Notes

June 12, 2018

1 Formulation and Algorithm

We are interested in the steady state, namely, the minimizer of the following energy functional:

$$E(m) = \frac{1}{2}\int_\Omega \left(D^2|\nabla m|^2 + \frac{\alpha}{\gamma}|m|^{2\gamma} + c^2|m\cdot\nabla p|^2 + c^2 r(x)|\nabla p|^2 \right) \mathrm{d}x, \tag{1.1}$$

with the following constrain:

$$-\nabla\cdot\big((r^2 I + m\otimes m)\nabla p\big) = S. \tag{1.2}$$

First by introducing a new variable q satisfying

$$q = (m\otimes m)s, \tag{1.3}$$

the the functional (1.4) can be written as

$$E(m) = \frac{1}{2}\int_\Omega \left(D^2|\nabla m|^2 + \frac{\alpha}{\gamma}|m|^{2\gamma} + q\cdot s + c^2 r(x)|\nabla p|^2 \right) \mathrm{d}x, \tag{1.4}$$

also for the constrain (1.2),

$$-r^2\nabla\cdot s - \nabla\cdot q = S. \tag{1.5}$$

.

So now we want to solve the following constrained optimization problem:

$$\begin{aligned} &\min_{m,\nabla p,q} \frac{1}{2}\int_\Omega \left(D^2|\nabla m|^2 + \frac{\alpha}{\gamma}|m|^{2\gamma} + q\cdot s + c^2 r(x)|\nabla p|^2 \right) \mathrm{d}x && (1.6)\\ \text{such that: } &-r^2\nabla\cdot s - \nabla\cdot q = S, && (1.7)\\ &q = (m\otimes m)s && (1.8) \end{aligned}$$

Algorithm: denote $E(m,s,q)$ as energy:

$$\begin{aligned} \frac{1}{2}\int_\Omega D^2|\nabla m|^2 + \frac{\alpha}{\gamma}|m|^{2\gamma} + q\cdot s + c^2 r(x)|\nabla p|^2 \\ + \lambda|r^2\nabla\cdot s + \nabla\cdot q + S|^2 + \lambda|q-(m\otimes m)s|^2 \,\mathrm{d}x. \end{aligned} \tag{1.9}$$

$$(m^{k+1}, s^{k+1}, q^{k+1}) = \arg\min_{m,s,q} E(m,s,q) - \langle p_m^k, m - m^k\rangle - \langle p_s^k, s - s^k\rangle - \langle p_q^k, q - q^k\rangle \tag{1.10}$$

$$p_q^{k+1} = p_q^k - \lambda\nabla(r^2\nabla\cdot s^{k+1} + \nabla\cdot q^{k+1} + S) - \lambda(q^{k+1} - (m^{k+1}\otimes m^{k+1})s^{k+1}) \tag{1.11}$$

$$p_s^{k+1} = p_s^k - \lambda\nabla(r^4\nabla\cdot s^{k+1} + r^2\nabla\cdot q^{k+1} + r^2 S) \tag{1.12}$$

$$-\lambda(m^{k+1}\otimes m^{k+1})((m^{k+1}\otimes m^{k+1})s^{k+1} - q^{k+1}) \tag{1.13}$$

$$p_m^{k+1} = p_m^k - \lambda(s^{k+1}\otimes m^{k+1} + s^{k+1}\cdot m^{k+1} I)((m^{k+1}\otimes m^{k+1})s^{k+1} - q^{k+1}) \tag{1.14}$$

Define

$$F(m) = \frac{1}{2}(\frac{\alpha}{\gamma}|m|^{2\gamma} + \lambda|q - (m\otimes m)s|^2), \tag{1.15}$$

