Three Forward PDE Problems with Urgent Need of Data Assimilation

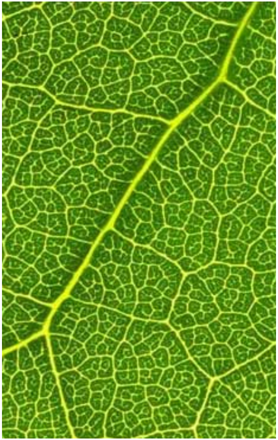
Peter Markowich (KAUST)

Part I: A PDE system modeling biological network formation
Part II: PDE models for price formation
Part III: A Brinkman-Forchheimer-Darcy model for porous media flow

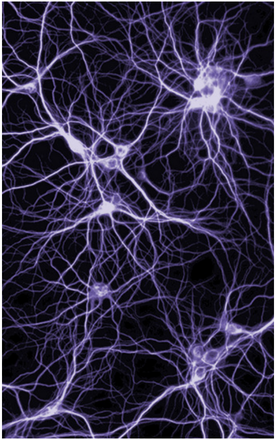
Part I: A PDE system modeling biological network formation *

*joint work with G. Albi (TU Munich, Germany), M. Artina (TU Munich, Germany), M. Fornasier (TU Munich, Germany), M. Burger (WWU Münster), J. Haskovec (Kaust), B. Perthame (Paris 6, France) and M. Schlottbom (WWU Münster, Germany).

Biological transportation networks



Leaf venation

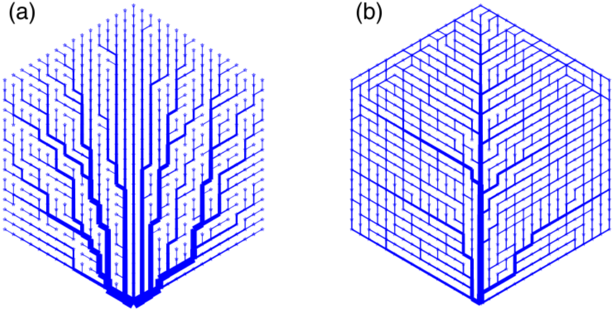


Neural network

Blood capillaries

Discrete modeling

- Static and dynamic discrete graph-based models, deterministic and (geometric) random graphs.
  Topological and geometric properties - loops, trees, connectivity, scale-free graphs [Barabasi&Albert'1999, Newman'2003, Watts&Strogatz'1998, . . .]



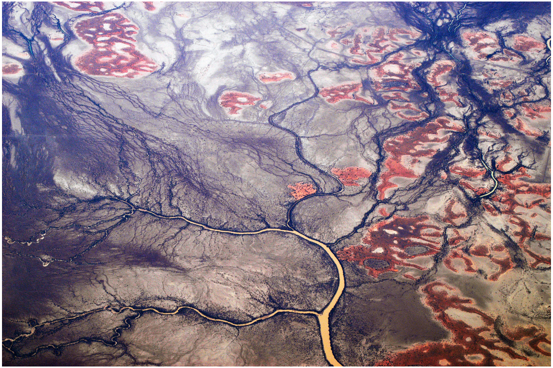
- Optimal mass transportation network modeling, based on a transportation cost law and Monge-Kantorovich theory [Bernot&Caselles&Morel'2009, Villani'2003&'2008, . . .]

Optimal mass transportation

$c(x, y)$ - cost of transport of a unit mass from x to y
Find a measure $\gamma(x, y)$ which minimizes the total transportation cost

$$C_\gamma := \int_{\Omega\times\Omega} c(x, y)\gamma(\mathrm{d}x, \mathrm{d}y)$$

where the marginals of γ are given.



New program

- Dynamic discrete network adaptation model by [Hu&Cai'2014] based on
  - Kirchoff's flux conservation law
  - Darcy's pressure law
  - local energy minimization
- PDE model derived as formal gradient flow of the continuous version of the energy functional.

The network formation model

- $\Omega \subseteq \mathbb{R}^d$, $d \leq 3$, bounded network domain (porous medium)
- $S = S(t,x)$ - scalar valued fluid sources
- $\varrho = \varrho(t,x)$ - scalar valued mass density
- $u = u(t,x)$ - vector valued flow velocity

Mass rate equation:

$$\partial_t \varrho + \nabla \cdot (\varrho u) = \varrho S$$

Quasi-incompressibility: constant density along particle trajectories

$$\frac{D\varrho}{Dt} := \partial_t \varrho + u \cdot \nabla \varrho = 0$$

Consequently:

$$\nabla \cdot u = S \quad \text{Kirchoff's law}$$

The network formation model

Darcy's law for porous media:

$$u = -\frac{1}{\mu}\mathbb{P}[m]\nabla p$$

- $m = m(t, x)$ vector valued conductance
- $\mu > 0$ fluid viscosity ($\mu := 1$)
- $\mathbb{P}[m] := I + m \otimes m$ permeability tensor

Principal directions of network flow:

- $m/|m|$ with principal permeability $1 + |m|^2$
- $m^\perp$ with principal permeability 1 (uniform background)

$$-\nabla \cdot [(I + m \otimes m)\nabla p] = S$$

The network formation model [Hu-Cai'2014]

Reaction diffusion system for the conductance m

$$\frac{\partial m}{\partial t} = \underbrace{D^2 \Delta m}_{\substack{\text{random effects} \\ \text{in the porous medium}}} + \underbrace{c^2 (m \cdot \nabla p) \nabla p}_{\substack{\text{activation (force)} \\ \text{term}}} - \underbrace{|m|^{2(\gamma-1)} m}_{\text{relaxation term}}$$

