Journal of Computational Physics 238 (2013) 154–168

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

A high order schema for the numerical solution of the fractional ordinary differential equations

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ARTICLE INFO

Article history:
Received 6 April 2012
Received in revised form 5 December 2012
Accepted 19 December 2012
Available online 3 January 2013

Keywords:
Fractional ordinary differential equations
High order schema
Stability and convergence analysis

ABSTRACT

In this paper we present a general technique to construct high order schemes for the numerical solution of the fractional ordinary differential equations (FODEs). This technique is based on the so-called block-by-block approach, which is a common method for the integral equations. In our approach, the classical block-by-block approach is improved in order to avoiding the coupling of the unknown solutions at each block step with an exception in the first two steps, while preserving the good stability property of the block-by-block schemes. By using this new approach, we are able to construct a high order schema for FODEs of the order $\alpha, \alpha > 0$. The stability and convergence of the schema is rigorously established. We prove that the numerical solution converges to the exact solution with order $3+\alpha$ for $0<\alpha \leqslant 1$, and order 4 for $\alpha > 1$. A series of numerical examples are provided to support the theoretical claims.

1. Introduction

The fractional ordinary differential equations (FODEs) of the form $D_t^\alpha u(t) = f(t, u(t))$ appears in many practical and scientific applications, where $\alpha > 0$, D_t^α is the fractional differential operator, which can be defined in some different but closely related ways. Some particular examples of this kind of problems can be found in control theory [32,38], viscoelastic materials [2,4,20,29], electrochemical processes [17], anomalous diffusion [7,8,15,19], signal processing [31], advection and dispersion of solutes in natural porous or fractured media [5,6], information theory [30], and etc. Some other related equations have also been widely used in many different fields, such as chaos [42], biochemistry [41], hydrology [36], finance [34], and so on. We can see [1,3,35] for some more recent developments.

Unlike integer order differential equations, the theoretical investigation of FDEs is very sparse in the literature. Diethelm et al. [11] considered the well-posedness of the initial value problems of FODEs. Kilbas et al. investigated the solution expression of Volterra integro-differential equations by using the generalized Mittag–Leffler function [18]. The recent development concerning the theoretical aspects of FODEs is given by Diethelm in the book [9].

For a general right hand side function f, it is usually difficult to obtain the analytical solution for a fractional differential equation. Thus there is a need to develop numerical methods for FODEs. While there exist enormous literatures on the numerical investigation for integer order differential equations, the investigation of numerical methods for FODEs are quite limited. This is probably due to the fact that the theoretical analysis of numerical methods of FODEs has been found to be very difficult. Nevertheless, we are seeing a growing interest for research of numerical methods in this topic. Langlands and Henry [22] considered the fractional diffusion equation, and analyze the L1 schema for the approximation of the

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http://dx.doi.org/10.1016/j.jcp.2012.12.013

fractional order time derivative. Sun and Wu [39] constructed a difference schema with L_∞ approximation for the fractional time derivative. Diethelm et al. proposed a predictor–corrector schema and a fractional Adams method for the numerical solution of FODEs [12,13]. Lin and Xu [24] analyzed a finite difference schema for the time discretization of the time fractional diffusion equation, and proved that the convergence in time is of $2-\alpha$ order. The convergence order of the above mentioned schemes is no more than 3. Lin and Liu [23] analyzed a linear multistep method and proved the stability and convergence of the method. Kumar and Agrawal [21] proposed a block-by-block method for a class of initial value problems of FODEs. Then Huang et al. [16] proved that the convergence of this method is of order at least 3. These methods take advantage of the fact that FODEs can be transformed into Volterra type integral equations. Therefore, the numerical schemes such as the block-by-block method for Volterra type integral equations [25,26,37,40] can be applied for the solution of FODEs.

In this paper, we construct and analyze a fractional high order numerical method for nonlinear FODEs. The method follows the idea of the block-by-block approach [21], which is a kind of linear multistep methods for the integral equations [27]. It is known that the p-block-by-block approach leads to a system of p unknowns $u^{mp+1}, u^{mp+2}, \cdots$, and u^{mp+p} at the block step $m+1$. The solution of this coupled p-system could be difficult and expensive, especially when the space variable is involved, i.e., when the unknowns are also functions in x. The main novelty of this paper is that by modifying the classical block-by-block technique, we obtain a high order schema allowing to solve the unknowns separably. The advantage of this new method is that the unknowns is completely decoupled at each block step except of the first two steps, while it preserves the same accuracy as the usual block-by-block schemes. This advantage would considerably reduce the computational cost if the schema is generalized to time FDEs, $D_t^\alpha u(t,x) = f(t,x,u(t,x),\Delta u(t,x))$ for example. Although the coefficients in the new schema is entirely different from the usual ones, we are able to derive the stability and error estimates. We prove that convergence order of the schema is $3+\alpha$ for $0<\alpha\leqslant 1$, and 4 for $\alpha>1$. To our knowledge, this is the first schema of such order with a rigorous convergence proof. Several numerical tests are conducted to support the theoretical results.

The outline of the paper is as follows: In Section 2 we describe the detailed construction of the high order schema based on a modified 2-block-by-block method. Then in Section 3, we derive an estimate for the local errors through a series of lemmes. The stability and convergence analysis is given in Section 4. Finally, some numerical experiments are provided in Section 5 to support the theoretical statement. Some concluding remarks are given in the final section.

2. A high order schema

We consider the following fractional ordinary differential equation

$$D_t^\alpha u(t) = f(t,u(t)), \quad 0\leqslant t\leqslant T,\ \alpha>0, \tag{2.1}$$

subject to the initial conditions:

$$u^{(k)}(0) = u_0^{(k)}, \quad k=0,1,\ldots,n-1, \tag{2.2}$$

where n is the integer such that $n-1<\alpha\leqslant n$, $u^{(k)}(t)$ means the kth derivative of u, and the real numbers $u_0^{(k)}, k=0,1,\ldots,n-1$, are assumed to be given. The fractional derivative $D_t^\alpha u(t)$ in (2.1) is defined, in the Caputo sense [33], by

$$D_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}u^{(n)}(\tau)d\tau, \tag{2.3}$$

where $\Gamma(\cdot)$ denotes Gamma function.

Some work on numerical methods for (2.1) have considered to directly discretize the fractional derivative (2.3) by using finite difference schemes [22,24,39]. However to our knowledge, this idea never leads to a finite difference schema of order higher than $2-\alpha$ for $0<\alpha<1$ with a rigorous proof. It is noted that Lubich [27,28] has considered a kind of high order finite difference schemes based on fractional multistep methods. It theoretically showed how to construct a high order schema for a fractional derivative. However as indicated by Diethelm et al. in [10] that the practical implementation of this approach can be very difficult, and an unsuitable choice of the starting values and starting weights may lead to unstable calculation. Some other work, e.g., [16,23], take advantages of the Volterra type integral representation of (2.1) so that the existing techniques [21] for integral equations can be applied. In this paper we will modify the classical block-by block approach to construct a high order schema for (2.1).

It has been proved [11] that the initial value problem (2.1), (2.2), if a continuous solution is admitted, is equivalent to the following Volterra integral equation

$$u(t) = g(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau, \tag{2.4}$$

where $g(t)$ is given as

$$g(t) = \sum_{k=0}^{n-1} u_0^{(k)}\frac{t^k}{k!}.$$

In order to construct the high order schema, we divide the interval $[0,T]$ into $2N$ equal sub-intervals with size $\Delta t = \frac{T}{2N}$, and denote $t_j = j\Delta t, j = 0,1,\cdots,2N$. The numerical solution of (2.4) at the point t_j is denoted by u_j. Set $g_j = g(t_j), f_j = f(t_j,u_j)$.

The idea is as follows. We start with computing the solution at first two steps. Let us first try to determine the value of $u(t)$ at t_1. Using quadratic interpolation, $f(t,u(t))$ can be approximated in the interval $[t_0,t_1]$ as

$$f(t,u(t)) \approx \varphi_{0,0}(t)f_0 + \varphi_{1,0}(t)f_{\frac{1}{2}} + \varphi_{2,0}(t)f_1, \quad \forall t \in [t_0,t_1], \tag{2.5}$$

where $t_{\frac{1}{2}} = t_0 + \frac{\Delta t}{2}, f_{\frac{1}{2}} = f(t_{\frac{1}{2}}, u(t_{\frac{1}{2}}))$, and $\varphi_{i,0}(t), i = 0,1,2$, are quadratic interpolating functions, defined by

$$\varphi_{0,0}(t) = \frac{2(t-t_{\frac{1}{2}})(t-t_1)}{\Delta t^2}, \quad \varphi_{1,0}(t) = \frac{-4(t-t_0)(t-t_1)}{\Delta t^2}, \quad \varphi_{2,0}(t) = \frac{2(t-t_0)(t-t_{\frac{1}{2}})}{\Delta t^2}. \tag{2.6}$$