$$E_1 = \int_\Omega F(m)\,\mathrm{d}x, \quad v(t) = \sqrt{E_1}. \tag{1.16}$$

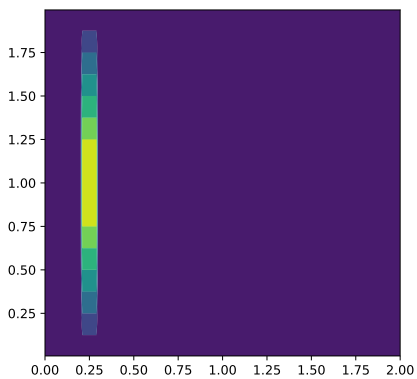
The dynamics of this system is

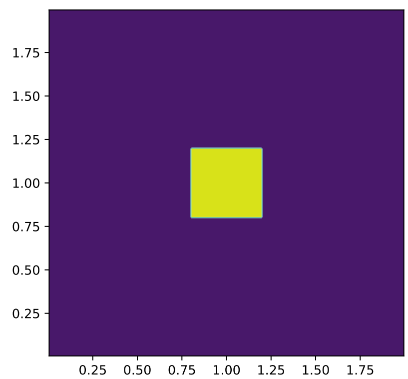
$$\frac{\partial m}{\partial t} = -D^2\Delta m + \frac{v(t)}{\sqrt{E_1(m)}} F'(m) \tag{1.17}$$

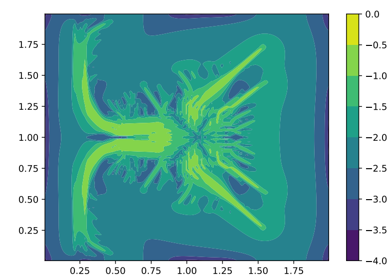
$$v_t(t) = \frac{1}{2\sqrt{E_1(m)}}\int_\Omega F'(m) m_t\,\mathrm{d}x. \tag{1.18}$$

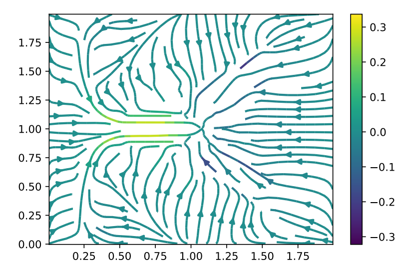
2 Introduction

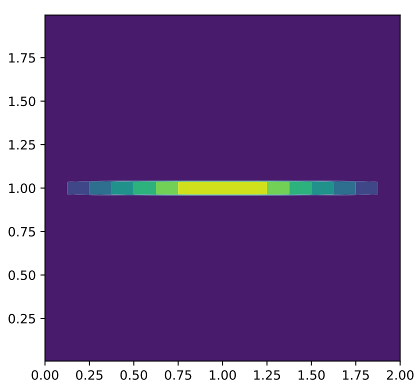
$dx = dy = 0.01$, $dt = 0.005$, $T = 1$. $r = 0.1$, $c = 50$, $D = 0.001$, $\rho = 10^{-12}$ and $\gamma = 0.75$.

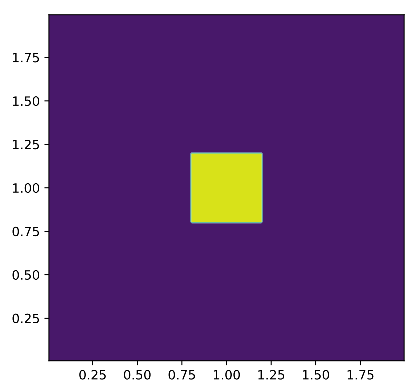

1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00 0.25 0.50 0.75 1.00 1.25 1.50 1.75 2.00

