Well-posedness for fractional time-space generalized Keller-Segel equation

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Abstract

ABSTRACT. In this paper, using the Mittag-Leffler function and Bessel function, we first construct the formula of the fundamental solution $P_{\alpha}^{\beta}(x,t)$ to the time-space Homogenous Fractional Diffusion Equation(HFDE) ${}_0^cD_t^{\beta}\rho+(-\triangle)^{\frac{\alpha}{2}}\rho=0$ where Caputo derivative ${}_0^cD_t^{\beta}\rho$ provides the memory effect of the past and $(-\triangle)^{\frac{\alpha}{2}}\rho$ represents the Lévy diffusion and study the asymptotic behavior of $P_{\alpha}^{\beta}(x,t)$ applying some equalities related to Bessel functions. Then, we obtain L^p estimate and weighted estimate of the solution to HFDE. As the application of the above estimates, we prove the existence of a class of time-space Fractional Diffusion Equation ${}_0^cD_t^{\beta}\rho+(-\triangle)^{\frac{\alpha}{2}}\rho+\nabla\cdot(\rho B(\rho))=0$, a generalization of a class of generalized Keller-Segel equation, here $B(\rho)=-s_{n,\gamma}\int_{R^n}\frac{x-y}{|x-y|^{n-\gamma+2}}\rho(y)dy$ is the general potential with a singular kernel which takes into account long rang interaction. Furthermore, we investigate the nonnegativity, mass conservation and finite time blowup of such mild solution.

Keywords: time-space fractional diffusion equation, mild solution, existence, finite time blow up.

1 Introduction

In this paper, we study the existence and finite time blowup of the time-space fractional diffusion equations. Fractional derivatives are integro-differential operators generalizing

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the definition of integer orders derivative to fractional orders. Recently, there are plenty of papers dealing with fractional derivative in numerous application areas such as physics, finance, hydrology, biomedical engineering, information theory, control theory, and so on [4, 20, 25, 29, 34, 37, 38, 43, 30, 36, 35]. [36] pointed out that standard diffusion paradigm relies on the restrictive assumptions including locality in space and time, Gaussianity and lack of long-rang correlations and breaks down in some particular examples, for instance, the confinement time scaling in low confinement mode plasmas, fast propagation and nonlocal transport phenomena, and the non-Gaussianity and long-range correlations of fluctuations. In order to overcome these restrictions of the standard diffusion paradigm, the authors proposed a model based on the use of fractional derivative operators which naturally appear in the macroscopic description of transport and is an important tool to describe the nonlocal effects in space and time. [35] demonstrated that fractional space derivative can be used to describe anomalous diffusion or dispersion when a particle plume, may be asymmetric, spreads at a rate inconsistent with the Brownian motion models. The appearance of the fractional space derivative in diffusion equation will lead to superdiffusion phenomenon. Fractional time derivative is connected with anomalous subdiffusion, where a cloud of particles spreads more slowly than a classical diffusion. [33] provided us that, in some real problems, the waiting times between particle jumps can not be adequately described by a single power law. Thus, a waiting time model, conditional power law, leads to a distributed-order fractional derivative in time. [15] showed that the time fractional diffusion equation describes the anomalous diffusion which shows the subdiffusive behavior caused by particle sticking and trapping phenomena.

In this paper, we consider the following nonlinear nonlocal time-space fractional equation which generalizes the well known Keller-Segel model

$$\begin{cases} {}_0^cD_t^\beta \rho + (-\triangle)^{\frac{\alpha}{2}}\rho + \nabla\cdot(\rho B(\rho)) = 0 \quad \text{in } (x,t)\in\mathbb{R}^n\times(0,\infty), \\ \rho(x,0) = \rho_0(x), \end{cases} \tag{1.1}$$

where $0<\beta<1, 1<\alpha\le 2$. ${}_0^cD_t^\beta$ is Caputo fractional derivative operator of order β and defined by

$$ {}_0^cD_t^\beta v(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\partial_s v(s)ds, \tag{1.2}$$

where Γ is the Gamma function and $\partial_t u(t)$ is the first order integer derivative of function $u(t)$ with respect to its independent variable t. According to Chapter V in [39], the nonlocal operator $(-\triangle)^{\frac{\alpha}{2}}$, known as the Laplacian of order $\frac{\alpha}{2}$, is given by means of the Fourier multiplier

$$D^\alpha \rho(x) \equiv (-\triangle)^{\frac{\alpha}{2}} \rho(x) = \mathfrak{F}^{-1}\big(|\xi|^\alpha \hat{\rho}(\xi)\big)(x).$$

where $\hat{\rho}(\xi) = \mathfrak{F}\big(\rho(x)\big)$ is the Fourier transformation of $\rho(x)$. For $\gamma \in (1, n]$, $n \geq 2$ and some constant $s_{n,\gamma} > 0$, the linear vector operator B can be formally represented as $B(u) = \nabla((-\triangle)^{-\frac{\gamma}{2}} u)$ and is explicitly expressed with convolution of a singular kernel as follows

$$B(u)(x) = -s_{n,\gamma} \int_{R^n} \frac{x-y}{|x-y|^{n-\gamma+2}} u(y)dy, \tag{1.3}$$

which is the attractive kernel as [26] pointed out.

This model is developed to describe the biological phenomenon chemotaxis not only with Lévy diffusion or anomalous diffusion but also the memory effect of the system. In the context of biological aggregation, $\rho(x,t)$ represents the density of some biology cells. Formaly, (1.1) is just to replace the time derivative of the classcial genenralized Keller-Segel model [6] with the Caputo fractional derivative. However, when the memory effect of the system is concerned, the nonlocality in time is incorporated in the fractional time derivative.

There are also many papers dealing with the Keller-Segel models with fractional laplacian operator. In [28], the uniqueness and stability under Wasserstein metric of classical Keller-Segel equation have been proved by connected it with a self-consistent stochastic process driven by Brownian motion. With the similar approach, [23] discussed the uniqueness and stability for nonlocal Keller-Segel equation by considering a self-consistent stochastic process driven by rotationally invariant α-stable Lévy process. Also, the comprehensive review of recent progress in terms of the theory of fractional laplacian operator can be found in [41, 42].

For the generalzied Keller-Segel equation [7], there are many results involved in. When $\gamma = 2$, Escudero [16] constructed the global in time solutions not only for the classical Brownian diffusion $\alpha = 2$ but also for the fractional diffusion $1 < \alpha < 2$. When $\alpha = 2, \gamma = 2$, Jäger and Luckhaus [24] analysed the global existence and explosion of the bacteria concentration in finite time in the bounded domain of two dimensional space. Blanchet, Dolbeault and Perthame [9] established the proof of the existence of weak solutions with subcritical mass without any symmetry assumption and the blow-up in finite time in the whole Euclidean space. In two dimensional space, $\gamma = 2$, Biler and Wu [5] investigated the cauchy problem with initial data u_0 in critical Besov spaces $\dot{B}^{1-\alpha}_{2,r}(\mathbb{R}^2)$ for $r \in [1, \infty]$ and

$1 < \alpha < 2$. Also in two dimensional space with $\gamma = 2$, Li, Rodrigo and Zhang [26] proved the wellposedness, continuation criteria and smoothness of local solutions. Biler and Karch [7] studied the existence and nonexistence of global in time mild solutions. They constructed the local mild solutions in $C([0,T], L^p(\mathbb{R}^n))$, for every $\max(\frac{n}{\alpha+\gamma-2}, \frac{2n}{n+\gamma-1}) < p \le n$, and global mild solution in

$$C([0,\infty), L^{\frac{n}{\alpha+\gamma-2}}(\mathbb{R}^n)) \cap \{u \in C((0,\infty), L^p(\mathbb{R}^n)) : \sup_{t>0} t^{\frac{n}{\alpha}(\frac{1}{p}-\frac{\alpha+\gamma-2}{n})} \|u(t)\|_p < \infty\}.$$

The rest of this paper is organized in four sections. In Section 2, we recall some notations, definitions and some known results needed later. In Section 3, we construct the fundamental solution to the corresponding homogenous equation of (1.1) and study the asymptotic behavior of this solution. After that, we obtain the L^p estimate and certain weighted estimate of the solution to homogenous equation. Section 4 is devoted to construction of the local, global and weighted existence of the mild solutions to inhomogenous equation (1.1) by applying the results derived in Section 3. Also, we provide the proof of the nonnegativity and mass conservation to the mild solution. In addition, the finite time blow up of mild solution to (1.1) is concerned in Section 5.

2 Notation and Preliminaries

In this section, we first introduce some special spaces and some special functions related to the Caputo derivative. Then we recall some important results which will be used later.

For $1 < p < \infty$, we use $\|u\|_p$ to denote the L^p-norm of a Lebesgue measurable function u in $L^p(\mathbb{R}^n)$ space. And $C[a,b]$ is the space of the continuous functions in the interval $[a,b]$.

For fix $\nu \ge 0$, we define weighted $L^\infty_\nu(\mathbb{R}^n)$ spaces as follows

$$L^\infty_\nu(\mathbb{R}^n) = \{v \in L^\infty(\mathbb{R}^n) : \|v\|_{L^\infty_\nu} \equiv \operatorname{ess}\sup_{x\in\mathbb{R}^n} (1+|x|)^\nu |v(x)| < \infty\}, \tag{2.1}$$

and its homogeneous space is

$$\dot{L}^\infty_\nu(\mathbb{R}^n) = \{v \in L^\infty(\mathbb{R}^n) : \|v\|_{\dot{L}^\infty_\nu} \equiv \operatorname{ess}\sup_{x\in\mathbb{R}^n} |x|^\nu |v(x)| < \infty\}. \tag{2.2}$$

Denote $X = C([0,T]; L^p(\mathbb{R}^n))$ to be supplemented with the usual norm $\|u\|_X \equiv \sup_{t\in[0,T]} \|u(t)\|_p$.

Denote

$$\tilde{X} = \{u \in C([0,\infty), L^{\frac{n}{\alpha+\gamma-2}}(\mathbb{R}^n)) : \sup_{t>0} t^{1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma)} \|u\|_p < \infty\}.$$

Define the Banach space

$$\hat{X} = C([0, \infty), L^{\frac{n}{\alpha+\gamma-2}}(\mathbb{R}^n)) \cap \tilde{X},$$

supplemented with the norm

$$\|u\|_{\hat{X}} \equiv \sup_{t>0} \|u(t)\|_{\frac{n}{\alpha+\gamma-2}} + \sup_{t>0} t^{1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma)} \|u\|_p.$$

In addition, we introduce the space $X_T = C([0, T], L^{\infty}_{\alpha+n}(\mathbb{R}^n))$ with the norm $\|u\|_{X_T} \equiv ess\,\sup_{x\in\mathbb{R}^n}(1 + |x|)^{\alpha+n|u|}$.

Then, we introduce the definitions of some special functions and some related identities.

For $\beta > 0, t \in [0, T], f(t) \in L^1[0, T]$, J_0^α, the Riemann-Liouville fractional integral operator of order β, is defined by

$$J_0^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds. \tag{2.3}$$

And if f is continuous, Theorem 3.7 in [13] gives that for $0 < \beta < 1$,

$${}_0^c D_t^\beta J_0^\beta f = f. \tag{2.4}$$

For complex number z and real number β, the Mittag-Leffler function $E_\beta(z)$ is defined by

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)} \tag{2.5}$$

and if $\beta > 0$, it is an entire function. For $s > 0$, as derived in (A.37) of [32], the Mittag-Leffler function can also be represented as

$$E_\beta(-s) = \int_0^\infty M_\beta(r) e^{-rs} dr, \tag{2.6}$$

where $M_\beta(r)$ is a special function of Wright type introduced in the Appendix A of [32] and for $\delta \geq 0$, the following identity holds

$$\int_0^{+\infty} r^\delta M_\nu(r) dr = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+1)}. \tag{2.7}$$

For $0 < \beta < 2$, the asymptotic expansion of $E_\beta(z)$ is

$$E_\beta(z) = \frac{1}{\beta} e^{z^{\frac{1}{\beta}}} - \sum_{r=1}^{N*} \frac{1}{\Gamma(1-\beta r)} \frac{1}{z^r} + o[\frac{1}{z^{N*+1}}], z \to \infty. \tag{2.8}$$

where $N* \in \mathbb{N}, N* \neq 1$ [21]. For $\nu > 0$, as in [18], we define the Bessel function of first kind of order ν to be

$$J_\nu(z) = (\frac{z}{2})^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)}(\frac{z}{2})^{2k}, \tag{2.9}$$

and it is an entire function for all z on the complex plane. For real number z, the asymptotic expansion of $J_\nu(z)$ (9.2.1 in [1], p364) is

$$J_\nu(z) = (\frac{2}{\pi z})^{\frac{1}{2}} \cos(z - \frac{\pi\nu}{2} - \frac{\pi}{4}) + o(\frac{1}{z}), z \to \infty. \tag{2.10}$$

In addition, we introduce

$$Y_\nu(z) = (\sin(\nu\pi))^{-1}[J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)],$$

$$I_\nu(z) = e^{\frac{-i\nu\pi}{2}} J_\nu(ze^{\frac{i\pi}{2}}),$$

to be the Bessel function of the second kind, Modified Bessel function of the first kind, respectively. Furthermore, we define

$$K_\nu(z) = \frac{\pi}{2}(\sin(\nu\pi))^{-1}[I_{-\nu}(z) - I_\nu(z)], \tag{2.11}$$

to be the Modified Bessel function of the third kind and define the Bessel function of the third kind to be

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z). \tag{2.12}$$

Equation (15) on page 5 of [18] gives the following formula which reflects the relation between the Modified Bessel function of the third kind (2.11) and the Bessel function of the third kind (2.12)

$$K_\nu(z) = \frac{i\pi}{2} e^{\frac{i\nu\pi}{2}} H_\nu^{(1)}(ze^{\frac{i\pi}{2}}). \tag{2.13}$$

Denote $\mathrm{Re}z$ to be the real part of complex number z. For $\mathrm{Re}(\mu \pm \nu) > 0, \mathrm{Re}\beta > 0$, one has

$$\int_0^\infty K_\nu(\beta t)t^{\mu-1}dt = 2^{\mu-2}\beta^{-\mu}\Gamma(\frac{\mu+\nu}{2})\Gamma(\frac{\mu-\nu}{2}), \tag{2.14}$$

which is the equality (27) in Page 51 of [18].

We recall some useful inequalities which will be used throughout the paper.

In order to obtain the estimate of the convolution operator $B(\rho)$, we need the following inequality.

Lemma 2.1[39] Let $0 < \ell < n$, $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\ell}{n}$. Then

$$\| \int_{R^n} \frac{f(y)}{|x-y|^{n-\ell}} dy \|_q \leq C\|f\|_p, \tag{2.15}$$

holds for all $f \in L^p(\mathbb{R}^n)$.

The following Lemma provides the main tool of the construction of the existence of the mild solution to (1.1).

