a different item ordering scheme.

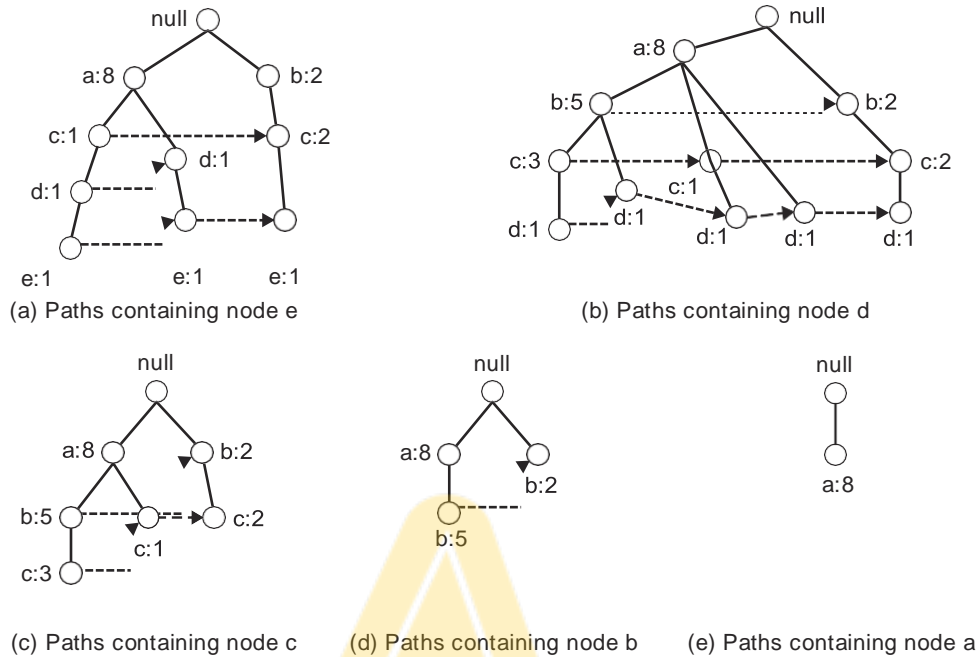
that contain  $\{c\}$ , and 40 transactions that contain  $\{b\}$ . Item  $e$  is now most frequent, followed by  $d$ ,  $c$ ,  $b$ , and  $a$ . With the augmented transactions, ordering by decreasing support counts will result in an FP-tree similar to Figure 6.25, while a scheme based on increasing support counts produces a smaller FP-tree similar to Figure 6.24(iv).

An FP-tree also contains a list of pointers connecting between nodes that have the same items. These pointers, represented as dashed lines in Figures 6.24 and 6.25, help to facilitate the rapid access of individual items in the tree. We explain how to use the FP-tree and its corresponding pointers for frequent itemset generation in the next section.

### Frequent Itemset Generation in FP-Growth Algorithm

FP-growth is an algorithm that generates frequent itemset from an FP-tree by exploring the tree in a bottom-up fashion. Given the example tree shown in Figure 6.24, the algorithm looks for frequent itemset ending in  $e$  first, followed by  $d$ ,  $c$ ,  $b$ , and finally,  $a$ . This bottom-up strategy for finding frequent itemset ending with a particular item is equivalent to the suffix-based approach described in Section 6.5. Since every transaction is mapped onto a path in the FP-tree, we can derive the frequent itemset ending with a particular item, say,  $e$ , by examining only the paths containing node  $e$ . These paths can be accessed rapidly using the pointers associated with node  $e$ . The extracted paths are shown in Figure 6.26(a). The details on how to process the paths to obtain frequent itemset will be explained later.



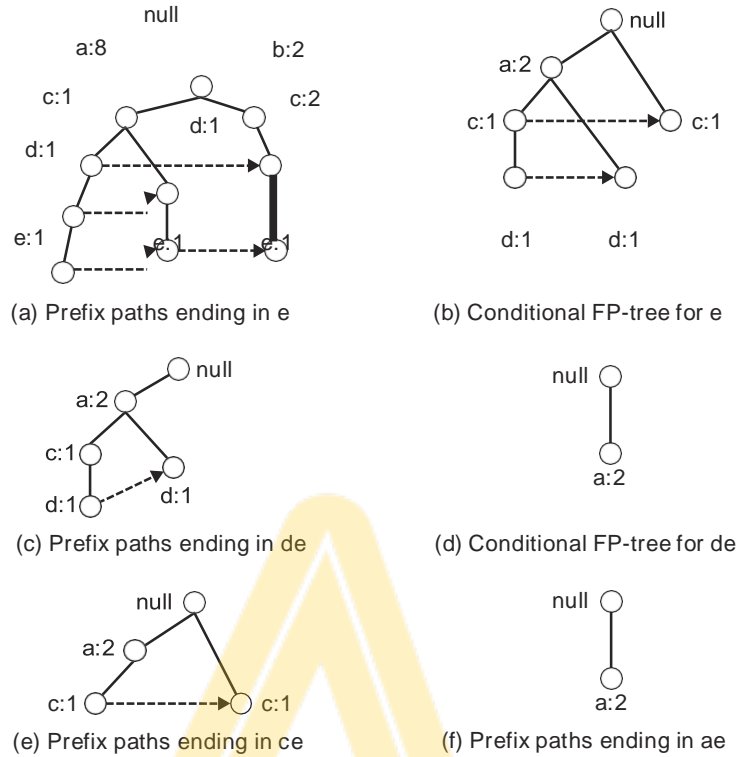


**Figure 6.26.** Decomposing the frequent itemset generation problem into multiple subproblems, where each subproblem involves finding frequent itemset ending in  $e$ ,  $d$ ,  $c$ ,  $b$ , and  $a$ .

