# Numerical Methods for (Viscous) Flows - Part II

R. Herbin

# 1. Viscous compressible and incompressible flows - Barotropic Navier Stokes equations

# Barotropic Navier Stokes equations

- Two conservation equations:

$$\partial_t \varrho + \mathrm{div}(\varrho \boldsymbol{u}) = 0,$$

$$\partial_t(\varrho \boldsymbol{u}) + \mathrm{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \mathrm{div}\boldsymbol{\tau} + \boldsymbol{\nabla} p = 0,$$

$$\varrho = \psi(p).$$

- For simplicity, we suppose $\boldsymbol{u} = 0$ on $\partial\Omega$.

- . . . application to more general situations: open boundary conditions, low Mach number flows, or compressible Navier-Stokes equations.

# 1. Viscous compressible and incompressible flows - Space discretization

# Colocated or staggered?

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- Positivity of density $\rightsquigarrow$ finite volume scheme on $\rho$:
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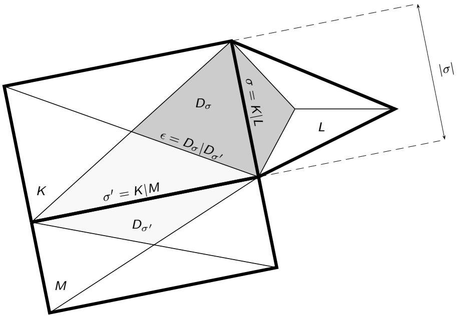
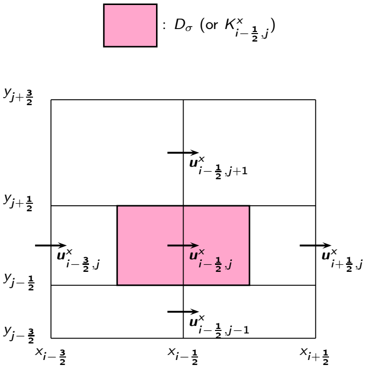
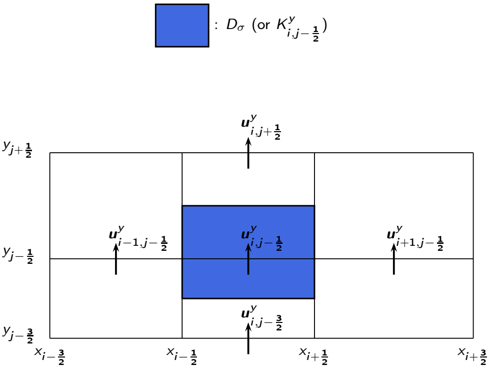
- Primal mesh : $\mathcal{D} = \{$ set of control volumes $\}$.
- Scalar variables defined at cell centers: $(p_K)_{K\in\mathcal{D}}$, $(\varrho_K)_{K\in\mathcal{D}}, \ldots$
- Velocity components defined at the (or some of the) edges : $(v_\sigma^{(i)})_{\sigma\in\mathcal{F}^{(i)}}$.
- Dual mesh(es) : $(D_\sigma^{(i)})_{\sigma\in\mathcal{F}^{(i)}}$.
- Normal velocity to the face $\sigma$ denoted by $\boldsymbol{v}_\sigma \cdot \boldsymbol{n}_\sigma$.




Example : Rannacher-Turek and Crouzeix-Raviart elements.

# The MAC mesh




The dual mesh for the MAC scheme, $x$ and $y$-component of the velocity.

# Discrete balance equations

- Mass balance:

$$\forall K \in \mathcal{D}, \quad \frac{|K|}{\delta t}\,(\varrho_K - \varrho_K^*) + \sum_{\sigma=K|L} F_{K,\sigma} = 0.$$

with:

$F_{K,\sigma} = |\sigma|\ \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}\ \varrho_\sigma^{\rm up}$, $\quad \varrho_\sigma^{\rm up}$: upwind approximation of the density on the face,

$\varrho_K$ linked to $p_K$ through the equation of state: $\varrho_K = \psi(p_K)$.

...so the discrete mass balance is a finite volume balance on the primal mesh.

- Momentum balance:

$$\forall \sigma \in \mathcal{E}_{\rm int}^{(i)},\ \text{for } 1 \leq i \leq d,$$

$$\frac{|D_\sigma|}{\delta t}\ \Big[\varrho_\sigma\ \boldsymbol{u}_\sigma^{(i)} - \varrho_\sigma^*(\boldsymbol{u}_\sigma^{(i)})^*\Big] + \sum_{\epsilon\in\bar{\mathcal{E}}(D_\sigma)} F_\epsilon\ \boldsymbol{u}_\epsilon^{(i)} + (T_{\rm dif})_\sigma^{(i)} + (\boldsymbol{\nabla} p)_\sigma^{(i)} = 0.$$

*i.e.* finite volume or finite volume/finite element balance on the dual mesh(es).

# First properties of the scheme

- Coercivity of the diffusion term. $\tilde{\boldsymbol{u}} = (\boldsymbol{u}_\sigma)_\sigma \in \mathcal{E}_{\text{int}}$
  Let:

$$T(\tilde{\boldsymbol{u}}) = \sum_{i=1}^{d} \sum_{\sigma \in \mathcal{E}_{\text{int}}} \left(T_{\text{dif}}(\tilde{\boldsymbol{u}})\right)_\sigma^{(i)} \; \boldsymbol{u}_\sigma^{(i)}.$$

  Then:

$$T(\tilde{\boldsymbol{u}}) \geq C \, \|\tilde{\boldsymbol{u}}\|_1^2,$$

  + discrete functional analysis: Poincaré inequality, Sobolev injections, compactness... : see Part I.

- $\boldsymbol{\nabla} - \mathrm{div}$ duality:

$$\sum_{\sigma \in \mathcal{E}_{\text{int}}} (\boldsymbol{\nabla} p)_\sigma \cdot \boldsymbol{u}_\sigma = -\sum_{K \in \mathcal{D}} p_K \sum_{\sigma = K|L} |\sigma| \; \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_\sigma.$$

- $\varrho_\sigma, \;\; F_\epsilon$ ?

# 1. Viscous compressible and incompressible flows - Stability: designing the convection term

# Kinetic energy identity

- Formally, at the continuous level, the momentum balance equation reads:

$$\partial_t\ (\varrho\, \boldsymbol{u}) + \mathrm{div}(\boldsymbol{u} \otimes q) - \mathrm{div}\boldsymbol{\tau}(\boldsymbol{u}) + \boldsymbol{\nabla} p = 0.$$

- Multiply the first equation by $\boldsymbol{u}$ and integrate over $\Omega$:

$$\underbrace{\int_\Omega \Big[\partial_t\ (\varrho\, \boldsymbol{u}) + \mathrm{div}(\boldsymbol{u} \otimes q)\Big] \cdot \boldsymbol{u}}_{T_k} + \int_\Omega \boldsymbol{\tau}(\boldsymbol{u}) : \boldsymbol{\nabla}\boldsymbol{u} - \int_\Omega p\, \mathrm{div}\boldsymbol{u} = 0,$$

- " kinetic energy identity"

$$T_k = \frac{d}{dt}\left(\int_\Omega \frac{1}{2}\varrho\, |\boldsymbol{u}|^2\right).$$

# Discrete kinetic energy identity

Assume:

- that a discrete mass balance holds over dual meshes:

$$\forall \sigma \in \mathcal{E}_{\rm int}, \qquad \frac{|D_\sigma|}{\delta t} \left[\varrho_\sigma - \varrho^*_\sigma\right] + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} F_\epsilon = 0,$$

- that the densities are positive:

$$\forall \sigma \in \mathcal{E}^{(i)}_{\rm int}, \qquad \varrho_\sigma > 0,\ \varrho^*_\sigma > 0.$$

Then:

$$\sum_{\sigma \in \mathcal{E}_{\rm int}} u_\sigma \left[\frac{|D_\sigma|}{\delta t} \left[\varrho_\sigma\, u_\sigma - \varrho^*_\sigma u^*_\sigma\right] + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} F_\epsilon\ u_\epsilon\right] \geq$$

$$\frac{1}{\delta t} \sum_{\sigma \in \mathcal{E}_{\rm int}} |D_\sigma| \left[\frac{1}{2}\, \varrho_\sigma\, (u_\sigma)^2 - \frac{1}{2}\, \varrho^*_\sigma\, (u^*_\sigma)^2\right],$$

where $u_\epsilon$ may stand for a centered or for an upwind discretization of $u$ at the edge $\epsilon$.