Substituting (2.5) into (2.4), then integrating the right hand side, we obtain an approximation

$$u(t_1) \approx g(t_1) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-\tau)^{\alpha-1}\left[\varphi_{0,0}(\tau)f_0 + \varphi_{1,0}(\tau)f_{\frac{1}{2}} + \varphi_{2,0}(\tau)f_1\right]d\tau = g(t_1) + \omega_1^{0,0}f_0 + \omega_1^{1,0}f_{\frac{1}{2}} + \omega_1^{2,0}f_1, \tag{2.7}$$

where

$$\omega_1^{i,0} = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-\tau)^{\alpha-1}\varphi_{i,0}(\tau)d\tau, \quad i = 0,1,2, \tag{2.8}$$

which can be exactly computed. The value of $f_{\frac{1}{2}}$ is approximated by using the interpolation

$$f_{\frac{1}{2}} \approx \frac{3}{8}f_0 + \frac{3}{4}f_1 - \frac{1}{8}f_2. \tag{2.9}$$

Substituting (2.9) into (2.7), we obtain the schema for the first step:

$$u_1 = g_1 + A_1^{0,0}f_0 + A_1^{1,0}f_1 + A_1^{2,0}f_2, \tag{2.10}$$

where

$$A_1^{0,0} = \omega_1^{0,0} + \frac{3}{8}\omega_1^{1,0}, A_1^{1,0} = \frac{3}{4}\omega_1^{1,0} + \omega_1^{2,0}, A_1^{2,0} = -\frac{1}{8}\omega_1^{1,0}. \tag{2.11}$$

Note that computing u_1 through (2.10) requires the values of f (or indirectly, the values of u) at t_1 and t_2. Particularly, the dependence of u_1 on f_2 means (2.10) has to be solved simultaneously with the schema of the second step.

Similarly, to compute the value of $u(t)$ at t_2, we approximate $f(t,u(t))$ on $[t_0,t_2]$ as

$$f(t,u(t)) \approx \psi_{0,0}(t)f_0 + \psi_{1,0}(t)f_1 + \psi_{2,0}(t)f_2, \quad \forall t \in [t_0,t_2], \tag{2.12}$$

where $\psi_{i,0}(t), i = 0,1,2$, are the quadratic interpolating functions:

$$\psi_{0,0}(t) = \frac{(t-t_1)(t-t_2)}{2\Delta t^2}, \quad \psi_{1,0}(t) = \frac{(t-t_0)(t-t_2)}{-\Delta t^2}, \quad \psi_{2,0}(t) = \frac{(t-t_0)(t-t_1)}{2\Delta t^2}.$$

Bringing (2.12) into (2.4) leads to the schema for the second step:

$$u_2 = g_2 + A_2^{0,0}f_0 + A_2^{1,0}f_1 + A_2^{2,0}f_2, \tag{2.13}$$

where

$$A_2^{i,0} = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2}(t_2-\tau)^{\alpha-1}\psi_{i,0}(\tau)d\tau, \quad i = 0,1,2. \tag{2.14}$$

As we have just mentioned, the first two step solutions u_1 and u_2 are coupled in (2.10) and (2.13), thus need to be solved simultaneously.

Now we construct the schema for the next steps. Assuming that $u_j, j = 0,1,\cdots,2m$, are already known, we want to derive an approximation to $u(t_{2m+1})$ and $u(t_{2m+2})$. By following the same lines as for the previous steps, we have

$$\begin{aligned} u(t_{2m+1}) &= g(t_{2m+1}) + \frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau \\ &= g(t_{2m+1}) + \frac{1}{\Gamma(\alpha)}\left[\int_0^{t_1}(t_{2m+1}-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau + \sum_{k=1}^{m}\int_{t_{2k-1}}^{t_{2k+1}}(t_{2m+1}-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau\right] \\ &\approx g(t_{2m+1}) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_{2m+1}-\tau)^{\alpha-1}\left[\varphi_{0,0}(\tau)f_0 + \varphi_{1,0}(\tau)f_{\frac{1}{2}} + \varphi_{2,0}(\tau)f_1\right]d\tau \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{2k-1}}^{t_{2k+1}}(t_{2m+1}-\tau)^{\alpha-1}\left[\varphi_{0,k}(\tau)f_{2k-1} + \varphi_{1,k}(\tau)f_{2k} + \varphi_{2,k}(\tau)f_{2k+1}\right]d\tau, \end{aligned} \tag{2.15}$$

where $\varphi_{i,0}(t), i = 0,1,2$, are defined in (2.6), and $\varphi_{i,k}(t), i = 0,1,2; k = 1,\dots,m$, are quadratic Lagrange polynomials associated with the points $t_{2k-1}, t_{2k}, t_{2k+1}$:

$$\varphi_{0,k}(t) = \frac{(t-t_{2k})(t-t_{2k+1})}{2\Delta t^2}, \quad \varphi_{1,k}(t) = \frac{(t-t_{2k-1})(t-t_{2k+1})}{-\Delta t^2}, \quad \varphi_{2,k}(t) = \frac{(t-t_{2k-1})(t-t_{2k})}{2\Delta t^2}.$$

Using (2.9), we arrive at the following schema:

$$u_{2m+1} = g_{2m+1} + A^{0,0}_{2m+1} f_0 + A^{1,0}_{2m+1} f_1 + A^{2,0}_{2m+1} f_2 + \sum_{k=1}^{m}\left[A^{0,k}_{2m+1} f_{2k-1} + A^{1,k}_{2m+1} f_{2k} + A^{2,k}_{2m+1} f_{2k+1}\right], \tag{2.16}$$

where

$$A^{0,0}_{2m+1} = \omega^{0,0}_{2m+1} + \frac{3}{8}\omega^{1,0}_{2m+1}, A^{1,0}_{2m+1} = \frac{3}{4}\omega^{1,0}_{2m+1} + \omega^{2,0}_{2m+1}, A^{2,0}_{2m+1} = -\frac{1}{8}\omega^{1,0}_{2m+1}, \tag{2.17}$$

$$A^{i,k}_{2m+1} = \frac{1}{\Gamma(\alpha)}\int_{t_{2k-1}}^{t_{2k+1}} (t_{2m+1}-\tau)^{\alpha-1}\varphi_{i,k}(\tau)d\tau, \quad i = 0,1,2;\ k = 1,\dots,m, \tag{2.18}$$

with

$$\omega^{i,0}_{2m+1} = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1} (t_{2m+1}-\tau)^{\alpha-1}\varphi_{i,0}(\tau)d\tau, \quad i = 0,1,2.$$

To compute $u(t_{2m+2})$, we use the following approximation:

$$\begin{aligned} u(t_{2m+2}) &= g(t_{2m+2}) + \frac{1}{\Gamma(\alpha)}\int_0^{t_{2m+2}} (t_{2m+2}-\tau)^{\alpha-1} f(\tau, u(\tau))d\tau \\ &= g(t_{2m+2}) + \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m}\int_{t_{2k}}^{t_{2k+2}} (t_{2m+2}-\tau)^{\alpha-1} f(\tau, u(\tau))d\tau \\ &\approx g(t_{2m+2}) + \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m}\int_{t_{2k}}^{t_{2k+2}} (t_{2m+2}-\tau)^{\alpha-1}\left[\psi_{0,k}(\tau)f_{2k} + \psi_{1,k}(\tau)f_{2k+1} + \psi_{2,k}(\tau)f_{2k+2}\right]d\tau, \end{aligned} \tag{2.19}$$

which leads to the schema at the step $2m+2$:

$$u_{2m+2} = g_{2m+2} + \sum_{k=0}^{m}\left[A^{0,k}_{2m+2} f_{2k} + A^{1,k}_{2m+2} f_{2k+1} + A^{2,k}_{2m+2} f_{2k+2}\right], \tag{2.20}$$

where

$$A^{i,k}_{2m+2} = \frac{1}{\Gamma(\alpha)}\int_{t_{2k}}^{t_{2k+2}} (t_{2m+2}-\tau)^{\alpha-1}\psi_{i,k}(\tau)d\tau, \quad i = 0,1,2;\ k = 0,1,\dots,m, \tag{2.21}$$

with $\psi_{i,k}(t)$ being quadratic Lagrange polynomials associated with the points $t_{2k}, t_{2k+1}, t_{2k+2}$.