- $c > 0$ - activation parameter, $D \geq 0$ - diffusivity
- $\gamma \geq 1/2$ - relaxation exponent
- homogeneous Dirichlet BC for m and p,

$$m|_{\partial\Omega} = 0, \qquad p|_{\partial\Omega} = 0 \qquad \forall t > 0$$

- $m(t = 0, x) = m^I(x), \quad x \in \Omega$

First observation

- Switching the sign of c^2, i.e.,

$$\frac{\partial m}{\partial t} = D^2 \Delta m - c^2 (m \cdot \nabla p) \nabla p - |m|^{2(\gamma-1)} m$$

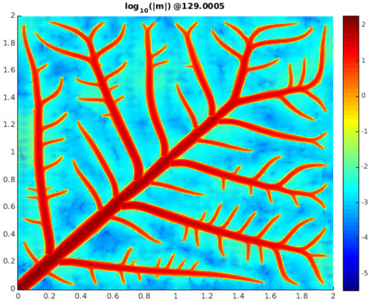
with p arbitrary, gives

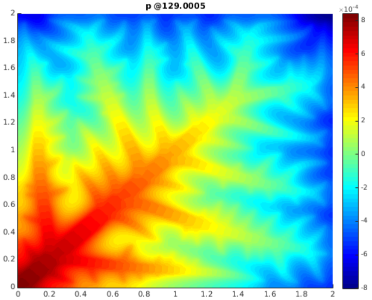
$$\frac{1}{2} \frac{d}{dt} \int_\Omega |m|^2 dx = - D^2 \int_\Omega |\nabla m|^2 dx - c^2 \int_\Omega |m \cdot \nabla p|^2 dx - \int_\Omega |m|^{2\gamma} \,\mathrm{d}x \leq -C \int_\Omega |m|^2 \,\mathrm{d}x.$$

$\Rightarrow$ the only stationary solution is $m \equiv 0$.

- Therefore, $+c^2 (m \cdot \nabla p) \nabla p$ is the **activation term.**

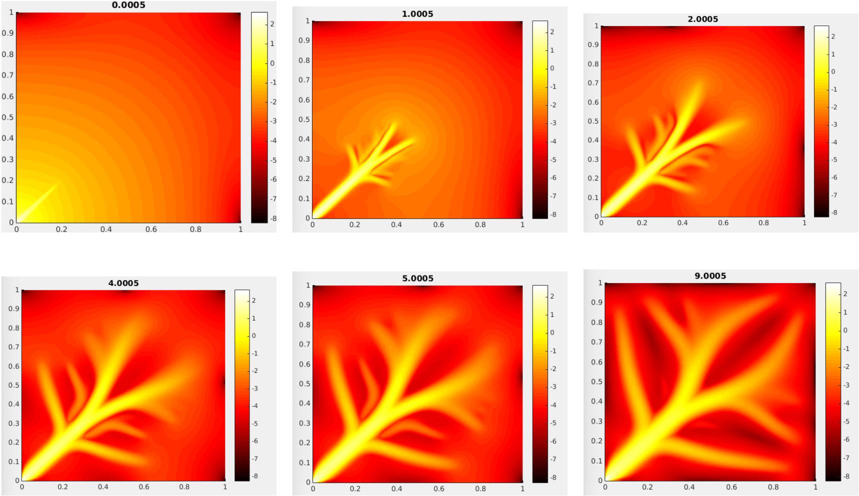
PDE simulation results (FEM)





M. Schlottbom (U. Münster)

Network formation



M. Schlottbom (U. Münster)

Gradient flow structure

$L^2(\Omega)$-gradient flow associated with the non-convex energy

$$\mathcal{E}(m) := \frac{1}{2}\int_\Omega \left(D^2|\nabla m|^2 + \frac{|m|^{2\gamma}}{\gamma} + c^2|m\cdot\nabla p[m]|^2 + c^2|\nabla p[m]|^2 \right) \mathrm{d}x$$

Observe: $\int_\Omega \left(D^2|\nabla m|^2 + \frac{|m|^{2\gamma}}{\gamma} \right) \mathrm{d}x$ is convex for $\gamma \geq 1/2$;
non-convexity of $\mathcal{E}$ due to the coupling with the Poisson equation.

Energy dissipation: Along smooth solutions $m,\ p = p[m]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(m) = -\int_\Omega \left(\frac{\partial m}{\partial t}(t,x) \right)^2 \mathrm{d}x.$$

Mathematical problems

Full mathematical model:

$$-\nabla\cdot[(I+m\otimes m)\nabla p]=S$$
$$\frac{\partial m}{\partial t}=D^2\Delta m+c^2(m\cdot\nabla p)\nabla p-|m|^{2(\gamma-1)}m$$

- Stronger regularity results than Lax-Milgram for the Poisson equation

$$\begin{aligned} -\nabla\cdot(A(x)\nabla p) =& \nabla\cdot F \qquad \text{in } \Omega \\ p =& 0 \qquad \text{on } \partial\Omega \end{aligned}$$

  require at least $A\in L^\infty(\Omega)$.
- While the divergence part is controlled, how to control the rotational part of $(m\otimes m)\nabla p$?
- Iterating between m and p in the system destroys the energy dissipation equation.