Lemma 2.2[7] Assume $(X, \|\cdot\|_X)$ be a Banach space and $H : X \times X \to X$ be a bounded bilinear form such that for all $u_1, u_2 \in X$ and constant $\eta > 0$,

$$||H(u_1, u_2)||_X \leq \eta \|u_1||_X \|u_2\|_X. \tag{2.16}$$

If $0 < \epsilon < \frac{1}{4\eta}$ and $v \in X$ such that $\|v\|_X \leq \epsilon$, then the equation $u = v + H(u, u)$ has a solution in X satisfying $\|u\|_X \leq 2\epsilon$. In addition, this solution is the unique one in $\overline{B}(0, 2\epsilon)$.

The proof of the nonnegativity of the mild solutions to (1.1) is based on the Stroock-Varopoulo's inequality (Lemma 2.3) and Lemma 2.6 which is related to Caputo derivative.

Lemma 2.3[11] Let $0 < \frac{\alpha}{2} < 1$, $p > 1$. Then

$$-\int_{\mathbb{R}^n} |f|^{p-2} f D^\alpha f dx \leq -\frac{4(p-1)}{p^2} \|D^{\frac{\alpha}{2}} f^{\frac{p}{2}}\|_2^2, \tag{2.17}$$

for any $f \in L^p(\mathbb{R}^n)$ such that $D^\alpha f \in L^p(\mathbb{R}^n)$.

In order to construct the proof of Lemma 2.6, we need the following two lemmas in advance.

Lemma 2.4 Let $0 < \alpha < 1, T > 0$. Assume $X(t) \in C[0, T]$. Suppose for $t \in [0, T]$, $\int_0^t (t-s)^{\alpha-1} |f(s)| ds < \infty$. If $X(t)$ is the solution of equation

$${}_0^c D_t^\alpha X(t) = f(t), \tag{2.18}$$

then $X(t)$ satisfies the following integral equation

$$X(t) = X(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds. \tag{2.19}$$

Furthermore, if $f \in L^p[0, T], p > \frac{1}{\alpha}$, then for any $s, t \in [0, T], s < t$, we have

$$|X(t) - X(s)| \leq \frac{(t-s)^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)} \Big(\frac{p-1}{\alpha p - 1} \Big)^{1 - \frac{1}{p}} \|f\|_p. \tag{2.20}$$

Conversely, if $f(t) \in C[0, T]$, $X(t) \in C[0, T]$, and

$$X(t) = X(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

then

$${}_0^c D_t^\alpha X(t) = f(t).$$

Proof: Multiply both sides of (2.18) with $(t-s)^{\alpha-1}$ and integrate from 0 to t to obtain

$$\begin{aligned}\int_0^t (t-s)^{\alpha-1} f(s)ds &= \int_0^t (t-s)^{\alpha-1} ({}_0^c D_t^\alpha X(s))ds \\ &= \int_0^t (t-s)^{\alpha-1} \Big(\frac{1}{\Gamma(1-\alpha)} \int_0^s \frac{X'(\tau)}{(s-\tau)^\alpha} d\tau\Big) ds \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t X'(\tau)\Big(\int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{-\alpha} ds\Big) d\tau,\end{aligned} \tag{2.21}$$

here we use (1.2) and Fubini theorem. The substitution $s = \tau + \theta(t-\tau)$ yields

$$\begin{aligned}\int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{-\alpha} ds &= \int_0^1 (1-\theta)^{\alpha-1}\theta^{-\alpha} d\theta \\ &= B(1-\alpha, \alpha) \\ &= \Gamma(1-\alpha)\Gamma(\alpha).\end{aligned} \tag{2.22}$$

Put (2.22) into (2.21) to get

$$\int_0^t (t-s)^{\alpha-1} f(s)ds = \Gamma(\alpha) \int_0^t X'(\tau)d\tau = \Gamma(\alpha)\big(X(t) - X(0)\big). \tag{2.23}$$

(2.23) implies

$$X(t) = X(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds.$$

For $1 < p < \infty$, using (2.19), for $s < t, 0 < \alpha < 1$, one has

$$\begin{aligned}|X(t) - X(s)| &\le \frac{1}{\Gamma(\alpha)} \Big| \int_0^s \big((t-\tau)^{\alpha-1} - (s-\tau)^{\alpha-1}\big) f(\tau)d\tau + \int_s^t (t-\tau)^{\alpha-1} f(\tau)d\tau \Big| \\ &\le \frac{1}{\Gamma(\alpha)} \Big(\int_0^s \big((s-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1}\big)|f(\tau)|d\tau \\ &+ \int_s^t (t-\tau)^{\alpha-1}|f(\tau)|d\tau \Big) \\ &\equiv \frac{1}{\Gamma(\alpha)}(I_1 + I_2).\end{aligned} \tag{2.24}$$

Hölder's inequality implies that

$$I_1 \le \Big(\int_0^s \big((s-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1}\big)^q d\tau \Big)^{\frac{1}{q}} \|f\|_p, \tag{2.25}$$

here $\frac{1}{q} + \frac{1}{p} = 1$. Substitution $s - \tau = \xi(t-s)$ gives that

$$\begin{aligned}&\Big(\int_0^s \big((s-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1}\big)^q d\tau \Big)^{\frac{1}{q}} \\ &= \Big(\int_0^{\frac{s}{t-s}} (t-s)^{1+(\alpha-1)q} \big(\xi^{\alpha-1} - (1+\xi)^{\alpha-1}\big)^q d\xi \Big)^{\frac{1}{q}} \\ &\le (t-s)^{\frac{1}{q}+(\alpha-1)} \Big(\Big(\int_0^{\frac{T}{t-s}} \xi^{(\alpha-1)q} d\xi\Big)^{\frac{1}{q}} + \Big(\int_0^{\frac{T}{t-s}} (1+\xi)^{(\alpha-1)q} d\xi\Big)^{\frac{1}{q}} \Big)\end{aligned} \tag{2.26}$$

If $0<(1-\alpha)q<1$, there is $C>0$ such that

$$\int_0^1 \xi^{(\alpha-1)q}d\xi = \mathfrak{B}(1+(\alpha-1)q,1) = \frac{1}{1+(\alpha-1)q}. \tag{2.27}$$

In addition,

$$\int_1^{\frac{T}{t-s}} \xi^{(\alpha-1)q}d\xi = \frac{1}{1+(\alpha-1)q}\big((\frac{T}{t-s})^{1+(\alpha-1)q}-1\big). \tag{2.28}$$

(2.27) and (2.28) lead to

$$\begin{aligned}\int_0^{\frac{T}{t-s}} \xi^{(\alpha-1)q}d\xi &= \int_0^1 \xi^{(\alpha-1)q}d\xi + \int_1^{\frac{T}{t-s}} \xi^{(\alpha-1)q}d\xi \\ &= \frac{1}{(1+(\alpha-1)q)^{\frac{1}{q}}}(\frac{T}{t-s})^{\frac{1}{q}+(\alpha-1)}.\end{aligned} \tag{2.29}$$

Furthermore,

$$\begin{aligned}\Big(\int_0^{\frac{T}{t-s}}(1+\xi)^{(\alpha-1)q}d\xi\Big)^{\frac{1}{q}} &= \frac{1}{(1+(\alpha-1)q)^{\frac{1}{q}}}\Big((1+\frac{T}{t-s})^{1+(\alpha-1)q}-1\Big)^{\frac{1}{q}} \\ &\le \frac{1}{(1+(\alpha-1)q)^{\frac{1}{q}}}\Big((1+\frac{T}{t-s})^{1+(\alpha-1)q}+1\Big)^{\frac{1}{q}} \\ &\le C(1+\frac{T}{t-s})^{\alpha-1+\frac{1}{q}}.\end{aligned} \tag{2.30}$$

Put (2.29) and (2.30) into (2.26) to yield

$$\begin{aligned}\Big(\int_0^s \big((s-\tau)^{\alpha-1}-(t-\tau)^{\alpha-1}\big)^q d\tau\Big)^{\frac{1}{q}} &\le C(t-s)^{\alpha-\frac{1}{p}}(1+\frac{T}{t-s})^{\alpha-\frac{1}{p}} \\ &\le C(t-s+T)^{\alpha-\frac{1}{p}}\end{aligned} \tag{2.31}$$

Insert (2.31) into (2.25) to obtain

$$I_1 \le C(t-s+T)^{\alpha-\frac{1}{p}}\|f\|_p. \tag{2.32}$$

Substitution $t-\tau=\xi(t-s)$ implies

$$\begin{aligned}I_2 = \Big(\int_s^t (t-\tau)^{(\alpha-1)q}d\tau\Big)^{\frac{1}{q}} &= (t-s)^{(\alpha-1)+\frac{1}{q}}\Big(\int_0^1 \xi^{(\alpha-1)q}d\xi\Big)^{\frac{1}{q}} \\ &= (t-s)^{(\alpha-1)+\frac{1}{q}}\Big(\mathfrak{B}((\alpha-1)q+1,1)\Big)^{\frac{1}{q}} \\ &= (t-s)^{\alpha-\frac{1}{p}}\Big(\frac{p-1}{\alpha p-1}\Big)^{1-\frac{1}{p}}.\end{aligned} \tag{2.33}$$

Plug (2.32) and (2.33) into (2.24) to get

$$|X(t)-X(s)| \le C(t-s+T)^{\alpha-\frac{1}{p}}\|f\|_p. \tag{2.34}$$

If $p=\infty$, the same argument as above yields

$$\begin{aligned}|X(t)-X(s)| &\le \frac{1}{\Gamma(\alpha)}\Big(\int_0^s \big((s-\tau)^{\alpha-1}-(t-\tau)^{\alpha-1}\big)d\tau\|f\|_\infty+\int_s^t(t-\tau)^{\alpha-1}d\tau\|f\|_\infty\Big)\\ &\le \frac{1}{\Gamma(1+\alpha)}\Big(s^\alpha-t^\alpha+2(t-s)^\alpha\Big)\|f\|_\infty.\end{aligned} \tag{2.35}$$

(2.34) and (2.35) verify the estimate (2.20).
On the other hand, due to the definition of Riemann-Liouville fractional integral (2.3) and the equality (2.4), apply ${}_0^cD_t^\alpha$ to both sides of (2.19), we obtain (2.18).□

Remark 2.1 Lemma 2.4 is dedicated to the expression of the solution of fractional ordinary differential equation with Caputo derivative and the L^∞ estimate of difference of the solution.

The following inequality is the Lemma 7.1.1 in [22] related to Riemann-Liouville fractional integral.

Lemma 2.5 Suppose $b\ge 0, \beta>0$ and $a(t)$ is a nonnegative function locally integrable on $0\le t<T$(some $T\le+\infty$) and suppose $u(t)$ is nonnegative and locally integrable on $0\le t<T$ with

$$u(t)\le a(t)+b\int_0^t (t-s)^{\beta-1}u(s)ds \tag{2.36}$$

on this interval, then for any $0\le t<T$,

$$u(t)\le a(t)+\int_0^t\big[\sum_{n=1}^{\infty}\frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\big]ds. \tag{2.37}$$

The proof of the following Grönwall inequality in terms of Caputo derivative will play an important role in the proof of the nonnegativity, mass conservation and finite time blow up of the solution to (1.1).

Lemma 2.6 Assume $0\le X(t)\in C[0,T]$. ${}_0^cD_t^\alpha X(t)=f(t)$ and $f(t)\le a+bX(t), a,b\ge 0$. For any $t\in[0,T)$, $\int_0^t(t-s)^{\alpha-1}|f|ds<\infty$. Then, when $b=0$,

$$X(t)\le X(0)+\frac{at^\alpha}{\Gamma(\alpha+1)}. \tag{2.38}$$

When $b\neq 0$,

$$X(t)\le X(0)E_\alpha(bt^\alpha)+\frac{a}{b}\big(E_\alpha(bt^\alpha)-1\big). \tag{2.39}$$

In particular, if $a=b=0$, that is $f(t)\le 0$, and $X(t)=0$, then for any $t\in[0,T)$, $X(t)\equiv 0$.
Proof: Fix $t\in[0,T)$. By Lemma 2.4, one has

$$\begin{aligned}X(t)&=X(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds\\&\le X(0)+\frac{a}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds+\frac{b}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}X(s)ds\\&\le X(0)+\frac{at^\alpha}{\Gamma(\alpha+1)}+\frac{b}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}X(s)ds\end{aligned}\tag{2.40}$$

By Lemma 2.5 and (2.40), we have

$$\begin{aligned}X(t)&\le X(0)+\frac{at^\alpha}{\Gamma(\alpha+1)}+\int_0^t\big[\sum_{n=1}^\infty\frac{b^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}(X(0)+\frac{as^\alpha}{\Gamma(\alpha+1)})\big]ds\\&\le X(0)E_\alpha(bt^\alpha)+a\big(\frac{t^\alpha}{\Gamma(1+\alpha)}+\sum_{n=1}^\infty\frac{b^nt^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)}\big).\end{aligned}\tag{2.41}$$

When $b=0$, (2.41) becomes

$$X(t)\le X(0)+\frac{at^\alpha}{\Gamma(\alpha+1)}.$$

When $b\ne 0$, (2.41) is

$$\begin{aligned}X(t)&\le X(0)E_\alpha(bt^\alpha)+\frac{a}{b}\Big(\frac{bt^\alpha}{\Gamma(1+\alpha)}+\sum_{n=1}^\infty\frac{b^{n+1}t^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)}\Big)\\&\le X(0)E_\alpha(bt^\alpha)+\frac{a}{b}\big(E_\alpha(bt^\alpha)-1\big).\end{aligned}$$

When $a=b=0$, (2.38) and $X(0)=0$ give that for any $t\in[0,T)$, $X(t)\le 0$. Combining $X,(t)\le 0$ with the assumption $X(t)\ge 0$, one derives for any $t\in[0,T)$ $X(t)\equiv 0$. $\square$

The following Lemmas are devoted to the proof of Mass conservation for the mild solutions to (1.1).

Lemma 2.7(Hardy-Littlewood sobolev inequality[27]) Assume $0<\lambda<n, p=\frac{2n}{2n-\lambda}$. Then

$$\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)f(y)}{|x-y|^\lambda}dxdy\Big|\le C\|f\|_p^2.\tag{2.42}$$

3 Fundamental solution to homogenous equation

In this section, we first construct the formula of the fundamental solution to the homogenous equations of (1.1)

$$\begin{cases}{}_0^cD_t^\beta u+(-\triangle)^{\frac{\alpha}{2}}u=0, & \text{in } (x,t)\in\mathbb{R}^n\times(0,\infty)\\ u(x,0)=u_0(x),\end{cases}\tag{3.1}$$

which is also called fractional diffusion equation with Caputo fractional derivative. Caputo derivative, as a variation of the Riemann-Liouville fractional derivative, was first introduced in [12] and is more suitable for initial-value problem. There have been many papers related to the study of fractional partial differential equation. Taylor [40] constructed the formulas and estimates for the solution to inhomogeneous fractional diffusion equations

$$ {}_0^cD_t^\beta u + Au - q(t) = 0, $$

where A is a positive self-adjoint operator and established the short time existence to fractional diffusion-reaction equations

$$ {}_0^cD_t^\beta u + (-\triangle)^\alpha u - F(u) = 0. $$

Zacher [44] considered the regularity of weak solutions to linear diffusion equations with Riemann-Liouville time fractional derivative in the bounded domain in $\mathbb{R}^n$.