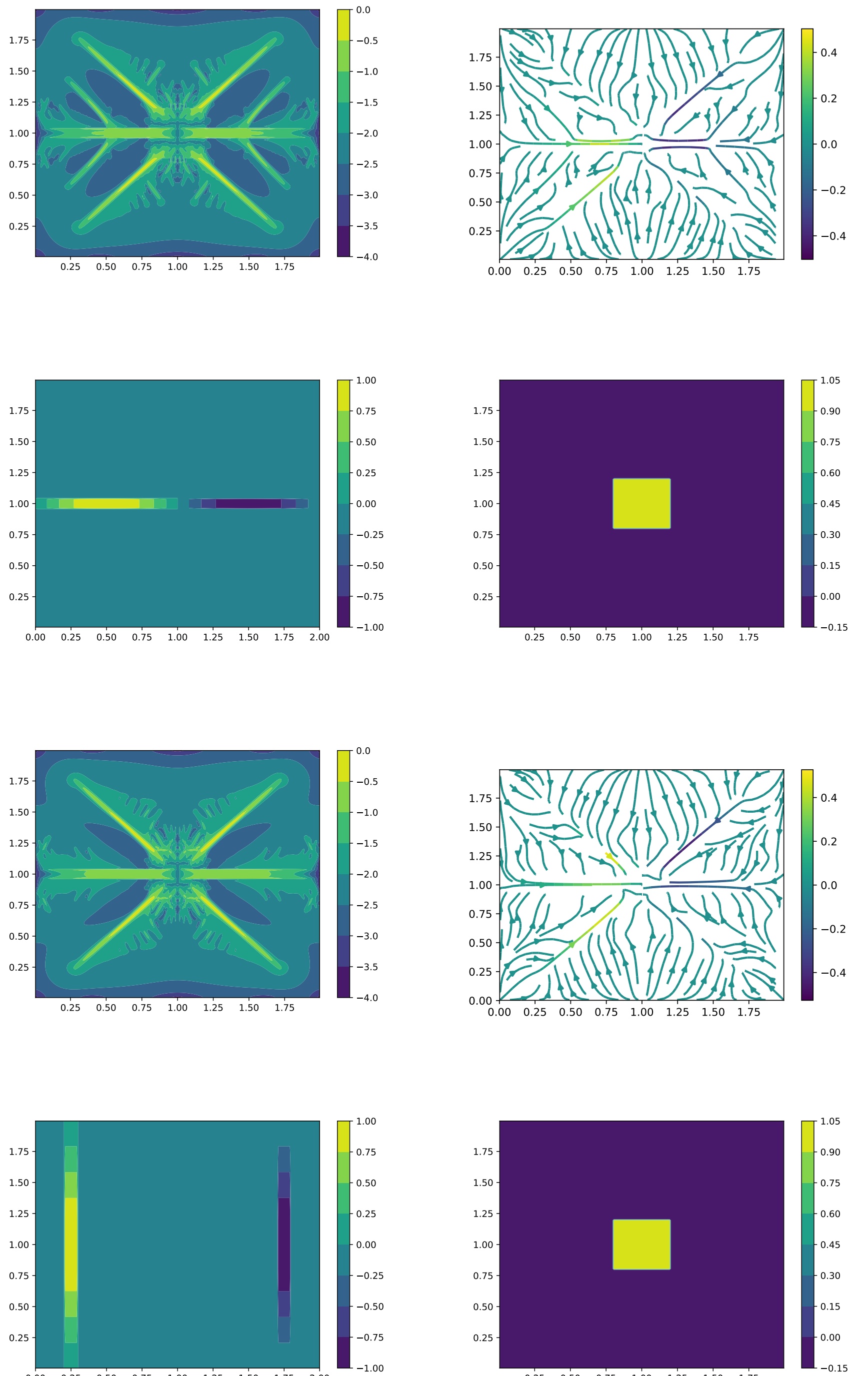

1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.25 0.50 0.75 1.00 1.25 1.50 1.75