Global existence of weak solutions for $\gamma > 1/2$

- Let $S \in L^2(\Omega)$, $m^I \in L^2(\Omega)$.
- Leray-Schauder fixed point theorem for the regularized system

$$
\begin{aligned}
&-\nabla \cdot [\nabla p + m((m \cdot \nabla p) * \eta_\varepsilon)] = S \\
&\frac{\partial m}{\partial t} = D^2 \Delta m + c^2 [(m \cdot \nabla p) * \eta_\varepsilon] \nabla p - |m|^{2(\gamma-1)} m
\end{aligned}
$$

  which preserves the energy dissipation, and the choice $\eta_\varepsilon(x) := (4\pi\varepsilon)^{-d/2} \exp(-|x|^2/4\varepsilon)$ guarantees

$$
\int_{\mathbb{R}^d} (m \cdot \nabla p)[(m \cdot \nabla p) * \eta_\varepsilon] \, \mathrm{d}x \geq 0
$$

- Limit $\varepsilon \to 0$ based on apriori estimates in the energy space.
- For $\gamma = 1/2$: relaxation term $m/|m|$ leads to sparse transient solutions

The case $\gamma = 1/2$

- Relaxation term $r(m) := \frac{m}{|m|}$ ... singularity in $m = 0$!
- Relaxation energy $R(m) := \int_\Omega |m| \,\mathrm{d}x$ convex!

We prove the existence of a weak solution of

$$\partial_t m = D^2 \Delta m + c^2 (m \cdot \nabla p[m]) \nabla p[m] - \tilde{r}(m)$$

with

$$\tilde{r}(m) \in \partial R(m) = \{r \in L^\infty(\Omega);\ r(x) = \begin{cases} m(x)/|m(x)| & \text{if } m(x) \neq 0, \\ |r(x)| \leq 1 & \text{if } m(x) = 0 \end{cases}\}$$

Conjecture:

$$\tilde{r}(m) = \begin{cases} \frac{m}{|m|} & \text{for } m \neq 0 \\ 0 & \text{for } m = 0 \end{cases}$$

- Compact support property of solutions (e.g., [Brezis'1974])
- Sparse networks!

Stationary states
and network formation

Long time convergence

Thm: Fix $T > 0$, a sequence $t_k \to \infty$ as $k \to \infty$, and for $\tau \in (0, T)$ define the time-shifts

$$m^{(t_k)}(\tau, x) := m(\tau + t_k, x), \qquad p^{(t_k)}(\tau, x) := p(\tau + t_k, x).$$

Then, after extraction of a subsequence,

$$m^{(t_k)} \to m^\infty \quad \text{strongly in } L^q(0, T; L^4(\Omega)) \text{ for any } q < \infty,$$
$$p^{(t_k)} \to p^\infty \quad \text{strongly in } L^2(0, T; H^1_0(\Omega)).$$

where (m^∞, p^∞) is a weak solution of the stationary system.

Proof: A priori estimates provided by energy dissipation, Aubion-Lions compactness theorem.

Stationary states

Trivial stationary state:

$$m_0 \equiv 0, \qquad -\Delta p_0 = S.$$

Q: Do **nontrivial** stationary states $(\bar{m}, \bar{p})$ exist?

- If $D \to \infty$, then the weak stationary solutions $\bar{m}^D \to 0$ and $\bar{p}^D \to p_0$ in $H_0^1(\Omega)$.
- In the 1d case, if D^2/c^2 is big enough, then $\bar{m}^D \equiv 0$ and $\bar{p}^D = p_0$

Existence of nontrivial stationary solutions

Trivial stationary solution branch: $m_0 = 0$, $-\Delta p_0 = S$ for all D, $c > 0$, $\gamma \geq \frac{1}{2}$.

The stationary system is equivalent to the fixed point problem

$$m = \beta L m + F(m, \beta)$$

with $\beta := c^2/D^2$ the bifurcation parameter and

- the linear part

$$Lm := (-\Delta)^{-1}(\nabla p_0 \otimes \nabla p_0) m, \qquad -\Delta p_0 = S$$

- the nonlinear part $F(m, \beta)$

Set of reciprocals of eigenvalues of L:

$$\mathcal{R}(L) := \{\beta \in \mathbb{R};\ \exists m \in \mathbb{X},\ m \neq 0 \text{ such that } m = \beta L m\}.$$

Spectral Theorem $\Rightarrow \mathcal{R}(L) = \{0 < \beta_1 < \beta_2 < \dots\}$.

Bifurcations off the branch of trivial stationary solutions

$X := L^\infty(\Omega) \cap VMO(\Omega) \ni m$

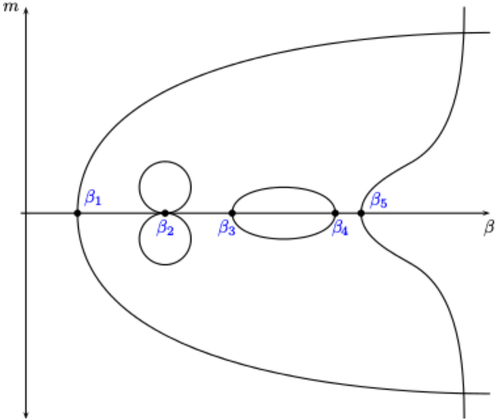
Thm: Let $\gamma \geq 1$. At every point $(m_0 \equiv 0, \beta_l > 0) \in \mathbb{X} \times \mathbb{R}$ for which $\beta_l \in \mathcal{R}(L)$ there is a bifurcation off the branch of trivial solutions $(m \equiv 0, \beta)$ of a solution branch of the stationary system. The branch

- **either** meets ∞ in $\mathbb{X} \times \mathbb{R}$
- **or** meets a point $(m_0 \equiv 0, \beta_k)$ where $\beta_k \in \mathcal{R}(L)$

Proof: Global bifurcation theorem by [Rabinowitz'71].