To summarize, that is by combining (2.10), (2.13), (2.16), and (2.20), we arrive at the following overall schema:

$$\begin{cases} u_1 = g_1 + A^{0,0}_1 f_0 + A^{1,0}_1 f_1 + A^{2,0}_1 f_2, \\ u_2 = g_2 + A^{0,0}_2 f_0 + A^{1,0}_2 f_1 + A^{2,0}_2 f_2, \\ u_{2m+1} = g_{2m+1} + A^{0,0}_{2m+1} f_0 + A^{1,0}_{2m+1} f_1 + A^{2,0}_{2m+1} f_2 + \sum_{k=1}^{m}\left[A^{0,k}_{2m+1} f_{2k-1} + A^{1,k}_{2m+1} f_{2k} + A^{2,k}_{2m+1} f_{2k+1}\right], \\ u_{2m+2} = g_{2m+2} + \sum_{k=0}^{m}\left[A^{0,k}_{2m+2} f_{2k} + A^{1,k}_{2m+2} f_{2k+1} + A^{2,k}_{2m+2} f_{2k+2}\right], \\ m = 1,\dots,N-1. \end{cases} \tag{2.22}$$

Remark 2.1. In a standard block-by-block approach, $u(t_{2m+1})$ is approximated by using the integral splitting:

$$u(t_{2m+1}) = g(t_{2m+1}) + \frac{1}{\Gamma(\alpha)}\left[\sum_{k=1}^{m}\int_{t_{2k-2}}^{t_{2k}} (t_{2m+1}-\tau)^{\alpha-1} f(\tau, u(\tau))d\tau + \int_{t_{2m}}^{t_{2m+1}} (t_{2m+1}-\tau)^{\alpha-1} f(\tau, u(\tau))d\tau\right], \tag{2.23}$$

instead of (2.15). Then the last integral in the right hand side of (2.23) is approximated by using quadratic interpolation based on the points t_{2m}, t_{2m+1}, and t_{2m+2}. By doing so, one would obtain a schema for u_{2m+1} depending on f_{2m+2} [21], and thus obtain a coupled system for u_{2m+1} and u_{2m+2}. It is clear that solving (2.22) is easier that solving a coupling system on u_{2m+1} and u_{2m+2} for all $m = 1,\dots,N-1$.

In the next two sections, we will give a stability and convergence analysis for the above schema. We start with an analysis for the local error estimation.

3. Estimation of the truncation errors

3.1. Auxiliary results

We present some lemmas which will be used later on. We hereafter denote by C a generic constant which may not be the same at different occurrences, but independent of all discretization parameters.

Theorem 3.1 (*Discrete Gronwall Inequality*, [14]). *Let $a_i, 0 \leqslant i \leqslant N$, be a sequence of non-negative real numbers satisfying*

$$a_i \leqslant b_i + M\Delta t^{\gamma}\sum_{j=0}^{i-1}(i-j)^{\gamma-1}a_j, \quad 1 \leqslant i \leqslant N, \tag{3.1}$$

where $0 < \gamma \leqslant 1, M > 0$ is bounded independently of Δt, and $b_i, 0 \leqslant i \leqslant N$, is a monotonic increasing sequence of non-negative real numbers. Then

$$a_i \leqslant b_i E_{\gamma}(M\Gamma(\gamma)(i\Delta t)^{\gamma}), \quad 0 \leqslant i \leqslant N, \tag{3.2}$$

where

$$E_{\gamma}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\gamma k+1)}$$

is the Mittag–Leffler function of order γ.

In particular, when $\gamma = 1$, (3.2) becomes

$$a_i \leqslant b_i \exp(Mi\Delta t), \quad 0 \leqslant i \leqslant N. \tag{3.3}$$

3.2. Truncation errors

Now we turn to derive an estimate for the truncation errors of the schema (2.22). We start with analyzing the local errors at the odd steps. To this end, we define the truncation error at the step $2m+1$ by

$$r_{2m+1}(\Delta t) := u(t_{2m+1}) - \bar{u}_{2m+1}, \tag{3.4}$$

where $\bar{u}_{2m+1}$ is an approximation to $u(t_{2m+1})$, evaluated by using the schema (2.16) with exact previous solutions, i.e.,

$$\begin{aligned}\bar{u}_{2m+1} = {} & g_{2m+1} + A^{0,0}_{2m+1}f(t_0, u(t_0)) + A^{1,0}_{2m+1}f(t_1, u(t_1)) + A^{2,0}_{2m+1}f(t_2, u(t_2)) + \sum_{k=1}^{m}[A^{0,k}_{2m+1}f(t_{2k-1}, u(t_{2k-1})) \\ & + A^{1,k}_{2m+1}f(t_{2k}, u(t_{2k})) + A^{2,k}_{2m+1}f(t_{2k+1}, u(t_{2k+1}))].\end{aligned} \tag{3.5}$$

Then we derive the following estimate for $r_{2m+1}(\Delta t)$ by using some ideas similar to that employed in [13].

Lemma 3.1. *Let $r_{2m+1}(\Delta t)$ being the truncation error defined in (3.4). If $f(\cdot, u(\cdot)) \in C^4[0, T]$, then it holds*

$$|r_{2m+1}(\Delta t)| \leqslant C\Delta t^{3+\alpha}$$

if $0 < \alpha \leqslant 1$, and

$$|r_{2m+1}(\Delta t)| \leqslant C\Delta t^{4}$$

if $\alpha > 1$.

Proof. By comparing (2.15), (2.16), and (3.5), we have

$$\begin{aligned}r_{2m+1}(\Delta t) = {} & u(t_{2m+1}) - \left\{g_{2m+1} + f(t_0, u(t_0))\omega^{0,0}_{2m+1} + \left[\frac{3}{8}f(t_0, u(t_0)) + \frac{3}{4}f(t_1, u(t_1)) - \frac{1}{8}f(t_2, u(t_2))\right]\omega^{1,0}_{2m+1} + f(t_1, u(t_1))\omega^{2,0}_{2m+1}\right. \\ & \left. + \sum_{k=1}^{m}\left[A^{0,k}_{2m+1}f(t_{2k-1}, u(t_{2k-1})) + A^{1,k}_{2m+1}f(t_{2k}, u(t_{2k})) + A^{2,k}_{2m+1}f(t_{2k+1}, u(t_{2k+1}))\right]\right\} \\ = {} & g(t_{2m+1}) + \frac{1}{\Gamma(\alpha)}\left[\int_0^{t_1}(t_{2m+1}-\tau)^{\alpha-1}f(\tau, u(\tau))d\tau + \sum_{k=1}^{m}\int_{t_{2k-1}}^{t_{2k+1}}(t_{2m+1}-\tau)^{\alpha-1}f(\tau, u(\tau))d\tau\right] \\ & - \left\{g(t_{2m+1}) + \frac{1}{\Gamma(\alpha)}\left[f(t_0, u(t_0))\int_0^{t_1}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{0,0}(\tau)d\tau + \left(\frac{3}{8}f(t_0, u(t_0)) + \frac{3}{4}f(t_1, u(t_1)) - \frac{1}{8}f(t_2, u(t_2))\right)\right.\right.\end{aligned}$$

$$\int_0^{t_1}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{1,0}(\tau)d\tau+f(t_1,u(t_1))\int_0^{t_1}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{2,0}(\tau)d\tau\Bigg]$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\Bigg[f(t_{2k-1},u(t_{2k-1}))\int_{t_{2k-1}}^{t_{2k+1}}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{0,k}(\tau)d\tau+f(t_{2k},u(t_{2k}))\int_{t_{2k-1}}^{t_{2k+1}}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{1,k}(\tau)d\tau$$
$$+f(t_{2k+1},u(t_{2k+1}))\int_{t_{2k-1}}^{t_{2k+1}}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{2,k}(\tau)d\tau\Bigg]\Bigg\}$$
$$=\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_{2m+1}-\tau)^{\alpha-1}\Big\{f(\tau,u(\tau))-[f(t_0,u(t_0))\varphi_{0,0}(\tau)+f(t_{\frac12},u(t_{\frac12}))\varphi_{1,0}(\tau)+f(t_1,u(t_1))\varphi_{2,0}(\tau)]+\Big[f(t_{\frac12},u(t_{\frac12}))$$
$$-\left(\frac{3}{8}f(t_0,u(t_0))+\frac{3}{4}f(t_1,u(t_1))-\frac{1}{8}f(t_2,u(t_2))\right)\Big]\varphi_{1,0}(\tau)\Big\}d\tau$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{2k-1}}^{t_{2k+1}}(t_{2m+1}-\tau)^{\alpha-1}\Big\{f(\tau,u(\tau))-[\varphi_{0,k}(\tau)f(t_{2k-1},u(t_{2k-1}))$$
$$+\varphi_{1,k}(\tau)f(t_{2k},u(t_{2k}))+\varphi_{2,k}(\tau)f(t_{2k+1},u(t_{2k+1}))]\Big\}d\tau$$
$$=\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_{2m+1}-\tau)^{\alpha-1}(R_0(\tau)+R_1(\tau)\varphi_{1,0}(\tau))d\tau+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{2k-1}}^{t_{2k+1}}(t_{2m+1}-\tau)^{\alpha-1}R_{2k-1}(\tau)d\tau,$$

where

$$R_0(\tau):=f(\tau,u(\tau))-\varphi_{0,0}(\tau)f(t_0,u(t_0))-\varphi_{1,0}(\tau)f(t_{\frac12},u(t_{\frac12}))-\varphi_{2,0}(\tau)f(t_1,u(t_1)),$$

$$R_1(\tau):=f(t_{\frac12},u(t_{\frac12}))-\left(\frac{3}{8}f(t_0,u(t_0))+\frac{3}{4}f(t_1,u(t_1))-\frac{1}{8}f(t_2,u(t_2))\right),$$

$$R_{2k-1}(\tau):=f(\tau,u(\tau))-\varphi_{0,k}(\tau)f(t_{2k-1},u(t_{2k-1}))-\varphi_{1,k}(\tau)f(t_{2k},u(t_{2k}))-\varphi_{2,k}(\tau)f(t_{2k+1},u(t_{2k+1})).$$