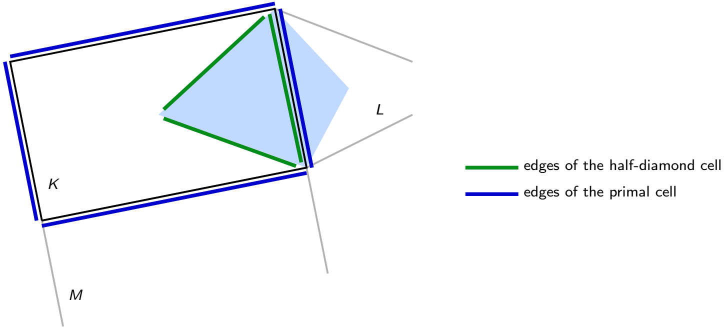
# Building a discrete mass balance over the dual cells (1/8)

Construct the fluxes $F_\epsilon$ on the dual mesh such that:

(H1) A discrete mass balance over the half-diamond cells is satisfied, in the following sense:

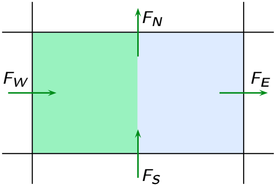
$$\forall K \in \mathcal{D},\ \forall \sigma \in \mathcal{E}(K), \qquad F_\sigma + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma),\ \epsilon \subset K} F_\epsilon = \xi_K^\sigma \Big[ \sum_{\sigma' \in \mathcal{E}(K)} F_{\sigma'} \Big],$$

with $\xi_K^\sigma \geq 0$, and $\displaystyle\sum_{\sigma \in \mathcal{E}(K)} \xi_K^\sigma = 1$.



# Dual discrete mass balance for the MAC scheme

Exemple of construction (MAC scheme):




- Mass balances on the primal cell:

$$\frac{|K|}{\delta t}\,(\varrho_K - \varrho_K^*) - F_W - F_S + F_E + F_N = 0,$$

- and:

$$F_E - \frac{F_W + F_E}{2} - \frac{F_S}{2} + \frac{F_N}{2} = \frac{1}{2}\Big[-F_W - F_S + F_E + F_N\Big]$$

$$-F_W + \frac{F_W + F_E}{2} - \frac{F_S}{2} + \frac{F_N}{2} = \frac{1}{2}\Big[-F_W - F_S + F_E + F_N\Big]$$

$$\text{(H1)}\quad \forall K \in \mathcal{D},\ \forall \sigma \in \mathcal{E}(K), \qquad F_\sigma + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma),\ \epsilon \subset K} F_\epsilon = \xi_K^\sigma \Big[\sum_{\sigma' \in \mathcal{E}(K)} F_{\sigma'}\Big].)$$

# Recover discrete mass balance over the dual cells

Assume:

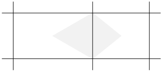
(H1) A discrete mass balance over the half-diamond cells is satisfied, in the following sense:

$$\forall K \in \mathcal{D},\ \forall \sigma \in \mathcal{E}(K), \qquad F_\sigma + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma),\ \epsilon \subset K} F_\epsilon = \xi_K^\sigma \Big[ \sum_{\sigma' \in \mathcal{E}(K)} F_{\sigma'} \Big].$$

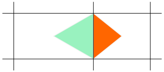
Let:

$$\forall \sigma = K|L \in \mathcal{E}_{\rm int}, \qquad |D_\sigma| = \xi_K^\sigma\, |K| + \xi_L^\sigma\, |L|, \quad |D_\sigma|\, \varrho_\sigma = \xi_K^\sigma\, |K|\, \varrho_K + \xi_L^\sigma\, |K|\, \varrho_L.$$

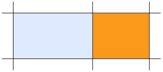
Then:



$$\frac{|D_\sigma|}{\delta t}\, [\varrho_\sigma - \varrho_\sigma^*] + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} F_\epsilon =$$



$$\xi_K^\sigma \frac{|K|}{\delta t}\, [\varrho_K - \varrho_K^*] + F_\sigma + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma),\ \epsilon \subset K} F_\epsilon \quad + \quad \xi_L^\sigma \cdots - F_\sigma + \cdots =$$



$$\xi_K^\sigma \Big[ \frac{|K|}{\delta t}\, [\varrho_K - \varrho_K^*] + \sum_{\sigma \in \mathcal{E}(K)} F_\sigma \Big] \quad + \quad \xi_L^\sigma \Big[ \ldots \Big] = 0.$$

# Control of the difference between the two meshes

Assume:

(H1) A discrete mass balance over the half-diamond cells is satisfied.

(H2) The dual fluxes are conservative.

(H3) The dual fluxes are uniformly bounded with respect to the primal ones.

Let:

$$v_K = \sum_{\sigma \in \mathcal{E}(K)} \xi_K^\sigma \, v_\sigma,$$

$$R = \sum_{\sigma \in \mathcal{E}_{\rm int}} v_\sigma \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} F_\epsilon \; u_\epsilon \quad - \quad \sum_{K \in \mathcal{D}} v_K \sum_{\sigma \in \mathcal{E}(K)} F_\sigma \; u_\sigma .$$

$$\Big( \text{ i.e. } \; R \approx \int_\Omega v \, \mathrm{div}(u\boldsymbol{q}) \, \mathrm{d}\boldsymbol{x} \quad - \quad \int_\Omega v \, \mathrm{div}(u\boldsymbol{q}) \, \mathrm{d}\boldsymbol{x} \; \Big)$$

Then:

$$|R| \leq C \; h \, \|F_\sigma\|_{\mathrm{L}^\infty} \, \|u\|_{\mathrm{H}^1} \, \|v\|_{\mathrm{H}^1}.$$

Proof: Use the conservativity □

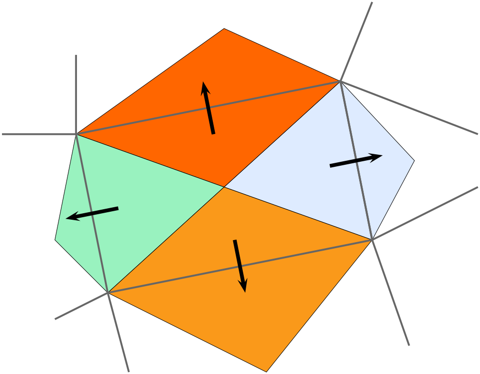
# An “inflated" gradient

## Lemma

*Let* $\varphi \in \mathrm{C}_c^\infty(\Omega)$*, and* $\boldsymbol{\nabla}_{\mathcal{D}}\varphi$ *be the piecewise constant function over the diamond cells such that:*

$$\forall \sigma = K|L \in \mathcal{E}_{\rm int}, \qquad (\boldsymbol{\nabla}_{\mathcal{D}}\varphi)_\sigma = \frac{|\sigma|}{|D_\sigma|}\,(\varphi_L - \varphi_K)\,\boldsymbol{n}_\sigma,$$

*where, for* $K \in \mathcal{D}$*,* $\varphi_K$ *is an approximation of* $\varphi$ *over* $K$*. Then* $\boldsymbol{\nabla}_{\mathcal{D}}\varphi$ *weakly converges to* $\boldsymbol{\nabla}\varphi$ *in any* $\mathrm{L}^p(\Omega)$*,* $p \in (1, +\infty)$*.*



# Case of the incompressible NS equations

Let us assume that we are solving stationary incompressible Navier-Stokes equations ($\psi(p) = 1$) on a sequence of quasi-uniform meshes. Then:

- estimates: $\boldsymbol{u}$ controlled in $\mathrm{H}^1$ discrete norm, so in $\mathrm{L}^6(\Omega)^d$, and thus $\boldsymbol{u}$ controlled as $h^{-d/6}$ in $\mathrm{L}^\infty$ norm.
- compactness: $\boldsymbol{u}$ converges in $\mathrm{L}^p(\Omega)^d$, $p \in [1, 6)$ to a function of $\mathrm{H}^1_0(\Omega)^d$.
- passing to the limit in the equation:

$$\sum_{\sigma \in \mathcal{E}_{\rm int}} \varphi_\sigma \sum_{\substack{\epsilon \in \bar{\mathcal{E}}(D_\sigma),\\ \epsilon = D_\sigma | D'_\sigma}} F_\epsilon \; u_\epsilon = \sum_{K \in \mathcal{D}} \varphi_K \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \; \boldsymbol{u}^i_\sigma \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_\sigma + R, \qquad |R| \leq C\, h^{1-d/6}.$$

Then, reordering:

$$\begin{aligned} \sum_{K \in \mathcal{D}} \varphi_K \sum_{\sigma \in \mathcal{E}(K)} \boldsymbol{u}^i_\sigma \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_\sigma &= -\sum_{\sigma} |D_\sigma| \; \boldsymbol{u}^i_\sigma \; \boldsymbol{u}_\sigma \cdot \frac{|\sigma|}{|D_\sigma|} (\varphi_L - \varphi_K) \boldsymbol{n}_\sigma, \\ &= -\int_\Omega \boldsymbol{u}^i_\sigma \boldsymbol{u}_\sigma \cdot \boldsymbol{\nabla}_{\mathcal{D}} \varphi \, \mathrm{d}\boldsymbol{x}. \end{aligned}$$