Note: Bifurcation occurs at all eigenvalues (even and odd multiplicity).

Bifurcations off the branch of trivial stationary solutions



How are networks generated?

- Small diffusion ($D \ll 1$) creates substrate layers in Ω
  BUT: $D = 0$ gives, for every fixed $x \in \Omega$, the ODE system

$$\partial_t m = \cancel{D^2 \Delta m} + c^2 (m \cdot \nabla p) \nabla p - |m|^{2(\gamma-1)} m$$

  nonlinearly coupled via ∇p to the Poisson equation. Thus: For $D = 0$ the support of m does **not** grow.

- The activation term propagates the thin substrate layers into a network if c^2 is large enough.

- The relaxation term makes the network stationary:
  - For $1/2 \le \gamma \le 1$: sparse stationary networks
  - For $\gamma > 1$: stationary network fills up Ω

Inverse problem

Goal: Identification of the source function $S = S(x)$.
Problem: It is hard/impossible to measure p or m.
Solution: Measure the flow velocity

$$u = -(I + m \otimes m)\nabla p$$

through

- tracking of marker density $\varrho = \varrho(t, x)$,

$$\partial_t \varrho + u \cdot \nabla \varrho = 0, \qquad t \in (0, T)$$

$\rightsquigarrow$ fluorescent microscopy;

- **or** tracking of particles $X_i(t)$,

$$\frac{\mathrm{d}X_i}{\mathrm{d}t}(t) = u(X_i(t), t), \qquad i = 1, \ldots, N, \; t \in (0, T),$$

Identification of S

- Tracking of marker density:
  With homogeneous Neumann BC, using $\nabla \cdot u = S$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho \,\mathrm{d}x = - \int_\Omega u \cdot \nabla \varrho \,\mathrm{d}x = \int_\Omega \varrho \nabla \cdot u \,\mathrm{d}x = \int_\Omega \varrho S \,\mathrm{d}x$$

  $\Rightarrow$ identification of S via linear functionals over time.

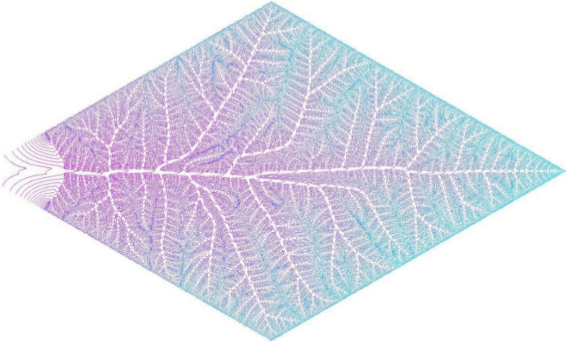
- Tracking of particles:
  Use the densities $\varrho(x,t) = \delta(X_i(t) - x)$,

$$S(X_i(t)) = \int_\Omega \varrho S \,\mathrm{d}x$$

  $\Rightarrow$ identification of S along particle trajectories.

Open problems in network formation

- Emergence of measure scaling limits as $D \to 0$ and $c \to \infty$ which are supported on sets of Hausdorff dimension $= 1$
  . . . very sparse networks



[Hu-Cai'2014]

- How are network junctions formed?
  Is the model capable of producing networks with loops?

Part II: PDE models for price formation †

† joint work with M. Burger (WWU Münster, Germany),L. Caffarelli (UT Austin, USA), J.F. Pietschmann (TU Darmstadt, Germany), T. Souganidis(University of Chicago, USA), J. Teichmann (ETH Zürich, Switzerlang) and M.T. Wolfram (RICAM, Austria).

The Lasry & Lions price formation model

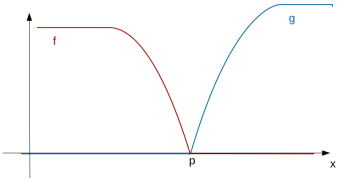
- Consider a market for one economical good, with a large number of buyers and a large number of vendors.
- Assume that each agent who has bought the good for a price x wants to resell immediately for at least a price of $x + a$, where $a > 0$ is a fixed transaction cost.
- Likewise every agent who has sold the good for the price x wants to rebuy for a price of at most $x - a$.
- Let $f = f(x, t)$ be the density of buyers at time t and $g = g(x, t)$ be the density of vendors at time t.

Price formation model Lasry & Lions

Lasry & Lions model the evolution of the agreed price $x = p(t) \in \mathbb{R}$ by

$$\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} = \lambda(t)\delta\left(x - p(t) + a\right), \quad \text{for } x < p(t)$$
$$\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} = \lambda(t)\delta\left(x - p(t) - a\right), \quad \text{for } x > p(t)$$

with a transaction rate $\lambda(t) = -\frac{\partial f}{\partial x}\left(p\left(t\right), t\right) = \frac{\partial g}{\partial x}\left(p\left(t\right), t\right)$.