By using Taylor theorem, it can be checked that for all $\tau\in[t_0,t_1]$ there exist $\xi_1(\tau),\xi(\tau)\in[t_0,t_1]$, such that

$$R_0(\tau)=\frac{f^{(3)}(\xi_1(\tau),u(\xi_1(\tau)))}{3!}(\tau-t_0)(\tau-t_{\frac12})(\tau-t_1),$$

$$R_1(\tau)=\frac{1}{16}\Delta t^3f^{(3)}(\xi(\tau),u(\xi(\tau))),$$

and that for all $\tau\in[t_{2k-1},t_{2k+1}]$ there exists $\xi_k(\tau)\in[t_{2k-1},t_{2k+1}]$, such that

$$R_{2k-1}(\tau)=\frac{f^{(3)}(\xi_k(\tau),u(\xi_k(\tau)))}{3!}(\tau-t_{2k-1})(\tau-t_{2k})(\tau-t_{2k+1}),\forall\tau\in[t_{2k-1},t_{2k+1}].$$

Therefore, we have

$$r_{2m+1}(\Delta t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_{2m+1}-\tau)^{\alpha-1}\frac{f^{(3)}(\xi_1(\tau),u(\xi_1(\tau)))}{3!}(\tau-t_0)(\tau-t_{\frac12})(\tau-t_1)d\tau+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_{2m+1}-\tau)^{\alpha-1}$$
$$\times\frac{1}{16}\Delta t^3f^{(3)}(\xi(\tau),u(\xi(\tau)))\varphi_{1,0}(\tau)d\tau+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{2k-1}}^{t_{2k+1}}(t_{2m+1}-\tau)^{\alpha-1}\frac{f^{(3)}(\xi_k(\tau),u(\xi_k(\tau)))}{3!}(\tau-t_{2k-1})(\tau-t_{2k})$$
$$(\tau-t_{2k+1})d\tau,\tag{3.6}$$

It remains to estimate the right hand side term by term. For the first term, denoted by $R1$, we have

$$|R1|\leqslant\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_{2m+1}-\tau)^{\alpha-1}\left|\frac{f^{(3)}(\xi_1(\tau),u(\xi_1(\tau)))}{3!}(\tau-t_0)(\tau-t_{\frac12})(\tau-t_1)\right|d\tau\leqslant\frac{M_1\Delta t^3}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_{2m+1}-\tau)^{\alpha-1}d\tau$$
$$\leqslant\frac{M_1\Delta t^4}{\Gamma(\alpha)}(t_{2m+1}-\tau^*)^{\alpha-1},\tag{3.7}$$

where $M_1=\sup_{t\in[0,T]}|f^{(3)}(t,u(t))|,\tau^*\in(t_0,t_1)$.

For the second term, denoted by $R2$, we have

$$|R2|\leqslant\frac{M_1\Delta t^3}{16\Gamma(\alpha)}\left|\int_{t_0}^{t_1}(t_{2m+1}-\tau)^{\alpha-1}\frac{(\tau-t_0)(\tau-t_1)}{-\frac14\Delta t^2}d\tau\right|\leqslant\frac{M_1\Delta t^3}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_{2m+1}-\tau)^{\alpha-1}d\tau\leqslant\frac{M_1\Delta t^4}{\Gamma(\alpha)}(t_{2m+1}-\tau^*)^{\alpha-1}.\tag{3.8}$$

The third term, $R3$, can be bounded by

$$
\begin{aligned}
|R3| &:= \left| \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{2k-1}}^{t_{2k+1}} (t_{2m+1}-\tau)^{\alpha-1} \frac{f^{(3)}(\xi_k(\tau), u(\xi_k(\tau)))}{3!} (\tau - t_{2k-1})(\tau - t_{2k})(\tau - t_{2k+1}) d\tau \right| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \left\{ \left| \int_{t_{2k-1}}^{t_{2k+1}} (t_{2m+1}-\tau)^{\alpha-1} \frac{f^{(3)}(\tilde{\xi}_k, u(\tilde{\xi}_k))}{3!} (\tau - t_{2k-1})(\tau - t_{2k})(\tau - t_{2k+1}) d\tau \right| \right. \\
&\quad \left. + \left| \int_{t_{2k-1}}^{t_{2k+1}} (t_{2m+1}-\tau)^{\alpha-1} \frac{f^{(3)}(\xi_k(\tau), u(\xi_k(\tau))) - f^{(3)}(\tilde{\xi}_k, u(\tilde{\xi}_k))}{3!} (\tau - t_{2k-1})(\tau - t_{2k})(\tau - t_{2k+1}) d\tau \right| \right\},
\end{aligned} \tag{3.9}
$$

where $\tilde{\xi}_k = t_{2k}$. For the first term in the right hand side of (3.9), we have

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \left| \int_{t_{2k-1}}^{t_{2k+1}} (t_{2m+1}-\tau)^{\alpha-1} \frac{f^{(3)}(\tilde{\xi}_k, u(\tilde{\xi}_k))}{3!} (\tau - t_{2k-1})(\tau - t_{2k})(\tau - t_{2k+1}) d\tau \right| \\
&\leqslant \frac{M_1}{\Gamma(\alpha)} \sum_{k=1}^{m} \left| \int_{t_{2k-1}}^{t_{2k+1}} (t_{2m+1}-\tau)^{\alpha-1} (\tau - t_{2k-1})(\tau - t_{2k})(\tau - t_{2k+1}) d\tau \right| \\
&= \frac{M_1}{\Gamma(\alpha)} \sum_{k=1}^{m} \left| \int_{t_{2k-1}}^{t_{2k}} (t_{2m+1}-\tau)^{\alpha-1} (\tau - t_{2k-1})(\tau - t_{2k})(\tau - t_{2k+1}) d\tau + \int_{t_{2k}}^{t_{2k+1}} (t_{2m+1}-\tau)^{\alpha-1} (\tau - t_{2k-1})(\tau - t_{2k})(\tau - t_{2k+1}) d\tau \right| \\
&= \frac{M_1}{\Gamma(\alpha)} \sum_{k=1}^{m} \left| (t_{2m+1}-\tilde{\tau}_k)^{\alpha-1} \int_{t_{2k-1}}^{t_{2k}} (\tau - t_{2k-1})(\tau - t_{2k})(\tau - t_{2k+1}) d\tau + (t_{2m+1}-\bar{\tau}_k)^{\alpha-1} \int_{t_{2k}}^{t_{2k+1}} (\tau - t_{2k-1})(\tau - t_{2k})(\tau - t_{2k+1}) d\tau \right| \\
&= \frac{M_1 \Delta t^4}{4\Gamma(\alpha)} \sum_{k=1}^{m} \left| (t_{2m+1}-\tilde{\tau}_k)^{\alpha-1} - (t_{2m+1}-\bar{\tau}_k)^{\alpha-1} \right| \\
&= \frac{M_1 \Delta t^4}{4(\alpha-1)\Gamma(\alpha)} \sum_{k=1}^{m} \left| (t_{2m+1}-\hat{\tau}_k)^{\alpha-2} (\bar{\tau}_k - \tilde{\tau}_k) \right| \leqslant \frac{M_1 \Delta t^4}{4(\alpha-1)\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{2k-1}}^{t_{2k+1}} (t_{2m+1}-\tau)^{\alpha-2} d\tau \leqslant \frac{M_1 \Delta t^4}{4\Gamma(\alpha)} (t_{2m+1}-t_1)^{\alpha-1}
\end{aligned} \tag{3.10}
$$

where $\tilde{\tau}_k \leqslant \hat{\tau}_k \leqslant \bar{\tau}_k$ and $t_{2k-1} \leqslant \tilde{\tau}_k \leqslant t_{2k} \leqslant \bar{\tau}_k \leqslant t_{2k+1}$.

For the second term in the right hand side of (3.9), it holds

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \left| \int_{t_{2k-1}}^{t_{2k+1}} (t_{2m+1}-\tau)^{\alpha-1} \frac{f^{(3)}(\xi_k(\tau), u(\xi_k(\tau))) - f^{(3)}(\tilde{\xi}_k, u(\tilde{\xi}_k))}{3!} (\tau - t_{2k-1})(\tau - t_{2k})(\tau - t_{2k+1}) d\tau \right| \\
&\leqslant \frac{M_2 \Delta t}{\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{2k-1}}^{t_{2k+1}} (t_{2m+1}-\tau)^{\alpha-1} |(\tau - t_{2k-1})(\tau - t_{2k})(\tau - t_{2k+1})| d\tau \leqslant \frac{M_2 \Delta t^4}{\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{2k-1}}^{t_{2k+1}} (t_{2m+1}-\tau)^{\alpha-1} d\tau \\
&\leqslant \frac{M_2 \Delta t^4}{\alpha\Gamma(\alpha)} (t_{2m+1}-t_1)^{\alpha} \leqslant \frac{M_2 \Delta t^4}{\alpha\Gamma(\alpha)} T^{\alpha},
\end{aligned} \tag{3.11}
$$

where $M_2 = \sup_{t\in[0,T]} |f^{(4)}(t, u(t))|$. In the above derivation we have used the fact that

$$
\left| \frac{f^{(3)}(\xi_k, u(\xi_k)) - f^{(3)}(\tilde{\xi}_k, u(\tilde{\xi}_k))}{3!} \right| \leqslant M_2 \Delta t, \text{for} \tilde{\xi}_k = t_{2k}, \forall \tau \in [t_{2k-1}, t_{2k+1}].
$$