Allen, Caffarelli and Vasseur [2] studied the regularity of the following nonlocal evolution equation with fractional time derivative

$$ {}_a^cD_t^\beta w(t,x) = \int [w(t,y) - w(t,x)]K(t,x,y)dy + f(t,x), $$

which K is the kernel with symmetry condition $K(t,x,y) = K(t,y,x)$ for any $x \neq y$ and an ellipticity condition. Also, Allen, Caffarelli and Vasseur [3] investigated the existence and regularity for solutions to the following porous medium equation

$$ D_t^\beta u - \operatorname{div}(u\nabla(-\triangle)^{-\sigma}u) - f(x,t) = 0, $$

with D_t^α is of Caputo-type and defined by

$$ D_t^\alpha u \equiv \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^t [u(x,t) - u(x,s)]K(s,x,t)ds. $$

If $K(x,t,s) = (t-s)^{-1-\alpha}$, the operator is exactly the Caputo derivative.

In order to construct the formula of the fundamental solution to (3.1), we need to recall the definition of Fourier transform and Laplace transform. For $u(x) \in L^1(\mathbb{R}^n)$. The Fourier transform of $u(x)$ is defined by

$$ \mathfrak{F}(u)(\xi) = \int_{\mathbb{R}^n} e^{ix\xi}u(x)dx. $$

And for Re$s > 0$, the Laplace transform of function $u(t)$ is given by

$$ \mathfrak{L}u(s) = \int_0^\infty u(t)e^{-st}dt. $$

Lemma 3.1 Denote $\mathfrak{F}^{-1}(\cdot)$ to be the inverse Fourier transform operator. Then the fundamental solution to (3.1), denoted by $P_\alpha^\beta(x,t)$, can be represented as $\mathfrak{F}^{-1}(E_\beta(-|\xi|^\alpha t^\beta))$.

Proof Take Fourier transform with respect to x in $\mathbb{R}^n$ to (3.1) to get

$$ {}_0^cD_t^\beta \mathfrak{F}(u)(\xi,t) + |\xi|^\alpha \mathfrak{F}(u)(\xi,t) = 0. \tag{3.2}$$

Apply Laplace transform with respect to time variable t to (3.2) and obtain

$$ s^\beta \mathfrak{L}(\mathfrak{F}(u)(\xi,s) - s^{\beta-1}\mathfrak{F}(u)(\xi,0) + |\xi|^\alpha \mathfrak{L}\mathfrak{F}(u)(\xi,s) = 0, \tag{3.3}$$

here we use the formula

$$\mathfrak{L}({}_0^cD_t^\beta u(s)) = s^\beta \mathfrak{L}u(s) - s^{\beta-1}u(0).$$

Since the Fourier transform of $\delta(x)$ is 1, then we get

$$\mathfrak{L}(\mathfrak{F}(u(\xi,s)) = \frac{s^{\beta-1}}{s^\beta + |\xi|^\alpha}, \tag{3.4}$$

For Re$s > |\lambda|^{1/\alpha}$, (A.28) in the appendix A of [32] shows that

$$\mathfrak{L}(E_\alpha(-\lambda t^\alpha)) = \frac{s^{\alpha-1}}{s^\alpha + \lambda}. \tag{3.5}$$

Application of Laplace inversion to (3.4) and (3.5) implies that

$$\mathfrak{F}(u(\xi,t)) = E_\beta(-|\xi|^\alpha t^\beta). \tag{3.6}$$

□

Remark 3.1 There are many references in construction of fundamental solutions of different fractional differential equations, such as [17], [31] and [32].

By Lemma 3.1 and variable transform $\eta = \xi t^{\frac{\beta}{\alpha}}$, we have that

$$\begin{aligned} P_\alpha^\beta(x,t) &= \int_{\mathbb{R}^n} E_\beta(-|\xi|^\alpha t^\beta) e^{i\xi x} d\xi \\ &= \int_{\mathbb{R}^n} E_\beta(-|\eta|^\alpha) e^{i\eta\cdot\frac{x}{t^{\frac{\beta}{\alpha}}}} t^{-\frac{n\beta}{\alpha}} d\eta \\ &= t^{-\frac{n\beta}{\alpha}} P_\alpha^\beta(\frac{x}{t^{\frac{\beta}{\alpha}}}), \end{aligned} \tag{3.7}$$

where $P_\alpha^\beta(x)$ is the inverse Fourier transform of $E_\beta(-|\xi|^\alpha)$.

Denote $r = |x|$. Since the Fourier transform of radial function is also a radial function, using the Fourier inversion theorem for radial functions [[8], Chapter II, §7], we have

$$P_\alpha^\beta(x) \equiv P_\alpha^\beta(r) = r^{-\frac{n-2}{2}} \cdot (2\pi)^{-\frac{n}{2}} \int_0^\infty t^{\frac{n}{2}} J_{\frac{n-2}{2}}(rt) E_\beta(-t^\alpha) dt. \tag{3.8}$$

In order to study the asymptotic behavior of $P_\alpha^\beta(x)$, we need the following formula.

Lemma 3.2 Let $\alpha > 0, \nu > 0, \alpha + 1 > \nu$. Then

$$\int_0^\infty t^\alpha J_\nu(t)dt = \frac{2^\alpha}{\pi}\sin(\frac{1+\alpha-\nu}{2}\pi)\Gamma(\frac{1+\alpha+\nu}{2})\Gamma(\frac{1+\alpha-\nu}{2}). \tag{3.9}$$

Proof Due to the definition of $H_\nu^{(1)}(t)$, (2.12), we have

$$\int_0^\infty t^\alpha J_\nu(t)dt = \mathrm{Re}\int_0^\infty t^\alpha H_\nu^{(1)}(t)dt. \tag{3.10}$$

And identity (2.13) gives that

$$\begin{aligned}\int_0^\infty t^\alpha K_\nu(t)dt &= \frac{i\pi}{2}e^{\frac{i\pi\nu}{2}}\int_0^\infty t^\alpha H_\nu^{(1)}(te^{\frac{i\pi}{2}})dt \\ &= \frac{i\pi}{2}e^{\frac{i\pi\nu}{2}}\int_0^\infty T^\alpha e^{-\frac{i\alpha\pi}{2}}e^{-\frac{i\pi}{2}}H_\nu^{(1)}(T)dT \\ &= \frac{i\pi}{2}e^{\frac{i\pi(\nu-\alpha-1)}{2}}\int_0^\infty t^\alpha H_\nu^{(1)}(t)dt,\end{aligned} \tag{3.11}$$

here we use the substitution $T = te^{\frac{i\pi}{2}}$. Therefore, (3.11) implies that

$$\int_0^\infty t^\alpha H_\nu^{(1)}(t)dt = \frac{2}{i\pi}e^{\frac{i\pi}{2}(1+\alpha-\nu)}\int_0^\infty t^\alpha K_\nu(t)dt. \tag{3.12}$$

(3.10), together with (3.12), shows that

$$\int_0^\infty t^\alpha J_\nu(t)dt = \mathrm{Re}\big(\frac{2}{i\pi}e^{\frac{i\pi}{2}(1+\alpha-\nu)}\int_0^\infty t^\alpha K_\nu(t)dt\big). \tag{3.13}$$

Using identity (2.14), one has

$$\int_0^\infty t^\alpha K_\nu(t)dt = 2^{\alpha-1}\Gamma(\frac{1+\alpha+\nu}{2})\Gamma(\frac{1+\alpha-\nu}{2}). \tag{3.14}$$

By (3.13) and (3.14), we end the proof of this Lemma. □

Under the instruction of the proof of Theorem 2.1 in [10], we obtain the following theorem which gives some information on the behavior of $P_\alpha^\beta(r)$ when r is large enough.

Theorem 3.1 Let $\alpha > 0, \beta > 0$. Then

$$\lim_{r\to\infty} r^{n+\alpha}P_\alpha^\beta(r) = \alpha\cdot 2^{\alpha-1}\cdot(\frac{1}{\pi})^{\frac{n}{2}+1}\cdot\sin(\frac{\pi\alpha}{2})\cdot\Gamma(\frac{n+\alpha}{2})\cdot\Gamma(\frac{\alpha}{2})\frac{\Gamma(2)}{\Gamma(\beta+1)}. \tag{3.15}$$

Proof Due to the formula (3.8) and variable substitution $T = rt$, we have

$$\begin{aligned}\lim_{r\to\infty} r^{n+\alpha}P_\alpha^\beta(r) &= (2\pi)^{-\frac{n}{2}}\lim_{r\to\infty} r^{\frac{n}{2}+\alpha+1}\int_0^\infty E_\beta(-t^\alpha)J_{\frac{n}{2}-1}(rt)t^{\frac{n}{2}}dt \\ &= (2\pi)^{-\frac{n}{2}}\lim_{r\to\infty} r^\alpha\int_0^\infty E_\beta(-(\frac{t}{r})^\alpha)J_{\frac{n}{2}-1}(t)t^{\frac{n}{2}}dt.\end{aligned} \tag{3.16}$$

By the definitions of the Mittag-Leffler function (2.5) and Bessel function of first kind (2.9), one has

$$J_{\frac{n}{2}}(0) = 0, E_\beta(0) = 1. \tag{3.17}$$

And the asymptotic behavior of $E_\beta(t)$ (2.8) and $J_\nu(t)$ (2.10) show that

$$\lim_{t\to\infty} E_\beta(-(\frac{t}{r})^\alpha) = 0, \lim_{t\to\infty} J_{\frac{n}{2}}(t) = 0. \tag{3.18}$$

Due to the formula (50) in page 11 of [18]

$$(t^\nu J_\nu(t))'_t = t^\nu J_{\nu-1}(t),$$

and

$$(E_\beta(-(\frac{t}{r})^\alpha))'_t = \int_0^\infty M_\beta(\tau)\tau e^{-\tau(\frac{t}{r})^\alpha} d\tau \cdot (-\alpha)\frac{t^{\alpha-1}}{r^\alpha},$$

integration by parts, together with (3.17) and (3.18), yields that

$$\begin{aligned}
&\int_0^\infty E_\beta(-(\frac{t}{r})^\alpha) J_{\frac{n}{2}-1}(t) t^{\frac{n}{2}} dt = \int_0^\infty E_\beta(-(\frac{t}{r})^\alpha)(t^{\frac{n}{2}} J_{\frac{n}{2}}(t))'_t dt \\
&= E_\beta(-(\frac{t}{r})^\alpha)(t^{\frac{n}{2}} J_{\frac{n}{2}}(t))|_0^\infty - \int_0^\infty (E_\beta(-(\frac{t}{r})^\alpha))'_t (t^{\frac{n}{2}} J_{\frac{n}{2}}(t)) dt \\
&= -\int_0^\infty t^{\frac{n}{2}} J_{\frac{n}{2}}(t)\Big(\int_0^\infty M_\beta(\tau)\tau e^{-\tau(\frac{t}{r})^\alpha} d\tau \cdot (-\alpha) t^{\alpha-1} r^{-\alpha}\Big) dt.
\end{aligned} \tag{3.19}$$

Put (3.19) into (3.16) to obtain

$$\begin{aligned}
\lim_{r\to\infty} r^{n+\alpha} P_\alpha^\beta(x) &= \alpha \cdot (2\pi)^{-\frac{n}{2}} \lim_{r\to\infty} \int_0^\infty t^{\frac{n}{2}+\alpha-1} J_{\frac{n}{2}}(t) \cdot (\int_0^\infty M_\beta(\tau)\tau e^{-\tau(\frac{t}{r})^\alpha} d\tau) dt \\
&= \alpha \cdot (2\pi)^{-\frac{n}{2}} \int_0^\infty t^{\frac{n}{2}+\alpha-1} J_{\frac{n}{2}}(t) dt \cdot \int_0^\infty M_\beta(\tau)\tau d\tau.
\end{aligned} \tag{3.20}$$

(3.9) in Lemma 3.2 shows that

$$\int_0^\infty t^{\frac{n}{2}+\alpha-1} J_{\frac{n}{2}}(t) dt = \frac{2^{\frac{n}{2}+\alpha-1}}{\pi} \sin(\frac{\alpha\pi}{2})\Gamma(\frac{n+\alpha}{2})\Gamma(\frac{\alpha}{2}). \tag{3.21}$$

Take $\delta = 1$ in (2.7), one has

$$\int_0^\infty M_\beta(\tau)\tau d\tau = \frac{\Gamma(2)}{\Gamma(\beta+1)}. \tag{3.22}$$

Put (3.21) and (3.22) into (3.20), one can deduce (3.15). □

Remark 3.2 Proposition 3.1 in [40] gives that for $0 < \beta < 1, r \geq 0$, $M_\beta(r) \geq 0$ and $M_\beta(r)$ is not identical to zero. With this observation, the definition (2.6) yields that for

$0 < \beta < 1$, $E_\beta(-s) > 0$. Thus, (3.8) shows that $0 < P_\alpha^\beta(x)$. Based on Theorem 3.1, one has that for all $x \in \mathbb{R}^n$, there is $C_1 > 0$ such that

$$0 < P_\alpha^\beta(x) < C_1(1+|x|)^{-(\alpha+n)}. \tag{3.23}$$

With the same argument as the above, we can investigate the asymptotic behavior of the derivative of $P_\alpha^\beta(r)$ as follows.