The definition of the transaction rate guarantees mass conservation:

$$\frac{d}{dt}\int_{-\infty}^{p(t)} f(x,t)dx = \frac{d}{dt}\int_{p(t)}^{\infty} g(x,t)dx = 0.$$

Price formation model by Lasry & Lions

Introduce the signed buyer-vendor density

$$v(x,t) = \begin{cases} f(x,t) & \text{for } x < p(t) \\ -g(x,t) & \text{for } x > p(t). \end{cases}$$

Then the system reduces to

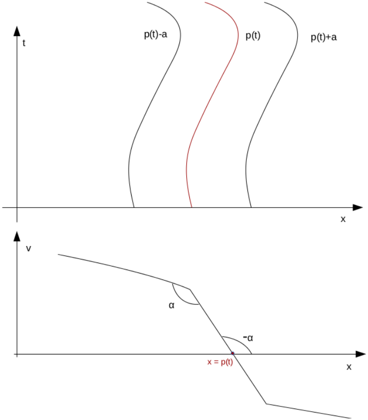
$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \lambda(t)\left(\delta(x - p(t) + a) - \delta(x - p(t) - a)\right)$$
$$v(x,t) > 0 \text{ if } x < p(t),\ \ v(x,t) < 0 \text{ if } x > p(t)$$
$$v(p(t),t) = 0, \lambda(t) = -\frac{\partial v}{\partial x}(p(t),t)$$

with initial conditions

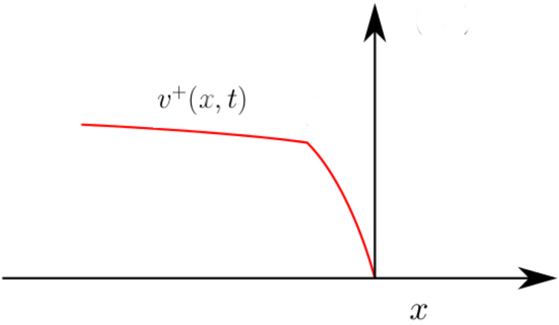
$$v(x,0) = f_I(x) - g_I(x),\ \ p(0) = p_0.$$

From Lasry & Lions to the heat equation

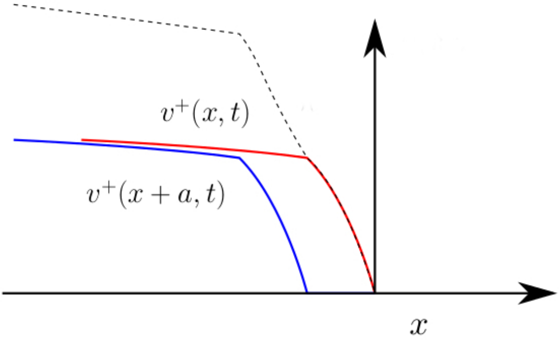
$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \lambda(t)\left(\delta(x - p(t) + a) - \delta(x - p(t) - a)\right)$$



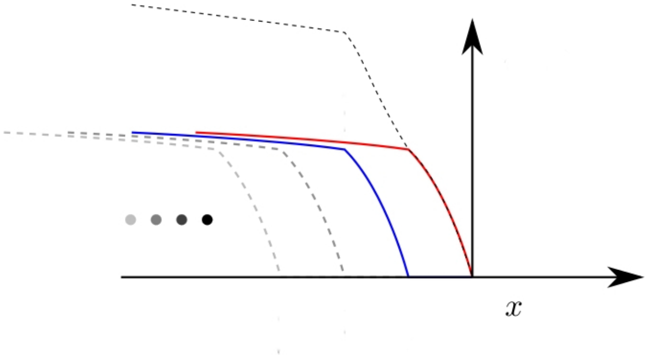
From Lasry & Lions to the heat equation



From Lasry & Lions to the heat equation



From Lasry & Lions to the heat equation



From Lasry & Lions to the heat equation

Apply construction to positive and negative part, i.e.

$$V(x,t) = \begin{cases} \sum_{n=0}^{\infty} v^{+}(x+na,t), & x < p(t), \\ -\sum_{n=0}^{\infty} v^{-}(x-na,t), & x > p(t). \end{cases}$$

The free boundary $p = p(t)$ corresponds to the zero-level set of V, which satisfies the heat equation

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}, \quad x \in \mathbb{R}, \ t > 0, \ \text{with } V_I(x) = \begin{cases} \sum_{n=0}^{\infty} v_I^{+}(x+na), & x < p_0, \\ -\sum_{n=0}^{\infty} v_I^{-}(x-na), & x > p_0. \end{cases}$$

Back-transformation to solution of L&L problem:

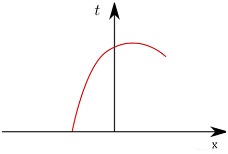
$$v(x,t) = \begin{cases} V^{+}(x,t) - V^{+}(x+a), & x < p(t), \\ -V^{-}(x,t) + V^{-}(x-a), & x > p(t). \end{cases}$$

Global existence theorem

Theorem (Global Existence)

There exists a unique solution $v = v(x,t)$ of the FBP for $t \in [0,\infty)$. Furthermore, $p \in \mathcal{C}([0,\infty))$.

Proof.



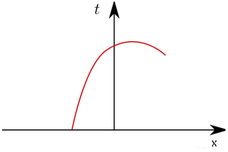
$V_x(p(t),t) < 0$ for all $t > 0$ (by the Hopf Lemma) and the min-max principle implies that $p = p(t)$ is graph of a function of time.

Global existence theorem

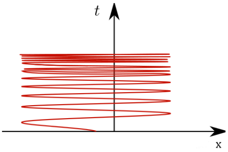
Theorem (Global Existence)

There exists a unique solution $v = v(x,t)$ of the FBP for $t \in [0,\infty)$. Furthermore, $p \in \mathcal{C}([0,\infty))$.

Proof.

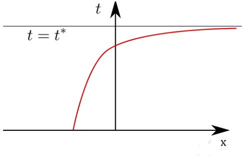


$V_x(p(t),t) < 0$ for all $t > 0$ (by the Hopf Lemma) and the min-max principle implies that $p = p(t)$ is graph of a function of time.



Contradicts x-analyticity of $V = V(x,t)$ (solution of the heat equation).