# 1. Viscous compressible and incompressible flows - The scheme

# Time semi-discretization

From $t^n$ to $t^{n+1}$, *i.e.* to compute $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}^{n+1}$, $\boldsymbol{u} = \boldsymbol{u}^{n+1}$, $p = p^{n+1}$ knowing $\boldsymbol{u}^* = \boldsymbol{u}^n$, $p^* = p^n$ and, for the density at the cells, $\varrho_K^* = \varrho_K^n = \psi(p_K^n)$:

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- From the mass balance at the previous time step, build a discrete mass balance over the dual cells, *i.e.* $(\varrho_\sigma^n)$, $(\varrho_\sigma^{n-1})$ and $F_\epsilon^n$ such that, for $1 \leq i \leq d$, $\forall \sigma \in \mathcal{E}_{\rm int}^{(i)}$:

$$\frac{|D_\sigma|}{\delta t}\left[\varrho_\sigma^n - \varrho_\sigma^{n-1}\right] + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} F_\epsilon^n = 0.$$

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- and set $\varrho_\sigma = \varrho_\sigma^n$, $\varrho_\sigma^* = \varrho_\sigma^{n-1}$ and $F_\epsilon = F_\epsilon^n$ in the algorithm:

$\forall \sigma \in \mathcal{E}_{\rm int}^{(i)}$, for $1 \leq i \leq d$,

$$\frac{|D_\sigma|}{\delta t}\left[\varrho_\sigma\, \tilde{\boldsymbol{u}}_\sigma^{(i)} - \varrho_\sigma^* (\boldsymbol{u}_\sigma^{(i)})^*\right] + \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} F_\epsilon\, \tilde{\boldsymbol{u}}_\epsilon^{(i)} + \left(T_{\rm dif}(\tilde{\boldsymbol{u}})\right)_\sigma^{(i)} + (\boldsymbol{\nabla} p^*)_\sigma^{(i)} = 0.$$

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- first order scheme (in time), but conservative.

# Stability of the scheme

- adding a renormalization step for the pressure (Guermond-Quartapelle, JCP, 2000),
- and exploiting (more or less) standard techniques of the analysis of pressure correction methods ...
- The scheme is:
  - unconditionally stable:

$$E_k^n + \sum_k \delta t \, \|\boldsymbol{u}^k\|_1^2 + \int_\Omega \varrho^n P(\varrho^n) \leq C,$$

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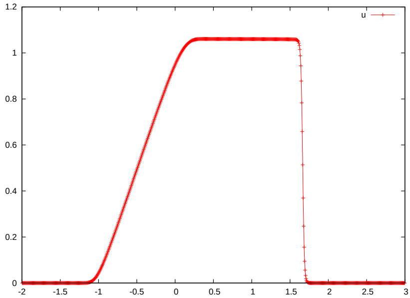
  - conserves the mass and preserves the positivity of the density.
  - From numerical experiments, roughly first order in time for regular solutions.

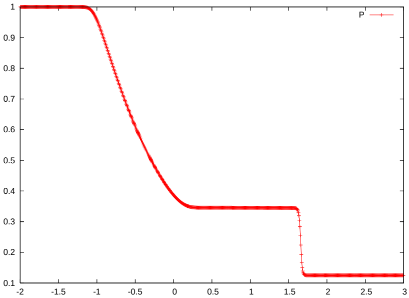
# 1. Viscous compressible and incompressible flows - Euler equations : computing solution with shocks

# Euler equation : computing solution with shocks

- with a residual (artificial) viscosity or with an upwind approximation for the velocity in the momentum balance equation, the scheme seems to converge to the correct (weak) solution,
- with a centered discretization in the momentum balance equation, we observe an odd-even decoupling.




Velocity




Pressure

*Obtained solution for the the Sod shock tube problem, with a residual viscosity, and 800 meshes.*

# 1. Viscous compressible and incompressible flows - Stability results

# Stability results

- Analysis for the constant density unstationary incompressible Stokes problem with Dirichlet boundary conditions: bounds of the splitting errror.
- Explicitation of the pressure discrete elliptic operator (. . . and so of the associated boundary conditions).
- Unconditionally stable pressure-correction algorithm for the barotropic diphasic flows.
- The stability proof is exploited to build artificial boundary conditions for external flows.
- A Crank-Nicolson version is under testing.
- Unconditionally stable pressure-correction algorithm for "complete" compressible Navier-Stokes equations.
- Convergence proofs for stationary problems:
  - Stokes problem with Crouzeix-Raviart and MAC scheme,
  - in progress for the MAC scheme and Navier-Stokes equations.

# 2. Convergence of the scheme, steady state NS equation, perfect gas - Continuous problem

# Steady compressible Navier-Stokes equations

- Steady compressible Navier-Stokes equations in a bounded open set of $\mathbb{R}^d$, $d = 2$ or 3, with EOS $p = \rho^\gamma$.

$$-\Delta u + (\rho u \cdot \nabla) u + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$\mathrm{div}(\rho u) = 0 \text{ in } \Omega, \ \rho \geq 0 \text{ in } \Omega, \ \int_\Omega \rho(x) dx = M,$$

$$p = \psi(\rho) = \rho^\gamma \text{ in } \Omega$$

$\gamma > 1$ if $d = 2$.
$\gamma > \frac{3}{2}$ id $d = 3$.

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- No uniqueness result.

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- Existence of (weak) solutions : works of P. L. Lions, E. Feirsel, A. Novotny...
- No uniqueness result.
- Aim : to prove the existence of solutions, passing to the limit on approximate solutions given by the same numerical scheme than that used in the industrial codes.
- First step to transient compressible NS

# Stationary compressible Navier Stokes equations

$\Omega$ is a bounded open set of $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz continuous boundary, $\gamma > 1$, $f \in L^2(\Omega)^d$ and $M > 0$

$$-\Delta u + (\rho u \cdot \nabla) u + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$\mathrm{div}(\rho u) = 0 \ \text{ in } \Omega, \ \rho \geq 0 \ \text{ in } \Omega, \ \int_\Omega \rho(x) dx = M,$$

$$p = \rho^\gamma \text{ in } \Omega$$

Functional spaces : $u \in H_0^1(\Omega)^d$, $p \in L^{\overline{q}}(\Omega)$, $\rho \in L^{\gamma\overline{q}}(\Omega)$.

If $d = 2$ or if $d = 3$ and $\gamma \geq 3$ : $\overline{q} = 2$.

If $d = 3$ and $\frac{3}{2} < \gamma < 3$ : $\overline{q} = \frac{3(\gamma-1)}{\gamma}$.

$\gamma = \frac{3}{2}$, $\frac{3\gamma-1}{\gamma} = 1$, $3(\gamma - 1) = \frac{3}{2}$

# Weak solution of the SCNS problem

Functional spaces : $u \in H_0^1(\Omega)^d$, $p \in L^{\overline{q}}(\Omega)$, $\rho \in L^{\gamma\overline{q}}(\Omega)$

- Momentum equation: $\overline{r} = \overline{q}' = \frac{\overline{q}}{\overline{q}-1}$ ($\overline{r} = 2$ or $\overline{r} = \frac{3(\gamma-1)}{2\gamma-3}$)

$$\int_\Omega \nabla u : \nabla v \,\mathrm{d}\boldsymbol{x} - \int_\Omega \rho u \otimes u : \nabla v \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \mathrm{div}(v) \,\mathrm{d}\boldsymbol{x}$$
$$= \int_\Omega f \cdot v \,\mathrm{d}\boldsymbol{x} \text{ for all } v \in W_0^{1,\overline{r}}(\Omega)^d.$$

- Mass equation:

$$\int_\Omega \rho u \cdot \nabla \varphi \,\mathrm{d}\boldsymbol{x} = 0 \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^d)$$

$$\rho \geq 0 \text{ a.e.}, \quad \int_\Omega \rho dx = M$$

- EOS: $p = \rho^\gamma$

# Main result

- Study of two discretizations for the momentum equation :
  $\rightsquigarrow$ MAC scheme (most commonly used scheme for incompressible and compressible Navier Stokes equations)
  $\rightsquigarrow$ Crouzeix-Raviart Finite Element (used by IRSN)
- Discretization of the mass equation (and total mass constraint) by classical upwind Finite Volume
- Existence of solution for the discrete problem
- Proof of the convergence (up to subsequence) of the solution of the discrete problem towards a weak solution of the continuous problem (no uniqueness result for this problem) as the mesh size goes to 0

# Steps for proving the convergence result

1. Estimates on the approximate solution $(u_n,\ p_n,\ \rho_n)$
2. Compactness result (convergence of the approximate solution, up to a subsequence)
3. Passage to the limit in the approximate equations

Main difficulty: Passage to the limit in the EOS ($p = \rho^\gamma$) since the EOS is a non linear function and Step 2 only leads to weak convergences of $p_n$ and $\rho_n$.

