We design a second order backward differentiation formula for the Caputo derivative

$$D_c^\gamma u = \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{u'(s)}{(t-s)^\gamma}ds.$$

We show that the derived scheme is of order $O(k^{3-\gamma})$ and is A-stable.

1 The scheme

Suppose we divide $[0,T]$ into N pieces and the time step is $k = T/N$. Let $t^n = nk$. We aim to approximate $D_c^\gamma u(t^{n+1})$. To be convenient, we denote $I_m = [t^m, t^{m+1}]$.

Suppose u_m is the nodal value of u at t^m. The Lagrange interpolation on I_m is given by:

$$L_m(t) = u_{m-1}\frac{(t-t^m)(t-t^{m+1})}{2k^2} - u_m\frac{(t-t^{m-1})(t-t^{m+1})}{k^2} + u_{m+1}\frac{(t-t^m)(t-t^{m-1})}{2k^2}$$

Hence, we use the approximation

$$u'(t) \approx L_m'(t) = u_{m-1}\frac{t-t^{m+1/2}}{k^2} - 2u_m\frac{t-t^m}{k^2} + u_{m+1}\frac{t-t^{m-1/2}}{k^2},\ t\in I_m. \tag{1}$$

For $t\in I_0$, L_0 will be the quadratic approximation using u_0, u_1, u_2, i.e., we use $L_1(t)$ on I_0.

The numerical scheme is given by

$$\mathcal{D}_k u(t^{n+1}) = \frac{1}{\Gamma(1-\gamma)}\sum_{m=0}^{n}\int_{I_m}\frac{L_m'(s)}{(t^{n+1}-s)^\gamma}ds = k^{-\gamma}\sum_{m=0}^{n+1} b_m^{n+1} u_{n+1-m}. \tag{2}$$

We call this formula 'backward differentiation' since $b_0^{n+1} > 0$. As we shall see the scheme tends to BDF2 as $\gamma \to 1$.

By the Taylor formula

$$f(t) = f(s) + (t-s)f'(s) + \frac{1}{2}(t-s)^2 f''(s) + \int_s^t \frac{1}{2}(t-\tau)^2 f'''(\tau)d\tau,$$

we find

$$R_m(s) = L_m'(s) - u_s(s) = \frac{s-t^{m+1/2}}{k^2}\int_s^{t^{m-1}}\frac{1}{2}(t^{m-1}-\tau)^2 u'''(\tau)d\tau$$
$$-2\frac{s-t^m}{k^2}\int_s^{t^m}\frac{1}{2}(t^m-\tau)^2 u'''(\tau)d\tau + \frac{s-t^{m-1/2}}{k^2}\int_s^{t^{m+1}}\frac{1}{2}(t^{m+1}-\tau)^2 u'''(\tau)d\tau \tag{3}$$

The total error will be

$$r = \frac{1}{\Gamma(1-\gamma)} \left(r_0 + \sum_{m=1}^{n} \int_{I_m} \frac{R_m(s)}{(t^{n+1}-s)^\gamma} ds \right)$$

where

$$r_0 = \int_{I_0} \frac{R_1(s)}{(t^{n+1}-s)^\gamma} ds.$$

1.1 The coefficients

In this section, we find the coefficients and prove some properties about the coefficients.

By a simple computation, we find

$$\int_{t^m}^{t^{m+1}} \frac{s-c}{(t^{n+1}-s)^\gamma} ds = -\frac{1}{1-\gamma}(t^{n+1}-s)^{1-\gamma}(s-c)|_{t^m}^{t^{m+1}} \\ - \frac{1}{(1-\gamma)(2-\gamma)}(t^{n+1}-s)^{2-\gamma}|_{t^m}^{t^{m+1}}.$$

We define the coefficients C_m^p for $m \geq 1$ through:

$$k^\gamma(1-\gamma) \int_{I_m} \frac{L_m'(s)}{(t^{n+1}-s)^\gamma} ds = C_m^{m+1} u_{m+1} + C_m^m u_m + C_m^{m-1} u_{m-1}.$$

Hence,

$$\begin{aligned} C_m^{m+1} = &\ -\tfrac{3}{2}(n-m)^{1-\gamma} + \tfrac{1}{2}(n-m+1)^{1-\gamma} - \tfrac{1}{2-\gamma}(n-m)^{2-\gamma} + \tfrac{1}{2-\gamma}(n-m+1)^{2-\gamma}, \\ C_m^m = &\ 2(n-m)^{1-\gamma} + \tfrac{2}{2-\gamma}(n-m)^{2-\gamma} - \tfrac{2}{2-\gamma}(n-m+1)^{2-\gamma}, \\ C_m^{m-1} = &\ -\tfrac{1}{2}(n-m)^{1-\gamma} - \tfrac{1}{2}(n-m+1)^{1-\gamma} - \tfrac{1}{2-\gamma}(n-m)^{2-\gamma} + \tfrac{1}{2-\gamma}(n-m+1)^{2-\gamma}. \end{aligned}$$

These are only valid for $m \geq 1$.

For $m = 0$, we compute $k^\gamma(1-\gamma)\int_0^{t^1} \frac{L_1'(t)}{(t^{n+1}-s)^\gamma} ds$ where

$$L_1'(t) = u_0 \frac{t - t^{3/2}}{k^2} - 2u_1 \frac{t-t^1}{k^2} + u_2 \frac{t-t^{1/2}}{k^2}.$$

and find

$$C_0^2 = -\frac{1}{2} n^{1-\gamma} - \frac{1}{2}(n+1)^{1-\gamma} - \frac{1}{2-\gamma}[n^{2-\gamma} - (n+1)^{2-\gamma}],,$$

$$C_0^1 = 2(n+1)^{1-\gamma} + \frac{2}{2-\gamma}[n^{2-\gamma} - (n+1)^{2-\gamma}],$$

$$C_0^0 = \frac{1}{2} n^{1-\gamma} - \frac{3}{2}(n+1)^{1-\gamma} - \frac{1}{2-\gamma}[n^{2-\gamma} - (n+1)^{2-\gamma}]$$

The derivative should be

$$D_c^\gamma u(t^{n+1}) \approx \frac{1}{k^\gamma}\sum_{m=0}^{n+1} b_m^{n+1} u_{n+1-m} =$$

$$\frac{1}{k^\gamma\Gamma(2-\gamma)}(C_0^0 u_0 + C_0^1 u_1 + C_0^2 u_2 + \sum_{m=1}^{n}(C_m^{m-1}u_{m-1} + C_m^m u_m + C_m^{m+1}u_{m+1}))$$

Hence, for $n \geq 1$, the derivative can be computed by

$$D_c^\gamma u(t^{n+1}) \approx k^{-\gamma} b^{n+1} * u.$$

For the discussions below, we will assume that u^0 and u^1 are given. In real simulation, we can compute u^1 for example, using $L_1(t)$ on both I_0, I_1 and then form a system of equations for u^1, u^2 and compute them all at once. The local truncation error is $O(k^{3-\gamma})$

1.1.1 The coefficients for $1 \leq n \leq 2$

The coefficients for $n = 1, 2$ can be computed directly.