Global existence theorem (II)



Need to exclude the existence of t^* s.t. $|p(t)|$ becomes unbounded as $t \to t^*$

Let G denote the 1D heat kernel, then we have

$$V(x,t) = -\int_{-\infty}^{x-p_0} G(t,z)V_I^-(x-z)\,dz + \int_{x-p_0}^{\infty} G(t,z)V_I^+(x-z)\,dz$$

$x \to +\infty$:	bounded from above by a negative constant	tends to 0
$x \to -\infty$:	tends to 0	bounded from below by a positive constant

$\Longrightarrow \exists$ unique x with $-\infty < x < \infty$ such that $V(x,t) = 0$.

Deterministic market size fluctuations

Recall: in the original L&L model a buyer immediately becomes a vendor and vice versa $\Rightarrow$ conservation of the number of buyers and vendors.

Mass fluctuations: choose fluctuation functions $a_l(t)$ and $a_r(t)$ with $a_l(0) = a_r(0) = 0$ and multiply the distribution of buyers v^+ and vendors v^- by $e^{a_l(t+\Delta t) - a_l(t)}$ and $e^{a_r(t+\Delta t) - a_r(t)}$ in each time step respectively.

Then the generalized Lasry& Lions model in the formal limit $\Delta t \to 0$ reads:

$$\begin{aligned} v_t(x,t) = v_{xx}(x,t) + \lambda(t)(\delta(x - p(t) + a) - \delta(x - p(t) - a)) \\ + (\dot{a}_l(t) v^+(x,t) - \dot{a}_r(t) v^-(x,t)). \end{aligned}$$

Market size fluctuations:

$$M_l(t) = e^{a_l(t)} M_l \quad \text{and} \quad M_r(t) = e^{a_r(t)} M_r.$$

Deterministic market size fluctuations

Recall the one-to-one transformation from the original L&L model to the heat equation:

$$V(x,t) = \begin{cases} \sum_{n=0}^{\infty} v^+(x+na,t), & x < p(t) \\ -\sum_{n=0}^{\infty} v^-(x-na,t), & x > p(t). \end{cases}$$

Then the transformed problem reads as:

$$V_t(x,t) = V_{xx}(x,t) + \dot{a}_l(t)V^+(x,t) - \dot{a}_r(t)V^-(x,t),$$
$$V(x,t=0) = V_I(x),$$

Existence

Theorem

Under suitable conditions on the initial data and the functions $a_l = a_l(t)$ and $a_r = a_r(t)$ the L&L system with deterministic market size fluctuations has a unique solution and the free boundary $p = p(t)$ (zero level set of V) is the graph of a locally bounded continuous function of time.

Long-time behavior for different buyer/vendor market sizes

- Original L&L model:

$$p(t) \sim q_\infty \sqrt{t} \text{ with } \operatorname{erf}(q_\infty) = 2\frac{M^l - M^r}{M^l + M^r}.$$

- Deterministic fluctuations: Let $l^\infty = e^{a_l^\infty}$ and $r^\infty = e^{a_r^\infty}$, $\alpha^l = \frac{l^\infty}{a} M^l$ and $\alpha^r = \frac{r^\infty}{a} M^r$.

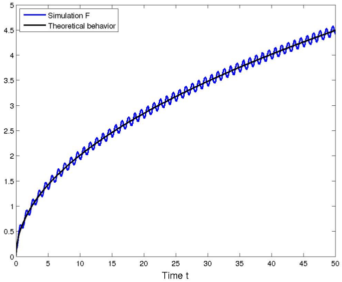
$$p(t) \sim q_\infty \sqrt{t} \text{ with } \operatorname{erf}(q_\infty) = 2\frac{\alpha^l - \alpha^r}{\alpha^l + \alpha^r}.$$

- Periodic fluctuations: a_l, a_r 1−periodic. Let $l^\infty = e^{\frac{1}{2}(\langle a_l \rangle - \langle a_r \rangle)}$ and $r^\infty = e^{\frac{1}{2}(\langle a_r \rangle - \langle a_l \rangle)}$ $\alpha^l = \frac{l^\infty}{a} M^l$ and $\alpha^r = \frac{r^\infty}{a} M^r$.

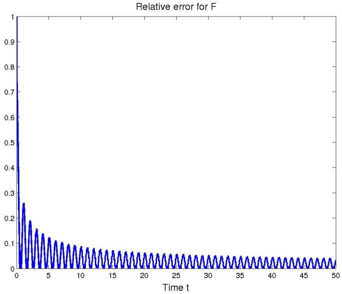
$$p(t) \sim q_\infty \sqrt{t} \text{ with } \operatorname{erf}(q_\infty) = 2\frac{\alpha^l - \alpha^r}{\alpha^l + \alpha^r}.$$

- Stochastic fluctuations: open.

Numerical simulations periodic case



(g) Evolution of the price



(h) Relative error

Figure : $\mathcal{O}(1)$-amplitude market size fluctuations result in $\mathcal{O}(\varepsilon)$-amplitude oscillations in the free boundary

Data assimilation

- Data assimilation based on an optimal control approach proposed by J.P. Puel[‡]
- Given measurements: price $p(t)$ and transaction rate $\lambda(t)$ on a time interval $t \in [0, T]$.
- Reconstruct the buyer-vendor distribution $v(x,t)$ at time $t = T$ to predict the price evolution for times $t > T$.