Bringing (3.10), (3.11) into (3.9) gives:

$$
|R3| \leqslant \frac{M_1 \Delta t^4}{4\Gamma(\alpha)} (t_{2m+1}-t_1)^{\alpha-1} + \frac{M_2 \Delta t^4}{\alpha\Gamma(\alpha)} T^{\alpha}. \tag{3.12}
$$

Combining (3.6), (3.7), (3.8), and (3.12) yields

$$
|r_{2m+1}(\Delta t)| \leqslant \frac{2M_1 \Delta t^4}{\Gamma(\alpha)} (t_{2m+1}-\tau^*)^{\alpha-1} + \frac{M_1 \Delta t^4}{4\Gamma(\alpha)} (t_{2m+1}-t_1)^{\alpha-1} + \frac{M_2 \Delta t^4}{\alpha\Gamma(\alpha)} T^{\alpha}. \tag{3.13}
$$

In virtue of (3.13), in the case $0 < \alpha \leqslant 1$ we have

$$
\begin{aligned}
|r_{2m+1}(\Delta t)| &\leqslant \frac{2M_1 \Delta t^4}{\Gamma(\alpha)} (t_{2m+1}-t_1)^{\alpha-1} + \frac{M_1 \Delta t^4}{4\Gamma(\alpha)} (t_{2m+1}-t_1)^{\alpha-1} + \frac{M_2 T^{\alpha}}{\alpha\Gamma(\alpha)} \Delta t^4 = \frac{CM_1 \Delta t^4}{\Gamma(\alpha)} (2m\Delta t)^{\alpha-1} + \frac{M_2 T^{\alpha}}{\alpha\Gamma(\alpha)} \Delta t^4 \\
&\leqslant \frac{CM_1 2^{\alpha-1} m^{\alpha-1}}{\Gamma(\alpha)} \Delta t^{3+\alpha} + \frac{M_2 T^{\alpha}}{\alpha\Gamma(\alpha)} \Delta t^4 \leqslant \frac{CM_1}{\Gamma(\alpha)} \Delta t^{3+\alpha} + \frac{M_2 T^{\alpha}}{\alpha\Gamma(\alpha)} \Delta t^4 \leqslant C\Delta t^{\alpha+3},
\end{aligned}
$$

where C only depends on M_1, M_2, α, and T.

In the case $\alpha > 1$, we get

$$
|r_{2m+1}(\Delta t)| \leqslant \frac{2M_1 T^{\alpha-1}}{\Gamma(\alpha)} \Delta t^4 + \frac{M_1 \Delta t^4}{4\Gamma(\alpha)} T^{\alpha-1} + \frac{M_2 T^{\alpha}}{\alpha\Gamma(\alpha)} \Delta t^4 \leqslant C\Delta t^4.
$$

The proof is completed. □

Similar to the truncation error at the odd steps, we define the truncation error at the even steps:

$$r_{2m+2}(\Delta t) := u(t_{2m+2}) - \bar{u}_{2m+2}, \tag{3.14}$$

with $\bar{u}_{2m+2}$ being an approximation to $u(t_{2m+2})$, evaluated by using (2.20) with exact previous solutions. Then an estimate for $r_{2m+2}(\Delta t)$ is given in the following lemma.

Lemma 3.2. *Let $r_{2m+2}(\Delta t)$ being the truncation error defined in (3.14). If $f(\cdot, u(\cdot)) \in C^4[0, T]$, then it holds*

$$|r_{2m+2}(\Delta t)| \leqslant C\Delta t^{3+\alpha}$$

if $0 < \alpha \leqslant 1$, and

$$|r_{2m+2}(\Delta t)| \leqslant C\Delta t^{4}$$

if $\alpha > 1$.

Proof. The proof is similar to Lemma 3.1, we omit the details. □

As a consequence of Lemmes 3.1 and 3.2, we conclude that the truncation errors of the overall schema is of order $3 + \alpha$ for $0 < \alpha \leqslant 1$, and order 4 for $\alpha > 1$. That is, the truncation errors $r_n(\Delta t), n = 1, 2, \ldots, 2N$, satisfy

$$r_n(\Delta t) \leqslant \begin{cases} C\Delta t^{3+\alpha}, & 0 < \alpha \leqslant 1, \\ C\Delta t^4, & \alpha > 1. \end{cases} \tag{3.15}$$

Remark 3.1. From (2.1), it is seen that the smoothness assumption $f(\cdot, u(\cdot)) \in C^4[0, T]$ is equivalent to require $D_t^{\alpha} u \in C^4[0, T]$. By virtue of the construction and analysis of our schema, which is based on high order interpolation approximation, this assumption does not seem to be removable or relaxable. One notices that, although such a condition can not be easily satisfied in practice, there exist some situations in which the solution of the problem under consideration is smooth (see [9], Section 6.4).

4. Stability and convergence analysis

For ease of notation, we reformulate the schema by introducing the following coefficients:

$$\begin{aligned} &\widehat{B}_i = \frac{A_1^{i,0}}{\Delta t^{\alpha}}, \quad \widetilde{B}_i = \frac{A_2^{i,0}}{\Delta t^{\alpha}}, \quad i = 0, 1, 2, \\ &\overline{B}_0 = \frac{A_{2m+1}^{0,0}}{\Delta t^{\alpha}}, \quad \overline{B}_1 = \frac{A_{2m+1}^{1,0} + A_{2m+1}^{0,1}}{\Delta t^{\alpha}}, \quad \overline{B}_2 = \frac{A_{2m+1}^{2,0} + A_{2m+1}^{1,1}}{\Delta t^{\alpha}}, \\ &B_0 = \frac{A_{2m+2}^{0,0}}{\Delta t^{\alpha}}, \quad B_{2k+1} = \frac{A_{2m+2}^{1,k}}{\Delta t^{\alpha}}, \quad k = 0, 1, \ldots, m, \\ &B_{2k} = \frac{A_{2m+2}^{2,k-1} + A_{2m+2}^{0,k}}{\Delta t^{\alpha}}, \; k = 1, 2, \ldots, m, \; B_{2m+2} = \frac{A_{2m+2}^{2,m}}{\Delta t^{\alpha}}. \end{aligned} \tag{4.1}$$

Noticing from (2.18) and (2.21) that $A_{2m+1}^{i,k} = A_{2m+2}^{i,k}, i = 0, 1, 2; k = 1, 2, \ldots, m$, then the schema (2.22) can be rewritten under an equivalent form as follows:

$$\begin{cases} u_1 = g_1 + \Delta t^{\alpha}\widehat{B}_0 f_0 + \Delta t^{\alpha}\widehat{B}_1 f_1 + \Delta t^{\alpha}\widehat{B}_2 f_2, \\ u_2 = g_2 + \Delta t^{\alpha}\widetilde{B}_0 f_0 + \Delta t^{\alpha}\widetilde{B}_1 f_1 + \Delta t^{\alpha}\widetilde{B}_2 f_2, \\ u_{2m+1} = g_{2m+1} + \Delta t^{\alpha}\sum_{j=0}^{2}\overline{B}_j f_j + \Delta t^{\alpha}\sum_{j=3}^{2m+1} B_{j+1} f_j, \\ u_{2m+2} = g_{2m+2} + \Delta t^{\alpha}\sum_{j=0}^{2m+2} B_j f_j, \quad m = 1, \ldots, N-1. \end{cases} \tag{4.2}$$

Lemma 4.1. *The coefficients B_j and $\overline{B}_j$, defined in (4.1), satisfy*

$$\begin{aligned} &|\overline{B}_j| \leqslant C(2m + 2 - j)^{\alpha-1}, \quad j = 0, 1, 2, \\ &|B_j| \leqslant C(2m + 3 - j)^{\alpha-1}, \quad j = 0, 1, \ldots, 2m + 2. \end{aligned} \tag{4.3}$$

Proof. (1) We first derive the estimate for $|\overline{B}_j|, j = 0, 1, 2$.

$$
\begin{aligned}
|\overline{B}_0| &= \left|\frac{A^{0,0}_{2m+1}}{\Delta t^\alpha}\right| = \left|\frac{\omega^{0,0}_{2m+1} + \frac{3}{8}\omega^{1,0}_{2m+1}}{\Delta t^\alpha}\right| = \frac{1}{\Delta t^\alpha \Gamma(\alpha)}\left|\int_{t_0}^{t_1}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{0,0}(\tau)d\tau + \frac{3}{8}\int_{t_0}^{t_1}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{1,0}(\tau)d\tau\right| \\
&= \frac{1}{\Delta t^\alpha \Gamma(\alpha)}\left|\int_{t_0}^{t_1}(t_{2m+1}-\tau)^{\alpha-1}\frac{2(\tau-t_{1/2})(\tau-t_1)}{\Delta t^2}d\tau - \frac{3}{2}\int_{t_0}^{t_1}(t_{2m+1}-\tau)^{\alpha-1}\frac{(\tau-t_0)(\tau-t_1)}{\Delta t^2}d\tau\right| \\
&= \frac{1}{\Delta t^\alpha \Gamma(\alpha)}\left|\int_{t_0}^{t_1}(t_{2m+1}-\tau)^{\alpha-1}\frac{(\frac{1}{2}\tau-t_1)(\tau-t_1)}{\Delta t^2}d\tau\right| \leqslant \frac{1}{\Delta t^{\alpha-1}\Gamma(\alpha)}(t_{2m+1}-\tau^*)^{\alpha-1}, \\
&\leqslant C(2m+2)^{\alpha-1} \;\; \tau^* \in (t_0, t_1).
\end{aligned}
$$