**Table 6.6.** The list of frequent itemsets ordered by their corresponding suffixes.

| Suffix | Frequent Itemset                                    |
|--------|-----------------------------------------------------|
| e      | {e}, {d,e}, {a,d,e}, {c,e}, {a,e}                   |
| d      | {d}, {c,d}, {b,c,d}, {a,c,d}, {b,d}, {a,b,d}, {a,d} |
| c      | {c}, {b,c}, {a,b,c}, {a,c}                          |
| b      | {b}, {a,b}                                          |
| a      | {a}                                                 |

After finding the frequent itemset ending in  $e$ , the algorithm proceeds to look for frequent itemset ending in  $d$  by processing the paths associated with node  $d$ . The corresponding paths are shown in Figure 6.26(b). This process continues until all the paths associated with nodes  $c$ ,  $b$ , and finally  $a$ , are processed. The paths for these items are shown in Figures 6.26(c), (d), and (e), while their corresponding frequent itemset are summarized in Table 6.6. FP-growth finds all the frequent itemset ending with a particular suffix by employing a divide-and-conquer strategy to split the problem into smaller subproblems. For example, suppose we are interested in finding all frequent



**Figure 6.27.** Example of applying the FP-growth algorithm to find frequent itemset ending in  $e$ . To do this, we must first check whether the itemset  $\{e\}$  itself is frequent. If it is frequent, we consider the subproblem of finding frequent itemset ending in  $de$ , followed by  $ce$ ,  $be$ , and  $ae$ . In turn, each of these subproblems are further decomposed into smaller subproblems. By merging the solutions obtained from the subproblems, all the frequent itemset ending in  $e$  can be found. This divide-and-conquer approach is the key strategy employed by the FP-growth algorithm. For a more concrete example on how to solve the subproblems, consider the task of finding frequent itemset ending with  $e$ . The first step is to gather all the paths containing node  $e$ . These initial paths are called prefix paths and are shown in Figure 6.27(a).

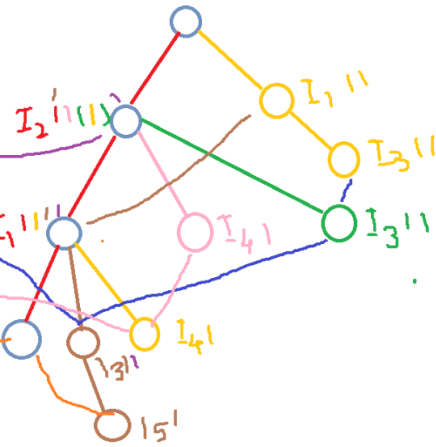
From the prefix paths shown in Figure 6.27(a), the support count for  $e$  is obtained by adding the support counts associated with node  $e$ . Assuming that the minimum support count is 2,  $\{e\}$  is declared a frequent itemset because its support count is 3.

Transactional Data for an *Alielectronics* Branch

| TID  | List of item IDs                 |
|------|----------------------------------|
| T100 | I1, I2, I5 $I_1 I_2 I_5$         |
| T200 | I2, I4                           |
| T300 | I2, I3                           |
| T400 | I1, I2, I4 $I_1 I_2 I_4$         |
| T500 | I1, I3                           |
| T600 | I2, I3                           |
| T700 | I1, I3                           |
| T800 | I1, I2, I3, I5 $I_1 I_2 I_3 I_5$ |
| T900 | I1, I2, I3 $I_1 I_2 I_3$         |

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|    |   |
|----|---|
| I2 | 7 |
| I1 | 6 |
| I3 | 6 |
| I4 | 2 |
| I5 | 2 |



Because  $\{e\}$  is frequent, the algorithm has to solve the subproblems of finding frequent itemset ending in  $de$ ,  $ce$ ,  $be$ , and  $ae$ . Before solving these subproblems, it must first convert the prefix paths into a conditional FP-tree, which is structurally similar to an FP-tree, except it is used to find frequent itemset ending with a particular suffix. A conditional FP-tree is obtained in the following way:

First, the support counts along the prefix paths must be updated because some of the counts include transactions that do not contain item  $e$ . For example, the rightmost path shown in Figure 6.27(a),  $\text{null} \rightarrow b:2 \rightarrow c:2 \rightarrow e:1$ , includes a transaction  $\{b, c\}$  that does not contain item  $e$ . The counts along the prefix path must therefore be adjusted to 1 to reflect the actual number of transactions containing  $\{b, c, e\}$ .

The prefix paths are truncated by removing the nodes for  $e$ . These nodes can be removed because the support counts along the prefix paths have been updated to reflect only transactions that contain  $e$  and the subproblems of finding frequent itemset ending in  $de$ ,  $ce$ ,  $be$ , and  $ae$  no longer need information about node  $e$ .

After updating the support counts along the prefix paths, some of the items may no longer be frequent. For example, the node  $b$  appears only once and has a support count equal to 1, which means that there is only one transaction that contains both  $b$  and  $e$ . Item  $b$  can be safely ignored from subsequent analysis because all itemset ending in  $be$  must be infrequent.

The conditional FP-tree for  $e$  is shown in Figure 6.27(b). The tree looks different than the original prefix paths because the frequency counts have been updated and the nodes  $b$  and  $e$  have been eliminated.

FP-growth uses the conditional FP-tree for  $e$  to solve the subproblems of finding frequent itemset ending in  $de$ ,  $ce$ , and  $ae$ . To find the frequent itemset ending in  $de$ , the prefix paths for  $d$  are gathered from the conditional FP-tree for  $e$  (Figure 6.27(c)). By adding the frequency counts associated with node  $d$ , we obtain the support count for  $\{d, e\}$ . Since the support count is equal to 2,  $\{d, e\}$  is declared a frequent itemset. Next, the algorithm constructs the conditional FP-tree for  $de$  using the approach described in step 3. After updating the support counts and removing the infrequent item  $c$ , the conditional FP-tree for  $de$  is shown in Figure 6.27(d). Since the conditional FP-tree contains only one item,

$a$ , whose support is equal to  $minsup$ , the algorithm extracts the frequent itemset  $\{a, d, e\}$  and moves on to the next subproblem, which is to generate frequent itemset ending in  $ce$ . After processing the prefix paths for  $c$ , only  $\{c, e\}$  is found to be frequent. The algorithm proceeds to solve the next subproblem and found  $\{a, e\}$  to be the only frequent itemset remaining.