# Simpler result: Approximation by “regular solutions"

Simpler result:

Assume that $u_n$, $p_n$, $\rho_n$ are weak solution to

$$-\Delta u_n + (\rho_n u_n \cdot \nabla) u_n + \nabla p_n = f_n \text{ in } \Omega, \quad u_n = 0 \text{ on } \partial\Omega,$$

$$\operatorname{div}(\rho_n u_n) = 0 \text{ in } \Omega, \ \rho_n \geq 0 \text{ in } \Omega, \ \int_\Omega \rho_n(x) dx = M_n,$$

$$p_n = \rho_n^\gamma \text{ in } \Omega$$

$d = 2$, $\gamma > 1$, or $d = 3$, $\gamma > \frac{3}{2}$,

And assume $f_n \to f$ weakly in $(L^2(\Omega))^d$ and $M_n \to M$.

Then, up to a subsequence,

- $u_n \to u$ in $L^2(\Omega)^d$ and weakly in $H_0^1(\Omega)^d$,
- $p_n \to p$ in $L^q(\Omega)$ for any $1 \leq q < \overline{q}$ and weakly in $L^{\overline{q}}(\Omega)$,
- $\rho_n \to \rho$ in $L^q(\Omega)$ for any $1 \leq q < \overline{q}\gamma$ and weakly in $L^{\gamma\overline{q}}(\Omega)$,

where $(u, p, \rho)$ is a weak solution of the compressible Stokes equations (with $f$ and $M$ as data)

$\overline{q} = 2$ ($d = 2$; $d = 3$, $\gamma \geq 3$) $\overline{q} = \frac{3(\gamma-1)}{\gamma}$ ($d = 3$, $\frac{3}{2} < \gamma < 3$)

# Preliminary lemma

For $\beta > 1$
$\rho \in L^{2\beta}(\Omega)$, $\rho \geq 0$ a.e. in $\Omega$, $u \in (H_0^1(\Omega))^d$ , $\mathrm{div}(\rho u) = 0$, then:

- (i) $$\int_\Omega \rho \mathrm{div}(u) dx = 0$$
- (ii) $$\int_\Omega \rho^\beta \mathrm{div}(u) dx = 0$$

- (ii) is used (with $\beta = \gamma$) in order to obtain some estimates on the approximate solutions.
- (i) The first part (for the approximate solutions and for the limit of the approximate solutions) is crucial for passing to the limit on the EOS

# Proof of the preliminary result, $\rho$ regular

For simplicity : $\rho \in C^1(\bar{\Omega})$, $\rho \geq \alpha$ a.e. in $\Omega$.

Take $\varphi = \rho^{\beta-1}$ as test function in $\mathrm{div}(\rho u) = 0$:

$$\int_\Omega \rho u \cdot \nabla \rho^{\beta-1} dx = (\beta - 1) \int_\Omega \rho^{\beta-1} u \cdot \nabla \rho dx = 0.$$

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In the discrete setting: similar proof, replacing the derivatives by discrete derivatives, and the integration by parts by summation by parts.

# Proof of the preliminary result, non regular $\rho$

Problem: preliminary lemma also needed for non regular functions because of the passage to the limit.

$\beta > 1$, $\rho \in L^{2\beta}(\mathbb{R}^d)$, and $u \in H^1(\mathbb{R}^d)^d$.

- Let $(r_n)_{n\in\mathbb{N}^\star}$ be a sequence of mollifiers

$$\begin{gathered} r \in C_c^\infty(\mathbb{R}^d,\mathbb{R}),\ \int_{\mathbb{R}^d} r dx = 1,\ r \geq 0 \text{ in } \mathbb{R}^d \\ \text{and, for } n \in \mathbb{N}^\star, x \in \mathbb{R}^d,\ r_n(x) = n^d r(nx). \end{gathered} \tag{1}$$

and, for $n \in \mathbb{N}^\star$, $\rho_n = \rho \star r_n$ and $(\rho u)_n = (\rho u) \star r_n$.

$\rho_n$ is regular, but $\mathrm{div}(\rho_n u) \neq 0$..., so we cannot conclude $\displaystyle\int_\Omega \rho_n \mathrm{div} u = 0$.

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- However $\mathrm{div}(\rho_n u_n) = 0$
- Idea: control $\mathrm{div}(\rho_n u_n) - \mathrm{div}(\rho_n u)$.
- Indeed, $[(\rho u)_n - \rho_n u] \to 0$ weakly in $W^{1,(2\beta)/(\beta+1)}(\mathbb{R}^d)^d$.
  In particular, $\mathrm{div}((\rho u)_n - \rho_n u) \to 0$ weakly in $L^{(2\beta)/(\beta+1)}(\mathbb{R}^d)$.

# Estimates on $u$

Taking $u_n$ as test function in $-\Delta u_n + (\rho_n u_n \cdot \nabla) u_n + \nabla p_n = f_n$:

$$\int_\Omega \nabla u_n : \nabla u_n dx - \int_\Omega \rho_n u_n \otimes u_n : \nabla u_n dx - \int_\Omega p_n \mathrm{div}(u_n) dx = \int_\Omega f_n \cdot u_n dx$$

But $p_n = \rho_n^\gamma$ a.e. and $\mathrm{div}(\rho_n u_n) = 0$, then (preliminary lemma) $-\int_\Omega p_n \mathrm{div}(u_n)\, dx = 0$.
Thanks to the mass equation, $\int_\Omega \rho_n u_n \otimes u_n : \nabla u_n \; \mathrm{d}\boldsymbol{x} = 0$.

This gives an estimate on $u_n$:

$$\|u_n\|_{(H_0^1(\Omega))^d} \leq C_1.$$

# Estimate on $p$, divergence lemma

Let $g \in L^{\overline{r}}(\Omega)$ s.t. $\int_\Omega g dx = 0$.

Then, there exists $v \in (W_0^{1,\overline{r}}(\Omega))^d$ s.t.

$$\mathrm{div}(v) = g \text{ a.e. in } \Omega,$$

$$\|v\|_{(W_0^{1,\overline{r}}(\Omega))^d} \leq C_2 \|q\|_{L^{\overline{r}}(\Omega)},$$

where $C_2$ only depends on $\Omega$ and $\overline{r}$.

$\overline{r} = 2$ (if $\overline{q} = 2$) or $\overline{r} = \frac{3(\gamma-1)}{2\gamma-3}$ (if $\overline{q} = \frac{3(\gamma-1)}{\gamma}$).

# Estimate on $p$, $d = 3$, $\gamma \geq 3$ ($\overline{q} = \overline{r} = 2$)

$m_n = \frac{1}{|\Omega|} \int_\Omega p_n dx$, $v_n \in H_0^1(\Omega)^d$, $\mathrm{div}(v_n) = p_n - m_n$.

Taking $v_n$ as test function in $-\Delta u_n + (\rho_n u_n \cdot \nabla) u_n + \nabla p_n = f_n$:

$$\int_\Omega \nabla u_n : \nabla v_n dx - \int_\Omega \rho_n u_n \otimes u_n : \nabla v_n dx - \int_\Omega p_n \mathrm{div}(v_n) dx = \int_\Omega f_n \cdot v_n dx.$$

Using $\int_\Omega \mathrm{div}(v_n) dx = 0$:

$$\int_\Omega (p_n - m_n)^2 dx = \int_\Omega (f_n \cdot v_n - \nabla u_n : \nabla v_n - \rho_n u_n \otimes u_n : \nabla v_n) dx.$$

Since $\|v_n\|_{(H_0^1(\Omega))^d} \leq C_2 \|p_n - m_n\|_{L^2(\Omega)}$, $\|u_n\|_{(H_0^1(\Omega))^d} \leq C_1$ and, for $d = 3$, $H_0^1(\Omega) \subset L^6(\Omega)$,

$$\int_\Omega \rho_n u_n \otimes u_n : \nabla v_n dx \leq \|\rho_n\|_6 \|u_n\|_6 \|v_n\|_{(H^1(\Omega))^d},$$

the preceding inequality leads to:

$$\|p_n - m_n\|_{L^2(\Omega)} \leq C_3 + C_3 \|\rho_n\|_6 \leq C_3 + C_3 \|\rho_n\|_{2\gamma} \leq C_3 + C_3 \|p_n\|_2^{\frac{1}{\gamma}},$$

since $2\gamma \geq 6$ and $p_n = \rho_n^\gamma$.