Consider L&L problem in the transformed variables:

$$V_t(x,t) - V_{xx}(x,t) = 0 \quad \text{for all } x \in \mathbb{R},\ t \geq 0$$
$$V(x,0) = V_I(x)$$

[‡] J.P. Puel, A nonstandard approach to a data assimilation problem and Tychonov regularization revisited, SICON, 48(2), 2009

Data assimilation

Let $r = r(t)$, $s = s(t)$ control variables. Solve the adjoint equation backwards for a given $\Psi = \Psi(x)$:

$$\partial_t \Phi + \partial_{xx} \Phi = \delta(x - p(t))r(t) + \delta'(x - p(t))s(t)$$
$$\Phi(x, T) = \Psi(x)$$

Let the optimal controls r_m and s_m be determined as minimizers of

$$\min_{r,s} \frac{1}{2} \int_{\mathbb{R}} \Phi(x, t = 0)^2 dx + \text{regularization terms}$$

and determine $V(x, t = T)$ from:

$$\int_{\mathbb{R}} V(x, t = T)\Psi(x)dx = \int_{\mathbb{R}} V_I(x)\Phi(x, t = 0)dx + \int_0^T s_m(t)\lambda(t)dt$$

Part III: A Brinkman-Forchheimer-Darcy model for porous media flow §

§ joint work with Edriss S. Titi (Texas A&M, USA) and Saber Trabelsi (KAUST, KSA).

Forchheimer equation as extension of Darcy's Law

- Darcy's flow model:

$$\nabla p = -\frac{\mu}{k}\, u,$$

u is the Darcy velocity, k is the permeability of the porous medium, μ is the dynamic viscosity of the fluid, and p the pressure.

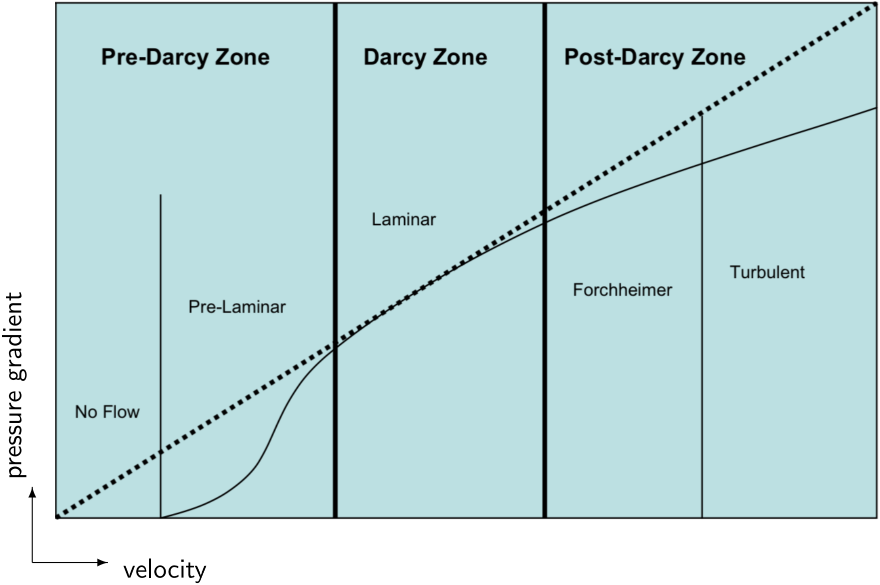
- The Darcy-Forchheimer law states

$$\nabla p = -\frac{\mu}{k}\, v - \gamma \varrho_f |v|\, v,$$

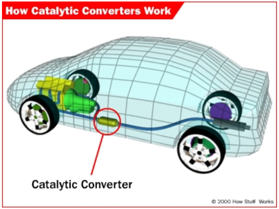
where $\gamma > 0$ is the so-called Forchheimer coefficient and v stands for the Forchheimer velocity, ϱ_f the density.

  - In a porous medium the nonlinear relaxation is caused by inertial effects.

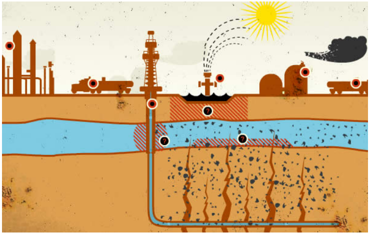
Flow regimes

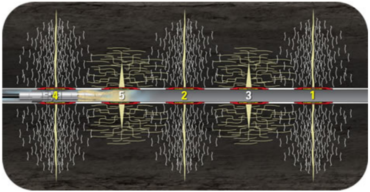


Application: catalytic converter



Application: naturally and hydraulically fractured reservoirs





The Brinkman-Forchheimer-extended Darcy model

Let $L > 0$, $\Omega = (0, L)^3$, $a > 0, b \in \mathbb{R}$ and consider the following generalization of the incompressible Navier-Stokes equation:

$$BFeD : \quad \begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p + a |u|^{2\alpha} u + b |u|^{2\beta} u = f, \\ \nabla \cdot u = 0, \ u|_{t=0} = u_0, \end{cases}$$

subject to either periodic boundary conditions or Dirichlet no-slip boundary conditions.

Difficulties:

- Dirichlet no-slip boundary conditions: need maximal regularity for the stationary problem.
- Periodic bc: we cannot use Poincaré inequality since there is no conservation of the mean value of u.

Weak solutions: Let $f \in L^\infty(\mathbb{R}^+; \mathbf{H})$, $u_0 \in \mathbf{H}$, $0 < \beta \leq \alpha$, $a > 0$ and $b \in \mathbb{R}$. Then system *BFeD* has weak solutions satisfying

- $u \in C^0(\mathbb{R}^+; \mathbf{H}_{\mathrm{weak}}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathbf{H}) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathbf{V}) \cap L^{2\alpha+2}_{\mathrm{loc}}(\mathbb{R}^+; L^{2\alpha+2}(\Omega))$.
- $\limsup_{t\to+\infty} \|u(t)\|_{L^2(\Omega)} \leq \varrho(\nu, a, b, \alpha, \beta)$.
- If $2\alpha \geq 3$, then the weak solutions depend continuously on u_0 and are unique.