Theorem 3.2 Let $\alpha > 0, \beta > 0$. Then

$$\lim_{r\to\infty} r^{n+\alpha+1}(P_\alpha^\beta(r))'_r = (\alpha+n)\cdot 2^\alpha \cdot (\frac{1}{\pi})^{\frac{n}{2}+1}\cdot \sin(\frac{\pi(\alpha+2)}{2})\cdot\Gamma(\frac{n+\alpha}{2})\cdot\Gamma(\frac{\alpha+2}{2})\frac{\Gamma(2)}{\Gamma(\beta+1)}. \tag{3.24}$$

Proof Due to substitution $T = rt$, one has

$$P_\alpha^\beta(r) = (2\pi)^{-\frac{n}{2}} r^{-n}\int_0^\infty E_\beta(-(\frac{t}{r})^\alpha) J_{\frac{n}{2}-1}(t) t^{\frac{n}{2}}\, dt. \tag{3.25}$$

Thus,

$$\begin{aligned}
(P_\alpha^\beta(r))'_r &= (2\pi)^{-\frac{n}{2}}\int_0^\infty \big(r^{-n}E_\beta(-(\frac{t}{r})^\alpha)\big)'_r J_{\frac{n}{2}-1}(t)t^{\frac{n}{2}}\,dt\\
&= (2\pi)^{-\frac{n}{2}}(-n)\int_0^\infty r^{-n-1}E_\beta(-(\frac{t}{r})^\alpha)J_{\frac{n}{2}-1}(t)t^{\frac{n}{2}}\,dt\\
&+ (2\pi)^{-\frac{n}{2}}\alpha\int_0^\infty r^{-n-\alpha-1}\cdot(\int_0^\infty M_\beta(\tau)\tau e^{-\tau(\frac{t}{r})^\alpha}d\tau)\cdot t^{\frac{n}{2}+\alpha}J_{\frac{n}{2}-1}(t)dt\\
&\equiv P_1 + P_2.
\end{aligned}\tag{3.26}$$

Since

$$\begin{aligned}
\lim_{r\to\infty} r^{n+\alpha+1}P_1 &= (-n)(2\pi)^{-\frac{n}{2}}\lim_{r\to\infty}\frac{\int_0^\infty E_\beta(-(\frac{t}{r})^\alpha)J_{\frac{n}{2}-1}(t)t^{\frac{n}{2}}\,dt}{r^{-\alpha}}\\
&= (-n)(2\pi)^{-\frac{n}{2}}\lim_{r\to\infty}\frac{\int_0^\infty E_\beta(-(\frac{t}{r})^\alpha)'_r\cdot(\alpha t^\alpha r^{-\alpha-1})J_{\frac{n}{2}-1}(t)t^{\frac{n}{2}}\,dt}{(-\alpha)r^{-\alpha-1}}\\
&= n(2\pi)^{-\frac{n}{2}}\cdot\lim_{r\to\infty}\int_0^\infty\big(\int_0^\infty M_\beta(\tau)\tau e^{-\tau(\frac{t}{r})^\alpha}d\tau\big)\cdot J_{\frac{n}{2}-1}(t)t^{\frac{n}{2}+\alpha}dt\\
&= n(2\pi)^{-\frac{n}{2}}\cdot\int_0^\infty M_\beta(\tau)\tau d\tau\cdot\int_0^\infty t^{\frac{n}{2}+\alpha}J_{\frac{n}{2}-1}(t)dt.
\end{aligned}\tag{3.27}$$

For P_2, one has

$$\begin{aligned}
\lim_{r\to\infty} r^{n+\alpha+1}P_2 &= \alpha(2\pi)^{-\frac{n}{2}}\lim_{r\to\infty}\int_0^\infty\big(\int_0^\infty M_\beta(\tau)\tau e^{-\tau(\frac{t}{r})^\alpha}d\tau\big)\cdot J_{\frac{n}{2}-1}(t)t^{\frac{n}{2}+\alpha}dt\\
&= \alpha(2\pi)^{-\frac{n}{2}}\cdot\int_0^\infty M_\beta(\tau)\tau d\tau\cdot\int_0^\infty t^{\frac{n}{2}+\alpha}J_{\frac{n}{2}-1}(t)dt
\end{aligned}\tag{3.28}$$

Using Lemma 3.2 and identity (2.7), (3.27) and (3.28) can deduce (3.24).□

Remark 3.3 Using Theorem 3.2, one has that for all $x \in \mathbb{R}^n$, there is $C_2 > 0$ such that

$$|\nabla P_\alpha^\beta(x)| \leq C_2(1+|x|)^{-(\alpha+n+1)}. \tag{3.29}$$

Under the help of (3.23) and (3.29) in Remark 3.2 and 3.3, we can obtain the following estimate for the solution to (3.1) which can be denoted by $S_\alpha^\beta(t)u_0 = P_\alpha^\beta(x,t) * u_0$.

Theorem 3.3 Let $u_0 \in L^q(\mathbb{R}^n)$. For $1 \leq q \leq p \leq \infty$, there exist $C_1, C_2 > 0$ such that

$$\|S_\alpha^\beta(t)u_0\|_p \leq C_1 t^{-\frac{n\beta}{\alpha}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_q. \tag{3.30}$$

$$\|\nabla S_\alpha^\beta(t)u_0\|_p \leq C_2 t^{-\frac{n\beta}{\alpha}(\frac{1}{q}-\frac{1}{p})-\frac{\beta}{\alpha}}\|u_0\|_q. \tag{3.31}$$

Proof: Young's inequality implies that for $1 \leq q \leq p \leq \infty$, if $\frac{1}{p}+1 = \frac{1}{r}+\frac{1}{q}$, then

$$\begin{aligned}\|S_\alpha^\beta(t)u_0\|_p &= \|P_\alpha^\beta(x,t) * u_0\|_p \\ &\leq \|P_\alpha^\beta(t)\|_r\|u_0\|_q.\end{aligned} \tag{3.32}$$

By the scaling property (3.7) and inequality (3.23), we have

$$\begin{aligned}\|P_\alpha^\beta(t)\|_r &= \|t^{-\frac{n\beta}{\alpha}}P_\alpha^\beta(\frac{x}{t^{\frac{\beta}{\alpha}}})\|_r \\ &\leq C_1 t^{-\frac{n\beta}{\alpha}}\Big(\int_{\mathbb{R}^n}(1+|\frac{x}{t^{\frac{\beta}{\alpha}}}|)^{-(\alpha+n)r}dx\Big)^{\frac{1}{r}} \\ &= C_1 t^{-\frac{n\beta}{\alpha}(1-\frac{1}{r})}\Big(\int_{\mathbb{R}^n}(1+|x|)^{-(\alpha+n)r}dx\Big)^{\frac{1}{r}} \\ &\leq C_1 t^{-\frac{n\beta}{\alpha}(\frac{1}{q}-\frac{1}{p})}.\end{aligned} \tag{3.33}$$

Putting (3.33) into (3.32), one can get (3.30). Since by calculation, we have

$$\nabla P_\alpha^\beta(x,t) = t^{-\frac{n\beta}{\alpha}-\frac{\beta}{\alpha}}\nabla P_\alpha^\beta(\frac{x}{t^{\frac{\beta}{\alpha}}}). \tag{3.34}$$

In addition, Young's inequality gives that for $1 \leq q \leq p \leq \infty$, if $\frac{1}{p}+1 = \frac{1}{r}+\frac{1}{q}$, then

$$\begin{aligned}\|\nabla S_\alpha^\beta(t)u_0\|_p &= \|\nabla P_\alpha^\beta(x,t) * u_0\|_p \\ &\leq \|\nabla P_\alpha^\beta(t)\|_r\|u_0\|_q.\end{aligned} \tag{3.35}$$

Due to (3.34) and (3.29), we obtain

$$\begin{aligned}\|\nabla P_\alpha^\beta(t)\|_r &= \|t^{-\frac{n\beta}{\alpha}-\frac{\beta}{\alpha}}\nabla P_\alpha^\beta(\frac{x}{t^{\frac{\beta}{\alpha}}})\|_r \\ &\leq C_2 t^{-\frac{n\beta}{\alpha}-\frac{\beta}{\alpha}}\Big(\int_{\mathbb{R}^n}(1+|\frac{x}{t^{\frac{\beta}{\alpha}}}|)^{-(\alpha+n+1)r}dx\Big)^{\frac{1}{r}} \\ &= C_2 t^{-\frac{n\beta}{\alpha}(1-\frac{1}{r})-\frac{\beta}{\alpha}}\Big(\int_{\mathbb{R}^n}(1+|x|)^{-(\alpha+n+1)r}dx\Big)^{\frac{1}{r}} \\ &\leq C_2 t^{-\frac{n\beta}{\alpha}(\frac{1}{q}-\frac{1}{p})-\frac{\beta}{\alpha}}.\end{aligned} \tag{3.36}$$

Insert (3.36) into (3.35) to yield (3.31).□

Furthermore, we construct the weighted $L^{\infty}_{\alpha+n}$ estimate for the solution to equation (3.1) by the method used in [6].

Theorem 3.4 Assume $u_0 \in L^{\infty}_{\alpha+n}(\mathbb{R}^n)$. Then, there exists $C > 0$ independent of u_0 and t such that

$$\|S^{\beta}_{\alpha}(t)u_0\|_{\infty} \leq C \min\big(t^{\frac{-n\beta}{\alpha}}\|u_0\|_1, \|u_0\|_{\infty}\big). \tag{3.37}$$

$$\|S^{\beta}_{\alpha}(t)u_0\|_{L^{\infty}_{\alpha+n}} \leq C\|u_0\|_{L^{\infty}_{\alpha+n}} + Ct^{\beta}\|u_0\|_1. \tag{3.38}$$

$$\|\nabla S^{\beta}_{\alpha}(t)u_0\|_{L^{\infty}_{\alpha+n}} \leq Ct^{\frac{-\beta}{\alpha}}\|u_0\|_{L^{\infty}_{\alpha+n}} + Ct^{\beta-\frac{\beta}{\alpha}}\|u_0\|_1. \tag{3.39}$$

Proof Applying Young's inequality to $P^{\beta}_{\alpha}(t) * u_0$, due to the following identities

$$\|P^{\beta}_{\alpha}(t)\|_1 = 1, \|P^{\beta}_{\alpha}(t)\|_{\infty} = t^{-\frac{n\beta}{\alpha}}\|P^{\beta}_{\alpha}\|_{\infty} < Ct^{-\frac{n\beta}{\alpha}}, t > 0,$$

one has

$$\|S^{\beta}_{\alpha}(t)u_0\|_{\infty} = C\|P^{\beta}_{\alpha}(t)\|_1\|u_0\|_{\infty} \leq C\|u_0\|_{\infty}. \tag{3.40}$$

or

$$\|S^{\beta}_{\alpha}(t)u_0\|_{\infty} = C\|P^{\beta}_{\alpha}(t)\|_{\infty}\|u_0\|_1 \leq Ct^{-\frac{n\beta}{\alpha}}\|u_0\|_1. \tag{3.41}$$

Combination of (3.40) and (3.41) leads to (3.37).
By calculation, (3.7) and (3.23) yield

$$\begin{aligned}
\|P^{\beta}_{\alpha}(x,t)\|_{\dot{L}^{\infty}_{\alpha+n}} &= \operatorname{ess}\sup_{x\in\mathbb{R}^n} |x|^{\alpha+n}|P^{\beta}_{\alpha}(x,t)| \\
&\leq C\operatorname{ess}\sup_{x\in\mathbb{R}^n} |x|^{\alpha+n}t^{-\frac{n\beta}{\alpha}}|P^{\beta}_{\alpha}(\frac{x}{t^{\frac{\beta}{\alpha}}})| \\
&\leq C\operatorname{ess}\sup_{x\in\mathbb{R}^n} |x|^{\alpha+n}t^{-\frac{n\beta}{\alpha}}\big(1+\frac{x}{t^{\frac{\beta}{\alpha}}}\big)^{-(\alpha+n)} \\
&\leq Ct^{\beta}
\end{aligned} \tag{3.42}$$

In order to prove (3.38) and (3.39), we introduce the following inequality which holds for $x, y \in \mathbb{R}^n$ and some constant $C > 0$

$$(1+|x|)^{\alpha+n} \leq C(1+|y|)^{\alpha+n} + C(|x-y|)^{\alpha+n}. \tag{3.43}$$

Using (3.43) and the identity $\|P_\alpha^\beta(t)\|_1 = 1$, Young inequality and (3.42) imply that

$$
\begin{aligned}
\|S_\alpha^\beta(t)u_0\|_{L^\infty_{\alpha+n}} &= \text{ess}\sup_{x\in\mathbb{R}^n} |S_\alpha^\beta(t)u_0|(1+|x|)^{n+\alpha} \\
&\leq C\text{ess}\sup_{x\in\mathbb{R}^n} \int_0^\infty P_\alpha^\beta(x-y,t)u_0(y)(1+|y|)^{\alpha+n}dy \\
&+ C\text{ess}\sup_{x\in\mathbb{R}^n} \int_0^\infty P_\alpha^\beta(x-y,t)|x-y|^{\alpha+n}u_0(y)dy \\
&= C\text{ess}\sup_{x\in\mathbb{R}^n} \|P_\alpha^\beta(t) * (1+|x|)^{\alpha+n}u_0\|_\infty \\
&+ C\text{ess}\sup_{x\in\mathbb{R}^n} \|P_\alpha^\beta(t)|x|^{\alpha+n} * u_0\|_\infty \\
&\leq C\|P_\alpha^\beta(t)\|_1\|u_0\|_{L^\infty_{\alpha+n}} + \|P_\alpha^\beta(t)\|_{\dot{L}^\infty_{\alpha+n}}\|u_0\|_1. \\
&\leq C\|u_0\|_{L^\infty_{\alpha+n}} + t^\beta\|u_0\|_1.
\end{aligned}
\tag{3.44}
$$

The scaling property (3.34) and the estimate (3.29) give that

$$
\begin{aligned}
\|\nabla P_\alpha^\beta(t)\|_1 &= \int_{\mathbb{R}^n} |\nabla P_\alpha^\beta(x,t)|dx \\
&= \int_{\mathbb{R}^n} t^{-\frac{n\beta}{\alpha}-\frac{\beta}{\alpha}}|\nabla P_\alpha^\beta(\frac{x}{t^{\frac{\beta}{\alpha}}})|dx \\
&\leq Ct^{\frac{-n\beta}{\alpha}-\frac{\beta}{\alpha}}\int_{\mathbb{R}^n}(1+|\frac{x}{t^{\frac{\beta}{\alpha}}}|)^{-(n+\alpha+1)}dx \\
&= Ct^{-\frac{\beta}{\alpha}}\int_{\mathbb{R}^n}(1+|x|)^{-(n+\alpha+1)}dx \\
&\leq Ct^{-\frac{\beta}{\alpha}},
\end{aligned}
\tag{3.45}
$$

here we used the transform of variable $\bar{x} = \frac{x}{t^{\frac{\beta}{\alpha}}}$. Also, by the definition of (2.2), we have

$$
\begin{aligned}
\|\nabla P_\alpha^\beta(t)\|_{\dot{L}^\infty_{\alpha+n}} &= \text{ess}\sup_{x\in\mathbb{R}^n} |x|^{\alpha+n}|\nabla P_\alpha^\beta(t)| \\
&= \text{ess}\sup_{x\in\mathbb{R}^n} |x|^{\alpha+n}t^{-\frac{n\beta}{\alpha}-\frac{\beta}{\alpha}}|\nabla P_\alpha^\beta(\frac{x}{t^{\frac{\beta}{\alpha}}})| \\
&\leq C\text{ess}\sup_{x\in\mathbb{R}^n} |x|^{\alpha+n}t^{-\frac{n\beta}{\alpha}-\frac{\beta}{\alpha}}\big(1+\frac{|x|}{t^{\frac{\beta}{\alpha}}}\big)^{-(\alpha+n+1)} \\
&\leq C\text{ess}\sup_{x\in\mathbb{R}^n} \frac{|x|^{\alpha+n}t^\beta}{(t^{\frac{\beta}{\alpha}}+|x|)^{\alpha+n+1}} \\
&\leq Ct^{\beta-\frac{\beta}{\alpha}}.
\end{aligned}
\tag{3.46}
$$

Applying (3.43) and Young's inequality, we get

$$
\begin{aligned}
\|\nabla S_\alpha^\beta(t)u_0\|_{L^\infty_{\alpha+n}} &= \operatorname{ess}\sup_{x\in\mathbb{R}^n} |\nabla S_\alpha^\beta(t)u_0|(1+|x|)^{\alpha+n} \\
&\le C\operatorname{ess}\sup_{x\in\mathbb{R}^n} |\nabla S_\alpha^\beta(t)u_0|[(1+|y|)^{\alpha+n}+(|x-y|)^{\alpha+n}] \\
&\le C\|\nabla P_\alpha^\beta(t)\|_1\|u_0\|_{L^\infty_{\alpha+n}} + C\|\nabla P_\alpha^\beta(t)\|_{\dot{L}^\infty_{\alpha+n}}\|u_0\|_1.
\end{aligned} \tag{3.47}
$$

Put (3.45) and (3.46) into (3.47) to yield (3.39).□

Remark 3.4 The estimate results in Remark 3.2 and Remark 3.3 is the same as the boundedness of the fundamental solution when the Caputo derivative is replace by the normal integer time derivative ∂t. And when we take $\beta = 1$ in the inequalities (3.30) and (3.31) in Theorem 3.3, and the inequalities (3.37) and (3.39) in Theorem 3.4, respectively, we obtain the same estimate as [7] derived for the case with integer time derivative ∂t and the results in [6].