# Estimate on $p$ and $\rho$

$$\|p_n - m_n\|_{L^2(\Omega)} \leq C_3 + C_3 \|p_n\|_2^{\frac{1}{\gamma}},$$

$$\int_\Omega p_n^{\frac{1}{\gamma}} \, dx = \int_\Omega \rho_n dx \leq \sup\{M_p, p \in \mathbb{N}\}.$$

Then:

$$\|p_n\|_{L^2(\Omega)} \leq C_4;$$

where $C_4$ only depends on the $L^2-$bound of $(f_n)_{n\in\mathbb{N}}$, the bound of $(M_n)_{n\in\mathbb{N}}$, $\gamma$ and $\Omega$.

$p_n = \rho_n^\gamma$ a.e. in $\Omega$, then:

$$\|\rho_n\|_{L^{2\gamma}(\Omega)} \leq C_5 = C_4^{\frac{1}{\gamma}}.$$

Case $d = 3$, $\frac{3}{2} < \gamma < 3$. Use $p_n^\theta$ with $\theta\overline{r} = \overline{q} = 1 + \theta$.

# Weak-convergence on $u_n$, $p_n$, $\rho_n$

Thanks to the estimates on $u_n$, $p_n$, $\rho_n$, it is possible to assume (up to a subsequence) that, as $n \to \infty$:

$$u_n \to u \text{ in } L^2(\Omega)^d \text{ and weakly in } H_0^1(\Omega)^d,$$

$$p_n \to p \text{ weakly in } L^{\overline{q}}(\Omega),$$

$$\rho_n \to \rho \text{ weakly in } L^{\overline{q}\gamma}(\Omega).$$

# Passage to the limit in the momentum equation

Weak momentum equation:
$v \in C_c^\infty(\Omega)^d$,

$$\int_\Omega \nabla u_n : \nabla v \,\mathrm{d}\boldsymbol{x} - \int_\Omega \rho_n u_n \otimes u_n : \nabla v \,\mathrm{d}\boldsymbol{x} - \int_\Omega p_n \mathrm{div}(v) \,\mathrm{d}\boldsymbol{x} = \int_\Omega f_n \cdot v \, dx.$$

$\nabla u_n \to \nabla u$ weakly in $L^2(\Omega)^d$

$\rho_n \to \rho$ weakly in $L^{\overline{q}\gamma}(\Omega)$, with $\overline{q}\gamma > \frac{3}{2}$,

$u_n \to u$ in $L^q(\Omega)$ for all $q < 6$ (and $\frac{2}{3} + \frac{1}{6} + \frac{1}{6} = 1$).
Then $\rho_n u_n \otimes u_n \to \rho u \otimes u$ weakly in $L^1(\Omega)$.

$p_n \to p$ weakly in $L^{\overline{q}}(\Omega)$

$f_n \to f$ weakly in $L^2(\Omega)^d$

Therefore

$$\int_\Omega \nabla u : \nabla v \, dx - \int_\Omega \rho u \otimes u : \nabla v \, dx - \int_\Omega p \mathrm{div}(v) \, dx = \int_\Omega f \cdot v \, dx.$$

# Passage to the limit in the mass equation

$v \in C_c^\infty(\mathbb{R}^d)$

$$\int_\Omega \rho_n u_n \cdot \nabla v = 0,$$

$\rho_n \to \rho$ weakly in $L^{\overline{q}\gamma}(\Omega)$, with $\overline{q}\gamma > \frac{3}{2}$, $u_n \to u$ in $L^q(\Omega)$ for all $q < 6$. Then $\rho_n u_n \to \rho u$ weakly in $L^1(\Omega)$. This gives $\int_\Omega \rho u \cdot \nabla v = 0$.

$L^1$-weak convergence of $\rho_n$ gives positivity of $\rho$ and convergence of mass:

$$\rho \geq 0 \text{ in } \Omega, \int_\Omega \rho(x) dx = M.$$

Conclusion: $(\rho, u, p)$ is solution of the momentum equation and of the mass equation ( + positivity of $\rho$ and total mass). It remains to prove $p = \rho^\gamma$.

# Passage to the limit in EOS

Question: $p = \rho^\gamma$ in $\Omega$ ?

$p_n$ and $\rho_n$ converge only weakly... and $\gamma > 1$

Idea (for $d = 2$ or $d = 3$, $\gamma \geq 3$): prove $\int_\Omega p_n \rho_n \to \int_\Omega p\rho$ and deduce a.e. convergence (of $p_n$ and $\rho_n$) and $p = \rho^\gamma$.

For $d = 3$, $\gamma \leq 3$, use $p_n \rho_n^\theta$.

In the sequel, we take $d = 3$, $\gamma > 3$.

# $\nabla : \nabla = \mathrm{divdiv} + \mathrm{curl} \cdot \mathrm{curl}$

For all $\bar{u}, \bar{v}$ in $H_0^1(\Omega)^d$,

$$\int_\Omega \nabla \bar{u} : \nabla \bar{v} = \int_\Omega \mathrm{div}(\bar{u})\mathrm{div}(\bar{v}) + \int_\Omega \mathrm{curl}(\bar{u}) \cdot \mathrm{curl}(\bar{v}).$$

Then, for all $\bar{v}$ in $H_0^1(\Omega)^d$, the momentum equation is

$$\int_\Omega \mathrm{div}(u_n)\mathrm{div}(\bar{v}) + \int_\Omega \mathrm{curl}(u_n) \cdot \mathrm{curl}(\bar{v}) - \int_\Omega (\rho_n u_n \otimes u_n) : \nabla \bar{v} dx \\ - \int_\Omega p_n \mathrm{div}(\bar{v}) = \int_\Omega f_n \cdot \bar{v}.$$

Choice of $\bar{v}$ ? $\bar{v} = \bar{v}_n$ with $\mathrm{curl}(\bar{v}_n) = 0$, $\mathrm{div}(\bar{v}_n) = \rho_n$ and $\bar{v}_n$ bounded in $H_0^1(\Omega)^d$ (unfortunately, 0 is impossible).

Then, up to a subsequence,

$\bar{v}_n \to v$ in $L^2(\Omega)^d$ and weakly in $H_0^1(\Omega)^d$,

$\mathrm{curl}(v) = 0$, $\mathrm{div}(v) = \rho$.

# Proof using $\bar{v}_n$ (1)

$$\int_\Omega \mathrm{div}(u_n)\mathrm{div}(\bar{v}_n) + \int_\Omega \mathrm{curl}(u_n)\cdot\mathrm{curl}(\bar{v}_n) - \int_\Omega p_n \mathrm{div}(\bar{v}_n) = \int_\Omega \rho_n u_n \otimes u_n : \nabla \bar{v}_n + \int_\Omega f_n \cdot \bar{v}_n.$$

But, $\mathrm{div}(\bar{v}_n) = \rho_n$ and $\mathrm{curl}(\bar{v}_n) = 0$. Then:

$$\int_\Omega (\mathrm{div}(u_n) - p_n)\rho_n = \int_\Omega \rho_n u_n \otimes u_n : \nabla \bar{v}_n + \int_\Omega f_n \cdot \bar{v}_n.$$

If we prove that $\int_\Omega \rho_n u_n \otimes u_n : \nabla \bar{v}_n \to \int_\Omega \rho u \otimes u : \nabla v$ then:

$$\lim_{n\to\infty} \int_\Omega (\mathrm{div}(u_n) - p_n)\rho_n = \int_\Omega \rho u \otimes u : \nabla v + \int_\Omega f \cdot v.$$

# Proof using $\bar{v}_n$ (2)

But, since $-\Delta u + (\rho u \cdot \nabla) u + \nabla p = f$:

$$\int_\Omega \mathrm{div}(u)\mathrm{div}(v) + \int_\Omega \mathrm{curl}(u) \cdot \mathrm{curl}(v) - \int_\Omega p\,\mathrm{div}(v) \\ = \int_\Omega \rho u \otimes u : \nabla v + \int_\Omega f \cdot v,$$

which gives (using $\mathrm{div}(v) = \rho$ and $\mathrm{curl}(v) = 0$):
$\displaystyle\int_\Omega (\mathrm{div}(u) - p)\rho = \int_\Omega \rho u \otimes u : \nabla v + \int_\Omega f \cdot v$. Then:

$$\lim_{n\to\infty} \int_\Omega (p_n - \mathrm{div}(u_n))\rho_n = \int_\Omega (p - \mathrm{div}(u))\rho.$$

Finally, the preliminary lemma gives, thanks to the mass equations, $\int_\Omega \rho_n \mathrm{div}(u_n) = 0$ and $\int_\Omega \rho\,\mathrm{div}(u) = 0$. Then,

$$\lim_{n\to\infty} \int_\Omega p_n \rho_n = \int_\Omega p\rho.$$

# Proof using $\bar{v}_n$ (3)

It remains to prove $\int_\Omega \rho_n u_n \otimes u_n : \nabla \bar{v}_n \to \int_\Omega \rho u \otimes u : \nabla v$.