For $n = 1$,

$$\Gamma(2-\gamma)b_0^2 = -\frac{1}{2}2^{1-\gamma} + \frac{1}{2-\gamma}2^{2-\gamma}$$
$$\Gamma(2-\gamma)b_1^2 = 2\cdot 2^{1-\gamma} - \frac{2}{2-\gamma}2^{2-\gamma}$$
$$\Gamma(2-\gamma)b_2^2 = -\frac{3}{2}2^{1-\gamma} + \frac{1}{2-\gamma}2^{2-\gamma}$$

For $n = 2$,

$$\Gamma(2-\gamma)b_0^3 = \frac{1}{2} + \frac{1}{2-\gamma}$$
$$\Gamma(2-\gamma)b_1^3 = -\frac{3}{2} - \frac{3}{2-\gamma} - \frac{1}{2}3^{1-\gamma} + \frac{1}{2-\gamma}3^{2-\gamma}$$
$$\Gamma(2-\gamma)b_2^3 = \frac{3}{2} + 2\cdot 3^{1-\gamma} + \frac{3}{2-\gamma} - \frac{2}{2-\gamma}3^{2-\gamma}$$
$$\Gamma(2-\gamma)b_3^3 = -\frac{1}{2} - \frac{3}{2}3^{1-\gamma} + \frac{1}{2-\gamma}(3^{2-\gamma} - 1)$$

1.1.2 The coefficients for $n \geq 3$

We now derive the formulas for $n \geq 3$.

Then, it is easy to find:

$$\begin{aligned}\Gamma(2-\gamma)b_0^{n+1} = C_n^{n+1} = \tfrac{1}{2} + \tfrac{1}{2-\gamma},\\ \Gamma(2-\gamma)b_1^{n+1} = C_{n-1}^n + C_n^n = -\tfrac{3}{2} + 2^{1-\gamma}\tfrac{1}{2} - \tfrac{3}{2-\gamma} + \tfrac{1}{2-\gamma}2^{2-\gamma}\end{aligned} \tag{4}$$

Then, for $3 \leq m \leq n-1$, we find

$$\begin{aligned}&\Gamma(2-\gamma)b_{n+1-m}^{n+1} = C_{m-1}^m + C_m^m + C_{m+1}^m = \\ &-\frac{1}{2}(n-m-1)^{1-\gamma} - \frac{3}{2}(n-m+1)^{1-\gamma} + \frac{3}{2}(n-m)^{1-\gamma} + \frac{1}{2}(n-m+2)^{1-\gamma}\\ &+\frac{1}{2-\gamma}(n-m+2)^{2-\gamma}+\frac{3}{2-\gamma}(n-m)^{2-\gamma}-\frac{3}{2-\gamma}(n-m+1)^{2-\gamma}-\frac{1}{2-\gamma}(n-m-1)^{2-\gamma}\end{aligned}$$

It follows that for $2 \leq m \leq n-2$

$$\begin{aligned}\Gamma(2-\gamma)b_m^{n+1} &= -\frac{1}{2}(m-2)^{1-\gamma} - \frac{3}{2}m^{1-\gamma} + \frac{3}{2}(m-1)^{1-\gamma} + \frac{1}{2}(m+1)^{1-\gamma}\\ &+ \frac{1}{2-\gamma}(m+1)^{2-\gamma} - \frac{3}{2-\gamma}m^{2-\gamma} + \frac{3}{2-\gamma}(m-1)^{2-\gamma} - \frac{1}{2-\gamma}(m-2)^{2-\gamma}\\ &=: h(m)\end{aligned} \tag{5}$$

Further, we find:

$$\begin{aligned}&\Gamma(2-\gamma)b_{n-1}^{n+1} = C_0^2 + C_1^2 + C_2^2 + C_3^2 = \\ &\quad -\frac{1}{2}n^{1-\gamma} - \frac{1}{2}(n+1)^{1-\gamma} - \frac{1}{2-\gamma}[n^{2-\gamma} - (n+1)^{2-\gamma}] + h(n-1)\end{aligned} \tag{6}$$

$$\begin{aligned}\Gamma(2-\gamma)b_n^{n+1} = C_2^1 + C_1^1 + C_0^1 = h(n) + \frac{3}{2}n^{1-\gamma} + \frac{3}{2}(n+1)^{1-\gamma}\\ + \frac{3}{2-\gamma}[n^{2-\gamma} - (n+1)^{2-\gamma}]\end{aligned} \tag{7}$$

and

$$\Gamma(2-\gamma)b_{n+1}^{n+1} = C_1^0+C_0^0 = -\frac{1}{2}(n-1)^{1-\gamma}-\frac{3}{2}(n+1)^{1-\gamma}-\frac{1}{2-\gamma}(n-1)^{2-\gamma}+\frac{1}{2-\gamma}(n+1)^{2-\gamma} \tag{8}$$

1.2 Properties of the coefficients

Theorem 1. *We have the following claims:*

1. *For all $n \geq 1$,*
$$\sum_{m=0}^{n+1} b_m^{n+1} = 0.$$
For all $n \geq 1$, $b_0^{n+1} > 0$, $b_1^{n+1} < 0$. For $n \geq 2$, $b_m^{n+1} < 0$, $m \geq 3$. $\exists \bar{\gamma}_0 \in (0,1)$ and $\bar{\gamma}_1 \in (\bar{\gamma}_0, 1)$ such that $b_2^{n+1} \leq 0$ for $n \geq 1, \gamma < \bar{\gamma}_0$ and $b_2^{n+1} > 0$ for $n \geq 1, \gamma > \bar{\gamma}_1$.

   There exists a sequence $b \in \mathbb{R}^{\mathbb{N}}$ such that if $n \geq 2$, $b_m^{n+1} = b_m$ for $m \leq n-2$.

2. *When $n \geq 1$, as $\gamma \to 1$,*
$$b_0^{n+1} \to 3/2, b_1^{n+1} \to -2, b_2^{n+1} \to 1/2, b_m^{n+1} \to 0$$
and the scheme tends to BDF2; as $\gamma \to 0$,
$$b_0^{n+1} \to 1, b_m^{n+1} \to 0, b_{n+1}^{n+1} \to -1.$$

3. *As $n \to \infty$, $b_n^{n+1} = O(n^{-1-\gamma})$ and $b_{n+1}^{n+1} = O(n^{-\gamma})$. b^{n+1} tends to b pointwise and in l^p, $p \geq 1$.*

   For the limiting sequence b, $b_0 > 0$, $b_m < 0$ for $m \geq 1, m \neq 2$, and
$$\sum_m b_m = 0, \; b_0 \geq -\frac{3}{4} b_1, \quad -b_1 \geq 4b_2.$$
$\exists \gamma_0 \in (0,1)$ such that $b_2 \leq 0$ if $\gamma \leq \gamma_0$ and $b_2 > 0$ for $\gamma > \gamma_0$. When $m \geq 3$, b_m increases to zero and $b_m = \frac{1}{\Gamma(-\gamma)} \frac{1}{m^{1+\gamma}} \left(1 + O(\frac{1}{m})\right)$ as $m \to \infty$.