4 Existence of mild solution and weighted solution to inhomogenous equation

4.1 Local existence and global existence of mild solution

This section is devoted to the study of the existence of mild solution and weighted mild solution to inhomogenous equation (1.1) based on the results of the L^p estimate and weighted estimate of corresponding homogenous equation. Furthermore, we will show the nonnegativity and mass conservation to the mild solution.

Using fundamental solution of homogenous equation (3.1), the mild solution to inhomogenous equation (1.1) can be represented as

$$
\rho(t) = S_\alpha^\beta(t)\rho_0 - \int_0^t \nabla\cdot S_\alpha^\beta(t-\tau)\rho(\tau)B(\rho)(\tau)d\tau \equiv v + H(\rho,\rho), \tag{4.1}
$$

where $S_\alpha^\beta(t)\rho_0 = P_\alpha^\beta(t) * \rho_0$. With the help of the estimates in the above section for $S_\alpha^\beta(t)\rho_0$, we are in the position to establish the existence of mild solution to (1.1) by means of abstract Banach fixed point theorem Lemma 2.2 used in [7].

Theorem 4.1(Existence of mild solution) For $n \ge 2$. Assume $0 < \beta < 1$, $1 < \alpha \le 2$ and $1 < \gamma \le n$.

1)Let $\max(\frac{2n}{n+\gamma-1}, \frac{n}{\alpha+\beta(\gamma-2)}) < p \le \frac{n}{\gamma-1}$. For $\rho_0 \in L^p(\mathbb{R}^n)$, there exists $T = T(\|\rho_0\|_p) > 0$

such that the unique local in time mild solution $\rho \in C([0,T], L^p(\mathbb{R}^n))$ to system (1.1) exists and the following estimate holds

$$\|\rho\|_p \leq 2C_1\|\rho_0\|_p. \tag{4.2}$$

where C_1 is the constant in the (3.30) of Thoerem 3.3.

2)Let $\alpha+\beta(\gamma-2) \leq n\beta$ and $\alpha+2\beta(\gamma-2) > 0$. Assume $\max(\frac{\beta n}{\alpha+\beta(\gamma-2)}, \frac{2n}{n+\gamma-1}, \frac{n}{n-\alpha+1}) < p \leq \min(\frac{n}{\gamma-1}, \frac{2n\beta}{\alpha+2\beta(\gamma-2)})$. There exists an $\epsilon > 0$ such that if $\rho_0 \in L^{\frac{n}{\alpha+\gamma-2}}(\mathbb{R}^n) \cap L^{\frac{n\beta}{\alpha+\beta(\gamma-2)}}(\mathbb{R}^n)$ satisfying $\|\rho_0\|_{\frac{n}{\alpha+\gamma-2}} \leq \frac{\epsilon}{2C_1}$ and $\|\rho_0\|_{\frac{n\beta}{\alpha+\beta(\gamma-2)}} \leq \frac{\epsilon}{2C_1}$, then (1.1) has a unique global in time mild solution such that $\rho \in C([0,\infty), L^p(\mathbb{R}^n))$.

Proof of Theorem 4.1 We first verify the local existence in the space X mentioned in Section 2. The formula (4.1) implies that $v = S_\alpha^\beta(t)\rho_0$ and

$$H(\rho,\tilde{\rho}) = -\int_0^t \nabla \cdot S_\alpha^\beta(t-\tau)\rho(\tau)B(\tilde{\rho})(\tau)d\tau.$$

(3.30) in Theorem 3.3 shows that

$$\|v\|_X = \sup_{t\in[0,T]} \|S_\alpha^\beta(t)\rho_0\|_p \leq C_1\|\rho_0\|_p. \tag{4.3}$$

Bochner theorem (see [19], p.650) and the commutativity of operators S_α^β and $\nabla\cdot$, and (3.31) in Theorem 3.3 deduce that for $1 \leq r \leq p \leq \infty$,

$$\begin{aligned}
\|H(\rho,\tilde{\rho})\|_X &= \sup_{t\in[0,T]} \|H(\rho,\tilde{\rho})\|_p \\
&\leq \sup_{t\in[0,T]} \int_0^t \|\nabla \cdot S_\alpha^\beta(t-\tau)\rho(B(\tilde{\rho}))(\tau)\|_p d\tau \\
&\leq C_2 \sup_{t\in[0,T]} \int_0^t (t-\tau)^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}} \|\rho B(\tilde{\rho})(\tau)\|_r d\tau
\end{aligned} \tag{4.4}$$

With the help of the definition of B in (1.3), applying Lemma 2.1, for $1 < p < q < \infty$ satisfying $\frac{1}{p} - \frac{\gamma-1}{n} = \frac{1}{q}$, one can deduce that

$$\|B(\tilde{\rho})\|_q \leq s_{n,\gamma}\|\frac{1}{|x|^{n-\gamma+1}} * \tilde{\rho}\|_q \leq C\|\tilde{\rho}\|_p \tag{4.5}$$

Hölder's inequality gives that

$$\|\rho B(\tilde{\rho})\|_r \leq C\|\rho\|_p\|B(\tilde{\rho})\|_q, \tag{4.6}$$

with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Insert (4.5) into (4.6) to obtain

$$\|\rho(B(\tilde{\rho}))\|_r \leq C\|\rho\|_p\|\tilde{\rho}\|_p, \tag{4.7}$$

here $r=\frac{np}{2n-p(\gamma-1)}$. Put (4.7) into (4.4), when $1-\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{\gamma-1}{n})-\frac{\beta}{\alpha}>0$, one has

$$
\begin{aligned}
\|H(\rho,\tilde{\rho})\|_X &\le C\sup_{t\in[0,T]}\int_0^t (t-\tau)^{-\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{\gamma-1}{n})-\frac{\beta}{\alpha}}d\tau\cdot\|\rho\|_X\|\tilde{\rho}\|_X \\
&\le CT^{1-\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{\gamma-1}{n})-\frac{\beta}{\alpha}}\int_0^1 (1-\tau)^{-\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{\gamma-1}{n})-\frac{\beta}{\alpha}}d\tau\|\rho\|_X\|\tilde{\rho}\|_X \\
&\le CT^{1-\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{\gamma-1}{n})-\frac{\beta}{\alpha}}\mathfrak{B}(1,1-\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{\gamma-1}{n})-\frac{\beta}{\alpha})\|\rho\|_X\|\tilde{\rho}\|_X \\
&\le C(T,\alpha,\beta,\gamma)\|\rho\|_X\|\tilde{\rho}\|_X,
\end{aligned}
\tag{4.8}
$$

here and hereafter $\mathfrak{B}(x,y)$ is the Beta function and defined by $\mathfrak{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt$. Then choosing sufficiently small $T>0$ in (4.8) such that $C(T,\alpha,\beta,\gamma)<\frac{1}{4C_1}$, by Lemma 2.2, (4.3) and (4.8) show that (1.1) has a solution in X and satisfying $\|\rho\|_p\le 2C_1\|\rho_0\|_p$.

Next, we consider the global existence of mild solution in space $\hat{X}$. In order to construct the global existence of mild solution to (1.1) with the help of Lemma 2.2, we need to estimate $\|H(\rho,\tilde{\rho})\|_{\hat{X}}$ and $\|v\|_{\hat{X}}$. By definition of v and the norm $\|\cdot\|_{\hat{X}}$, we have

$$
\|v\|_{\hat{X}}=\sup_{t>0}\|S_\alpha^\beta(t)\rho_0\|_{\frac{n}{\alpha+\gamma-2}}+\sup_{t>0}t^{1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma)}\|S_\alpha^\beta\rho_0\|_p. \tag{4.9}
$$

(3.30) in Theorem 3.3 implies that

$$
\sup_{t>0}\|S_\alpha^\beta(t)\rho_0\|_{\frac{n}{\alpha+\gamma-2}}\le C_1\|\rho_0\|_{\frac{n}{\alpha+\gamma-2}}. \tag{4.10}
$$

and

$$
\begin{aligned}
\sup_{t>0}t^{1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma)}\|S_\alpha^\beta\rho_0\|_p &\le C_1\sup_{t>0}t^{1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma)}t^{-\frac{n\beta}{\alpha}(\frac{\alpha+\beta(\gamma-2)}{n\beta}-\frac{1}{p})}\|\rho_0\|_{\frac{n\beta}{\alpha+\beta(\gamma-2)}} \\
&\le C_1\|\rho_0\|_{\frac{n\beta}{\alpha+\beta(\gamma-2)}}.
\end{aligned}
\tag{4.11}
$$

Combine (4.10), (4.11) with (4.9) to obtain

$$
\|v\|_{\hat{X}}\le C_1\|\rho_0\|_{\frac{n}{\alpha+\gamma-2}}+C_1\|\rho_0\|_{\frac{n\beta}{\alpha+\beta(\gamma-2)}}. \tag{4.12}
$$

In addition, we estimate

$$
\begin{aligned}
\|H(\rho,\tilde{\rho})\|_{\hat{X}} &= \sup_{t>0}\|\int_0^t\nabla\cdot S_\alpha^\beta(t-\tau)\rho(\tau)B(\tilde{\rho})(\tau)d\tau\|_{\frac{n}{\alpha+\gamma-2}} \\
&\quad+\sup_{t>0}t^{1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma)}\|\int_0^t\nabla\cdot S_\alpha^\beta(t-\tau)\rho(\tau)B(\tilde{\rho})(\tau)d\tau\|_p.
\end{aligned}
\tag{4.13}
$$

Lemma 2.1 yields that

$$
\|B(\tilde{\rho})\|_{\frac{n}{\alpha-1}}\le C\|\tilde{\rho}\|_{\frac{n}{\alpha+\gamma-2}}. \tag{4.14}
$$

Höler's inequality and (4.14) show that for $1 < \frac{np}{n+p(\alpha-1)} \leq p$,

$$\begin{aligned} \|\rho(B(\tilde{\rho}))\|_{\frac{np}{n+p(\alpha-1)}} &\leq C\|\rho\|_p\|B(\tilde{\rho})\|_{\frac{n}{\alpha-1}} \\ &\leq C\|\rho\|_p\|\tilde{\rho}\|_{\frac{n}{\alpha+\gamma-2}}. \end{aligned} \tag{4.15}$$

For $q = \frac{np}{n+p(\alpha-1)}$, (3.31) in Theorem 3.3, (4.14) and (4.15) imply that

$$\begin{aligned} &\|\nabla \cdot S_\alpha^\beta(t-\tau)(\rho B(\tilde{\rho}))(\tau)\|_{\frac{n}{\alpha+\gamma-2}} \\ &\leq C_2(t-\tau)^{-\frac{n\beta}{\alpha}\cdot\frac{n+p(1-\gamma}{np})-\frac{\beta}{\alpha}}\|\rho B(\tilde{\rho})(\tau)\|_{\frac{np}{n+p(\alpha-1)}} \\ &\leq C(t-\tau)^{-\frac{\beta}{\alpha}\cdot\frac{n+p(1-\gamma)}{p}-\frac{\beta}{\alpha}}\|\rho\|_p\|\tilde{\rho}\|_{\frac{n}{\alpha+\gamma-2}}, \end{aligned} \tag{4.16}$$

holds to be true if $1 \leq \frac{np}{n+p(\alpha-1)} \leq \frac{n}{\alpha+\gamma-2}$. Then with the help of Bochner theorem and (4.16), we obtain that if $\frac{\beta n+\beta p(2-\gamma)}{\alpha p} > 0, 1-\frac{\beta n+\beta p(2-\gamma)}{\alpha p} > 0$

$$\begin{aligned} &\sup_{t>0}\|\int_0^t \nabla \cdot S_\alpha^\beta(t-\tau)\rho B(\tilde{\rho})(\tau)d\tau\|_{\frac{n}{\alpha+\gamma-2}} \\ &\leq \sup_{t>0}\int_0^t \|\nabla \cdot S_\alpha^\beta(t-\tau)\rho B(\tilde{\rho})(\tau)\|_{\frac{n}{\alpha+\gamma-2}}d\tau \\ &\leq C\sup_{t>0}\int_0^t (t-\tau)^{-\frac{\beta}{\alpha}\cdot\frac{n+p(1-\gamma)}{p}-\frac{\beta}{\alpha}}\|\rho\|_p\|\tilde{\rho}\|_{\frac{n}{\alpha+\gamma-2}}d\tau \\ &\leq C\sup_{t>0}\int_0^t (t-\tau)^{-\frac{\beta}{\alpha}\cdot\frac{n+p(2-\gamma)}{p}}\tau^{-1+\frac{\beta}{\alpha}(\frac{n+p(2-\gamma)}{p})}d\tau\|\rho\|_{\hat{X}}\|\tilde{\rho}\|_{\hat{X}} \\ &\leq C\mathfrak{B}(\frac{\beta n+\beta p(2-\gamma)}{\alpha p}, 1-\frac{\beta n+\beta p(2-\gamma)}{\alpha p})\|\rho\|_{\hat{X}}\|\tilde{\rho}\|_{\hat{X}}. \end{aligned} \tag{4.17}$$