We remark that (since $\mathrm{div}(\rho_n u_n) = 0$)

$$\int_\Omega \rho_n u_n \otimes u_n : \nabla \bar{v}_n = \int_\Omega (\rho_n u_n \cdot \nabla) u_n \cdot \bar{v}_n,$$

and the sequence $((\rho_n u_n \cdot \nabla) u_n)_{n\in\mathbb{N}}$ is bounded in $L^r(\Omega)^d$ with $\frac{1}{r} = \frac{1}{2} + \frac{1}{6} + \frac{1}{2\gamma}$ (and $r > \frac{6}{5}$ since $\gamma > 3$).

Then, up to a subsequence $(\rho_n u_n \cdot \nabla) u_n \to G$ weakly in $L^r(\Omega)^d$. and (since $v_n \to v$ in $L^s(\Omega)^d$ for all $s < 6$),

$$\int_\Omega (\rho_n u_n \cdot \nabla) u_n \cdot \bar{v}_n \to \int_\Omega G \cdot v$$

But, $G = (\rho u \cdot \nabla) u$, since for a fixed $w \in H_0^1(\Omega)^d$,

$$\int_\Omega (\rho_n u_n \cdot \nabla) u_n \cdot w = \int_\Omega \rho_n u_n \otimes u_n : \nabla w \to \int_\Omega \rho u \otimes u : \nabla w.$$

# Error in the preceding proof

In the preceding proof, we used $\bar{v}_n$ such that $\mathrm{curl}(\bar{v}_n) = 0$, $\mathrm{div}(\bar{v}_n) = \rho_n$ and $\bar{v}_n$ bounded in $H_0^1(\Omega)^d$.

Unfortunately, it is impossible to have $\bar{v}_n \in H_0^1(\Omega)^d$ but only $\bar{v}_n \in H^1(\Omega)^d$.

# Curl-free test function

Let $w_n \in H_0^1(\Omega)$, $-\Delta w_n = \rho_n$,
One has $w_n \in H^2_{loc}(\Omega)$ since, for $\varphi \in C_c^\infty(\Omega)$, one has $\Delta(w_n\varphi) \in L^2(\Omega)$ and

$$\sum_{i,j=1}^{d} \int_\Omega \partial_i\partial_j(w_n\varphi)\,\partial_i\partial_j(w_n\varphi) = \sum_{i,j=1}^{d} \int_\Omega \partial_i\partial_i(w_n\varphi)\,\partial_j\partial_j(w_n\varphi)$$
$$= \int_\Omega (\Delta(w_n\varphi))^2 = C_\varphi < \infty$$

Then, taking $v_n = \nabla w_n$

- $v_n \in (H^1_{loc}(\Omega))^d$,
- $\mathrm{div}(v_n) = \rho_n$ a.e. in $\Omega$,
- $\mathrm{curl}(v_n) = 0$ a.e. in $\Omega$,
- $H^1_{loc}(\Omega)$-estimate on $v_n$ with respect to $\|\rho_n\|_{L^2(\Omega)}$.

Then, up to a subsequence, as $n \to \infty$, $v_n \to v$ in $L^2_{loc}(\Omega)$ and weakly in $H^1_{loc}(\Omega)$, $\mathrm{curl}(v) = 0$, $\mathrm{div}(v) = \rho$.

# Proof of $\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n\varphi \to \int_\Omega (p - \mathrm{div}(u))\rho\varphi$

Let $\varphi \in C_c^\infty(\Omega)$ (so that $v_n\varphi \in H_0^1(\Omega)^d$)). Taking $\bar{v} = v_n\varphi$:

$$\int_\Omega \mathrm{div}(u_n)\mathrm{div}(v_n\varphi) + \int_\Omega \mathrm{curl}(u_n)\cdot\mathrm{curl}(v_n\varphi) - \int_\Omega p_n\mathrm{div}(v_n\varphi)$$
$$= \int_\Omega \rho_n u_n \otimes u_n : \nabla(v_n\varphi) + \int_\Omega f_n \cdot (v_n\varphi).$$

Using a proof similar to that given if $\varphi = 1$ (with additionnal terms involving $\varphi$), we obtain :

$$\lim_{n\to\infty} \int_\Omega (p_n - \mathrm{div}(u_n))\rho_n\varphi = \int_\Omega (p - \mathrm{div}(u))\rho\varphi,$$

## Proof of $\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n \to \int_\Omega (p - \mathrm{div}(u))\rho$

Lemma : $F_n \to F$ in $D'(\Omega)$, $(F_n)_{n\in\mathbb{N}}$ bounded in $L^q$ for some $q > 1$. Then $F_n \to F$ weakly in $L^1$.

With $F_n = (p_n - \mathrm{div}(u_n))\rho_n$, $F = (p - \mathrm{div}(u))\rho$ and since $p_n - \mathrm{div}(u_n)$ is bounded in $L^2(\Omega)$ and $\rho_n$ is bounded in $L^r(\Omega)$ with some $r > 2$, the lemma gives

$$\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n \to \int_\Omega (p - \mathrm{div}(u))\rho.$$

## Proving $\int_\Omega p_n \rho_n \to \int_\Omega p\rho$

$$\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n \to \int_\Omega (p - \mathrm{div}(u))\rho.$$

But thanks to the mass equations, the preliminary lemma gives:

$$\int_\Omega \mathrm{div}(u_n)\rho_n = 0, \ \int_\Omega \mathrm{div}(u)\rho = 0;$$

Then:

$$\lim_{n\to\infty} \int_\Omega p_n \rho_n = \int_\Omega p\rho.$$

# a.e. convergence of $\rho_n$ and $p_n$

Let $G_n = (\rho_n^\gamma - \rho^\gamma)(\rho_n - \rho) \in L^1(\Omega)$ and $G_n \geq 0$ a.e. in $\Omega$. Futhermore $G_n = (p_n - \rho^\gamma)(\rho_n - \rho) = p_n \rho_n - p_n \rho - \rho^\gamma \rho_n + \rho^\gamma \rho$ and:

$$\int_\Omega G_n = \int_\Omega p_n \rho_n - \int_\Omega p_n \rho - \int_\Omega \rho^\gamma \rho_n + \int_\Omega \rho^\gamma \rho.$$

Using the weak convergence in $L^2(\Omega)$ of $p_n$ and $\rho_n$ and $\lim_{n\to\infty} \int_\Omega p_n \rho_n = \int_\Omega p \rho$:

$$\lim_{n\to\infty} \int_\Omega G_n = 0,$$

Then (up to a subsequence), $G_n \to 0$ a.e. and then $\rho_n \to \rho$ a.e. (since $y \mapsto y^\gamma$ is an increasing function on $\mathbb{R}_+$). Finally:

$\rho_n \to \rho$ in $L^q(\Omega)$ for all $1 \leq q < 2\gamma$,

$p_n = \rho_n^\gamma \to \rho^\gamma$ in $L^q(\Omega)$ for all $1 \leq q < 2$,

and $p = \rho^\gamma$.

# Generalizations

- (Easy) Complete Diffusion term: $-\mu\Delta u - \frac{\mu}{3}\nabla(\mathrm{div}\, u)$, with $\mu \in \mathbb{R}_+^\star$ given, instead of $-\Delta u$.
- more general EOS: $\psi$ non decreasing $\psi(s) > as^\gamma - b$, with $\gamma > 1$, $a, b > 0$.
- (Ongoing work) Navier-Stokes Equations with $d = 3$ and $\frac{3}{2}\gamma < \gamma \leq 3$. (probably sharp result with respect to $\gamma$ without changing the diffusion term or the EOS)
- (Open question) Other boundary conditions. Addition of an energy equation ?
- (Open question) Evolution equation (Stokes and Navier-Stokes)

# Preliminary lemma with the numerical scheme (1)

Roughly speaking, upwinding replaces $\mathrm{div}(\rho u) = 0$ and $\int_\Omega \rho dx = M$ by

$$\mathrm{div}(\rho u) - h\mathrm{div}(|u|\nabla\rho) + h^\alpha(\rho - \rho^\star) = 0$$

with $\rho^\star = \frac{M}{|\Omega|}$

This equation as (for a given $u$) a solution $\rho > 0$ and we prove

$$\int_\Omega \rho_n^\gamma \mathrm{div}_n\, u_n dx \leq Ch^\alpha,$$

$$\int_\Omega \rho_n \mathrm{div}_n\, u_n dx \leq Ch^\alpha.$$

$C$ depends on $\Omega$, $M$ and $\gamma$

$Ch^\alpha$ is due to $h^\alpha(\rho - \rho^\star)$

$\leq$ is due to upwinding

The first inequality leads to the estimate on the approx. solution.