This example illustrates the divide-and-conquer approach used in the FP-growth algorithm. At each recursive step, a conditional FP-tree is constructed by updating the frequency counts along the prefix paths and removing all infrequent items. Because the subproblems are disjoint, FP-growth will not generate any duplicate itemset. In addition, the counts associated with the nodes allow the algorithm to perform support counting while generating the common suffix itemset.

FP-growth is an interesting algorithm because it illustrates how a compact representation of the transaction data set helps to efficiently generate frequent itemset. In addition, for certain transaction data sets, FP-growth outperforms the standard *Apriori* algorithm by several orders of magnitude. The run-time performance of FP-growth depends on the compaction factor of the dataset. If the resulting conditional FP-trees are very bushy (in the worst case, a full prefix tree), then the performance of the algorithm degrades significantly because it has to generate a large number of subproblems and merge the results returned by each subproblem.

## Evaluation of Association Patterns

Association analysis algorithms have the potential to generate a large number of patterns. For example, although the data set shown in Table 6.1 contains only six items, it can produce up to hundreds of association rules at certain support and confidence thresholds. As the size and dimensionality of real commercial databases can be very large, we could easily end up with thousands or even millions of patterns, many of which might not be interesting. Sifting through the patterns to identify the most interesting ones is not a trivial task because “one person’s trash might be another person’s treasure.” It is therefore important to establish a set of well-accepted criteria for evaluating the quality of association patterns.

The first set of criteria can be established through statistical arguments. Patterns that involve a set of mutually independent items or cover very few transactions are considered uninteresting because they may capture spurious relationships in the data. Such patterns can be eliminated by applying an



objective interestingness measure that uses statistics derived from data to determine whether a pattern is interesting. Examples of objective interestingness measures include support, confidence, and correlation.

The second set of criteria can be established through subjective arguments. A pattern is considered subjectively uninteresting unless it reveals unexpected information about the data or provides useful knowledge that can lead to

profitable actions. For example, the rule  $\{Butter\} \rightarrow \{Bread\}$  may not be interesting, despite having high support and confidence values, because the relationship represented by the rule may seem rather obvious. On the other hand, the rule  $\{Diapers\} \rightarrow \{Beer\}$  is interesting because the relationship is quite unexpected and may suggest a new cross-selling opportunity for retailers. Incorporating subjective knowledge into pattern evaluation is a difficult task because it requires a considerable amount of prior information from the domain experts.

The following are some of the approaches for incorporating subjective knowledge into the pattern discovery task.

**Visualization** This approach requires a user-friendly environment to keep the human user in the loop. It also allows the domain experts to interact with the data mining system by interpreting and verifying the discovered patterns.

**Template-based approach** This approach allows the users to constrain the type of patterns extracted by the mining algorithm. Instead of reporting all the extracted rules, only rules that satisfy a user-specified template are returned to the users.

**Subjective interestingness measure** A subjective measure can be defined based on domain information such as concept hierarchy (to be discussed in Section 7.3) or profit margin of items. The measure can then be used to filter patterns that are obvious and non-actionable. Readers interested in subjective interestingness measures may refer to resources listed in the bibliography at the end of this chapter.

### Objective Measures of Interestingness

An objective measure is a data-driven approach for evaluating the quality of association patterns. It is domain-independent and requires minimal input from the users, other than to specify a threshold for filtering low-quality patterns. An objective measure is usually computed based on the frequency

**Table 6.7.** A 2-way contingency table for variables  $A$  and  $B$ .

|                | $B$      | $\overline{B}$ |          |
|----------------|----------|----------------|----------|
| $A$            | $f_{11}$ | $f_{10}$       | $f_{1+}$ |
| $\overline{A}$ | $f_{01}$ | $f_{00}$       | $f_{0+}$ |
|                | $f_{+1}$ | $f_{+0}$       | $N$      |

counts tabulated in a contingency table. Table 6.7 shows an example of a contingency table for a pair of binary variables,  $A$  and  $B$ . We use the notation  $A(B)$  to indicate that  $A(B)$  is absent from a transaction. Each entry  $f_{ij}$  in this  $2 \times 2$  table denotes a frequency count. For example,  $f_{11}$  is the number of times  $A$  and  $B$  appear together in the same transaction, while  $f_{01}$  is the number of transactions that contain  $B$  but not  $A$ . The row sum  $f_{1+}$  represents the support count for  $A$ , while the column sum  $f_{+1}$  represents the support count for  $B$ . Finally, even though our discussion focuses mainly on asymmetric binary variables, note that contingency tables are also applicable to other attribute types such as symmetric binary, nominal, and ordinal variables.

**Limitations of the Support-Confidence Framework** Existing association rule mining formulation relies on the support and confidence measures to eliminate uninteresting patterns. The drawback of support was previously described in Section 6.8, in which many potentially interesting patterns involving low support items might be eliminated by the support threshold. The drawback of confidence is more subtle and is best demonstrated with the following example.

**Example 6.3.** Suppose we are interested in analyzing the relationship between people who drink tea and coffee. We may gather information about the beverage preferences among a group of people and summarize their responses into a table such as the one shown in Table 6.8.

**Table 6.8.** Beverage preferences among a group of 1000 people.

|                  | <i>Coffee</i> | $\overline{Coffee}$ |      |
|------------------|---------------|---------------------|------|
| <i>Tea</i>       | 150           | 50                  | 200  |
| $\overline{Tea}$ | 650           | 150                 | 800  |
|                  | 800           | 200                 | 1000 |



## Limitation of the Support-Confidence Framework

- Are  $s$  and  $c$  interesting in association rules: " $A \Rightarrow B$ " [ $s$ ,  $c$ ]?
- Example: Suppose one school may have the following statistics on # of students who may play basketball and/or eat cereal:

|                | play-basketball | not play-basketball | sum (row) |
|----------------|-----------------|---------------------|-----------|
| eat-cereal     | 400 ✓           | 350                 | 750       |
| not eat-cereal | 200 ✓✓          | 50                  | 250       |
| sum(col.)      | 600             | 400                 | 1000 ✓    |