Furthermore, we estimate the second term in (4.13) as follows. Bochner theorem and (3.31) in Theorem 3.3 follow that

$$\begin{aligned} &\sup_{t>0} t^{1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma)}\|\int_0^t \nabla \cdot S_\alpha^\beta(t-\tau)\rho(\tau)B(\tilde{\rho})(\tau)d\tau\|_p \\ &\leq C_2\sup_{t>0} t^{1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma)}\int_0^t (t-\tau)^{-\frac{n\beta}{\alpha}(\frac{1}{q}-\frac{1}{p})-\frac{\beta}{\alpha}}\|\rho B(\tilde{\rho})(\tau)\|_q d\tau \end{aligned} \tag{4.18}$$

with $1 \leq q \leq p \leq \infty$. Since by Lemma 2.1, for $\frac{1}{p} - \frac{\gamma-1}{n} = \frac{1}{r}$, we have

$$\|B(\tilde{\rho})(\tau)\|_r \leq C\|\tilde{\rho}\|_p. \tag{4.19}$$

Combination with Höler's inequality and (4.19) gives that

$$\|\rho B(\tilde{\rho})(\tau)\|_q \leq C\|\rho\|_p\|B(\tilde{\rho})\|_r \leq C\|\rho\|_p\|\tilde{\rho}\|_p, \tag{4.20}$$

here $\frac{1}{q} = \frac{1}{r} + \frac{1}{p}$, $\frac{1}{p} - \frac{\gamma-1}{n} = \frac{1}{r}$ and $\frac{1}{q} - \frac{1}{p} = \frac{n-p(\gamma-1)}{np}$ hold.
Thus, putting (4.20) into (4.18), we get

$$\begin{aligned}
&\sup_{t>0} t^{1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma)} \| \int_0^t \nabla \cdot S_\alpha^\beta(t-\tau)\rho(\tau)B(\tilde{\rho})(\tau)d\tau\|_p \\
&\leq C_2 \sup_{t>0} t^{1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma)} \int_0^t (t-\tau)^{-\frac{n\beta}{\alpha}\cdot\frac{n-p(\gamma-1)}{np}-\frac{\beta}{\alpha}} \|\rho\|_p \|\tilde{\rho}\|_p d\tau \\
&\leq C \sup_{t>0} t^{1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma)} \int_0^t (t-\tau)^{-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma)} \tau^{-2(1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma))} d\tau \|\rho\|_{\hat{X}} \|\tilde{\rho}\|_{\hat{X}} \\
&\leq C\mathfrak{B}(1-2(1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma)), 1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma))\|\rho\|_{\hat{X}}\|\tilde{\rho}\|_{\hat{X}},
\end{aligned} \tag{4.21}$$

here $1-2(1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma)) > 0$ and $1-\frac{\beta}{\alpha}(\frac{n}{p}+2-\gamma) > 0$. Combine (4.17) and (4.21) with (4.13) to obtain

$$\|H(\rho, \tilde{\rho}\|_{\hat{X}} \leq \hat{C}\|\rho\|\|\tilde{\rho}\|_{\hat{X}}. \tag{4.22}$$

Therefore, when we choose $\epsilon = C_1(\|\rho_0\|_{\frac{n}{\alpha+\gamma-2}} + \|\rho_0\|_{\frac{n\beta}{\alpha+\beta(\gamma-2)}})$, if $0 < \epsilon < \frac{1}{4\hat{C}}$, then Lemma 2.2 verifies the global existence of (1.1) in space $\hat{X}$ and $\|\rho\|_{\hat{X}} \leq 2\epsilon$.□

Remark 4.1 It is worthy to note that when $\beta \neq 1$, $\frac{n\beta}{\alpha+\beta(\gamma-2)} \neq \frac{n}{\alpha+\gamma-2}$, thus, we need the extra initial condition $\|\rho_0\|_{\frac{n\beta}{\alpha+\beta(\gamma-2)}} < \infty$.

4.2 Existence of weighted mild solution

In this subsection, we discuss the existence of weighted mild solution $\int_{\mathbb{R}^n} |x|^\nu \rho(x,t)dx$ of (1.1) based on which we can discuss the finite time blow by means of the extinction of the function $\int_{\mathbb{R}^n} \varphi_\nu(x)\rho(x,t)dx$ in a finite time where smooth function $\varphi_\nu(x)$ is like $|x|^\nu$ for large $|x|$.

First, we provide the L^∞ estimate of linear operator B using the X_T-norm.

Lemma 4.1 Assume $u \in X_T$. Let $1 < \gamma \leq n$, $1 < \alpha \leq 2$, $n \geq 2$. $B(u)$ is defined as (1.3). Then

$$\|B(u)\|_\infty \leq C\|u\|_{X_T}, \tag{4.23}$$

holds to be true.

Proof: For given $x \in \mathbb{R}^n$. Denote $\Omega_1 = \{y \in \mathbb{R}^n : |y| \leq \frac{|x|}{2}\}$, $\Omega_2 = \{y \in \mathbb{R}^n : |x-y| \leq \frac{|x|}{2}\}$

and $\Omega_3 = \{y \in \mathbb{R}^n : |x-y| \geq \frac{|x|}{2}, |y| \geq \frac{|x|}{2}\}$. According to (1.3), we have

$$\begin{aligned}
B(u)(x,t) &= -s_{n,\gamma} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n-\gamma+2}} u(y,t)dy \\
&= -s_{n,\gamma} \int_{\Omega_1} \frac{x-y}{|x-y|^{n-\gamma+2}} u(y,t)dy - s_{n,\gamma} \int_{\Omega_2} \frac{x-y}{|x-y|^{n-\gamma+2}} u(y,t)dy \\
&\quad - s_{n,\gamma} \int_{\Omega_3} \frac{x-y}{|x-y|^{n-\gamma+2}} u(y,t)dy \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{4.24}$$

If $y \in \Omega_2$, for $1 < \gamma \leq n$, $1 < \alpha \leq 2$, $n \geq 2$, we get

$$\begin{aligned}
|I_2| &\leq s_{n,\gamma} \int_{\Omega_2} \frac{|x-y|}{|x-y|^{n-\gamma+2}} \cdot (1+|y|)^{-(n+\alpha)} \cdot \big((1+|y|)^{(n+\alpha)}|u(y,t)|\big) dy \\
&\leq C \int_{\Omega_2} \frac{1}{|x-y|^{n-\gamma+1}} dy \cdot \|u\|_{X_T} \\
&\leq C\|u\|_{X_T}.
\end{aligned} \tag{4.25}$$

If $y \in \Omega_3$, one has

$$\frac{|x|}{2} \leq |x-y| \leq |x| + |y| \leq 3|y|. \tag{4.26}$$

Then, for $1 < \gamma \leq n$, $1 < \alpha \leq 2$, $n \geq 2$, from (4.26) one can deduce

$$\begin{aligned}
|I_3| &\leq C \int_{\Omega_3} \frac{|x-y|}{|x-y|^{n-\gamma+2}} (1+|y|)^{-(n+\alpha)} dy \cdot \|u\|_{X_T} \\
&\leq C \int_{\Omega_3} \frac{1}{|x-y|^{n-\gamma+1}} \cdot \frac{1}{|y|^{n+\alpha}} dy \cdot \|u\|_{X_T} \\
&\leq C \int_{\Omega_3} \frac{1}{|x-y|^{2n+\alpha-\gamma+1}} dy \cdot \|u\|_{X_T} \\
&\leq C\|u\|_{X_T}.
\end{aligned} \tag{4.27}$$

For $y \in \Omega_1$, one has

$$\frac{|x|}{2} \leq |x| - \frac{|x|}{2} \leq |x| - |y| \leq |x-y| \Rightarrow \frac{1}{|x-y|} \leq \frac{2}{|x|} \leq \frac{1}{|y|}. \tag{4.28}$$

Then, (4.28) gives

$$\begin{aligned}
|I_1| &\leq s_{n,\gamma} \int_{\Omega_1} \frac{1}{|x-y|^{n-\gamma+1}} \cdot (1+|y|)^{-(n+\alpha)} dy \cdot \|u\|_{X_T} \\
&\leq C \int_{\Omega_1} \frac{1}{|y|^{n-\gamma+1}} dy \cdot \|u\|_{X_T} \\
&\leq C\|u\|_{X_T}.
\end{aligned} \tag{4.29}$$

Plug (4.25), the estimate of operator B in (4.27) and (4.29) into (4.24) to get (4.23).$\Box$

Based on Lemma 4.1, we establish the existence of weighted mild solution in X_T.

Theorem 4.2 For $n \geq 2$. Let $0 < \beta < 1, 1 < \alpha \leq 2$ and $1 < \gamma \leq n$. Assume that ρ is the solution to equation (1.1) constructed in Theorem 4.1. If $\rho_0 \in L^\infty_{\alpha+n}(\mathbb{R}^n)$. Then $\rho \in C([0,T]; L^\infty_{\alpha+n}(\mathbb{R}^n))$. Especially, for $\nu < \alpha$, $\int_{\mathbb{R}^n} |x|^\nu \rho(x,t)dx < \infty$.

Proof: By applying Theorem 3.4 in Banach space X_T and using Lemma 2.2, we can construct the existence of solution to (1.1) in X_T.

(3.38) in Theorem 3.4 implies that $v = S^\beta_\alpha(\cdot)\rho_0 \in X_T$. Therefore, we need to show that for all $\rho, \tilde{\rho} \in X_T$, there is a constant $C > 0$ independent of $\rho, \tilde{\rho}$, the following estimate of bilinear form $H(\rho, \tilde{\rho}) = \int_0^t S^\beta_\alpha(t-\tau)\nabla(\rho B(\tilde{\rho}))d\tau$

$$\|H(\rho,\tilde{\rho})\|_{X_T} \leq C\max(T^{1-\frac{\beta}{\alpha}}, T^{1+\beta-\frac{\beta}{\alpha}})\|\rho\|_{X_T}\|\tilde{\rho}\|_{X_T} \tag{4.30}$$

holds. By the definition of $H(\rho,\tilde{\rho})$ and (3.39), we have

$$\begin{aligned}
\|H(\rho,\tilde{\rho})\|_{X_T} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}(1+|x|)^{\alpha+n}\left|\int_0^t S^\beta_\alpha(t-\tau)\nabla(\rho B(\tilde{\rho}))(\tau)d\tau\right| \\
&\leq \int_0^t \|\nabla S^\beta_\alpha(t-\tau)(\rho B(\tilde{\rho}))(\tau)\|_{L^\infty_{\alpha+n}}d\tau \\
&\leq C\int_0^t (t-\tau)^{-\frac{\beta}{\alpha}}\|(\rho B(\tilde{\rho}))(\tau)\|_{L^\infty_{\alpha+n}} + (t-\tau)^{\beta-\frac{\beta}{\alpha}}\|(\rho B(\tilde{\rho}))(\tau)\|_1 d\tau
\end{aligned} \tag{4.31}$$

(4.23) in Lemma 4.1 implies that

$$\begin{aligned}
\|\rho B(\tilde{\rho})(\tau)\|_{L^\infty_{\alpha+n}} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}(1+|x|)^{n+\alpha}|\rho(\tau)B(\tilde{\rho})(\tau)| \\
&\leq C\|\rho\|_{L^\infty_{\alpha+n}}\|B(\tilde{\rho})(\tau)\|_\infty \\
&\leq C\|\rho\|_{X_T}\|\tilde{\rho}\|_{X_T}.
\end{aligned} \tag{4.32}$$

By simple calculation, we have

$$\|\rho\|_1 \leq C\|\rho\|_{X_T}. \tag{4.33}$$

Thus, (4.33) and (4.23) imply that

$$\begin{aligned}
\|(\rho B(\tilde{\rho}))(\tau)\|_1 &\leq C\|\rho\|_1\|B(\tilde{\rho})(\tau)\|_\infty \\
&\leq C\|\rho\|_{X_T}\|\tilde{\rho}\|_{X_T}.
\end{aligned} \tag{4.34}$$

Plug (4.32) and (4.34) into (4.31), one has

$$\|H(\rho,\tilde{\rho})\|_{X_T} \leq Ct^{1-\frac{\beta}{\alpha}}\|\rho\|_{X_T}\|\tilde{\rho}\|_{X_T} + Ct^{1+\beta-\frac{\beta}{\alpha}}\|\rho\|_{X_T}\|\tilde{\rho}\|_{X_T}. \tag{4.35}$$

Then, (4.35) verify (4.30) to be true. $\Box$

4.3 Nonnegativity property

In this section, we consider the problem that if $\rho_0 \geq 0$ can imply $\rho(x,t) \geq 0$.

Theorem 4.3(Nonnegativity) For $n \geq 2$, assume $0 < \beta < 1, 1 < \alpha \leq 2$ and $1 < \gamma \leq 2$. Suppose the assumption in Theorem 4.1 to be held and ρ to be the solution to (1.1). If any $x \in \mathbb{R}^n$, $\rho_0 \geq 0$, then for any $t \in [0,T]$, $\rho(x,t) \geq 0$.

Proof Denote $\rho_- \equiv \rho_-(x,t) = \min(\rho, 0)$ and $\rho \equiv \rho(x,t)$. Thus, $\rho_-(x,0) = 0$. With the help of the integration by parts formula (2.1) listed in [3], one has

$$ {}_0^cD_t^\beta \rho = \frac{1}{\Gamma(1-\beta)}\big(\frac{\rho-\rho_0(x)}{t^\beta} + \beta\int_0^t \frac{\rho-\rho(x,s)}{t^{\beta+1}}ds\big). \tag{4.36} $$

Denote $\rho(x,s) = \rho_+(x,s)+\rho_-(x,s)$. Multiplying both sides of (1.1) with ρ_- and integrating with respect to x in $\mathbb{R}^n$ to obtain

$$ \int_{\mathbb{R}^n} ({}_0^cD_t^\beta \rho)\rho_- dx = -\int_{\mathbb{R}^n} \rho_-(-\triangle)^{\alpha/2}\rho dx - \int_{\mathbb{R}^n} \rho_-(\nabla\cdot(\rho(B(\rho)))dx. \tag{4.37} $$

(4.36) implies that

$$ \begin{aligned} \int_{\mathbb{R}^n} ({}_0^cD_t^\beta \rho)\rho_- dx = \frac{1}{\Gamma(1-\beta)}\Big(&\int_{\mathbb{R}^n} \frac{|\rho_-|^2}{t^\beta}dx + \beta\int_{\mathbb{R}^n}\int_0^t \frac{|\rho_+(x,s)\rho_-|}{t^{\beta+1}}dxds \\ &+\beta\int_{\mathbb{R}^n}\int_0^t \frac{(\rho_- - \rho_-(x,s))\rho_-}{t^{\beta+1}}dxds\Big). \end{aligned} \tag{4.38} $$

For $a, b > 0$, the formula $(a-b)a = \frac{a^2-b^2+(a-b)^2}{2}$ shows that

$$ \begin{aligned} \beta\int_{\mathbb{R}^n}\int_0^t \frac{(\rho_- - \rho_-(x,s))\rho_-}{t^{\beta+1}}dxds = &\frac{\beta}{2}\int_{\mathbb{R}^n}\int_0^t \frac{|\rho_-|^2-|\rho_-(x,s)|^2}{t^{\beta+1}}dsdx \\ &+\frac{\beta}{2}\int_{\mathbb{R}^n}\int_0^t \frac{(\rho_- - \rho_-(x,s))^2}{t^{\beta+1}}. \end{aligned} \tag{4.39} $$

