A PDE System Modeling Biological Network Formation

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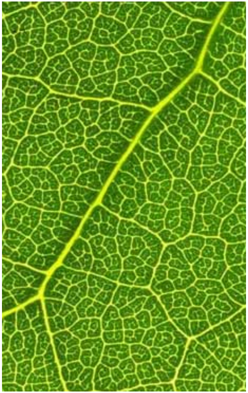
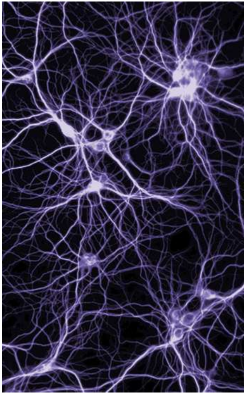
Biological transport networks



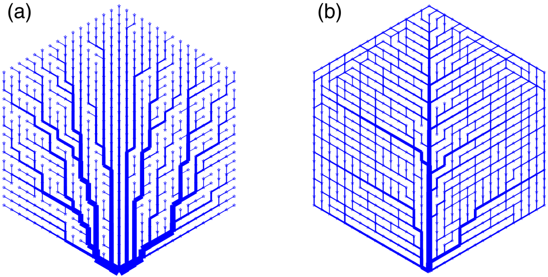
Leaf venation



Neural network

Blood capillaries

- Static and dynamic discrete graph-based models, deterministic and (geometric) random graphs.
  Topological and geometric properties - loops, trees, connectivity, scale-free graphs [Barabasi&Albert'1999, Newman'2003, Watts&Strogatz'1998, . . .]



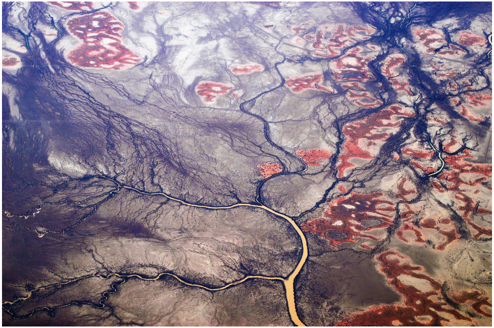
- Optimal mass transportation network modeling, based on a transportation cost law and Monge-Kantorovich theory [Bernot&Caselles&Morel'2009, Villani'2003&'2008, . . .]

Optimal mass transportation

$c(x, y)$ - cost of transport of a unit mass from x to y
Find a measure $\gamma(x, y)$ which minimizes the total transportation cost

$$C_\gamma := \int_{\Omega\times\Omega} c(x, y)\gamma(\,\mathrm{d}x,\,\mathrm{d}y)$$

where the marginals of γ are given.



River Branching

New program

- Dynamic discrete network adaptation model by [Hu&Cai'2014] based on
  - Kirchoff's law
  - Darcy's pressure law
  - local energy minimization
- PDE model derived as gradient flow of the continuous version of the energy functional.

The network formation model

- $\Omega \subseteq \mathbb{R}^d$, $d \leq 3$, bounded network domain (porous medium)
- $p = p(t, x)$ - scalar valued pressure
- $m = m(t, x)$ - vector valued conductance
- $S = S(x)$ - scalar valued source term

Kirchoff's conservation law: total outflux = the volume integral of sources.

$$-\nabla \cdot \underbrace{[(I + m \otimes m)\nabla p]}_{\text{flux}} = S \qquad \text{(Darcy's law)}$$

- $\mathbb{P}[m] := I + m \otimes m$ permeability tensor.
  Principal directions of network flow:
  1. $\frac{m}{|m|}$ with principal permeability $1 + |m|^2$
  2. $m^{\perp}$ with principal permeability 1 (uniform background)

The network formation model [Hu-Cai'2014]