Proof. If we set $u = 1$, then $u_s(s) - L_m'(s) = 0$ for all $m \geq 1$. Hence,
$$k^{-\gamma} \sum_{m=0}^{n+1} b_m^{n+1} = \mathcal{D}_k 1 = D_c^{\gamma} 1 = 0$$
for any $n \geq 1$.

By the explicit formulas, $b_2^2 = \frac{2^{1-\gamma}}{\Gamma(2-\gamma)}(\frac{2}{2-\gamma} - \frac{3}{2})$ and the inside is monotone which increases from $-1/2$ to $1/2$. For $n \geq 4$, the sign of $b_2^{n+1} = \frac{1}{\Gamma(2-\gamma)} h(2)$ is given by the discussion below. $b_2^4 = \frac{1}{\Gamma(2-\gamma)}[h(2) - \frac{1}{2} 3^{1-\gamma} - \frac{1}{2} 4^{1-\gamma} - \frac{1}{2-\gamma}(3^{2-\gamma} -$

$4^{2-\gamma})]$. The extra terms adding to $h(2)$ is nonpositive and is zero for $\gamma = 0, \gamma = 1$. By the property of $h(2)$ below, we find that b_2^4 is negative near $\gamma = 0$ and positive near $\gamma = 1$. Similarly, we find $\Gamma(2-\gamma)b_2^3 = h(2) + \frac{3}{2}(2^{1-\gamma} + 3^{1-\gamma}) + \frac{3}{2-\gamma}(2^{2-\gamma} - 3^{2-\gamma})$. The argument is similar as b_2^4. The existence of $\bar{\gamma}_0$ and $\bar{\gamma}_1$ is proved.

The signs of other coefficients when $n = 1, n = 2$ can be checked directly since the explicit expressions are simple enough. The other claims about $n = 1, n = 2$ in the theorem can also be checked directly.

We now focus on $n \geq 3$. $b_0^{n+1} > 0$ is obvious by the expressions of b_0^{n+1}. If $n \geq 2$, we have

$$\Gamma(2-\gamma)b_1^{n+1} = -1 + \frac{2^{1-\gamma}}{2} + (-\frac{1}{2} - \frac{3}{2-\gamma} + \frac{2^{2-\gamma}}{2-\gamma}) < 0.$$

The existence of b is obvious since b_m^{n+1} is independent of n when $n \geq 1$ and $m \leq n-1$.

Since $\Gamma(2-\gamma) \geq 0$ and $\Gamma(2-\gamma) \to 1$ as $\gamma \to 1$ or $\gamma \to 0$, the properties of b_m can be reduced to considering

$$\begin{aligned} h(m) = &-\frac{1}{2}(m-2)^{1-\gamma} - \frac{3}{2}m^{1-\gamma} + \frac{3}{2}(m-1)^{1-\gamma} + \frac{1}{2}(m+1)^{1-\gamma} \\ &+ \frac{1}{2-\gamma}(m+1)^{2-\gamma} - \frac{3}{2-\gamma}m^{2-\gamma} + \frac{3}{2-\gamma}(m-1)^{2-\gamma} - \frac{1}{2-\gamma}(m-2)^{2-\gamma}. \end{aligned}$$

Then,

$$h(2) = -\frac{3}{2}2^{1-\gamma} + \frac{3}{2} + \frac{1}{2}3^{1-\gamma} + \frac{1}{2-\gamma}3^{2-\gamma} + \frac{3}{2-\gamma} - \frac{3}{2-\gamma}2^{2-\gamma}.$$

As $\gamma \to 0$, $h(2) \to 0$ and $h(2) \to 1/2$ as $\gamma \to 1$. Consider $H_2(\gamma) = (2-\gamma)h(2) = 3(\gamma-6)2^{-\gamma} + (12-3\gamma/2)3^{-\gamma} + (6-3\gamma/2)$. It's easy to see $H_2'(0) < 0$. Further,

$$\begin{aligned} H_2''(\gamma) &= 3^{-\gamma}[3\ln 3 + (12-3\gamma/2)(\ln 3)^2 - 6(3/2)^{\gamma}\ln 2 + (3/2)^{\gamma}(\gamma-6)(\ln 2)^2] \\ &\geq 3^{-\gamma}[3\ln 3 + (12-3\gamma/2)(\ln 3)^2 - 9\ln 2 + \frac{3}{2}(\gamma-6)(\ln 2)^2] \end{aligned}$$

The inside is a linear function which is positive at both $\gamma = 0$ and $\gamma = 1$. Hence, $\exists \gamma_0 > 0$ such that $H_2(\gamma) < 0$ when $\gamma \in (0, \gamma_0)$ and $H_2(\gamma) > 0$ when $\gamma > \gamma_0$. Then, $h(2)$ has the same sign since $2 - \gamma > 0$.

For $m \geq 3$, we find $h(m) \to 0$ as $\gamma \to 0$ or $\gamma \to 1$. Setting $g = \frac{1}{2-\gamma}x^{2-\gamma} + \frac{1}{2}x^{1-\gamma}$, we have $h(m) = -g(m-2) + 3g(m-1) - 3g(m) + g(m+1)$.

$g''' < 0$ for $x \geq 1$. Hence,

$$h(m) = \int_{m-2}^{m-1} g' dx - 2\int_{m-1}^{m} g' dx + \int_{m}^{m+1} g' dx < 0,$$

because $g''' < 0$ implies that g' is concave. $h(m)$ increases for $m \geq 4$ since $g'''' > 0$ for $x > 3/2$. Direct computation shows that $b_3 < b_4$.Need to confirm. It is clear that $\lim_{m\to\infty} b_m = 0$. By Taylor expansion about $m - 1/2$, we find

$$h(m) = -(1-\gamma)\gamma m^{-1-\gamma}(1 + O(1/m)),\ m \to \infty.$$

Now ,consider $n-1$, $n = 3$ and $n = 4$ are different, though the formula is uniform.

Consider $h_1(n-1)$ where $n \geq 4$. $h_1(2)$ corresponds to b_2^{3+1} which has been discussed.

Then, $h_2(n)$ and $h_3(n+1)$

The last four terms equal to

$$\frac{1}{2}n^{1-\gamma} + \frac{1}{2}(n+1)^{1-\gamma} - \int_{n}^{n+1} x^{1-\gamma} dx < 0$$

because $x^{1-\gamma}$ is a concave function. If $n = 2$, as $\gamma \to 1$, $\tilde{h}(n) \to 1/2$ and as $\gamma \to 0$, $\tilde{h}(n) \to 0$. Hence, it is clear that $\tilde{h}(2)$ is negative if γ is close to 0 and positive if γ is close to 1. If $n \geq 3$, $\tilde{h}(n) < 0$. By Taylor expansion about $n + 1/2$, the last four terms add to $O(n^{-1-\gamma})$. It is also clear that when $n \geq 3$, as $\gamma \to 0, 1$, $\tilde{h}(n) \to 0$.