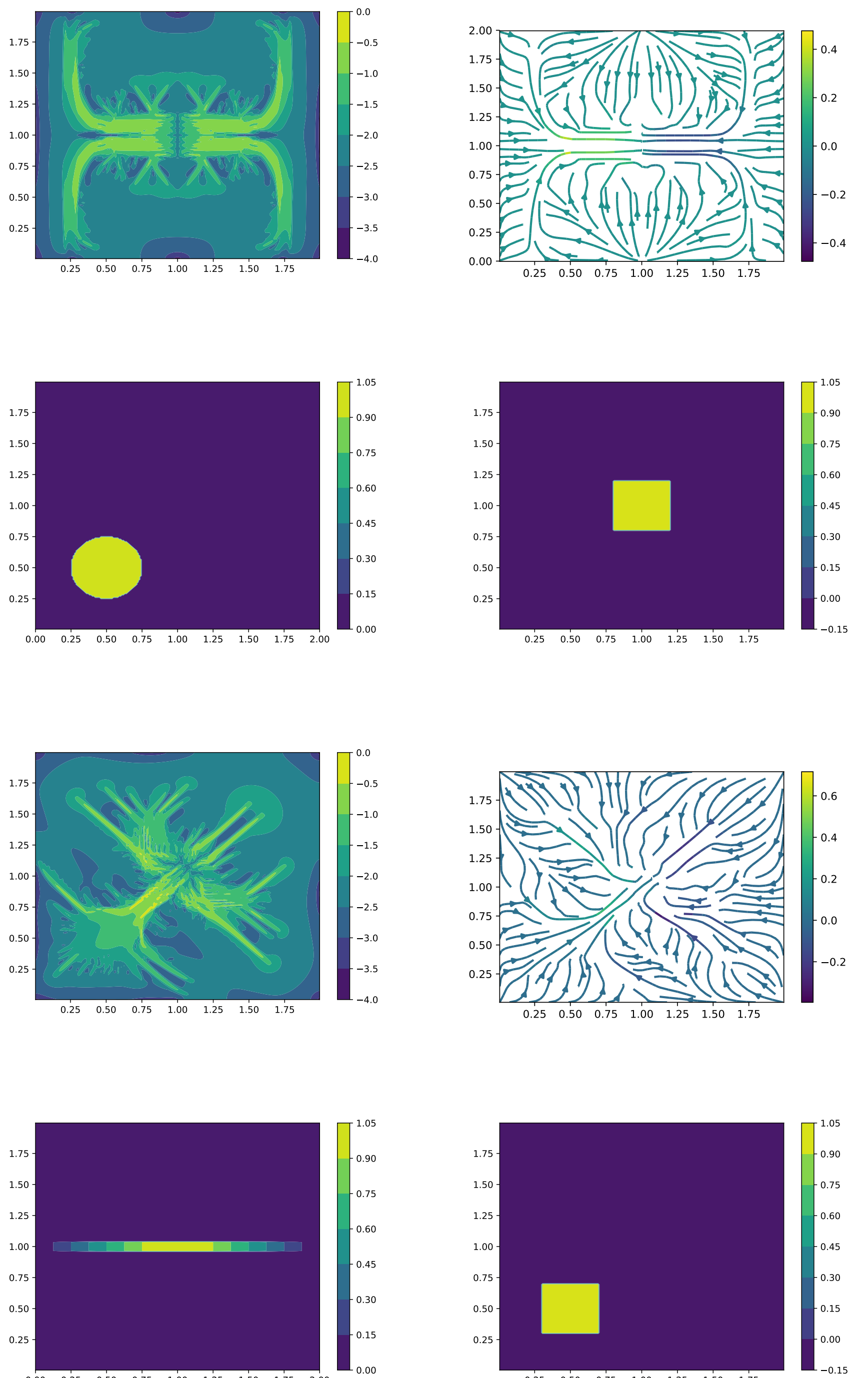

1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.25 0.50 0.75 1.00 1.25 1.50 1.75
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
−3.5
−4.0

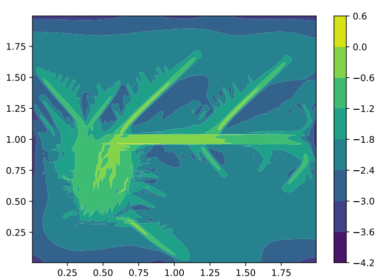

1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.25 0.50 0.75 1.00 1.25 1.50 1.75
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3

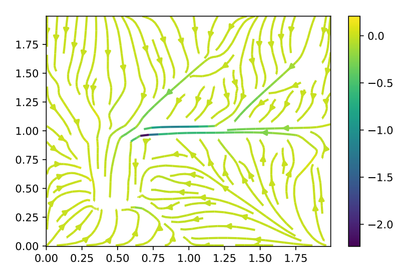

1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00 0.25 0.50 0.75 1.00 1.25 1.50 1.75 2.00


1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.25 0.50 0.75 1.00 1.25 1.50 1.75






1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.25 0.50 0.75 1.00 1.25 1.50 1.75
0.6
0.0
−0.6
−1.2
−1.8
−2.4
−3.0
−3.6
−4.2


1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.00 0.25 0.50 0.75 1.00 1.25 1.50 1.75
0.0
−0.5
−1.0
−1.5
−2.0