Plugging (4.39) into (4.38) to obtain

$$ \begin{aligned} \int_{\mathbb{R}^n} ({}_0^cD_t^\beta \rho)\rho_- dx = \frac{1}{\Gamma(1-\beta)}\Big(&\int_{\mathbb{R}^n} \frac{|\rho_-|^2}{t^\beta}dx + \beta\int_{\mathbb{R}^n}\int_0^t \frac{|\rho_+(x,s)\rho_-|}{t^{\beta+1}}dxds \\ &+\frac{\beta}{2}\int_{\mathbb{R}^n}\int_0^t \frac{|\rho_-|^2-|\rho_-(x,s)|^2}{t^{\beta+1}}dsdx \\ &+\frac{\beta}{2}\int_{\mathbb{R}^n}\int_0^t \frac{(\rho_- - \rho_-(x,s))^2}{t^{\beta+1}}dsdx\Big). \end{aligned} \tag{4.40} $$

Also, due to (4.36), one has

$$ \int_{\mathbb{R}^n} ({}_0^cD_t^\beta |\rho_-|^2)dx = \frac{1}{\Gamma(1-\beta)}\Big(\int_{\mathbb{R}^n} \frac{|\rho_-|^2}{t^\beta}dx + \beta\int_{\mathbb{R}^n}\int_0^t \frac{|\rho_-|^2-|\rho_-(x,s)|^2}{t^{\beta+1}}dsdx\Big). \tag{4.41} $$

Thus, by (4.41), we can deduce

$$\frac{\beta}{2}\int_{\mathbb{R}^n}\int_0^t \frac{|\rho_-|^2-|\rho_-|^2(x,s)}{t^{\beta+1}}dsdx = \frac{\Gamma(1-\beta)}{2}\int_{\mathbb{R}^n}({}_0^cD_t^\beta|\rho_-|^2)dx - \frac{1}{2}\int_{\mathbb{R}^n}\frac{|\rho_-|^2}{t^\beta}dx. \tag{4.42}$$

Put (4.42) into (4.40) to derive

$$\begin{aligned}\int_{\mathbb{R}^n}({}_0^cD_t^\beta\rho)\rho_- dx = &\frac{1}{2\Gamma(1-\beta)}\int_{\mathbb{R}^n}\frac{|\rho_-|^2}{t^\beta}dx + \frac{1}{2}\int_{\mathbb{R}^n}({}_0^cD_t^\beta|\rho_-|^2)dx\\ &+ \frac{\beta}{\Gamma(1-\beta)}\int_{\mathbb{R}^n}\int_0^t\frac{|\rho_+(x,s)\rho_-|}{t^{\beta+1}}dxds\\ &+ \frac{\beta}{2\Gamma(1-\beta)}\int_{\mathbb{R}^n}\int_0^t\frac{(\rho_- -\rho_-(x,s))^2}{t^{\beta+1}}dsdx.\end{aligned} \tag{4.43}$$

Combining (4.37) with (4.43), one yields

$$\begin{aligned}\int_{\mathbb{R}^n}({}_0^cD_t^\beta|\rho_-|^2)dx = &-2\int_{\mathbb{R}^n}\rho_-(-\triangle)^{\alpha/2}\rho dx - 2\int_{\mathbb{R}^n}\rho_-(\nabla\cdot(\rho(B(\rho)))dx\\ &- \frac{1}{\Gamma(1-\beta)}\int_{\mathbb{R}^n}\frac{|\rho_-|^2}{t^\beta}dx\\ &- \frac{2\beta}{\Gamma(1-\beta)}\int_{\mathbb{R}^n}\int_0^t\frac{|\rho_+(x,s)\rho_-|}{t^{\beta+1}}dxds\\ &- \frac{\beta}{\Gamma(1-\beta)}\int_{\mathbb{R}^n}\int_0^t\frac{(\rho_- -\rho_-(x,s))^2}{t^{\beta+1}}dsdx.\end{aligned} \tag{4.44}$$

With the help of Lemma 2.3, we can estimate the first term of the right hand side of equation (4.44) as follows

$$\begin{aligned}-2\int_{\mathbb{R}^n}\rho_-((-\triangle)^{\frac{\alpha}{2}}\rho)dx &= -2\int_{\mathbb{R}^n}\rho_-(D^\alpha\rho)dx\\ &\le -2\|(-\triangle)^{\frac{\alpha}{4}}\rho_-\|_2^2\\ &\le 0.\end{aligned} \tag{4.45}$$

For the second term of the right hand side of equation (4.44), integration by parts and $\rho_-\le 0$ imply that

$$\begin{aligned}-2\int_{\mathbb{R}^n}\rho_-(\nabla\cdot(\rho(B(\rho)))dx &= 2\int_{\mathbb{R}^n}\rho B(\rho)\nabla\rho_- dx\\ &= \int_{\mathbb{R}^n}B(\rho)\nabla|\rho_-|^2dx\\ &= -\int_{\mathbb{R}^n}\nabla\cdot(\nabla((-\triangle)^{-\gamma/2}\rho_-))|\rho_-|^2dx\\ &= -\int_{\mathbb{R}^n}((-\triangle)^{1-\gamma/2}\rho_-)\rho_-|\rho_-|dx\\ &\le -\int_{\mathbb{R}^n}(D^{2-\gamma}\rho_-)\rho_-|\rho_-|dx.\end{aligned} \tag{4.46}$$

For $0 < \gamma < 2$, Lemma 2.3 shows that

$$\begin{aligned} -\int_{\mathbb{R}^n} (D^{2-\gamma}\rho_-)\rho_-|\rho_-|dx &\leq -\frac{8}{9}\|D^{2-\gamma}\rho_-^{\frac{3}{2}}\|_2^2 \\ &\leq 0. \end{aligned} \tag{4.47}$$

When $\gamma = 2$,

$$\begin{aligned} -\int_{\mathbb{R}^n} (D^{2-\gamma}\rho_-)\rho_-|\rho_-|dx &= -\int_{\mathbb{R}^n} |\rho_-|^2|\rho_-|dx \\ &\leq 0. \end{aligned} \tag{4.48}$$

Plug (4.45) and (4.47), (4.48) into (4.44) to obtain

$${}_0^cD_t^\beta \int_{\mathbb{R}^n} |\rho_-|^2 dx \leq 0. \tag{4.49}$$

Define $X(t) = \int_{\mathbb{R}^n} |\rho_-|^2 dx$. Then $X(t) \geq 0$ and $X(0) = 0$. Since $\rho(x,t) \in C[0,T]$, then $X(t) \in C[0,T]$. Therefore, (4.49) and Lemma 2.6 show that for any $t \in [0,T]$, $\rho_- = 0$. This result ends the proof. □

4.4 Mass conservation

In this subsection, we consider whether the integral function $\int_{\mathbb{R}^n} \rho(x,t)dx$ can keep constant in some interval. To this end, we introduce the cutoff function $0 \leq \psi(x) \leq 1, \psi(x) \in C_0^\infty(\mathbb{R}^n)$ and

$$\psi(x) = \begin{cases} 1, & \text{if } |x| \geq 1, \\ 0, & \text{if } |x| \geq 2. \end{cases} \tag{4.50}$$

Define $\psi_R(x) = \psi(\frac{x}{R})$. From the definition of $\psi_R(x)$ and (4.50), in [23], the authors deduced that

$$\psi_R(x) \to 1 \text{ as } R \to \infty, \tag{4.51}$$

and there exist two constants C_3 and C_4 such that for $x \in \mathbb{R}^n$

$$|\nabla \psi_R(x)| \leq \frac{C_3}{R}, \tag{4.52}$$

$$|D^\alpha \psi_R(x)| \leq \frac{C_4}{R^\alpha}. \tag{4.53}$$

Theorem 4.4 (Mass Conservation) For $n \geq 2$, assume $0 < \beta < 1, 1 < \alpha \leq 2$ and $1 < \gamma \leq 2$. Suppose the assumption in theorem 4.1 to be held and ρ to be the solution to (1.1). For $T > 0$. Then for any $t \in [0,T]$,

$$\int_{\mathbb{R}^n} \rho dx = \int_{\mathbb{R}^n} \rho_0 dx. \tag{4.54}$$

Proof First assume T is small enough. Fix $t \in [0, T]$. Multiply both sides of (1.1) with $\psi_R(x)$ and integrate with respect to x in $\mathbb{R}^n$ to get

$$ {}_0^c D_t^\beta \Big(\int_{\mathbb{R}^n} \rho \psi_R(x) dx \Big) = - \int_{\mathbb{R}^n} (-\triangle)^{\frac{\alpha}{2}} \rho \psi_R(x) dx - \int_{\mathbb{R}^n} \nabla \cdot (\rho B(\rho)) \psi_R(x) dx = I + II. \quad (4.55) $$

Integration by parts and (4.53) imply that

$$ |I| = | \int_{\mathbb{R}^n} \rho(x,t) D^\alpha \psi_R(x) dx| \leq \frac{C}{R^\alpha} \|\rho\|_1. \quad (4.56) $$

With the help of integration by parts and definition of $B(\rho)$, we estimate the second term of (4.55) as follows,

$$ \begin{aligned} II &= \int_{\mathbb{R}^n} \rho B(\rho) \cdot \nabla \psi_R(x) dx \\ &= - \int_{\mathbb{R}^n} \rho(x,t) \Big(s_{n,\gamma} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n-\gamma+2}} \rho(y,t) dy \Big) \nabla \psi_R(x) dx \\ &= -s_{n,\gamma} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n-\gamma+2}} \rho(y,t) \rho(x,t) \nabla \psi_R(x) dx dy \end{aligned} \quad (4.57) $$

Thus, using (4.52) and (2.42) in Lemma 2.7, one can deduce

$$ |II| \leq \frac{C}{R} \|\rho\|^2_{\frac{2n}{n+\gamma-1}}. \quad (4.58) $$

By interpolation inequality and the inequality $\|\rho\|_{\frac{n}{\gamma-1}} \leq 2C_1 \|\rho_0\|_{\frac{n}{\gamma-1}}$ in Theorem 4.1, we get

$$ \|\rho\|^2_{\frac{2n}{n+\gamma-1}} \leq C \|\rho\|_1 \|\rho\|_{\frac{n}{\gamma-1}} \leq C \|\rho\|_1. \quad (4.59) $$

Insert (4.59) into (4.58), we have

$$ |II| \leq \frac{C}{R} \|\rho\|_1. \quad (4.60) $$

Put (4.56) and (4.60) into (4.55), due to the nonnegativity of solution ρ in theorem 4.3, one derives that

$$ -(\frac{C}{R^\alpha} + \frac{C}{R}) \|\rho\|_1 \leq {}_0^c D_t^\beta \Big(\int_{\mathbb{R}^n} \rho \psi_R(x) dx \Big) \leq (\frac{C}{R^\alpha} + \frac{C}{R}) \|\rho\|_1. \quad (4.61) $$

Let $R \to \infty$. Then (4.61) implies

$$ 0 \leq {}_0^c D_t^\beta \Big(\int_{\mathbb{R}^n} \rho dx \Big) \leq 0. \quad (4.62) $$

By Lemma 2.6, we have for any $t \in [0, T)$

$$ \int_{\mathbb{R}^n} \rho dx = \int_{\mathbb{R}^n} \rho_0 dx. $$

If T is not small. We can repeat the above process and construct the same result in $[0, \infty)$.□

5 Blow up of solutions

In this section, we consider the finite time blowup of the solutions to (1.1) by means of the method applied in [7]. To this end, we first consider the scaling property of the solutions to (1.1).

Let $\rho^\lambda(x,t) = \lambda^a \rho(\lambda x, \lambda^b t)$, for $\lambda > 0$. Since

$$\begin{aligned}
{}_0^c D_t^\beta \rho^\lambda(x,t) &= \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \partial_s(\rho^\lambda(x,s)) ds \\
&= \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \lambda^a \partial_2(\rho(\lambda x, \lambda^b s)) \lambda^b ds \\
&= \frac{\lambda^a}{\Gamma(1-\beta)} \int_0^{\lambda^b t} (t - \lambda^{-b} z)^{-\beta} \partial_z(\rho(\lambda x, z)) dz \\
&= \frac{\lambda^a}{\Gamma(1-\beta)} \int_0^{\lambda^b t} (\lambda^b t - z)^{-\beta} \lambda^{b\beta} \partial_z(\rho(\lambda x, z)) dz \\
&= \lambda^{a+b\beta} ({}_0^c D_t^\beta \rho(\lambda x, \lambda^b t)),
\end{aligned} \tag{5.1}$$

here we use the variable substitution $z = \lambda^b s$. And

$$(-\triangle)^{\frac{\alpha}{2}} \rho(x,t) = \lambda^a (-\triangle)_1^{\frac{\alpha}{2}} \rho(\lambda x, \lambda^b t) = \lambda^{a+\alpha} (-\triangle)_1^{\frac{\alpha}{2}} \rho(\lambda x, \lambda^b t). \tag{5.2}$$

$$\nabla(\rho^\lambda B(\rho^\lambda)) = \lambda^{2a+2-\gamma} \nabla \rho(\lambda\, x, \lambda^b t) B(\rho(\lambda\, x, \lambda^b t)). \tag{5.3}$$

Compare the index of λ of (5.1), (5.2) and (5.3) to obtain $a = \alpha + \gamma - 2$ and $b = \frac{\alpha}{\beta}$. Therefore, (1.1) has the following scaling property:

$$\rho^\lambda(x,t) = \lambda^{\alpha+\gamma-2} \rho(\lambda x, \lambda^{\frac{\alpha}{\beta}} t), \lambda > 0, \tag{5.4}$$

in the sense that if $\rho(x,t)$ is a solution to (1.1), then ρ^λ is so. Also, with calculation, we have that

$$\begin{aligned}
\|\lambda^{\alpha+\gamma-2} \rho(\lambda x)\|_{\frac{n}{\alpha+\gamma-2}} &= \Big(\int_{\mathbb{R}^n} |\lambda^{\alpha+\gamma-2} \rho(\lambda x)|^{\frac{n}{\alpha+\gamma-2}} dx \Big)^{\frac{\alpha+\gamma-2}{n}} \\
&= \Big(\int_{\mathbb{R}^n} |\rho(x)|^{\frac{n}{\alpha+\gamma-2}} dx \Big)^{\frac{\alpha+\gamma-2}{n}} \\
&= \|\rho(x)\|_{\frac{n}{\alpha+\gamma-2}},
\end{aligned} \tag{5.5}$$

here we use the variable transform $\bar{x} = \lambda x$. For $1 < \nu \leq 2$, we introduce an auxiliary function defined as follows

$$\varphi(x) = \varphi_\nu(x) = (1 + |x|^2)^{\frac{\nu}{2}} - 1. \tag{5.6}$$

Then, by the observation of [6], we have the following Lemma which shows that $\varphi(x)$ is essentially equivalent to $|x|^\nu$.