Now, consider $\bar{h}(n+1)$ for $n \geq 2$ where

$$\bar{h}(n+1) = -(n-1)^{1-\gamma}\frac{1}{2} + \frac{1}{2}n^{1-\gamma} - \frac{1}{2-\gamma}(n-1)^{2-\gamma} + \frac{1}{2-\gamma}n^{2-\gamma} - (n+1)^{1-\gamma}.$$

As $\gamma \to 1$, $\bar{h}(n+1) \to 0$ and as $\gamma \to 1$, $\bar{h}(n+1) \to -1$.

We find

$$\begin{aligned}\bar{h}(n+1) &= \int_{n-1}^{n} x^{1-\gamma} dx - (n+1)^{1-\gamma} + \frac{1}{2}n^{1-\gamma} - \frac{1}{2}(n-1)^{1-\gamma} \\ &< -(n+1)^{1-\gamma} + \frac{3}{2}n^{1-\gamma} - \frac{1}{2}(n-1)^{1-\gamma} < 0\end{aligned}$$

Further, as $n \to \infty$, by Taylor expansion about n, we find that $\bar{h}(n+1) = O(n^{-\gamma})$.

Now, we compute directly

$$\Gamma(2-\gamma)(b_0+\frac{3}{4}b_1) = -\frac{5}{8}-\frac{5}{4}\frac{1}{2-\gamma}+\frac{3}{4}2^{-\gamma}+\frac{3}{2-\gamma}2^{-\gamma} =: g_1(\gamma)$$
$$\Gamma(2-\gamma)(|b_1|-4b_2) = -\frac{9}{2}-\frac{8-\gamma}{2-\gamma}6\cdot 3^{-\gamma}+\frac{6-\gamma}{2-\gamma}11\cdot 2^{-\gamma}-\frac{9}{2-\gamma} =: g_2(\gamma)$$

$g_1 = \frac{3}{2-\gamma}(2^{-\gamma}-\frac{1}{2})+\frac{1}{4(2-\gamma)}+\frac{3}{4}2^{-\gamma}-\frac{5}{8} > \frac{1}{4(2-\gamma)}+\frac{3}{4}2^{-\gamma}-\frac{5}{8} := \tilde{g}_1$. $\tilde{g}_1$ is a convex function and $\tilde{g}_1(1)=0$, $\tilde{g}_1'(1)<0$, then $\tilde{g}_1>0$. $g_2(0)=g_2(1)=0$. Note that $[(2-\gamma)g_2]'' = 3^{-\gamma}[-12\ln 3-6(8-\gamma)\ln(3)^2+22{\cdot}1.5^{\gamma}\ln 2+11(6-\gamma)1.5^{\gamma}(\ln 2)^2]$. Clearly, $-12\ln 3-6(8-\gamma)\ln(3)^2+22\cdot 1.5^{\gamma}\ln 2+11(6-\gamma)1.5^{\gamma}(\ln 2)^2 < -12\ln 3-6(8-\gamma)\ln(3)^2+33\ln 2+16.5(6-\gamma)(\ln 2)^2$. The right hand side is a linear function and the values of this linear function at two endpoints are negative. Hence, $[(2-\gamma)g_2]''<0$. Hence, $(2-\gamma)g_2$ is concave and $g_2>0$.

Then, the third part of the theorem and the second part of the theorem for $n\ge 2$ are proved. □

Numerical computation shows $\gamma_0\in(0.3,0.4)$ maybe, plot the figure.

To implement the scheme numerically, one needs to compute the sequence b^{n+1} since $D_c^{\gamma}u(t^{n+1}) = k^{-\gamma}b^{n+1}*u$. When moving from b^{n+1} to b^{n+2}, we only need to change the last two components and add one more component.

To summarize, $\gamma\to 1$, we have BDF2. For $\gamma=0$, the derivative becomes $u_{n+1}-u_0$ which is the exact value of integral $\int_0^{t^{n+1}} u_s ds$, and it makes sense.

1.3 Accuracy

Direct estimate shows

$$r_0 \le Ck^2\int_0^k \frac{1}{((n+1)k-s)^{\gamma}}ds \le C_1 k^{3-\gamma}.$$

where C_1 is independent of n,γ. If we use the same estimate, $|R_m|\le Ck^2$, then we obtain

$$r \le \frac{1}{\Gamma(1-\gamma)}(r_0+Ck^2\int_{t^1}^{t^{n+1}}\frac{1}{(t^{n+1}-s)^{\gamma}}ds) \le \frac{C}{\Gamma(2-\gamma)}T^{1-\gamma}k^2$$

The scheme is at least uniformly second order. Actually, by careful computation, we find

Theorem 2. *$\exists C>0$ independent of γ and N such that*

$$r \le Ck^{3-\gamma}.$$

Proof. Change the order of integration:

$$\frac{1}{\Gamma(1-\gamma)}\int_{t^m}^{t^{m+1}}\frac{R_m(s)}{(t^{n+1}-s)^\gamma}ds = I_1^m + I_2^m$$

where

$$I_1^m = \frac{1}{\Gamma(1-\gamma)}\frac{1}{2k^2}\int_{t^{m-1}}^{t^m} u'''(\tau)(t^{m+1}-\tau)^2\int_{t^m}^{t^{m+1}}\frac{t^{m+1/2}-s}{(t^{n+1}-s)^\gamma}dsd\tau,$$

and $I_2^m = \frac{1}{2k^2\Gamma(1-\gamma)}\int_{t^m}^{t^{m+1}} u'''(\tau)P_m(\tau)d\tau$,

$$\begin{aligned}P_m(\tau) = (t^{m-1}-\tau)^2&\int_\tau^{t^{m+1}}\frac{t^{m+1/2}-s}{(t^{n+1}-s)^\gamma}ds\\ -2(t^m-\tau)^2&\int_\tau^{t^{m+1}}\frac{t^m-s}{(t^{n+1}-s)^\gamma}ds + (t^{m+1}-\tau)^2\int_{t^m}^\tau\frac{s-t^{m-1/2}}{(t^{n+1}-s)^\gamma}ds.\end{aligned}$$

Note that $\int_{t^m}^{t^{m+1}}\frac{t^{m+1/2}-s}{(t^{n+1}-s)^\gamma}ds$ is negative for all m, we find

$$|\sum_m I_1^m| \le \sum_m \frac{Ck}{\Gamma(1-\gamma)}\int_{t^m}^{t^{m+1}}\frac{s-t^{m+1/2}}{(t^{n+1}-s)^\gamma}ds$$

where $C \sim \sup|u'''|$ is independent of n, m, γ.

$$\int_{t^m}^{t^{m+1}}\frac{s-t^{m+1/2}}{(t^{n+1}-s)^\gamma}ds = \frac{k^{2-\gamma}}{1-\gamma}[\frac{1}{2-\gamma}((n-m+1)^{2-\gamma}-(n-m)^{2-\gamma})-\frac{1}{2}((n-m+1)^{1-\gamma}+(n-m)^{1-\gamma})]$$

This number $\le \frac{Ck^{2-\gamma}}{1-\gamma}(n-m+1/2)^{-1-\gamma}$ by Taylor expansion about $n-m+1/2$, where $C = \sup_\gamma \gamma(1-\gamma)$ is independent of m, γ, n. Hence,

$$|\sum_{m\ge1} I_1^m| \le \sum_{m=1}^n \frac{1}{\Gamma(1-\gamma)}\frac{Ck^{3-\gamma}}{1-\gamma}(n-m+1/2)^{-1-\gamma} \le C_1k^{3-\gamma}$$

where C_1 is independent of γ and n.