This proves the first estimate of (4.3) for $j = 0$. For $j = 1$, we have

$$
\begin{aligned}
|\overline{B}_1| &= \left|\frac{A^{1,0}_{2m+1} + A^{0,1}_{2m+1}}{\Delta t^\alpha}\right| = \left|\frac{\frac{3}{4}\omega^{1,0}_{2m+1} + \omega^{2,0}_{2m+1} + A^{0,1}_{2m+1}}{\Delta t^\alpha}\right| \\
&= \frac{1}{\Delta t^\alpha}\left|\frac{3}{4}\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{1,0}(\tau)d\tau + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{2,0}(\tau)d\tau + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_3}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{0,1}(\tau)d\tau\right| \\
&\leqslant \frac{1}{\Delta t^\alpha \Gamma(\alpha)}\left\{\int_{t_0}^{t_1}(t_{2m+1}-\tau)^{\alpha-1}|\varphi_{1,0}(\tau)|d\tau + \int_{t_0}^{t_1}(t_{2m+1}-\tau)^{\alpha-1}|\varphi_{2,0}(\tau)|d\tau + \int_{t_1}^{t_3}(t_{2m+1}-\tau)^{\alpha-1}|\varphi_{0,1}(\tau)|d\tau\right\} \\
&\leqslant \frac{1}{\Delta t^\alpha \Gamma(\alpha)}\left\{(t_{2m+1}-\bar{\xi}_1\Delta t)^{\alpha-1}\Delta t + (t_{2m+1}-\bar{\xi}_2\Delta t)^{\alpha-1}\Delta t + (t_{2m+1}-\bar{\xi}_3\Delta t)^{\alpha-1}\Delta t\right\} \\
&= \frac{1}{\Gamma(\alpha)}\left\{(2m+1-\bar{\xi}_1)^{\alpha-1} + (2m+1-\bar{\xi}_2)^{\alpha-1} + (2m+1-\bar{\xi}_3)^{\alpha-1}\right\},
\end{aligned}
$$

where $0 \leqslant \bar{\xi}_1 \leqslant 1, 0 \leqslant \bar{\xi}_2 \leqslant 1$ and $1 \leqslant \bar{\xi}_3 \leqslant 3$. Therefore it holds

$$
|\overline{B}_1| \leqslant C(2m+1)^{\alpha-1}. \tag{4.4}
$$

This is nothing than the first estimate of (4.3) for $j = 1$.

In a similar way, we can derive

$$
|\overline{B}_2| \leqslant C(2m)^{\alpha-1}.
$$

This completes the proof of the first inequality of (4.3).2) Now we turn to derive the estimate for $|B_j|, j = 0, 1, \ldots, 2m+2$. We distinguish two cases: odd and even j.

- Odd case: $j = 2k+1, k = 0, 1, \ldots, m$.

$$
\begin{aligned}
|B_{2k+1}| &= \frac{1}{\Delta t^\alpha \Gamma(\alpha)}\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}\psi_{1,k}(\tau)d\tau\right| \leqslant \frac{1}{\Delta t^{\alpha+2}\Gamma(\alpha)}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}|(\tau-t_{2k})(\tau-t_{2k+2})|d\tau \\
&\leqslant \frac{1}{\Delta t^{\alpha-1}\Gamma(\alpha)}(t_{2m+2}-\hat{\xi}_k)^{\alpha-1}.
\end{aligned}
$$

where $t_{2k} \leqslant \hat{\xi}_k \leqslant t_{2k+2}$. Similar to (4.4), we get

$$
|B_{2k+1}| \leqslant C(2m-2k+2)^{\alpha-1}, k = 0, 1, \ldots, m.
$$

- Even case: $j = 2k, k = 0, 1, \ldots, m+1$.
  - For $k = 0$, we have

$$
\begin{aligned}
|B_0| &= \left|\frac{A^{0,0}_{2m+2}}{\Delta t^\alpha}\right| = \frac{1}{\Gamma(\alpha)\Delta t^\alpha}\left|\int_{t_0}^{t_2}(t_{2m+2}-\tau)^{\alpha-1}\psi_{0,0}(\tau)d\tau\right| \leqslant \frac{1}{2\Gamma(\alpha)\Delta t^{\alpha+2}}\int_{t_0}^{t_2}(t_{2m+2}-\tau)^{\alpha-1}|(\tau-t_1)(\tau-t_2)|d\tau \\
&\leqslant \frac{1}{\Delta t^{\alpha-1}\Gamma(\alpha)}(t_{2m+2}-\xi_0)^{\alpha-1}, \leqslant C(2m+3)^{\alpha-1}, \;\; \xi_0 \in (t_0, t_2).
\end{aligned}
$$

  - For $k = 1, \ldots, m$, we have

$$
\begin{aligned}
|B_{2k}| &= \left|\frac{A^{2,k-1}_{2m+2} + A^{0,k}_{2m+2}}{\Delta t^\alpha}\right| = \frac{1}{\Gamma(\alpha)\Delta t^\alpha}\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}\psi_{2,k-1}(\tau)d\tau + \int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}\psi_{0,k}(\tau)d\tau\right| \\
&\leqslant \frac{C}{\Delta t^{\alpha-1}\Gamma(\alpha)}\{(t_{2m+2}-\eta_k)^{\alpha-1} + (t_{2m+2}-\zeta_k)^{\alpha-1}\}, \\
&\leqslant C(2m-2k+3)^{\alpha-1} \;\; \eta_k, \zeta_k \in (t_{2k}, t_{2k+2}).
\end{aligned}
$$

– For $k = m + 1$,

$$B_{2m+2} = \frac{A^{2,m}_{2m+2}}{\Delta t^{\alpha}} = \frac{1}{\Gamma(\alpha)\Delta t^{\alpha}} \int_{t_{2m}}^{t_{2m+2}} (t_{2m+2} - \tau)^{\alpha-1} \psi_{2,m}(\tau) d\tau = \frac{2^{\alpha}(2-\alpha)}{\Gamma(\alpha+3)}. \tag{4.5}$$

Thus

$$|B_{2m+2}| \leqslant C.$$

In summary, combining all above results gives the estimates (4.3). □

As for integer order differential equations, it is indicative to study the stability property of the schema (4.2) with

$$f(t, u(t)) := \lambda u(t), \tag{4.6}$$

where λ is a real number.

Theorem 4.1. *Let*

$$M_0 = \max\{|u_0|, |u_0^{(1)}|, \cdots, |u_0^{(k)}|, \cdots, |u_0^{(n-1)}|\}. \tag{4.7}$$

Then the schema (4.2) with f given in (4.6) is stable with respect to the initial values under the condition

$$\frac{2^{\alpha}(2-\alpha)}{\Gamma(\alpha+3)} \Delta t^{\alpha} \lambda \neq 1. \tag{4.8}$$

That is, if (4.8) is satisfied, then

$$|u_j| \leqslant CM_0, \quad j = 1, 2, \ldots, 2N, \tag{4.9}$$

where C only depends on λ, α, and T.

Proof. First, we prove $|u_{2m+1}| \leqslant CM_0$ for $m = 1, 2, \ldots, N-1$. From (4.7), we have

$$|g(t)| = \left| \sum_{k=0}^{n-1} u_0^{(k)} \frac{t^k}{k!} \right| \leqslant M_0 \left| \sum_{k=0}^{n-1} \frac{t^k}{k!} \right| \leqslant M_0 \exp(t) \leqslant M_0 \exp(T). \tag{4.10}$$

Plugging $f(t, u) = \lambda u$ into (4.2) gives

$$u_{2m+1} = g_{2m+1} + \lambda \Delta t^{\alpha} \sum_{j=0}^{2} \overline{B}_j u_j + \lambda \Delta t^{\alpha} \sum_{j=3}^{2m+1} B_{j+1} u_j. \tag{4.11}$$

Then a rearrangement of (4.11) yields

$$(1 - \lambda \Delta t^{\alpha} B_{2m+2}) u_{2m+1} = g_{2m+1} + \lambda \Delta t^{\alpha} \sum_{j=0}^{2} \overline{B}_j u_j + \lambda \Delta t^{\alpha} \sum_{j=3}^{2m} B_{j+1} u_j. \tag{4.12}$$

By using (4.3) and (4.8), we obtain

$$|u_{2m+1}| \leqslant CM_0 \exp(T) + C|\lambda| \Delta t^{\alpha} \sum_{j=0}^{2m} (2m + 2 - j)^{\alpha-1} |u_j|. \tag{4.13}$$

When $0 < \alpha \leqslant 1$, it holds $(2m + 2 - j)^{\alpha-1} \leqslant (2m + 1 - j)^{\alpha-1}$, thus

$$|u_{2m+1}| \leqslant CM_0 \exp(T) + C|\lambda| \Delta t^{\alpha} \sum_{j=0}^{2m} (2m + 1 - j)^{\alpha-1} |u_j|. \tag{4.14}$$

Applying the discrete Gronwall Theorem 3.1 to (4.14) gives

$$|u_{2m+1}| \leqslant CE_{\alpha}\big(C|\lambda|\Gamma(\alpha)((2m+1)\Delta t)^{\alpha}\big) M_0 \exp(T) \leqslant CE_{\alpha}\big(C|\lambda|\Gamma(\alpha)T^{\alpha}\big) M_0 \exp(T) \leqslant CM_0, \tag{4.15}$$

where C depends on λ, α, and T.