Strong solutions: Let $f \in L^\infty(\mathbb{R}^+; \mathbf{H})$, $u_0 \in \mathbf{V}$, $\alpha > 1$, $0 < \beta \leq \alpha \leq 2$, $a > 0$ and $b \in \mathbb{R}$. Then system *BFeD* has a **unique** strong solution

- $u \in C^0_{\mathrm{b}}(\mathbb{R}^+; \mathbf{V}) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^2(\Omega) \cap \mathbf{V}) \cap L^{2\alpha+2}_{\mathrm{loc}}(\mathbb{R}^+; L^{2\alpha+2}(\Omega))$ and $\partial_t u \in L^2_{\mathrm{loc}}(\mathbb{R}^+; L^2(\Omega))$.
- $\limsup_{t\to+\infty} \|\nabla u(t)\|_2 \leq \kappa(\nu, a, b, \alpha, \beta)$.
- If $u_0 \in L^{2\alpha+2}(\Omega) \cap \mathbf{V}$, then for all $\alpha > 1$ we have $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^{2\alpha+2}(\Omega)$ and $\partial_t u \in L^2_{\mathrm{loc}}(\mathbb{R}^+; L^2(\Omega))$.

Notation: $\mathcal{V} = \{u \text{ is a trigonometric polynomial } : \nabla \cdot u = 0\}$,
$\mathbf{H} :=$ closure of $\mathcal{V}$ in $L^2(\Omega)$ and $\mathbf{V} :=$ closure of $\mathcal{V}$ in $H^1(\mathbb{R}^3)$.

Classical Data Assimilation

- Data assimilation is the process by which observations are incorporated into a computer model of a real system. The goal is to use low spatial resolution observational measurements to find a corresponding reference solution from which future predictions can be made.

- Applications of data assimilation arise in many fields of geosciences, perhaps most importantly in weather forecasting and hydrology.

- Using satellite observations, in the late 1960's Charney, Halem and Jastraw (1969) inserted observational measurements, at coarse scales, directly into a model as the latter was integrated in time. Continuous data assimilation is born.

Feedback Control

- Developed by Azouani, and Titi (2013) and Azouani, Olson and Titi (2013). Based on ideas from control theory. Rather than inserting the measurements directly into the model, a feedback control term is introduced, which forces the solution towards the reference solution corresponding to the observations.

The idea

Suppose that $u(t)$ represents a solution of a dynamical system

$$\frac{d}{dt}\,u = F(u)$$

with missing initial condition $u(t = 0) = u_0$.

Let $\mathcal{I}_h(u(t)), t \in [0, T]$ represent an interpolation of the observational measurements of the unknown reference solution u at time t at a coarse spatial resolution of mesh size h.

Use $\mathcal{I}_h(u)$ as a feedback control term

$$\frac{d}{dt}\,v = F(v) - \mu\left(\mathcal{I}_h(v) - \mathcal{I}_h(u)\right), \quad v(t = 0) = v_0 \quad \text{arbitrary,}$$

where $\mu > 0$ is a relaxation parameter.

Data Assimilation for BFeD

Let h be the resolution parameter, $\mu > 0$ be the relaxation parameter and $b = 0$ for simplicity

$$\begin{cases} \partial_t v - \nu \Delta v + (v \cdot \nabla) v + \nabla q + a |v|^{2\alpha} v = f + \mu \left(\mathcal{I}_h(u) - \mathcal{I}_h(v) \right), \\ \nabla \cdot v = 0, \; v|_{t=0} = v_0 \text{ arbitrary} \end{cases}$$

subject to periodic boundary conditions.

We consider interpolants which satisfy:

- $\mathcal{I}_h : H^1(\mathbb{R}^3) \to L^2(\Omega)$ such that

$$\|\psi - \mathcal{I}_h(\psi)\|_2^2 \leq c_0 h^2 \|\nabla \psi\|_2^2.$$

  Examples: projection onto low frequency Fourier modes, finite volume element averaging

Mathematical results

- Let $\mu > 0$ and h small enough such that $2\mu\, c_0\, h^2 \leq \nu$ and $\alpha > 0$, then the system above has weak solutions. These solutions are unique if $2\alpha \geq 3$.
- Let $\mu > 0$ and h small enough such that $2\mu\, c_0\, h^2 \leq \nu$ and $1 < \alpha \leq 2$, then the system above has a unique strong solution.
- Let $\mu > 0$ large enough and h small enough, then there exists $\gamma > 0$ such that the strong solution $v(t)$ satisfies

$$\|u(t) - v(t)\|_2 \leq e^{-\gamma t}.$$

- Let $\mu > 0$ large enough and h small enough and v_0 belongs to the ball of $\dot{H}^1(\Omega)$ of radius $r = 3\kappa$, then there exists $\gamma > 0$ such that the strong solution $v(t)$ satisfies

$$\|u(t) - v(t)\|_{\dot{H}^1} \leq e^{-\gamma t}.$$

A model is by definition that in which nothing has to be changed, that which works perfectly; whereas reality, as we see clearly, does not work and constantly falls to pieces; so we must force it, more or less roughly, to assume the form of the model.

Italo Calvino, 'The Model of Models', in *Mr. Palomar* (1983).