2-way contingency table

- Association rule mining may generate the following:  $400/1000$ 
  - $\text{play-basketball} \Rightarrow \text{eat-cereal}$  [40%, 66.7%] (higher  $s$  &  $c$ )  $400/600$
- But this strong association rule is misleading: The overall % of students eating cereal is 75% > 66.7%, a more telling rule:
  - $\neg \text{play-basketball} \Rightarrow \text{eat-cereal}$  [35%, 87.5%] (high  $s$  &  $c$ )

The information given in this table can be used to evaluate the association rule  $\{Tea\} \rightarrow \{Coffee\}$ . At first glance, it may appear that people who drink tea also tend to drink coffee because the rule's support (15%) and confidence (75%) values are reasonably high. This argument would have been acceptable except that the fraction of people who drink coffee, regardless of whether they drink tea, is 80%, while the fraction of tea drinkers who drink coffee is only 75%. Thus knowing that a person is a tea drinker actually decreases her probability of being a coffee drinker from 80% to 75%! The rule  $\{Tea\} \rightarrow \{Coffee\}$  is therefore misleading despite its high confidence value.

The pitfall of confidence can be traced to the fact that the measure ignores the support of the itemset in the rule consequent. Indeed, if the support of coffee drinkers is taken into account, we would not be surprised to find that many of the people who drink tea also drink coffee. What is more surprising is that the fraction of tea drinkers who drink coffee is actually less than the overall fraction of people who drink coffee, which points to an inverse relationship between tea drinkers and coffee drinkers.

Because of the limitations in the support-confidence framework, various objective measures have been used to evaluate the quality of association patterns. Below, we provide a brief description of these measures and explain some of their strengths and limitations.

**Interest Factor** The tea-coffee example shows that high-confidence rules can sometimes be misleading because the confidence measure ignores the support of the itemset appearing in the rule consequent. One way to address this problem is by applying a metric known as lift:

$$\text{Lift} = \frac{c(A \rightarrow B)}{s(B)}, \quad (6.4)$$

which computes the ratio between the rule's confidence and the support of the itemset in the rule consequent. For binary variables, lift is equivalent to another objective measure called interest factor, which is defined as follows

$$I(A, B) = \frac{s(A, B)}{s(A) \times s(B)} = \frac{Nf_{11}}{f_{1+}f_{+1}}. \quad (6.5)$$

Interest factor compares the frequency of a pattern against a baseline frequency computed under the statistical independence assumption. The baseline frequency for a pair of mutually independent variables is

$$\frac{f_{11}}{N} = \frac{f_{1+}}{N} \times \frac{f_{+1}}{N}, \quad \text{or equivalently, } f_{11} = \frac{f_{1+}f_{+1}}{N}. \quad (6.6)$$

**Table 6.9.** Contingency tables for the word pairs  $\{p, q\}$  and  $\{r, s\}$ .

|           | $p$ | $\bar{p}$ |      |
|-----------|-----|-----------|------|
| $q$       | 880 | 50        | 930  |
| $\bar{q}$ | 50  | 20        | 70   |
|           | 930 | 70        | 1000 |

|           | $r$ | $\bar{r}$ |      |
|-----------|-----|-----------|------|
| $s$       | 20  | 50        | 70   |
| $\bar{s}$ | 50  | 880       | 930  |
|           | 70  | 930       | 1000 |

This equation follows from the standard approach of using simple fractions as estimates for probabilities. The fraction  $f_{11}/N$  is an estimate for the joint probability  $P(A, B)$ , while  $f_{1+}/N$  and  $f_{+1}/N$  are the estimates for  $P(A)$  and  $P(B)$ , respectively. If  $A$  and  $B$  are statistically independent, then  $P(A, B) = P(A) \times P(B)$ , thus leading to the formula shown in Equation 6.6. Using Equations 6.5 and 6.6, we can interpret the measure as follows:

$$I(A, B) = \begin{cases} = 1, & \text{if } A \text{ and } B \text{ are independent;} \\ > 1, & \text{if } A \text{ and } B \text{ are positively correlated;} \\ < 1, & \text{if } A \text{ and } B \text{ are negatively correlated.} \end{cases} \quad (6.7)$$

For the tea-coffee example shown in Table 6.8,  $I = \frac{0.15}{0.21 \times 0.28} = 0.9375$ , thus suggesting a slight negative correlation between tea drinkers and coffee drinkers.

**Limitations of Interest Factor** We illustrate the limitation of interest factor with an example from the text mining domain. In the text domain, it is reasonable to assume that the association between a pair of words depends on the number of documents that contain both words. For example, because of their stronger association, we expect the words *data* and *mining* to appear together more frequently than the words *compiler* and *mining* in a collection of computer science articles.

Table 6.9 shows the frequency of occurrences between two pairs of words,

$\{p, q\}$  and  $\{r, s\}$ . Using the formula given in Equation 6.5, the interest factor for  $\{p, q\}$  is 1.02 and for  $\{r, s\}$  is 4.08. These results are somewhat troubling for the following reasons. Although  $p$  and  $q$  appear together in 88% of the documents, their interest factor is close to 1, which is the value when  $p$  and  $q$  are statistically independent. On the other hand, the interest factor for  $\{r, s\}$  is higher than  $\{p, q\}$  even though  $r$  and  $s$  seldom appear together in the same document. Confidence is perhaps the better choice in this situation because it considers the association between  $p$  and  $q$  (94.6%) to be much stronger than that between  $r$  and  $s$  (28.6%).

**Correlation Analysis** Correlation analysis is a statistical-based technique for analyzing relationships between a pair of variables. For continuous variables, correlation is defined using Pearson's correlation coefficient (see Equation 2.10 on page 77). For binary variables, correlation can be measured using the  $\phi$ -coefficient, which is defined as

$$\phi = \frac{f_{11}f_{00} - f_{10}f_{01}}{\sqrt{(f_{1+}f_{+1}f_{0+}f_{+0})}}. \quad (6.8)$$

The value of correlation ranges from  $-1$  (perfect negative correlation) to  $+1$  (perfect positive correlation). If the variables are statistically independent, then  $\phi = 0$ . For example, the correlation between the tea and coffee drinkers given in Table 6.8 is  $-0.0625$ .