For I_2^m, in the appendix, we show that $P_m(\tau) \le 0$ for all $m \ge 1$ and $\tau \in I_m$. Applying the integral mean value theorem, we obtain $\int_{t^m}^{t^{m+1}} u'''(\tau)P_m(\tau)d\tau = u'''(\xi)\int_{I_m} P_m(\tau)d\tau$. Direct computation shows

$$\begin{aligned}(1-\gamma)\int P_m(\tau)d\tau &= \frac{k^3}{2}k^{2-\gamma}(n-m)^{1-\gamma} + \frac{5}{3}\frac{1}{2-\gamma}(n-m)^{2-\gamma}\\ &+ (\frac{k^3}{3})k^{2-\gamma}[\frac{1}{2}(n-m+1)^{1-\gamma} + \frac{1}{2-\gamma}(n-m+1)^{2-\gamma}]\\ &+ \frac{2k^2}{(2-\gamma)(3-\gamma)}(n-m)^{3-\gamma}k^{3-\gamma} - \frac{2k^2}{(2-\gamma)(3-\gamma)}(n-m+1)^{3-\gamma}k^{3-\gamma}\end{aligned}$$

We do Taylor expansion about $n-m+1/2$. We find $2-\gamma$, $1-\gamma$, $-\gamma$ all cancel out. The nonzero power will be $(n-m+1/2)^{-1-\gamma}$, and hence summable.

Hence,

$$|\sum_{m\geq 1} I_2^m| \leq \sum_{1\leq m\leq n} \frac{1}{k^2\Gamma(1-\gamma)}\frac{C_2}{1-\gamma}k^{5-\gamma}(n-m+1/2)^{-1-\gamma}$$

Hence,

$$r \leq \frac{1}{\Gamma(1-\gamma)}r_0 + C_3k^{3-\gamma} \leq Ck^{3-\gamma},$$

where C is uniform for $\gamma \in [0,1]$ since $\Gamma(1-\gamma) \geq 1$. $\square$

1.4 Stability region

Applying the scheme to the model problem $D_c^\gamma u = \lambda u$. We define $z = k^\gamma \lambda$. Setting $u^n = \zeta^n$ yields the 'characteristic polynomial'

$$(b_0 - z)\zeta^n = -\sum_{m=1}^{n} b_m^n \zeta^{n-m}$$

Letting $\eta = 1/\zeta$, we have

$$b_0 - z + \sum_{m=1}^{n} b_m^n \zeta^{-m} = 0.$$

By the stability theory of the normal ODEs, we need $|\zeta| \leq 1$ or $|\eta| \geq 1$. Note that the above is not actually the characteristic function because there is dependence on n. As $n \to \infty$, the sequence b^n tends to b. This then motivates the definition of the power series

$$f(\eta, z) = b_0 - z + \sum_{m=1}^{\infty} b_m \eta^m.$$

The the radius of convergence is 1 and the series also converges on the boundary of the unit disk since $b_m = O(m^{-1-\gamma})$. We define the stability region to be the set of those z such that f has no zero in the unit disk.

The scheme is called A-stable if the stability region contains the whole left half plane. Clearly, the A-stability is reduced to study the zeros of $f(\eta, 0)$ which is actually the generating function of b.

Theorem 3. *The second backward differentiation $\mathcal{D}_k$ is A-stable.*

In the case $b_2 \le 0$, the scheme is A-stable. This is because all coefficients except b_0 are all negative. $|b_0 - z| \le \sum_{m=1}^{\infty}(-b_m^{\infty})|\eta|^m \le b_0$ if $|\eta| \le 1$.

Now, assume $b_2 > 0$. It would suffice to show $\inf_{z\in D(0,1)} Re(f(\eta,0)) \ge 0$. Recall that $b_1 < 0, b_2 > 0$

Since $-|\sum_{m=3}^{\infty} b_m\eta^m| \ge -\sum_{m=3}^{\infty}|b_m| = \sum_{m=3}^{\infty} b_m = -(b_0+b_1+b_2)$, we find $Re(f(\eta,0)) \ge b_0 + Re(b_1\eta + b_2\eta^2) - (b_0+b_1+b_2)$

Setting $\eta = r\cos\theta$ results in the function

$$h(r,\theta) = |b_1|(1 - r\cos\theta) - b_2(1 - r^2\cos(2\theta)).$$

This function is positive for $\cos\theta \le 0$ because $|b_1| \ge 2b_2$. For $\cos\theta \ge 0$, $h_r = -|b_1|\cos\theta + 2b_2 r\cos(2\theta) \le 0$ since $r\cos(2\theta) \le \cos(\theta)$. The minimum value is achieved on $r = 1$.

Set $h(\theta) = |b_1|(1-\cos\theta) - b_2(1-\cos(2\theta))$ with $0 \le \theta \le \pi/2$. We find that $h(0) = 0$, which is desired since $f(1,0) = 0$. $h' = \sin\theta(|b_1| - 4b_2\cos\theta)$. Since $|b_1| > 4b_2$, $h > 0$ when $\theta \ne 0$. Hence, $h(\theta) > 0$ for $0 < \theta \le \pi/2$.

2 some stability results for FODEs

In this section, we prove a result that may be useful for stability analysis that involves the difference schemes for fractional ODEs.

Given two sequences a and c. The convolution is defined as $(a*c)_n = \sum_{m=0}^{n} a_m c_{n-m}$. The generating function of $a*c$ is simply $F_a(z)F_c(z)$. The convolution identity is $\delta_d = (1,0,0,\dots)$. Hence, for a sequence $b(b_0 \ne 0)$ with generating function $F(z)$, the generation function of the convolution inverse of $b^{(-1)}$ is given by $1/F(z)$.

Lemma 1. *Suppose $b = \{b_n\}$ is a sequence satisfying $\sum_{n=0}^{\infty} b_n = 0$ and $b_0 > 0$. Suppose further that $b_n = \frac{1}{\Gamma(-\gamma)}n^{-1-\gamma}(1+O(\frac{1}{n}))$, where $0<\gamma<1$. Let $F(z)$ be its generating function $F(z) = \sum_{n=0}^{\infty} b_n z^n$. Then,*

1. $$F(z) - (1-z)^{\gamma} = (1-z)G(z)$$
   where $G(z)$ is bounded in the unit disk.