When $\alpha > 1$, we have $\Delta t^{\alpha-1}(2m + 2 - j)^{\alpha-1} \leqslant T^{\alpha-1}$. Therefore we derive from (4.13)

$$|u_{2m+1}| \leqslant CM_0 \exp(T) + C|\lambda| T^{\alpha-1} \Delta t \sum_{j=0}^{2m} |u_j|. \tag{4.16}$$

Once again, applying Theorem 3.1 with $\gamma = 1$ to (4.16) yields

$$|u_{2m+1}| \leqslant C\exp\left(C|\lambda|T^{\alpha}\right)\exp(T)M_0 \leqslant CM_0. \tag{4.17}$$

In a similar way, we can derive

$$|u_{2m+2}| \leqslant CM_0, m = 1,2,\ldots,N-1.$$

The proof of (4.9) is completed. □

Remark 4.1. It is observed from the proof that the constant C in (4.9) may be large if Δt is taken close to $1/\left(|\frac{2^{\alpha}(2-\alpha)}{\Gamma(\alpha+3)}|L\right)^{1/\alpha}$. However it is worth to note that this constant decreases as Δt decreases. For example, if

$$\Delta t \leqslant \frac{1}{\left(2|\frac{2^{\alpha}(2-\alpha)}{\Gamma(\alpha+3)}|L\right)^{1/\alpha}},$$

then $C \leqslant 2$.

This means that for reasonably small Δt, the constant C can be upper bounded by 2.

In the error analysis, we consider the general $f(t,u)$ and assume it satisfies the following Lipschitz condition with respect to the second variable: there exists a constant L, such that

$$|f(t,u_1) - f(t,u_2)| \leqslant L|u_1 - u_2|, \quad \forall u_1,u_2 \in R. \tag{4.18}$$

Theorem 4.2. *Let u be the exact solution of (2.1), (2.2), $\{u_j\}_{j=0}^{2N}$ be the numerical solution of (4.2). If $f(\cdot,u(\cdot)) \in C^4[0,T]$ and the time step size Δt satisfies*

$$|\frac{2^{\alpha}(2-\alpha)}{\Gamma(\alpha+3)}|\Delta t^{\alpha}L < 1, \tag{4.19}$$

then the following error estimates hold:

$$|u(t_j) - u_j| \leqslant C\Delta t^{3+\alpha}, \quad j = 1,2,\ldots,2N \tag{4.20}$$

if $0 < \alpha \leqslant 1$, and

$$|u(t_j) - u_j| \leqslant C\Delta t^{4}, \quad j = 1,2,\ldots,2N \tag{4.21}$$

if $\alpha > 1$, where C only depends on f, α, T and L.

Proof. Let $e_j = u(t_j) - u_j, j = 0,1,\ldots,2N$. It is readily seen that $e_0 = 0$, and $e_j, j \geqslant 1$, satisfy

$$\begin{cases} e_1 = \Delta t^{\alpha}\sum_{i=0}^{2}\widehat{B}_i[f(t_i,u(t_i)) - f(t_i,u_i)] + r_1(\Delta t), \\ e_2 = \Delta t^{\alpha}\sum_{i=0}^{2}\widetilde{B}_i[f(t_i,u(t_i)) - f(t_i,u_i)] + r_2(\Delta t), \\ e_{2m+1} = \Delta t^{\alpha}\sum_{i=0}^{2}\overline{B}_i[f(t_i,u(t_i)) - f(t_i,u_i)] + \Delta t^{\alpha}\sum_{i=3}^{2m+1}B_{i+1}[f(t_i,u(t_i)) - f(t_i,u_i)] + r_{2m+1}(\Delta t), \quad m \geqslant 1, \\ e_{2m+2} = \Delta t^{\alpha}\sum_{i=0}^{2m+2}B_i[f(t_i,u(t_i)) - f(t_i,u_i)] + r_{2m+2}(\Delta t), \quad m \geqslant 1, \end{cases} \tag{4.22}$$

where the coefficients B_i are defined in (4.1). By a direct calculation, we know $\widehat{B}_i, \widetilde{B}_i, i = 0,1,2$, are bounded. Then from (4.3) and the assumption (4.18), we obtain

$$\begin{cases} |e_1| \leqslant LC\Delta t^{\alpha}\sum_{i=0}^{2}|e_i| + |r_1(\Delta t)|, \\ |e_2| \leqslant LC\Delta t^{\alpha}\sum_{i=0}^{2}|e_i| + |r_2(\Delta t)|, \\ |e_{2m+1}| \leqslant LC\Delta t^{\alpha}\sum_{i=0}^{2m}(2m+2-i)^{\alpha-1}|e_i| + L\Delta t^{\alpha}|B_{2m+2}||e_{2m+1}| + |r_{2m+1}(\Delta t)|, \\ |e_{2m+2}| \leqslant LC\Delta t^{\alpha}\sum_{i=0}^{2m+1}(2m+3-i)^{\alpha-1}|e_i| + L\Delta t^{\alpha}|B_{2m+2}||e_{2m+2}| + |r_{2m+2}(\Delta t)|. \end{cases} \tag{4.23}$$

By combining the first and second inequalities of (4.23), and taking into account (4.19), we obtain

$$|e_1| \leqslant C(|r_1(\Delta t)| + |r_2(\Delta t)|), \quad |e_2| \leqslant C(|r_1(\Delta t)| + |r_2(\Delta t)|).$$

The above result, together with Lemmas 3.1 and 3.2, lead to (4.20) and (4.21) for $j = 1, 2$.

Next we prove (4.20) and (4.21) for $j > 2$. Under condition (4.19), we derive from the last two inequalities of (4.23):

$$\begin{cases} |e_{2m+1}| \leqslant LC\Delta t^{\alpha} \sum_{i=0}^{2m} (2m+2-i)^{\alpha-1}|e_i| + C|r_{2m+1}(\Delta t)|, \\ |e_{2m+2}| \leqslant LC\Delta t^{\alpha} \sum_{i=0}^{2m+1} (2m+3-i)^{\alpha-1}|e_i| + C|r_{2m+2}(\Delta t)|. \end{cases}$$

They can be unified as

$$|e_j| \leqslant LC\Delta t^{\alpha} \sum_{i=0}^{j-1} (j+1-i)^{\alpha-1}|e_i| + C|r_j(\Delta t)|, \quad j > 2. \tag{4.24}$$

- For $0 < \alpha \leqslant 1$, we have from (4.24)

$$|e_j| \leqslant LC\Delta t^{\alpha} \sum_{i=0}^{j-1} (j-i)^{\alpha-1}|e_i| + C|r_j(\Delta t)|, \quad j > 2. \tag{4.25}$$

Then the discrete Gronwall Theorem 3.1 applied to (4.25) gives

$$|e_j| \leqslant C|r_j(\Delta t)|E_{\alpha}(LC\Gamma(\alpha)(j\Delta t)^{\alpha}) \leqslant C|r_j(\Delta t)|E_{\alpha}(LC\Gamma(\alpha)T^{\alpha}).$$

Combining the above estimate with Lemmas 3.1 and 3.2 yields

$$|e_j| \leqslant C\Delta t^{3+\alpha}, \quad j = 3, \ldots, 2N.$$

This proves (4.20).

- For $\alpha > 1$, it holds $(j+1-i)^{\alpha-1}\Delta t^{\alpha-1} \leqslant CT^{\alpha-1}$ for all $j > i$. Thus we have from (4.24):

$$|e_j| \leqslant LCT^{\alpha-1}\Delta t \sum_{i=0}^{j-1} |e_i| + C|r_j(\Delta t)|. \tag{4.26}$$

Therefore, as a result of Theorem 3.1, Lemmas 3.1 and 3.2, we obtain

$$|e_j| \leqslant C\Delta t^4, \quad j = 3, \ldots, 2N.$$

The proof is then complete. □

5. Numerical results

We carry out a number of numerical experiments to verify the theoretical results obtained in the previous sections. Precisely, our main purpose is to check the convergence behavior of the numerical solution with respect to the step size Δt.