**Limitations of Correlation Analysis** The drawback of using correlation can be seen from the word association example given in Table 6.9. Although the words  $p$  and  $q$  appear together more often than  $r$  and  $s$ , their  $\phi$ -coefficients are identical, i.e.,  $\phi(p, q) = \phi(r, s) = 0.232$ . This is because the  $\phi$ -coefficient gives equal importance to both co-presence and co-absence of items in a transaction. It is therefore more suitable for analyzing symmetric binary variables. Another limitation of this measure is that it does not remain invariant when there are proportional changes to the sample size.

**IS Measure**  $IS$  is an alternative measure that has been proposed for handling asymmetric binary variables. The measure is defined as follows:

Note that  $IS$  is large when the interest factor and support of the pattern are large. For example, the value of  $IS$  for the word pairs  $\{p, q\}$  and  $\{r, s\}$  shown in Table 6.9 are  $0.946$  and  $0.286$ , respectively. Contrary to the results given by interest factor and the  $\phi$ -coefficient, the  $IS$  measure suggests that the association between  $\{p, q\}$  is stronger than  $\{r, s\}$ , which agrees with what we expect from word associations in documents.

It is possible to show that  $IS$  is mathematically equivalent to the cosine measure for binary variables (see Equation 2.7 on page 75). In this regard, we

**Table 6.10.** Example of a contingency table for items  $p$  and  $q$ .

|           | $q$ | $\bar{q}$ |      |
|-----------|-----|-----------|------|
| $p$       | 800 | 100       | 900  |
| $\bar{p}$ | 100 | 0         | 100  |
|           | 900 | 100       | 1000 |

consider  $A$  and  $B$  as a pair of bit vectors,  $A \cdot B = s(A, B)$  the dot product between the vectors, and  $|A| = \sqrt{s(A)}$  the magnitude of vector  $A$ . Therefore:

$$IS(A, B) = \frac{s(A, B)}{\sqrt{s(A) \times s(B)}} = \frac{A \cdot B}{|A| \times |B|} = \cosine(A, B). \quad (6.10)$$

The  $IS$  measure can also be expressed as the geometric mean between the confidence of association rules extracted from a pair of binary variables:

$$IS(A, B) = \frac{s(A, B)}{s(A)} \times \frac{s(A, B)}{s(B)} = \quad (6.11)$$

Because the geometric mean between any two numbers is always closer to the smaller number, the  $IS$  value of an itemset  $\{p, q\}$  is low whenever one of its rules,  $p \rightarrow q$  or  $q \rightarrow p$ , has low confidence.

**Limitations of IS Measure** The  $IS$  value for a pair of independent itemset,  $A$  and  $B$ , is

$$IS_{\text{indep}}(A, B) =$$

Since the value depends on  $s(A)$  and  $s(B)$ ,  $IS$  shares a similar problem as the confidence measure—that the value of the measure can be quite large, even for uncorrelated and negatively correlated patterns. For example, despite the large  $IS$  value between items  $p$  and  $q$  given in Table 6.10 (0.889), it is still less than the expected value when the items are statistically independent ( $IS_{\text{indep}} = 0.9$ ).

### Alternative Objective Interestingness Measures

Besides the measures we have described so far, there are other alternative measures proposed for analyzing relationships between pairs of binary variables. These measures can be divided into two categories, symmetric and asymmetric measures. A measure  $M$  is symmetric if  $M(A \rightarrow B) = M(B \rightarrow A)$ . For example, interest factor is a symmetric measure because its value is identical for the rules  $A \rightarrow B$  and  $B \rightarrow A$ . In contrast, confidence is an asymmetric measure since the confidence for  $A \rightarrow B$  and  $B \rightarrow A$  may not be the same. Symmetric measures are generally used for evaluating itemset, while asymmetric measures are more suitable for analyzing association rules. Tables 6.11 and 6.12 provide the definitions for some of these measures in terms of the frequency counts of a  $2 \times 2$  contingency table.

### Consistency among Objective Measures

Given the wide variety of measures available, it is reasonable to question whether the measures can produce similar ordering results when applied to a set of association patterns. If the measures are consistent, then we can choose any one of them as our evaluation metric. Otherwise, it is important to understand what their differences are in order to determine which measure is more suitable for analyzing certain types of patterns.

**Table 6.11.** Examples of symmetric objective measures for the itemset  $\{A, B\}$ .

| Measure (Symbol)            | Definition                                                                                                               |
|-----------------------------|--------------------------------------------------------------------------------------------------------------------------|
| Correlation ( $\phi$ )      | $\frac{Nf_{11} - f_{1+}f_{+1}}{\sqrt{f_{1+}f_{+1}f_{0+}f_{+0}}}$                                                         |
| Odds ratio ( $a$ )          | $\frac{f_{11}f_{00}}{f_{10}f_{01}}$                                                                                      |
| Kappa ( $\kappa$ )          | $\frac{Nf_{11} + Nf_{00} - f_{1+}f_{+1} - f_{0+}f_{+0}}{N^2 - f_{1+}f_{+1} - f_{0+}f_{+0}}$                              |
| Interest ( $I$ )            | $Nf_{11}, f_{1+}f_{+1}$                                                                                                  |
| Cosine ( $IS$ )             | $\frac{f_{11}}{\sqrt{f_{1+}f_{+1}}}$                                                                                     |
| Piatetsky-Shapiro ( $PS$ )  | $\frac{f_{11}}{N} - \frac{f_{1+}f_{+1}}{N^2}$                                                                            |
| Collective strength ( $S$ ) | $\frac{f_{11} + f_{00}}{f_{1+}f_{+1} + f_{0+}f_{+0}} \times \frac{N - f_{1+}f_{+1} - f_{0+}f_{+0}}{N - f_{11} - f_{00}}$ |
| Jaccard ( $\mathcal{J}$ )   | $\frac{f_{11}}{f_{1+} + f_{+1} - f_{11}}$                                                                                |
| All-confidence ( $h$ )      | $\min \frac{f_{11}}{f_{1+}}, \frac{f_{11}}{f_{+1}}$                                                                      |