2. *Let $b^{(-1)}$ be the convolution inverse of b. Its generatating function $F(z)^{-1} \sim (1-z)^{-\gamma}$ and thus $b_n^{(-1)} = [z^n]F(z)^{-1} \sim \frac{1}{\Gamma(\gamma)}n^{\gamma-1}$*

Proof. Let $H(z) = F(z) - (1-z)^{\gamma}$. Then, $H(1) = 0$. Consider the Taylor series of H:

$$H(z) = \sum_{n=0}^{\infty} d_n z^n.$$

By the asymptotic behavior of b and the asymptotic behavior of the coefficients of $(1-z)^\gamma$, we find

$$d_n = O(\frac{1}{n^{2+\gamma}}).$$

Now consider

$$\frac{H(z)}{1-z} = \frac{H(z)-H(1)}{1-z} = \sum \frac{d_n(z^n-1)}{1-z}.$$

Then, for each term, $|d_n z^n - d_n|(1-z)^{-1} \le Cnd_n$ where C is independent of n and $z \in D(0,1)$. Since $\sum n|d_n|$ converges. Hence, $G(z) = H(z)/(1-z)$ is bounded. The first claim is proved.

Using the first claim, we find that

$$F(z)^{-1} = (1-z)^{-\gamma}\frac{1}{1+(1-z)^{1-\gamma}G(z)}$$

This implies that as $z \sim 1$, $F(z)^{-1} \sim (1-z)^{-\gamma}$.

By the lemma in the book of 'analytical combinartorics', we have $b_n^{(-1)} = [z^n]F(z)^{-1} \sim \frac{1}{\Gamma(\gamma)}n^{\gamma-1}$. $\square$

Remark 1. *Actually, we should have $G(1) = 0$ also. Further, it should be the case that*

$$F(z) - (1-z)^\gamma = (1-z)^{1+\gamma-\epsilon}G(z)$$

and G is still bounded, but we don't need these.

The following discrete Gronwall inequality is important:

Lemma 2. *discrete gronwall*

Now, we prove an important lemma

Lemma 3. *Suppose $b = \{b_n\}$ is a sequence satisfying the properties above. Let $E = \{E^n\}$ be a nonnegative sequence. Let $b^{(-1)}$ be the convolution inverse of b, which has generating function $1/F(z)$. Let $k = T/N$ where N is a big integer.*

If one of the following two conditions is satisfied:
(i). $\exists \gamma_n \ge 0, \beta \ge 0$, $\gamma_n \le Ck^{-\gamma}(1+n)^{-\gamma}$

$$k^{-\gamma}|(b*E)_{n+1}| \le \gamma_n + \beta E^n, \forall 0 \le n \le N.$$

(ii). $b_n^{(-1)} \geq 0$ *and* $\exists \gamma_n \geq 0, \beta \geq 0$, $\gamma_n \leq Ck^{-\gamma}(1+n)^{-\gamma}$

$$k^{-\gamma}(b * E)_{n+1} \leq \gamma_n + \beta E^n, \forall 0 \leq n \leq N.$$

Then,

$$E^n \leq C(T)E_\gamma(C_1(nk)^\gamma), \ \forall 0 \leq n \leq N.$$

Proof. Let $H_n = k^{-\gamma}(b * E)_n$ and consider the sequence $H = \{H_n\}$. Then clearly, we have $E^n = k^\gamma(b^{(-1)} * H)_n$ for all $n \geq 0$.

By the asymptotic behavior of $b_n^{(-1)}$, there exists C independent of n such that $|b_n^{(-1)}| \leq C\frac{1}{\Gamma(\gamma)}(n+1)^{\gamma-1}$ for all $n \geq 0$.

Now, consider only $0 \leq n \leq N$:

If the first case happens, we have

$$E^n \leq k^\gamma \sum_{i=0}^{n} |b_{n-j}^{(-1)}||H_j| \leq |b_n^{(-1)}|b_0 E_0 + Ck^\gamma \sum_{j=1}^{n} \frac{1}{\Gamma(\gamma)}(n-j+1)^{\gamma-1}(\gamma_{j-1} + \beta E^{j-1})$$
$$\leq C + k^\gamma \frac{C_1}{\Gamma(\gamma)} \sum_{j=0}^{n-1}(n-j)^{\gamma-1}E^j, \ \forall 0 \leq n \leq N$$

Here, we have used the fact $k^\gamma \sum (n-j)^{\gamma-1}k^{-\gamma}(1+j)^{-\gamma} \sim const$

if the second case happens, then, we have

$$E^n \leq k^\gamma \sum_{j=0}^{n} b_{n-j}^{(-1)}H_j \leq b_n^{(-1)}b_0 E_0 + Ck^\gamma \sum_{j=1}^{n} \frac{1}{\Gamma(\gamma)}(n-j+1)^{\gamma-1}(\gamma_{j-1} + \beta E^{j-1})$$
$$\leq C + \frac{C_1}{\Gamma(\gamma)} \sum_{j=0}^{n-1}(n-j)^{\gamma-1}E^j, \ \forall 0 \leq n \leq N$$

The discrete Gronwall inequality guarantees that E^n has the bound as indicated. ☐

In The paper by Xu.., a scheme is designed. There, the coefficients satisfy: $a_0^{n+1} > 0$, $a_m^{n+1} < 0, m \geq 1$. a_m^{n+1} agrees with the limiting sequence for $m \leq n$. $a_{n+1}^{n+1} = O(n^{-\gamma})$. We denote the first order scheme as

$$(\mathscr{D}_k u)^{n+1} = k^{-\gamma}(a^{n+1} * u)_{n+1}.$$

As we have seen, the difference scheme is $(b^{n+1} * u)_{n+1}$ where b^{n+1} is a sequence with length $n+1$.

Theorem 4. *If E^n is a nonnegative sequence satisfying*

$$(\mathscr{D}_k E)^{n+1} \leq \lambda E^n$$

then

$$E^n \leq E^0 C(T) E_\gamma(C_1(nk)^\gamma), \forall 0 \leq n \leq N.$$

Proof. Let $a = (a_0, a_1, \ldots)$ be the limit sequence.

As we know $a_0 > 0$ and $a_m < 0$ for all $m \geq 1$. Then, we find that all components of $a^{(-1)}$ are positive.

By what has been computed, a^{n+1} is exactly the same as the subsequence of a for the first n components.

$$k^{-\gamma}(a^{n+1} * E)_{n+1} = k^{-\gamma}(a * E)_{n+1} + k^{-\gamma}(a^{n+1}_{n+1} - a_{n+1})E^0 \leq CE^n, \ \forall 0 \leq n \leq N.$$

Note that $a^{n+1}_{n+1} - a_{n+1} = \sum_{m=n+2}^{\infty} a_m = O(n^{-\gamma})$.

Applying the variant version of the lemma above shows that the first order KS scheme is stable. □

Corollary 1. *Consider the FODE $D_c^\gamma u = A(u)$. Suppose $\mathscr{A}_k u$ is a first order approximation for $A(u)$ such that $\exists \lambda \geq 0$,*

$$\langle u^{n+1}, (\mathscr{A}_k u)^{n+1} \rangle \leq \lambda (u^n)^2.$$

The scheme given by

$$(\mathcal{D}_k u)^{n+1} = (\mathscr{A}_k u)^{n+1}, \ n \geq 1$$

is stable.