# Preliminary lemma with the numerical scheme (2)

For the passage to the limit on the EOS

$$\int_\Omega \rho_n \mathrm{div}_\mathcal{D}\, u_n dx \leq Ch^\alpha$$

$$\int_\Omega \rho \mathrm{div}\, u dx = 0$$

give $\lim_{n\to\infty} \int_\Omega p_n \rho_n dx \leq \int_\Omega p\rho dx = 0$,
which is sufficient to prove the a.e. convergence (up to a subsequence) of $p_n$ and $\rho_n$

# Passage to the limit in the EOS with the Mac scheme

Miracle with the Mac scheme:

There exists a discrete counterpart of $\int_\Omega \nabla u : \nabla v dx = \int_\Omega (\mathrm{div}(u)\mathrm{div}(v) + \mathrm{curl}(u) \cdot \mathrm{curl}(v)) dx$

# Passage to the limit in the EOS with Crouzeix-Raviart

No discrete counterpart with Crouzeix-Raviart. Two possible solutions

- Use the continuous equality. This is possible with an additional regularization term in the mass equation (not needed from the numerical point of view, only needed to prove the convergence), less diffusive than the upwinding.
- Discretize $\int_\Omega (\mathrm{div}(u)\mathrm{div}(v) + \mathrm{curl}(u) \cdot \mathrm{curl}(v)) dx$ instead of $\int_\Omega \nabla u : \nabla v dx$. Better for passing to the limit in the EOS but the discretized momentum equation is not coercive (with Crouzeix-Raviart Finite Element). One needs a penalization term in the discrete momentum equation (crucial from the numerical point of view). cf. Karlsen-Karper work for the compressible Stokes problem.

# 3. Incompressible Navier-Stokes equations - The Stokes problem

# Linear Stokes equations

- Steady state Stokes equations: $u : \Omega \to \mathbb{R}^d$, $p : \Omega \to \mathbb{R}$

$$
\begin{aligned}
&-\nu \Delta u + \nabla p = f \text{ in } \Omega,\\
&\mathrm{div} u = 0 \text{ in } \Omega,\\
&u = 0 \text{ on } \partial\Omega.
\end{aligned}
$$

- Weak formulation $E(\Omega) := \{v \in (H_0^1(\Omega))^d, \mathrm{div} v = 0\}$.

$$
\left\{
\begin{array}{l}
u = (u^{(1)}, \dots, u^{(d)})^t \in E(\Omega),\\
\nu \displaystyle\int_\Omega \nabla u : \nabla v \; dx = \int_\Omega f \cdot v \; dx, \quad \forall v \in E(\Omega).
\end{array}
\right.
$$

with $\displaystyle\int_\Omega \nabla u : \nabla v \; dx = \sum_{i=1,d} \int_\Omega \nabla u^{(i)} \cdot \nabla v^{(i)} dx.$

# Discrete divergence and gradient, colocated unknowns

- $H_{\mathcal{D}}(\Omega) \subset L^2(\Omega)$ piecewise constant functions on $K$ cells.
- For $u \in (H^1)^d$, $\displaystyle\int_K \operatorname{div} \boldsymbol{u}\, dx = \sum_{L \in \mathcal{N}_K} \int_{\sigma_{KL}} \boldsymbol{u} \cdot \boldsymbol{n}_{K,\sigma_{KL}} d\gamma(x)$

  Centred discretization of $\boldsymbol{u} \cdot \boldsymbol{n}$ on $\sigma = K|L \rightsquigarrow$

$$\text{For } \boldsymbol{u} \in (H_{\mathcal{D}})^d,\ \operatorname{div}_{\mathcal{D}} \boldsymbol{u} = \frac{1}{|K|} \sum_{\sigma = K|L \subset \partial K} |\sigma|\ \boldsymbol{n}_{K,\sigma} \cdot \frac{(\boldsymbol{u}_K + \boldsymbol{u}_L)}{2}.$$

- $$\int_\Omega \operatorname{div}_{\mathcal{D}} \boldsymbol{u}\ p = -\int_\Omega \boldsymbol{u} \cdot \nabla_{\mathcal{D}} p,$$

$$\rightsquigarrow (\nabla_{\mathcal{D}} p)_K = \frac{1}{|K|} \sum_{L \in \mathcal{N}_K} |\sigma| \boldsymbol{n}_{K,\sigma} \frac{(p_L - p_K)}{2},$$

# FV scheme

- $H_{\mathcal{D}}(\Omega) \subset L^2(\Omega)$ piecewise constant functions on $K$ cells
  $E_{\mathcal{D}}(\Omega) = \{u \in (H_{\mathcal{D}}(\Omega))^d, \mathrm{div}_{\mathcal{D}}(u) = 0\}$,

$$\begin{aligned} & u \in E_{\mathcal{D}}(\Omega), \\ & \nu[u, v]_{\mathcal{D}} = \int_{\Omega} f(x) \cdot v(x) dx, \ \forall v \in E_{\mathcal{D}}(\Omega). \end{aligned}$$

- Penalized version

$$\begin{aligned} & (u, p) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega), \\ & \nu[u, v]_{\mathcal{D}} - \int_{\Omega} p(x) \mathrm{div}_{\mathcal{D}}(v)(x) \ dx = \int_{\Omega} f(x) \cdot v(x) \ dx, \ \forall v \in H_{\mathcal{D}}(\Omega)^d, \\ & \int_{\Omega} \mathrm{div}_{\mathcal{D}}(u)(x) q(x) \ dx = -\langle p, q \rangle_{\mathcal{D},\lambda}, \ \forall q \in H_{\mathcal{D}}(\Omega). \end{aligned}$$

$$\langle v, w \rangle_{\mathcal{D},\lambda} = \frac{1}{2} \sum_{K \in \mathcal{D}} \sum_{L \in \mathcal{N}_K} \lambda_{K|L} T_{K,L} (v_L - v_K)(w_L - w_K), \text{ with } \lambda : \mathcal{E} \to \mathbb{R}.$$

# Stabilization

- $H^1$ penalization

$$\langle v, w\rangle_{\mathcal{D},\lambda} = \frac{1}{2}\sum_{K\in\mathcal{D}}\sum_{L\in\mathcal{N}_K} \lambda_{K|L} T_{K,L}(v_L - v_K)(w_L - w_K),$$

  with $\lambda : \mathcal{E} \to \mathbb{R}$.

- Stabilization *a la* Brezzi Pitkäranta:
  $\lambda_{K|L} = \beta h_{\mathcal{D}}^{\alpha}$, $\alpha \in (0, 2)$.
- Stabilization by clusters $\mathcal{C}_K$:

$$\lambda_{K|L} = \begin{array}{ll} 0 & \mathcal{C}_K \neq \mathcal{C}_L, \\ \gamma, & \mathcal{C}_K = \mathcal{C}_L \end{array}$$

  $\gamma \geq 0$.

- System equivalent to. . .

# Penalized FV scheme

$$-\nu\left(\sum_{L\in\mathcal{N}_K}\tau_{KL}(u_L-u_K)+\sum_{s\in\mathcal{E}_K\cap\mathcal{E}_{\text{ext}}}\tau_{K,s}(-u_K)\right)+$$
$$\sum_{L\in\mathcal{N}_K}\frac{|\sigma_{K,L}|\boldsymbol{n}_{K,\sigma_{KL}}}{2}\,(p_L-p_K)=\int_K f(x)\,dx,\ \forall K\in\mathcal{D},$$

$$\sum_{L\in\mathcal{N}_K}\frac{|\sigma_{K,L}|\boldsymbol{n}_{K,\sigma_{KL}}}{2}\cdot(u_K+u_L)$$
$$+\sum_{L\in\mathcal{N}_K}\lambda_{KL}\tau_{KL}(p_L-p_K)=0,\ \forall K\in\mathcal{D}.$$

# Convergence of diffusion term

If $\|u^{(m)}\|_{\mathcal{D}_{\boldsymbol{m}}} \leq C$ for all $m \in \mathbb{N}$. Then:

$\exists u^* \in H_0^1(\Omega)$ and a subsequence of $(\ u^{(m)})_{m\in\mathbb{N}}$ such that:

$u^{(m)} \to u^*$ as $m \to +\infty$ in $L^2(\Omega)$,

Convergence of the discrete diffusion operator: $\forall \varphi \in C_c^\infty(\Omega)$,

$\lim\limits_{m\to+\infty} [u^{(m)}, P_{\mathcal{D}_{\boldsymbol{m}}}\varphi]_{\mathcal{D}_{\boldsymbol{m}}} = \int_\Omega \nabla u^* \cdot \nabla\varphi \, dx.$

# Estimates

- Estimate on velocity
  $\nu|u_{\mathcal{D}}|_{\mathcal{D}} \leq \mathrm{diam}(\Omega)\|f\|_{(L^2(\Omega))^d}$,
- Estimate on pressure (thanks to penalisation).
  $\nu\ |p_{\mathcal{D}}|^2_{\mathcal{D},\lambda} \leq \mathrm{diam}(\Omega)\|f\|_{(L^2(\Omega))^d}$.
- Hence existence and uniqueness of $u$ and $p$.
- $L^2$ estimate on pressures $\|p_{\mathcal{D}}\|_{L^2(\Omega)} \leq C\|f\|_{(L^2(\Omega))^d}$
  Proof: use function $v \in H^1_0(\Omega)^d$ (Necas) such that $\mathrm{div} v = p_{\mathcal{D}}$ and $\|v\|_{H^1_0(\Omega)^d} \leq C\|p_{\mathcal{D}}\|_{L^2(\Omega)}$

# Passage to the limit in the scheme

- Estimates on the velocity translates
- Kolmogorov: convergence to some $\tilde{u} \in H_0^1$, up to a subsequence.
- Convergence of $p$ to some $\tilde{p}$ weakly in $L^2$.
  $(\tilde{u}, \tilde{p})$ weak solution to the scheme ?
- YES ! Take $\varphi \in C_c^\infty(\Omega)$, and $v = P_{\mathcal{D}}(\varphi)$ in the scheme, and pass to the limit as $h_{\mathcal{D}}$ tends to 0, using weak consistency of the divergence (consistency of the normal fluxes) and weak convergence of the gradient.