**Table 6.12.** Examples of asymmetric objective measures for the rule  $A \rightarrow B$ .

| Measure (Symbol)              | Definition                                                                                                                                                                                                                                                                                                      |
|-------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Goodman-Kruskal ( $\lambda$ ) | $\frac{2}{\sum_j \max_k f_{jk} - \max_k f_{+k}} \cdot \frac{N - \max_k f_{+k}}{N}$                                                                                                                                                                                                                              |
| Mutual Information ( $M$ )    | $\sum_i \sum_j \frac{f_{ij}}{N} \log \frac{Nf_{ij}}{f_{i+}f_{+j}} - \sum_i \frac{f_{i+}}{N} \log \frac{f_{i+}}{N}$                                                                                                                                                                                              |
| J-Measure ( $J$ )             | $\frac{f_{11}}{N} \log \frac{Nf_{11}}{f_{1+}f_{+1}} + \frac{f_{10}}{N} \log \frac{Nf_{10}}{f_{1+}f_{+0}}$                                                                                                                                                                                                       |
| Gini index ( $G$ )            | $\frac{f_{1+}}{N} \times \left( \frac{f_{11}}{f_{1+}} \right)^2 + \left( \frac{f_{10}}{f_{1+}} \right)^2 - \left( \frac{f_{+1}}{N} \right)^2$<br>$+ \frac{f_{0+}}{N} \times \left[ \left( \frac{f_{01}}{f_{0+}} \right)^2 + \left( \frac{f_{00}}{f_{0+}} \right)^2 \right] - \left( \frac{f_{+0}}{N} \right)^2$ |
| Laplace ( $L$ )               | $\frac{f_{11} + 1}{f_{1+} + 2} \cdot \frac{f_{+1} + 2}{N + 2}$                                                                                                                                                                                                                                                  |
| Conviction ( $V$ )            | $\frac{f_{1+}f_{+0}}{Nf_{10}}$                                                                                                                                                                                                                                                                                  |
| Certainty factor ( $F$ )      | $\frac{\frac{f_{11}}{f_{1+}} - \frac{f_{+1}}{N}}{\frac{f_{11}}{f_{1+}} - \frac{f_{+1}}{N}} \cdot 1 - \frac{f_{+1}}{N}$                                                                                                                                                                                          |
| Added Value ( $AV$ )          | $\frac{f_{11}}{f_{1+}} - \frac{f_{+1}}{N}$                                                                                                                                                                                                                                                                      |

**Table 6.13.** Example of contingency tables.

| Example  | $f_{11}$ | $f_{10}$ | $f_{01}$ | $f_{00}$ |
|----------|----------|----------|----------|----------|
| $E_1$    | 8123     | 83       | 424      | 1370     |
| $E_2$    | 8330     | 2        | 622      | 1046     |
| $E_3$    | 3954     | 3080     | 5        | 2961     |
| $E_4$    | 2886     | 1363     | 1320     | 4431     |
| $E_5$    | 1500     | 2000     | 500      | 6000     |
| $E_6$    | 4000     | 2000     | 1000     | 3000     |
| $E_7$    | 9481     | 298      | 127      | 94       |
| $E_8$    | 4000     | 2000     | 2000     | 2000     |
| $E_9$    | 7450     | 2483     | 4        | 63       |
| $E_{10}$ | 61       | 2483     | 4        | 7452     |

Suppose the symmetric and asymmetric measures are applied to rank the ten contingency tables shown in Table 6.13. These contingency tables are chosen to illustrate the differences among the existing measures. The ordering produced by these measures are shown in Tables 6.14 and 6.15, respectively (with 1 as the most interesting and 10 as the least interesting table). Although some of the measures appear to be consistent with each other, there are certain measures that produce quite different ordering results. For example, the rankings given by the  $\phi$ -coefficient agree with those provided by  $\kappa$  and collective strength, but are somewhat different than the rankings produced by interest

**Table 6.14.** Rankings of contingency tables using the symmetric measures given in Table 6.11.

|          | $\varphi$ | $a$ | $\kappa$ | $I$ | $IS$ | $PS$ | $S$ | $\zeta$ | $h$ |
|----------|-----------|-----|----------|-----|------|------|-----|---------|-----|
| $E_1$    | 1         | 3   | 1        | 6   | 2    | 2    | 1   | 2       | 2   |
| $E_2$    | 2         | 1   | 2        | 7   | 3    | 5    | 2   | 3       | 3   |
| $E_3$    | 3         | 2   | 4        | 4   | 5    | 1    | 3   | 6       | 8   |
| $E_4$    | 4         | 8   | 3        | 3   | 7    | 3    | 4   | 7       | 5   |
| $E_5$    | 5         | 7   | 6        | 2   | 9    | 6    | 6   | 9       | 9   |
| $E_6$    | 6         | 9   | 5        | 5   | 6    | 4    | 5   | 5       | 7   |
| $E_7$    | 7         | 6   | 7        | 9   | 1    | 8    | 7   | 1       | 1   |
| $E_8$    | 8         | 10  | 8        | 8   | 8    | 7    | 8   | 8       | 7   |
| $E_9$    | 9         | 4   | 9        | 10  | 4    | 9    | 9   | 4       | 4   |
| $E_{10}$ | 10        | 5   | 10       | 1   | 10   | 10   | 10  | 10      | 10  |

**Table 6.15.** Rankings of contingency tables using the asymmetric measures given in Table 6.12.

|          | $\lambda$ | $M$ | $J$ | $G$ | $L$ | $V$ | $F$ | $AV$ |
|----------|-----------|-----|-----|-----|-----|-----|-----|------|
| $E_1$    | 1         | 1   | 1   | 1   | 4   | 2   | 2   | 5    |
| $E_2$    | 2         | 2   | 2   | 3   | 5   | 1   | 1   | 6    |
| $E_3$    | 5         | 3   | 5   | 2   | 2   | 6   | 6   | 4    |
| $E_4$    | 4         | 6   | 3   | 4   | 9   | 3   | 3   | 1    |
| $E_5$    | 9         | 7   | 4   | 6   | 8   | 5   | 5   | 2    |
| $E_6$    | 3         | 8   | 6   | 5   | 7   | 4   | 4   | 3    |
| $E_7$    | 7         | 5   | 9   | 8   | 3   | 7   | 7   | 9    |
| $E_8$    | 8         | 9   | 7   | 7   | 10  | 8   | 8   | 7    |
| $E_9$    | 6         | 4   | 10  | 9   | 1   | 9   | 9   | 10   |
| $E_{10}$ | 10        | 10  | 8   | 10  | 6   | 10  | 10  | 8    |

factor and odds ratio. Furthermore, a contingency table such as  $E_{10}$  is ranked lowest according to the  $\varphi$ -coefficient, but highest according to interest factor.