Corollary 2. *Consider the FODE*

$$D_c^\gamma u = \lambda u.$$

The first order explicit scheme given by

$$(\mathscr{D}_k u)^{n+1} = \lambda u^n,$$

is stable for $\lambda \geq 0$.

The first order implicit scheme

$$(\mathscr{D}_k u)^{n+1} = \lambda u^{n+1},$$

is stable for $\lambda \leq 0$.

Proof. Consider the explicit scheme and $\lambda \geq 0$. Since both the FODE and the scheme are linear. We can consider $u_0 \geq 0$. By the sign of the coefficients, we find that u^n are all nonnegative. The theorem then implies ....

For the implicit scheme, since $\langle u^{n+1}, \lambda u^{n+1}\rangle \leq 0$. Using Corollary..., we find that ... $\square$

Remark 2. *The second order explicit scheme is given by*

$$(\mathcal{D}_k u)^{n+1} = \lambda(2u^n - u^{n-1}).$$

To show that the scheme is stable, we may need the positivity, but it is not clear if we can prove this. Monotonicity?...

The following result is about l^2-stability for the second order scheme.

Theorem 5. *Consider the FODE $D_c^\gamma u = A(u)$. Suppose $A_k u$ is a second order approximation for $A(u)$ such that $\exists \lambda \geq 0$,*

$$\langle u^{n+1}, (A_k u)^{n+1}\rangle \leq \lambda(\|u^n\|^2 + \|u^{n-1}\|^2).$$

The scheme given by

$$(\mathcal{D}_k u)^{n+1} = (A_k u)^{n+1},\ n \geq 1$$

is l^2-stable.

Proof. Now considering left hand side. The issue appears since $b_2 > 0$ if $\gamma > \gamma_0$.

The key point is to write

$$b_0 u^{n+1} + b_1 u^n + b_2 u^{n-1} = -\frac{b_1}{2}(\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}) + (b_0 + \frac{3}{4}b_1)u^{n+1} + (b_2 + \frac{b_1}{4})u^{n-1}$$

Define the new sequence $c_0^{n+1} = b_0 + \frac{3}{4}b_1 > 0$, $c_1^{n+1} = 0$, $c_2^{n+1} = b_2 + \frac{1}{4}b_2 < 0$, $c_m^{n+1} = b_m < 0, m \geq 3$. $\sum_m c_m^{n+1} = 0$ still holds. We also use c to mean the limiting sequence for c^{n+1}. By the technique used in...

$$\langle u^{n+1}, \sum_m c_m^{n+1} u^{n+1-m}\rangle \geq \frac{1}{2}\sum_m c_m^{n+1}\|u^{n+1-m}\|^2.$$

Further,

$$\langle u^{n+1}, \frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}\rangle \geq \frac{1}{4}(\|u^{n+1}\|_2^2 - \|u^n\|_2^2 + \|2u^{n+1} - u^n\|_2^2 - \|2u^n - u^{n-1}\|_2^2)$$

Since $\langle u^{n+1}, (\mathcal{D}_k u)^{n+1}\rangle \le \lambda(\|u^n\|^2 + \|u^{n+1}\|^2)$ and $-b_1 > 0$, we have

$$
\begin{aligned}
&-\frac{b_1}{8}k^{-\gamma}(\|u^{n+1}\|_2^2 - \|u^n\|_2^2 + \|2u^{n+1} - u^n\|_2^2 - \|2u^n - u^{n-1}\|_2^2) \\
&\qquad + \frac{1}{2}k^{-\gamma}\left(c_0\|u^{n+1}\|^2 + c_2\|u^{n-1}\|^2 + \sum_{m=3}^{n+1} c_m\|u^{n+1-m}\|^2\right) \\
&\le k^{-\gamma}(b_n - b_n^{n+1})\|u^1\|^2 + k^{-\gamma}(b_{n+1} - b_{n+1}^{n+1})\|u^0\|^2 + \lambda(\|u^n\|^2 + \|u^{n-1}\|^2)
\end{aligned}
$$

Here, it is important to notice that

$$
b_n^{n+1} - b_n = \frac{1}{2}n^{1-\gamma} + \frac{1}{2}(n+1)^{1-\gamma} + \frac{1}{2-\gamma}n^{2-\gamma} - \frac{1}{2-\gamma}(n+1)^{2-\gamma} = O(n^{-1-\gamma})
$$
$$
b_{n+1}^{n+1} - b_{n+1} = O(n^{-\gamma})
$$

Set $F_n = \frac{1}{n+1}\sum_{m=0}^{n}\|u^m\|_2^2$. Taking the summation to $n+1$ and diving by $n+2$, we have

$$
\begin{aligned}
&-\frac{b_1}{8(n+2)}k^{-\gamma}((\|u^{n+1}\|_2^2 - \|u^0\|_2^2 + \|2\rho^{n+1} - \rho^n\|_2^2 - \|2\rho^1 - \rho^0\|_2^2) \\
&\qquad + \frac{1}{2}k^{-\gamma}\left(c_0F_{n+1} + c_2\frac{n}{n+2}F_{n-1} + \sum_{m=3}^{n+1} c_m\frac{n+2-m}{n+2}F_{n+1-m}\right) \\
&\le k^{-\gamma}\frac{1}{n+2}\sum_{m=0}^{n+1}((b_m - b_m^{m+1}))\|\rho_1\|^2 + k^{-\gamma}\frac{1}{n+2}\sum_{m=0}^{m+1}(b_{m+1} - b_{m+1}^{m+1})\|\rho_0\|^2 + 2\lambda F_n
\end{aligned}
$$

Direct estimation tells us that $\frac{1}{n+1}\sum_{n=0}^{n+1}((b_n - b_n^{n+1}))\|\rho_1\|^2 = O(n^{-\gamma})$ and $\frac{1}{n+1}\sum_{n=0}^{n+1}(b_{n+1} - b_{n+1}^{n+1}) = O(n^{-\gamma})$. Noting the signs of the coefficients, we have

$$
\begin{aligned}
&\frac{1}{2}k^{-\gamma}\left(c_0F_{n+1} + c_2F_{n-1} + \sum_{m=3}^{n+1} c_mF_{n+1-m}\right) \le \\
&\qquad -\frac{|b_1|}{8|n+2|k^{\gamma}}\|u^{n+1}\|^2 + C_2k^{-\gamma}(n+1)^{-\gamma}(\|u^0\|_2^2 + \|u^1\|_2^2) + 2\lambda F_n.
\end{aligned}
$$

Using Lemma ..., we can show that F is bounded since $c = (b_0 + \frac{3}{4}b_1, 0, b_2 + \frac{b_1}{4}, b_3, \ldots)$ satisfies the conditions in the lemma. $\square$

The condition on A_k says that A_k should have a kind of negativity. For example, if $A_k(u) = -\delta u^{n+1} + B(u^{n-1}, u^n)$ where $\delta > 0$ and B is a bounded bilinear operator, then, the condition is satisfied.