# Transient isothermal incompressible Navier Stokes

Continuous problem

$$u_t - \nu\Delta u + u \cdot \nabla u + \frac{1}{\rho}\nabla p = f$$
$$\mathrm{div} u = 0,$$

+ B.C., + I.C. $\quad (u \cdot \nabla u)_k = \sum_{i=1,d} u^{(i)} \partial_i u^{(k)}.$

Time discretization by the Crank Nicolson scheme

$$\frac{u^{n+1} - u^n}{\delta t} + \nu\Delta u^{n+\frac{1}{2}} + u^{n+\frac{1}{2}} \cdot \nabla u^{n+\frac{1}{2}} + \frac{1}{\rho}\nabla p^{n+\frac{1}{2}} = f^{n+\frac{1}{2}}$$
$$\mathrm{div} u^{n+\frac{1}{2}} = 0$$

with $u^{n+\frac{1}{2}} = \frac{1}{2}(u^n + u^{n+1})$, $p^{n+\frac{1}{2}} = \frac{1}{2}(p^n + p^{n+1})$

# The weak continuous problem

Continuous problem
$E(\Omega) = \{v \in (H_0^1(\Omega))^d ; \mathrm{div} v = 0\}$
$u \in L^2(0, T; E(\Omega)) \cap L^\infty(0, T; L^2(\Omega)^d)$,

$$
\begin{aligned}
&-\int_0^T \int_\Omega u \cdot \partial_t \varphi \, dx \, dt - \int_\Omega u_0(x) \cdot \varphi(x, 0) \, dx \\
&+\nu \int_0^T \int_\Omega \nabla u : \nabla \varphi \, dx \, dt \; + \int_0^T \int_\Omega (u \cdot \nabla u)) \cdot \varphi) \, dx \, dt \\
&\qquad\qquad = \int_0^T \int_\Omega f(x) \cdot \varphi \, dx \, dt \\
&\qquad\qquad \forall \varphi \in L^2(0, T; E(\Omega)) \cap C_c^\infty(\Omega \times (-\infty, T))^d.
\end{aligned}
$$

For $d = 3$, $u \in L^2(0, T; E(\Omega)) \cap L^\infty(0, T; L^2(\Omega)^d)$ yields $(u \cdot \nabla) u \in L^{4/3}(0, T, (E(\Omega))')$ so that :

$$u_t \in L^{4/3}(0, T, (E(\Omega))').$$

# Finite volume scheme

Discretization of the nonlinear convection term:

$$\int_K (u \cdot \nabla) u(x,t)dx = \int_{\partial K} (u \cdot \boldsymbol{n}_K) u(x) d\gamma(x) =$$

$$\sum_{\sigma \subset \partial K} \int_{\sigma_{KL}} (u \cdot \boldsymbol{n}_{K,\sigma}) u(x) d\gamma(x)$$

$$\rightsquigarrow \sum_{\sigma \subset \partial K} \frac{m_{K|L} \boldsymbol{n}_{K,K|L}}{2} \cdot (u_K + u_L) \frac{u_K + u_L}{2}$$

(omitting the penalization terms)

$$b_{\mathcal{D}}(u,v,w) = \sum_{\sigma \subset \partial K} \frac{m_{K|L} \boldsymbol{n}_{K,K|L}}{2} \cdot (u_K + u_L) \frac{v_K + v_L}{2} w_K$$

# Finite volume scheme for incompressible NS

$$
\begin{aligned}
&(u_{\mathcal{D}}, p_{\mathcal{D}}) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega),\\
&\int_\Omega \frac{u_{\mathcal{D}}^{n+1} - u_{\mathcal{D}}^n}{\delta t} v \, dx + \nu [u_{\mathcal{D}}, v]_{\mathcal{D}} + b_{\mathcal{D}}(u_{\mathcal{D}}, u_{\mathcal{D}}, v)\\
&- \int_\Omega p_{\mathcal{D}}(x) \mathrm{div}_{\mathcal{D}}(v)(x) \; dx = \int_\Omega f(x) \cdot v(x) \; dx, \forall v \in H_{\mathcal{D}}(\Omega)^d,\\
&\int_\Omega \mathrm{div}_{\mathcal{D}}(u_{\mathcal{D}})(x) q(x) \; dx = -\langle p_{\mathcal{D}}, q \rangle_{\mathcal{D},\lambda}, \; \forall q \in H_{\mathcal{D}}(\Omega).
\end{aligned}
$$

(All non specified $u_{\mathcal{D}}$ and $p_{\mathcal{D}}$ are $u_{\mathcal{D}}^{n+\frac{1}{2}}$ and $p_{\mathcal{D}}^{n+\frac{1}{2}}$).

# Estimates for the continuous NS problem ($d = 3$)

$$u \in L^2(0, T, E(\Omega)) \rightsquigarrow$$

$$\|u(\cdot + \eta, \cdot) - u(\cdot, \cdot)\|_{L^2(0,T,(L^2(\Omega))^3)} \leq C\eta$$

$$E(\Omega) = \{v \in (H_0^1(\Omega))^3; \operatorname{div} v = 0\}$$

$$u \in L^2(0, T, E(\Omega)) \text{ and } u_t \in L^{\frac{4}{3}}(0, T, (E(\Omega))') \rightsquigarrow$$

$$\|u(\cdot, \cdot + \tau) - u(\cdot, \cdot)\|_{L^{\frac{4}{3}}(0,T,(L^2(\Omega))^3))} \leq C\tau^{\frac{1}{2}}.$$

# A simple continuous estimate

$$u \in L^2(0, T, E(\Omega)) \text{ and } u_t \in L^1(0, T, (E(\Omega))') \rightsquigarrow$$

$$\|u(\cdot, \cdot + \tau) - u(\cdot, \cdot)\|_{L^1(0,T,(L^2(\Omega))^3))} \leq C\tau^{\frac{1}{2}}.$$

BUT NO $L^2(0, T, (L^2(\Omega)^3)$ estimate.

# Estimates for the discrete NS problem

Approximate solution $u_{\mathcal{D}} \in H_{\mathcal{D}}(\Omega \times (0, T))$ = set of piecewise constant functions on $K \times (t_n, t_{n+1})$

Estimate on $u_{\mathcal{D}}$: $\|u_{\mathcal{D}}\|_{L^\infty((0,T),(L^2(\Omega))^d)} \leq C$, $\|u_{\mathcal{D}}\|_{L^2((0,T),H_{\mathcal{D}}(\Omega))} \leq C$

Estimate on the space translates:

$\|u_{\mathcal{D}}(\cdot + \eta, \cdot) - u_{\mathcal{D}}(\cdot, \cdot)\|_{L^2((0,T,(L^2(\Omega)^d)} \leq C\big(|\eta|(|\eta| + h_{\mathcal{D}})\big)^{\frac{1}{2}}$

Estimate on the time translates: $\|u_{\mathcal{D}}(\cdot, \cdot + \tau) - u_{\mathcal{D}}(\cdot, \cdot)\|_{L^1(0,T,(L^2(\Omega))^d)} \leq C\tau^{\frac{1}{2}}$

(Estimate in $L^{\frac{4}{3}}(0, T, (L^2(\Omega))^d)$ possible, but not $L^2(0, T, (L^2(\Omega))^d)$ )

# Convergence Navier-Stokes

$\rightsquigarrow$ Convergence of $u_{\mathcal{D}}$ to $\bar{u} \in L^2((0,T), E(\Omega))$ in $L^1((0,T), L^2(\Omega))$.

Passage to the limit in the scheme: $\bar{u} = u$, weak solution of the NS equations.