#### Properties of Objective Measures

The results shown in Table 6.14 suggest that a significant number of the measures provide conflicting information about the quality of a pattern. To understand their differences, we need to examine the properties of these measures.

**Inversion Property** Consider the bit vectors shown in Figure 6.28. The 0/1 bit in each column vector indicates whether a transaction (row) contains a particular item (column). For example, the vector A indicates that item  $a$



| A | B | C | D | E | F |
|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 |

(a) (b) (c)

**Figure 6.28.** Effect of the inversion operation. The vectors *C* and *E* are inversions of vector *A*, while the vector *D* is an inversion of vectors *B* and *F*.

belongs to the first and last transactions, whereas the vector *B* indicates that item *b* is contained only in the fifth transaction. The vectors *C* and *E* are in fact related to the vector *A*—their bits have been inverted from 0's (absence) to 1's (presence), and vice versa. Similarly, *D* is related to vectors *B* and *F* by inverting their bits. The process of flipping a bit vector is called inversion. If a measure is invariant under the inversion operation, then its value for the vector pair (*C*, *D*) should be identical to its value for (*A*, *B*). The inversion property of a measure can be tested as follows.

**Definition 6.6 (Inversion Property).** An objective measure *M* is invariant under the inversion operation if its value remains the same when exchanging the frequency counts  $f_{11}$  with  $f_{00}$  and  $f_{10}$  with  $f_{01}$ .

Among the measures that remain invariant under this operation include the  $\phi$ -coefficient, odds ratio,  $\kappa$ , and collective strength. These measures may not be suitable for analyzing asymmetric binary data. For example, the  $\phi$ -coefficient between *C* and *D* is identical to the  $\phi$ -coefficient between *A* and *B*, even though items *c* and *d* appear together more frequently than *a* and *b*. Furthermore, the  $\phi$ -coefficient between *C* and *D* is less than that between *E* and *F* even though items *e* and *f* appear together only once! We had previously raised this issue when discussing the limitations of the  $\phi$ -coefficient on page

375. For asymmetric binary data, measures that do not remain invariant under the inversion operation are preferred. Some of the non-invariant measures include interest factor, *IS*, *PS*, and the Jaccard coefficient.

**Null Addition Property** Suppose we are interested in analyzing the relationship between a pair of words, such as *data* and *mining*, in a set of documents. If a collection of articles about ice fishing is added to the data set, should the association between *data* and *mining* be affected? This process of adding unrelated data (in this case, documents) to a given data set is known as the null addition operation.

**Definition 6.7 (Null Addition Property).** An objective measure *M* is invariant under the null addition operation if it is not affected by increasing  $f_{00}$ , while all other frequencies in the contingency table stay the same.

For applications such as document analysis or market basket analysis, the measure is expected to remain invariant under the null addition operation. Otherwise, the relationship between words may disappear simply by adding enough documents that do not contain both words! Examples of measures that satisfy this property include cosine (*IS*) and Jaccard ( $\mathcal{J}$ ) measures, while those that violate this property include interest factor, *PS*, odds ratio, and the  $\phi$ -coefficient.



|           | $B$ | $\bar{B}$ |      |
|-----------|-----|-----------|------|
| $A$       | 700 | 100       | 800  |
| $\bar{A}$ | 100 | 100       | 200  |
|           | 800 | 200       | 1000 |

|           | $B$ | $\bar{B}$ |      |
|-----------|-----|-----------|------|
| $A$       | 700 | 100       | 800  |
| $\bar{A}$ | 110 | 1100      | 1200 |
|           | 800 | 1200      | 2000 |

no-changes  
null values

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**Scaling Property** Table shows the contingency tables for gender and the grades achieved by students enrolled in a particular course in 1993 and 2004. The data in these tables showed that the number of male students has doubled since 1993, while the number of female students has increased by a factor of 3. However, the male students in 2004 are not performing any better than those in 1993 because the ratio of male students who achieve a high grade to those who achieve a low grade is still the same, i.e., 3:4. Similarly, the female students in 2004 are performing no better than those in 1993. The association between grade and gender is expected to remain **unchanged** despite changes in the sampling distribution.

**Table 6.16.** The grade-gender example.

|      | Male | Female |     |
|------|------|--------|-----|
| High | 30   | 20     | 50  |
| Low  | 40   | 10     | 50  |
|      | 70   | 30     | 100 |

|      | Male | Female |     |
|------|------|--------|-----|
| High | 60   | 60     | 120 |
| Low  | 80   | 30     | 110 |
|      | 140  | 90     | 230 |

(a) Sample data from 1993.

(b) Sample data from 2004.

**Table 6.17.** Properties of symmetric measures.

| Symbol    | Measure                | Inversion | Null Addition | Scaling |
|-----------|------------------------|-----------|---------------|---------|
| $\varphi$ | $\varphi$ -coefficient | Yes       | No            | No      |
| $a$       | odds ratio             | Yes       | No            | Yes     |
| $\kappa$  | Cohen's                | Yes       | No            | No      |
| $I$       | Interest               | No        | No            | No      |
| $IS$      | Cosine                 | No        | Yes           | No      |
| $PS$      | Piatetsky-Shapiro's    | Yes       | No            | No      |
| $S$       | Collective strength    | Yes       | No            | No      |
| $\zeta$   | Jaccard                | No        | Yes           | No      |
| $h$       | All-confidence         | No        | No            | No      |
| $s$       | Support                | No        | No            | No      |

**Definition 6.8 (Scaling Invariance Property).** An objective measure  $M$  is invariant under the row/column scaling operation if  $M(T) = M(T')$ , where  $T$  is a contingency table with frequency counts  $[f_{11}; f_{10}; f_{01}; f_{00}]$ ,  $T'$  is a contingency table with scaled frequency counts  $[k_1 k_3 f_{11}; k_2 k_3 f_{10}; k_1 k_4 f_{01}; k_2 k_4 f_{00}]$ , and  $k_1, k_2, k_3, k_4$  are positive constants.