Corollary 3. *The implicit second order scheme for $D_c^\gamma u = \lambda u, \lambda \le 0$ is l^2-stable.*

This agrees with the A-stability analysis.

3 An FPDE

Consider the following fractional Keller-Segel equation:

$$D_c^\gamma \rho = \Delta\rho - \nabla\cdot(\rho\nabla c)$$
$$-\Delta c = \rho$$

Integrating by parts,

$$\frac{u(t)-u(0)}{t^\gamma} + \gamma\int_0^t \frac{u(t)-u(s)}{(t-s)^{1+\gamma}}ds = \dots$$

It is clear that the difference scheme is a kind of approximation for this. $b_0 \approx \frac{1}{t^\gamma} + \gamma\int_0^{t-k}\frac{1}{(t-s)^{1+\gamma}}ds$, etc..

Since

$$\langle u, \Delta u - \nabla\cdot(\rho\nabla c)\rangle = -\|\nabla u\|_2^2 + \frac{1}{2}\|\rho^3\|_1,$$

and $\|\rho^3\|_1 \le C\|\rho\|_1\|\nabla\rho\|_2^2$ by Gargolidardo-Nirenberg inequality for dimension 2, we see that if the initial mass is small, then,

$$\langle u(t), \frac{u(t)-u(0)}{t^\gamma} + \gamma\int_0^t \frac{u(t)-u(s)}{(t-s)^{1+\gamma}}ds\rangle \le 0.$$

Since $a(a-b) \ge \frac{1}{2}(a^2-b^2)$, then $\|u(t)\|_2^2$ decays.

Clearly, in numerics, if we approximate $-\Delta c = \rho$ with ρ in previous time steps. The Gargoliardo inequality can't be applied and we don't have the decay claim. Now, we explore the explicit schemes.

3.1 First order scheme

Consider the first order time discrete approximation:

$$\frac{1}{k^\gamma}(b_0\rho^{n+1} - \sum_{m=1}^{n+1}|b_m|\rho^{n+1-m}) = \Delta\rho^{n+1} - \nabla\cdot(\rho^{n+1}\nabla c^{n+1}) \tag{9}$$
$$-\Delta c^{n+1} = \rho^n \tag{10}$$

where $b_0 = \sum_{m=1}^{n+1}|b_m|$.

If we multiply ρ^{n+1} and integrate, we obtain on right hand side:

$$-\|\nabla\rho^{n+1}\|_2^2 + \frac{1}{2}\|\rho^n(\rho^{n+1})^2\|_1$$

For right hand side, we have

$$\leq -\|\nabla\rho^{n+1}\|_2^2 + C\|\rho^n\|_2\|\rho^{n+1}\|_2\|\nabla\rho^{n+1}\|_2 \leq -\|\nabla\rho^{n+1}\|_2^2 + \frac{1}{2}\lambda\|\rho^n\|_2^2 + C\|\rho^{n+1}\|_2^2\|\nabla\rho^{n+1}\|_2^2$$

By assuming the small data, then we have

$$\frac{1}{k^\gamma}(b_0\|\rho^{n+1}\|_2^2 - \sum_{m=1}^{n+1}|b_m|\|\rho^{n+1-m}\|^2) \leq \lambda\|\rho^n\|_2^2$$

Applying theorem... we obtain:

Proposition 1. *The first order scheme for the KS is stable with small initial data.*

3.2 Second order scheme

For $\gamma < 0.3x$, b has similar properties with the first order scheme. Using similar techniques and the variant lemma, we conclude

Proposition 2. *The second order scheme for the KS is l^2-stable if the initial data is small.*

Just check the right hand side and Applying Theorem..., we have....

3.3 Spatial discretization

If $\gamma \leq \gamma_0$, it preserves positivity..

A The negativity of P_m

Lemma 4. $P_m(\tau) \leq 0$ *for all* $m \geq 1$ *and* $\tau \in I_m$.

Proof.

$$P_m(\tau) = (t^{m-1} - \tau)^2 k^{2-\gamma}[\frac{1}{2}(n-m)^{1-\gamma} + \frac{1}{2-\gamma}(n-m)^{2-\gamma}] + \\ (-2)(t^m - \tau)^2 k^{2-\gamma}[(n-m)^{1-\gamma} + \frac{1}{2-\gamma}(n-m)^{2-\gamma}] \\ + (t^{m+1} - \tau)^2 k^{2-\gamma}[\frac{1}{2}(n-m+1)^{1-\gamma} + \frac{1}{2-\gamma}(n-m+1)^{2-\gamma}] \\ - \frac{2k^2}{2-\gamma}(t^{n+1} - \tau)^{2-\gamma}$$

$P_m(\tau = t^{m+1}) = 0$ and $P_m(\tau = t^m) = k^2 \int_{I_m} \frac{t^{m+1/2} - s}{(t^{n+1} - s)^\gamma} ds < 0$.
We find that

$$(1-\gamma)P' = 2(\tau - t^{m-1}) k^{2-\gamma}[\frac{1}{2}(n-m)^{1-\gamma} + \frac{1}{2-\gamma}(n-m)^{2-\gamma}] + \\ (-2)2(\tau - t^m) k^{2-\gamma}[(n-m)^{1-\gamma} + \frac{1}{2-\gamma}(n-m)^{2-\gamma}] \\ + 2(\tau - t^{m+1}) k^{2-\gamma}[\frac{1}{2}(n-m+1)^{1-\gamma} + \frac{1}{2-\gamma}(n-m+1)^{2-\gamma}] \\ + 2k^2(t^{n+1} - \tau)^{1-\gamma}$$

We can show that $P'(t^m) < 0$. $P'(t^{m+1}) = 0$.
Further,

$$(1-\gamma)P''(t^{m+1})/k^{2-\gamma} = -3(n-m)^{1-\gamma} - \frac{2}{2-\gamma}(n-m)^{2-\gamma} + \\ + [(n-m+1)^{1-\gamma} + \frac{2}{2-\gamma}(n-m+1)^{2-\gamma}] - 2(1-\gamma)(n-m)^{-\gamma} \\ = (n-m+1)^{1-\gamma} - 3(n-m)^{1-\gamma} + 2\int_{n-m}^{n-m+1} x^{1-\gamma} dx - 2(1-\gamma)(n-m)^{-\gamma}$$

Using the inequality $x^{1-\gamma} \le (n-m)^{1-\gamma} + (1-\gamma)(n-m)^{-\gamma}(x - (n-m))$ since the function is concave, we find

$$(1-\gamma)P''(t^{m+1})/k^{2-\gamma} \le (n-m+1)^{1-\gamma} - (n-m)^{1-\gamma} - (1-\gamma)(n-m)^{-\gamma} \le 0.$$

Hence, P'' could be positive on $\tau \in (t^m, t_0)$ and negative on (t_0, t^{m+1}) or all negative on (t^m, t^{m+1}). Together with the fact $P'(t^{m+1}) = 0$, we know either P either first decreases and then increases or increases for all time. Since $P(t^{m+1}) = 0$ and $P(t^m) < 0$, we find that $P \le 0, \tau \in (t^m, t^{m+1})$. $\square$