Example 1. We consider the initial value problem (2.1), (2.2) with

$$f(t, u(t)) = \frac{\Gamma(4+\alpha)}{6} t^3 + t^{3+\alpha} - u(t).$$

It can be verified that the exact solution is $u(t) = t^{3+\alpha}$.

All the results presented in this example correspond to the numerical solution captured at $T = 1$. In Tables 1,2 we list the maximum errors $\max_i|u(t_i) - u_i|$ as a function of Δt for several α. Also shown are the corresponding rates. From Table 1, it is observed that for all α smaller or equal to 1, the convergence rate is close to $3 + \alpha$. For all $\alpha > 1$, as shown in Table 2, the error decay rates are all close to 4. This is in a good agreement with the theoretical prediction.

Example 2. We consider the problem (2.1), (2.2) with the following right hand side function:

$$f(t, u(t)) = \frac{\Gamma(5+\alpha)}{24T^{4+\alpha}} t^4 + \eta\left(\frac{t}{T}\right)^{8+2\alpha} - \eta u^2(t), \tag{5.1}$$

where η is a constant. Note that the function f is nonlinear with respect to u. The corresponding exact solution is $u(t) = (\frac{t}{T})^{4+\alpha}$.

Table 1
Maximum errors and decay rate as functions of Δt with $\alpha = 0.2, 0.5$ and 1.0.

Δt	$\alpha = 0.2$	Rate	$\alpha = 0.5$	Rate	$\alpha = 1.0$	Rate
$\frac{1}{10}$	5.8970E−005	–	1.0094E−004	–	9.3656E−005	–
$\frac{1}{20}$	6.6398E−006	3.1508	9.5872E−006	3.3962	6.0468E−006	3.9531
$\frac{1}{40}$	7.4472E−007	3.1564	8.9417E−007	3.4225	3.8420E−007	3.9763
$\frac{1}{80}$	8.3430E−008	3.1581	8.2222E−008	3.4430	2.4212E−008	3.9881
$\frac{1}{160}$	9.4477E−009	3.1425	7.4797E−009	3.4585	1.5195E−009	3.9940
$\frac{1}{320}$	1.0658E−009	3.1481	6.7500E−010	3.4700	9.5169E−011	3.9970

Table 2
Maximum errors and decay rate as functions of Δt with $\alpha = 1.5$ and 2.0.

Δt	$\alpha = 1.5$	Rate	$\alpha = 2.0$	Rate
$\frac{1}{10}$	2.7796E−004	–	5.9626E-004	–
$\frac{1}{20}$	1.8079E−005	3.9425	4.0515E−005	3.8794
$\frac{1}{40}$	1.1514E−006	3.9729	2.6317E−006	3.9444
$\frac{1}{80}$	7.2657E−008	3.9861	1.6756E−007	3.9732
$\frac{1}{160}$	4.5649E−009	3.9925	1.0569E−008	3.9869
$\frac{1}{320}$	2.8616E−010	3.9957	6.6352E−010	3.9935

Table 3
Maximum errors and decay rate with $\alpha = 0.2, 0.5$ and 1.0 for Example 2.

Δt	$\alpha = 0.2$	Rate	$\alpha = 0.5$	Rate	$\alpha = 1.0$	Rate
$\frac{1}{10}$	1.6558E−004	–	2.2974E−004	–	5.9594E−005	–
$\frac{1}{20}$	1.9982E−005	3.0507	2.2161E−005	3.3739	3.4490E−006	4.1109
$\frac{1}{40}$	2.2771E−006	3.1335	2.0734E−006	3.4179	2.0823E−007	4.0499
$\frac{1}{80}$	2.5436E−007	3.1622	1.9054E−007	3.4439	1.2802E−008	4.0237
$\frac{1}{160}$	2.8099E−008	3.1783	1.7293E−008	3.4618	7.9398E−010	4.0112
$\frac{1}{320}$	3.0849E−009	3.1872	1.5566E−009	3.4738	4.9434E−011	4.0054

Table 4
Maximum errors and decay rate with $\alpha = 1.5$ and 2.0 for Example 2.

Δt	$\alpha = 1.5$	Rate	$\alpha = 2.0$	Rate
$\frac{1}{10}$	3.4624E−004	–	7.2823E-004	–
$\frac{1}{20}$	2.0468E−005	4.0803	4.1853E−005	4.1210
$\frac{1}{40}$	1.2495E−006	4.0339	2.4659E−006	4.0852
$\frac{1}{80}$	7.7757E−008	4.0063	1.4887E−007	4.0500
$\frac{1}{160}$	4.8796E−009	3.9941	9.1318E−009	4.0270
$\frac{1}{320}$	3.0701E−010	3.9904	5.6520E−010	4.0141

First we take $T = \eta = 1$ and repeat the same calculation as in the Example 1 by using the proposed schema. Tables 3, 4 show the maximum errors and decay rates as a function of the time step size for several α ranging from 0.2 to 2. Once again these results confirm that the convergence of the numerical solution is of order $3 + \alpha$ for $0 < \alpha \leqslant 1$, and order 4 for $\alpha > 1$.

Finally, we check the sharpness of the convergence condition (4.19) on the time step. We take $\alpha = 0.5, T = 1, \eta = -3$ in the second example. In this case it can be directly verified that the Lipschitz condition (4.18) is satisfied with $L = 6$, thus the convergence condition (4.19) predicts a critical time step close to 0.068. We present the obtained numerical results in Table 5. It is found that the time step sizes smaller than $\frac{1}{16}$ produce convergent solutions, while the calculation with $\Delta t = \frac{1}{16}$ blows up. We see here a reasonably good agreement between the estimated and numerical critical time steps.

Example 3. This last test aims at illustrating the role of the regularity of the solution. To this end let's consider the exact solution $u(t) = |t^2 - \left(\frac{T}{2}\right)^2|$ with the corresponding right hand side function f:

Table 5
Stability investigation for Example 2 with $\alpha = 0.5, T = 1, \eta = -3$.

Δt	Error	Rate
$\frac{1}{16}$	NaN	
$\frac{1}{18}$	1.2484445491137155171340559690E−002	
$\frac{1}{20}$	6.2704692242968425631300125426E−003	
$\frac{1}{40}$	3.4796754089425219710342967831E−004	4.1715
$\frac{1}{80}$	3.0190884879626470267133052629E−005	3.5267
$\frac{1}{160}$	2.7212367122370355677983869551E−006	3.4718
$\frac{1}{320}$	2.4477057456319015100523047715E−007	3.4748
$\frac{1}{640}$	2.1907639158651824860429164945E−008	3.4819
$\frac{1}{1280}$	1.9533974311202977718688861978E−009	3.4874
$\frac{1}{2560}$	1.7371674153756386862280622506E−010	3.4912
$\frac{1}{5120}$	1.5420524848444669274777575942E−011	3.4938
$\frac{1}{10240}$	1.3671100998645792325557625268E−012	3.4956

Table 6
Maximum errors and decay rates with $\alpha = 0.5, 0.8$ for the less smooth solution.

Δt	$\alpha = 0.5$	Rate	$\alpha = 0.8$	Rate
$\frac{1}{10}$	4.7188E−002	–	9.9848E−002	–
$\frac{1}{20}$	2.4668E−002	0.9357	5.0476E−002	0.9841
$\frac{1}{40}$	1.2752E−002	0.9519	2.5418E−002	0.9897
$\frac{1}{80}$	6.5398E−002	0.9635	1.2772E−002	0.9929
$\frac{1}{160}$	3.3318E−003	0.9729	6.4067E−003	0.9953
$\frac{1}{320}$	1.6889E−003	0.9802	3.2099E−003	0.9970

$$f(t,u) = \begin{cases} \frac{-2}{\Gamma(3-\alpha)} t^{2-\alpha} + \left| t^2 - \left(\frac{T}{2}\right)^2 \right| - u(t), & t \leqslant \frac{T}{2}, \\ \frac{2}{\Gamma(3-\alpha)} \left[T(2-\alpha)\left(t - \frac{T}{2}\right)^{1-\alpha} + 2\left(t - \frac{T}{2}\right)^{2-\alpha} - t^{2-\alpha} \right] & \\ + |t^2 - \left(\frac{T}{2}\right)^2| - u(t), & t > \frac{T}{2}. \end{cases}$$

Clearly, both the solution and f are only C^0 continuous. The errors of the numerical solution at $T = 1$ are shown in Table 6. The error decay rates listed in the figure indicate that the convergence order is reduced to 1 for this C^0 solution. This test confirms that the derived convergence rates in Theorem 4.2 are valid only for smooth enough solutions.

6. Concluding remarks

We presented a high order schema for the fractional differential equations. The schema was constructed by using the so-called block-by-block approach, which has been frequently employed for traditional integer order differential equations. We established an error estimate for the numerical solution, showing that the proposed schema is of order $3 + \alpha$ for $0 < \alpha \leqslant 1$, and order 4 for $\alpha > 1$. The carried out numerical tests confirmed the theoretical prediction. In the future, we plan to follow this idea to construct higher order schemes, as well as to apply this kind of methods to fractional partial differential equations with spatial derivatives.

Acknowledgements

The research of C. Xu was partially supported by National NSF of China(Grant Nos. 11071203 and 91130002).

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