From Table 6.17, notice that only the odds ratio ( $a$ ) is invariant under the row and column scaling operations. All other measures such as the  $\varphi$ -coefficient,  $\kappa$ ,  $IS$ , interest factor, and collective strength ( $S$ ) change their values when the rows and columns of the contingency table are rescaled. Although we do not discuss the properties of asymmetric measures (such as confidence, J-measure, Gini index, and conviction), it is clear that such measures do not preserve their values under inversion and row/column scaling operations, but are invariant under the null addition operation.

### Measures beyond Pairs of Binary Variables

The measures shown in Tables are defined for pairs of binary variables). However, many of them, such as support and all-confidence, are also applicable to larger-sized itemsets. Other measures, such as interest factor,  $IS$ ,  $PS$ , and Jaccard coefficient, can be extended to more than two variables using the frequency tables tabulated in a multidimensional contingency table. An example of a three-dimensional contingency table for  $a$ ,  $b$ , and  $c$  is shown in Table. Each entry  $f_{ijk}$  in this table represents the number of transactions that contain a particular combination of items  $a$ ,  $b$ , and  $c$ . For example,  $f_{101}$  is the number of transactions that contain  $a$  and  $c$ , but not  $b$ . On the other hand, a marginal frequency

**Table 6.18.** Example of a three-dimensional contingency table.

| $c$       | $b$       | $\bar{b}$ |           | $\bar{c}$ | $b$       | $\bar{b}$ |           |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $a$       | $f_{111}$ | $f_{101}$ | $f_{1+1}$ | $a$       | $f_{110}$ | $f_{100}$ | $f_{1+0}$ |
| $\bar{a}$ | $f_{011}$ | $f_{001}$ | $f_{0+1}$ | $\bar{a}$ | $f_{010}$ | $f_{000}$ | $f_{0+0}$ |
|           | $f_{+11}$ | $f_{+01}$ | $f_{++1}$ |           | $f_{+10}$ | $f_{+00}$ | $f_{++0}$ |

such as  $f_{1+1}$  is the number of transactions that contain  $a$  and  $c$ , irrespective of whether  $b$  is present in the transaction.

Given a  $k$ -itemset  $\{i_1, i_2, \dots, i_k\}$ , the condition for statistical independence can be stated as follows:

$$f_{i_1 i_2 \dots i_k} = \frac{f_{i_1+ \dots +} \times f_{+i_2 \dots +} \times \dots \times f_{++ \dots i_k}}{N^{k-1}}, \quad (6.12)$$

With this definition, we can extend objective measures such as interest factor and  $PS$ , which are based on deviations from statistical independence, to more than two variables:

$$I = \frac{N^{k-1} \times f_{i_1 i_2 \dots i_k}}{f_{i_1+ \dots +} \times f_{+i_2 \dots +} \times \dots \times f_{++ \dots i_k}}$$

$$PS = \frac{f_{i_1 i_2 \dots i_k}}{N} - \frac{f_{i_1+ \dots +} \times f_{+i_2 \dots +} \times \dots \times f_{++ \dots i_k}}{N^k}$$

Another approach is to define the objective measure as the maximum, minimum, or average value for the associations between pairs of items in a pattern. For example, given a  $k$ -itemset  $X = \{i_1, i_2, \dots, i_k\}$ , we may define the  $\varphi$ -coefficient for  $X$  as the average  $\varphi$ -coefficient between every pair of items  $(i_p, i_q)$  in  $X$ . However, because the measure considers only pairwise associations, it may not capture all the underlying relationships within a pattern.

Analysis of multidimensional contingency tables is more complicated because of the presence of partial associations in the data. For example, some associations may appear or disappear when conditioned upon the value of certain variables. This problem is known as Simpson's paradox and is described in the next section. More sophisticated statistical techniques are available to analyze such relationships, e.g., loglinear models, but these techniques are beyond the scope of this book.

**Table 6.19.** A two-way contingency table between the sale of high-definition television and exercise machine.

| Buy HDTV | Buy Exercise Machine |     |     |
|----------|----------------------|-----|-----|
|          | Yes                  | No  |     |
| Yes      | 99                   | 81  | 180 |
| No       | 54                   | 66  | 120 |
|          | 153                  | 147 | 300 |

**Table 6.20.** Example of a three-way contingency table.

| Customer Group   | Buy HDTV | Buy Exercise Machine |    | Total |
|------------------|----------|----------------------|----|-------|
|                  |          | Yes                  | No |       |
| College Students | Yes      | 1                    | 9  | 10    |
|                  | No       | 4                    | 30 | 34    |

|               |     |    |    |     |
|---------------|-----|----|----|-----|
| Working Adult | Yes | 98 | 72 | 170 |
|               | No  | 50 | 36 | 86  |

### Simpson's Paradox

It is important to exercise caution when interpreting the association between variables because the observed relationship may be influenced by the presence of other confounding factors, i.e., hidden variables that are not included in the analysis. In some cases, the hidden variables may cause the observed relationship between a pair of variables to disappear or reverse its direction, a phenomenon that is known as Simpson's paradox. We illustrate the nature of this paradox with the following example.

|        | COMPANY A       | COMPANY B       |
|--------|-----------------|-----------------|
| Test 1 | $63/90 = 70\%$  | $8/10 = 80\%$   |
| Test 2 | $4/10 = 40\%$   | $45/90 = 50\%$  |
|        | $67/100 = 67\%$ | $53/100 = 53\%$ |