Lemma 5.1 For $\varepsilon > 0$ and suitable $C(\varepsilon) > 0$, for $x \in \mathbb{R}^n$,

$$\varphi(x) \leq |x|^\nu \leq \varepsilon + C(\varepsilon)\varphi(x) \tag{5.7}$$

and for $1 < \alpha < 2, 1 < \nu < \alpha$,

$$(-\triangle)^{\frac{\alpha}{2}}\varphi(x) \in L^\infty(\mathbb{R}^n). \tag{5.8}$$

Furthermore, for $1 < \nu \leq 2$, there exists $K = K(\nu)$ such that the following inequality

$$\frac{|x-y|^2}{1+|x|^{2-\nu}+|y|^{2-\nu}} \leq \frac{1}{K}\big(\nabla\varphi(x) - \nabla\varphi(y)\big) \cdot (x-y), \tag{5.9}$$

holds true for all $x, y \in \mathbb{R}^n$.

Also, the following Lemma is needed in the proof of finite time blowup.

Lemma 5.2 Assume $0 \leq X(t) \in C[0,T]$ and for $a, b, c, d, r > 0$,

$${}_0^cD_t^\beta X(t) \leq a - \frac{b}{(c+dX(t))^r}.$$

If

$$a - \frac{b}{(c+dX(0))^r} < 0,$$

then, for any $t \in [0,T]$,

$$X(t) \leq X(0). \tag{5.10}$$

Proof The continuity of $X(t)$ implies that $F(t) = a - \frac{b}{(c+dX(t))^r}$ is also a continuous function in $[0,T]$. Then, due to $F(0) = a - \frac{b}{(c+dX(0))^r} < 0$, there exists $\delta > 0$, for any $t \in [0,\delta]$, $F(t) = a - \frac{b}{(c+dX(t))^r} < 0$. By Lemma 2.6, one has, for any $t \in [0,\delta)$, (5.10) holds. Then we claim that (5.10) holds for any $t \in [0,T]$. If not, there is $t^* \in [\delta,T]$ such that for $t \leq t^*$, $X(t) \leq X(0)$ and for any $\epsilon > 0$, there exists t_0, for $t^* < t_0 < t^* + \epsilon$, $X(t_0) > X(0)$. But since $X(t)$ is continuous at t^*, then there is $0 < \delta^*$ such that for any $t^* < t < t^* + \delta^*$, $X(t) \leq X(0)$. The contradiction ends the proof.$\square$

The following generalized Gronwall type inequality is the first part of theorem 11 in [14] is used to prove Lemma 5.4.

Lemma 5.3 Let $X(t)$ and $K(t)$ be continuous and $a(t), b(t)$ Riemann integrable functions on $J = [\alpha,\beta]$ with $b(t)$ and $K(t)$ nonnegative on J.

If

$$X(t) \leq a(t) + b(t)\int_\alpha^t K(s)X(s)ds, t \in J, \tag{5.11}$$

then

$$X(t) \le a(t) + b(t) \int_\alpha^t a(s)K(s)\exp\big(\int_s^t b(r)K(r)dr\big)ds, t \in J. \tag{5.12}$$

Moreover, equality holds in (5.12) for a subinterval $J_1 = [\alpha, \beta_1]$ of J if equality holds in (5.11) for $t \in J_1$.

Lemma 5.4 Let $0 < \beta < 1$. Assume $X(t) \in C[0, \infty)$. If for $t \in [0, \infty\infty)$, there is $C < 0$ such that ${}_0^cD_t^\beta X(t) \le C$. Then, there exists $t_0 \in (0, \infty)$ such that $X(t_0) = 0$.

Proof Using the identity (4.36) and the assumption ${}_0^cD_t^\beta X(t) \le C$, we have

$$\frac{X(t) - X(0)}{t^\beta} + \beta \int_0^t \frac{X(t) - X(s)}{t^{\beta+1}} ds \le C\Gamma(1-\beta). \tag{5.13}$$

By calculation, (5.13) is equivalent to

$$X(t) \le \frac{C\Gamma(1-\beta)t^\beta + X(0)}{1+\beta} + \frac{\beta}{(1+\beta)t}\int_0^t X(s)ds. \tag{5.14}$$

By Lemma 5.3, we deduce

$$X(t) \le \frac{C\Gamma(1-\beta)t^\beta + X(0)}{1+\beta} + \frac{\beta}{(1+\beta)^2 t^{\frac{1}{1+\beta}}}\int_0^t \frac{C\Gamma(1-\beta)s^\beta + X(0)}{s^{\frac{\beta}{1+\beta}}} ds \equiv G(t). \tag{5.15}$$

Let $t \to +\infty$. The application of L'Hopital rule and $C < 0$ lead to $G(t) \to -\infty$ which means there exists some $t_0 \in (0, \infty]$, $X(t_0) = 0$. $\Box$

Therefore, with the help of scaling property (5.4), auxiliary function with estimate (5.7), (5.8), (5.9) and (5.10), we develop the approach in the proof of Theorem 2.3 in [6] to deal with the corresponding equation with fractional Caputo derivative in time. For the sake of completeness, we give the detail of the proof in the following.

Theorem 5.1 For $n \ge 2$, assume $0 < \beta < 1, 1 < \alpha \le 2$ and $1 < \gamma \le 2$. Suppose the assumption of Theorem 4.1 to be held and ρ to be the solution to (1.1). $\rho_0 > 0$. Then if one of the following conditions holds, the mild solution of (1.1) will blow up in a finite time.

(1)If $\alpha = \nu = \gamma = n = 2$, $\rho_0 \in L^1(\mathbb{R}^n, (1+|x|^2)dx)$ and $M_0 = \int_{\mathbb{R}^n} \rho_0(x)dx > \frac{2n}{s_{n,\gamma}}$;

(2)Assume $1 < \alpha \le 2, 1 < \gamma \le 2$ satisfying $\alpha + \gamma < n + 2$. If for some $1 < \nu < \alpha$, $\rho_0 \in L^1(\mathbb{R}^n, (1+|x|)^\nu dx)$ and if there exists certain sufficiently small constant $C > 0$ independent of ρ_0 such that

$$\frac{\int_{\mathbb{R}^n} |x|^\nu \rho_0(x)dx}{\int_{\mathbb{R}^n} \rho_0(x)dx} \le C\big(\int_{\mathbb{R}^n} \rho_0(x)dx\big)^{\frac{\nu}{n+2-\alpha-\gamma}}. \tag{5.16}$$

Proof For $1 < \nu \leq \alpha$. Define

$$\omega = \omega(t) = \int_{\mathbb{R}^n} \varphi(x)\rho(x,t)dx.$$

Multiply both sides of the first equation of (1.1) with $\varphi(x)$ and integrate with respect to x in $\mathbb{R}^n$ to obtain

$$\begin{aligned}
{}_0^cD_t^\beta \omega(t) &= -\int_{\mathbb{R}^n} (-\triangle)^{\alpha/2}\rho(x,t)\varphi(x)dx + \int_{\mathbb{R}^n} \rho(x,t)B(\rho)(x,t)\cdot\nabla\varphi(x)dx \\
&= -\int_{\mathbb{R}^n} (-\triangle)^{\alpha/2}\varphi(x)\rho(x,t)dx \\
&- \frac{s_{n,\gamma}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)\frac{\rho(x,t)\rho(y,t)}{|x-y|^{n-\gamma+2}}dxdy
\end{aligned} \tag{5.17}$$

here we use the formula of integration by parts, the definition of operator B and the symmetrization for the double integral.

(1)Under the first assumption, denote $M_2 = \int_{\mathbb{R}^n} |x|^2\rho(x,t)dx$ and $M_0 = \int_{\mathbb{R}^n} \rho_0 dx$. For $\alpha = \nu = \gamma = n = 2$, (5.17) is changed to be

$${}_0^cD_t^\beta M_2(t) = (2n - s_{n,n}M_0)M_0,$$

here the mass conservation, the result of Theorem 4.4, has been applied. Therefore, we obtain a fractional ordinary differential equation of $M_2(t)$

$$\begin{cases} {}_0^cD_t^\beta M_2(t) = (2n - s_{n,n}M_0)M_0, \\ M_2(x,0) = M_2(0), \end{cases} \tag{5.18}$$

Due to the identity ${}_0^cD_t^\beta(t^\beta) = \beta\Gamma(\beta)$, we obtain that the solution of (5.18) is

$$M_2(t) = M_2(0) - \frac{M_0(s_{n,n}M_0 - 2n)}{\beta\Gamma(\beta)}t^\beta. \tag{5.19}$$

Therefore, when $M_0 > \frac{2n}{s_{n,n}}$, (5.19) implies $M_2(T) = 0$ for some T satisfying $T^\beta = \frac{\beta\Gamma(\beta)M_2(0)}{M_0(s_{n,n}M_0-2n)} > 0$ and it is a contradiction with the global existence of nonnegative solutions of (1.1).

(2) When $1 < \gamma \leq 2, 1 < \nu < \alpha$. Define

$$J(t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\rho(x,t)\rho(y,t)}{|x-y|^{n-\gamma}(1+|x|^{2-\nu}+|y|^{2-\nu})}dxdy.$$

(5.9) in Lemma 5.1 gives

$$J(t) \leq \frac{1}{K}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\nabla\varphi(x)-\nabla\varphi(y)\big)\cdot(x-y)\frac{\rho(x,t)\rho(y,t)}{|x-y|^{n-\gamma+2}}dxdy. \tag{5.20}$$

For $M>0$, one has the following identity

$$\begin{aligned}M^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\rho(x,t)\rho(y,t)dxdy\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\rho(x,t)\rho(y,t)\frac{|x-y|^s}{(1+|x|^{2-\nu}+|y|^{2-\nu})^\delta}\frac{(1+|x|^{2-\nu}+|y|^{2-\nu})^\delta}{|x-y|^s}dxdy,\end{aligned}\tag{5.21}$$

here $s,\delta>0$ are two constants. Hölder's inequality implies that for $p>1$ and $p'=\frac{p}{p-1}$ such that

$$p=\frac{1}{\delta}, s=(n-\gamma)\delta, sp'+(2-\nu)\delta p'=\nu,\tag{5.22}$$

we have

$$\begin{aligned}M^2 &\le J(t)^{\frac{1}{p}}\\ &\cdot\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\rho(x,t)\rho(y,t)|x-y|^{sp'}\big(1+|x|^{2-\nu}+|y|^{2-\nu}\big)^{\delta p'}dxdy\Big)^{\frac{1}{p'}}.\end{aligned}\tag{5.23}$$

Relation (5.22) and inequality (5.7) in Lemma 5.1 show that there exists a constant $C_0>0$ satisfying

$$|x-y|^{sp'}(1+|x|^{2-\nu}+|y|^{2-\nu})^{\delta p'}\le C_0(1+\varphi(x)+\varphi(y)).\tag{5.24}$$

Put (5.24) into (5.23), one get

$$M^2\le C_0^{\frac{1}{p'}}J(t)^{\frac{1}{p}}(M^2+2M\omega(t))^{\frac{1}{p'}}.\tag{5.25}$$

(5.25) and (5.20) imply that

$$K(M^2C_0^{-\frac{1}{p'}}(M^2+2M\omega(t))^{-\frac{1}{p'}})^p\le\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(\nabla\varphi(x)-\nabla\varphi(y)\big)\cdot(x-y)\frac{\rho(x,t)\rho(y,t)}{|x-y|^{n-\gamma+2}}dxdy.\tag{5.26}$$

From (5.8) and (5.26), (5.17) can be changed to be

$${}_0^cD_t^\beta\omega(t)\le C_5M-C_6\frac{M^{2p}}{(M^2+2M\omega(t))^{\frac{p}{p'}}},\tag{5.27}$$

here $C_5=\|(-\triangle)^{\alpha/2}\varphi\|_\infty$ and $C_6>0$.
If $M=M_0$ is large enough such that

$$C_5M_0-C_6\frac{M_0^{2p}}{M_0^{\frac{2p}{p'}}}<0,\tag{5.28}$$

then there exists $C_7=C_7(M_0)>0$ such that for $0<\omega(0)\le C_7$, one has

$$C_5M_0-C_6\frac{M_0^{2p}}{(M_0^2+2M_0\omega(0))^{\frac{p}{p'}}}<0.\tag{5.29}$$

Due to (5.27), (5.32), Lemma 5.2 implies that $\omega(t) \leq \omega(0)$ which leads to

$$
{}_0^cD_t^\beta \omega(t) \leq C_5 M_0 - C_6 \frac{M_0^{2p}}{(M_0^2 + 2M_0\omega(0))^{\frac{p}{p'}}} < 0. \tag{5.30}
$$

Therefore, by Lemma 5.4, from (5.30), one can deduce that there exists $0 < T^* < \infty$ such that $\omega(T^*) = 0$ which is a contradiction with the global existence of nonnegative solutions of (1.1). Then if we assume that

$$
\int_{\mathbb{R}^n} |x|^\nu \rho_0(x)dx \leq C_7, \int_{\mathbb{R}^n} \rho_0(x)dx = M_0, \tag{5.31}
$$

(5.7) in Lemma 5.1 gives $0 < \omega(0) \leq C_7$ which follows that (5.30) holds true.

If $M = \int_{\mathbb{R}^n} \rho(x)dx = \int_{\mathbb{R}^n} \rho_0(x)dx$ does not satisfy the condition (5.28) of M_0, we can discuss it as follows.

The scaling property (5.4) of (1.1) gives that if let $\lambda^{\alpha+\gamma-2-n} = \frac{M_0}{M}$, transformation of variables imply that

$$
\begin{aligned}
\int_{\mathbb{R}^n} \rho^\lambda(x,t)dx &= \int_{\mathbb{R}^n} \lambda^{\alpha+\gamma-2} \rho(\lambda x, \lambda^{\frac{\alpha}{\beta}} t)dx \\
&= \lambda^{\alpha+\gamma-2-n} \int_{\mathbb{R}^n} \rho(x, \lambda^{\frac{\alpha}{\beta}} t)dx \\
&= M_0 \\
&= \int_{\mathbb{R}^n} \rho^\lambda(x,0)dx.
\end{aligned} \tag{5.32}
$$

And since

$$
\begin{aligned}
\int_{\mathbb{R}^n} |x|^\nu \rho^\lambda(x,0)dx &= \int_{\mathbb{R}^n} |x|^\nu \lambda^{\alpha+\gamma-2} \rho(\lambda x, 0)dx \\
&= \lambda^{\alpha+\gamma-2-n-\nu} \int_{\mathbb{R}^n} |x|^\nu \rho_0(x)dx \\
&= (\frac{M_0}{M})^{1-\frac{\nu}{\alpha+\gamma-2-n}} \int_{\mathbb{R}^n} |x|^\nu \rho_0(x)dx
\end{aligned} \tag{5.33}
$$

Thus, in this case, the condition of the initial data (5.31) is changed to be

$$
M_0 = \int_{\mathbb{R}^n} \rho^\lambda(x,0)dx,
$$

and

$$
\int_{\mathbb{R}^n} |x|^\nu \rho_0(x)dx \leq C_4 M_0^{-1+\frac{\nu}{\alpha+\gamma-2-n}} \big(\int_{\mathbb{R}^n} \rho_0(x)dx\big)^{1-\frac{\nu}{\alpha+\gamma-2-n}}.
$$

□

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