Reaction diffusion system for the conductance m

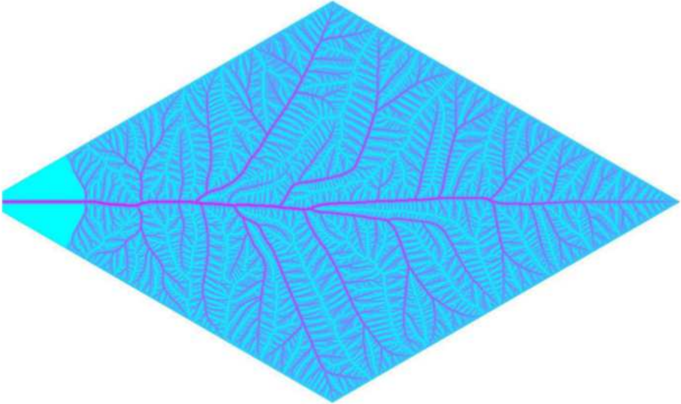
$$\frac{\partial m}{\partial t} = \underbrace{D^2 \Delta m}_{\substack{\text{random effects} \\ \text{in the porous medium}}} + \underbrace{c^2 (m \cdot \nabla p) \nabla p}_{\substack{\text{activation (force)} \\ \text{term}}} - \underbrace{|m|^{2(\gamma-1)} m}_{\text{relaxation term}}$$

- $c > 0$ - activation parameter, $D \geq 0$ - diffusivity
- $\gamma \in \mathbb{R}$ - relaxation exponent;
  in this talk: $\gamma > 1/2$ (transient sols.), $\gamma \geq 1$ (stationary sols.)
- homogeneous Dirichlet BC for m and p,

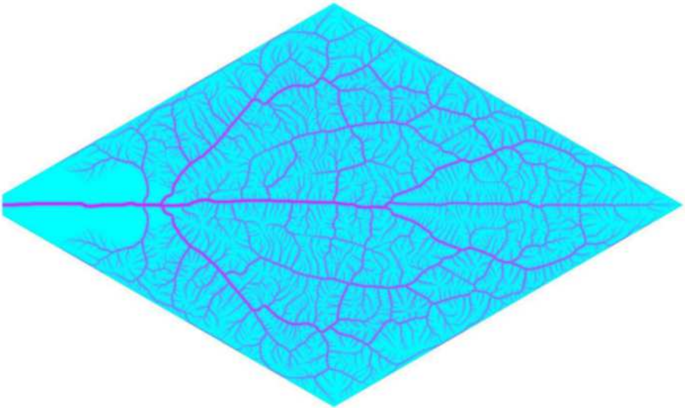
$$m|_{\partial\Omega} = 0, \qquad p|_{\partial\Omega} = 0 \qquad \forall t > 0$$

- $m(t = 0, x) = m^I(x), \quad x \in \Omega$

PDE simulation results - Trees (D. Hu, 2014)



PDE simulation results - Loops (D. Hu, 2014)



First observation

- Switching the sign of c^2, i.e.,

$$\frac{\partial m}{\partial t} = D^2 \Delta m - c^2 (m \cdot \nabla p) \nabla p - |m|^{2(\gamma-1)} m$$

with p arbitrary, gives

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |m|^2 \,\mathrm{d}x = & - & D^2 \int_\Omega |\nabla m|^2 \,\mathrm{d}x - c^2 \int_\Omega |m \cdot \nabla p|^2 \,\mathrm{d}x \\ & - & \int_\Omega |m|^{2\gamma} \,\mathrm{d}x \le -C \int_\Omega |m|^2 \,\mathrm{d}x. \end{aligned}$$

$\Rightarrow$ the only stationary solution is $m \equiv 0$.

- Therefore, $+c^2 (m \cdot \nabla p) \nabla p$ is the **activation term.**

$L^2(\Omega)$-gradient flow associated with the non-convex energy

$$\mathcal{E}(m) := \frac{1}{2}\int_\Omega \left(D^2|\nabla m|^2 + \frac{|m|^{2\gamma}}{\gamma} + c^2|m\cdot\nabla p[m]|^2 + c^2|\nabla p[m]|^2 \right) \mathrm{d}x.$$

Observe: $\int_\Omega \left(D^2|\nabla m|^2 + \frac{|m|^{2\gamma}}{\gamma} \right) \mathrm{d}x$ is convex for $\gamma \geq 1/2$;
non-convexity due to the coupling with the Poisson equation.

Energy dissipation: Along smooth solutions m, $p = p[m]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(m) = -\int_\Omega \left(\frac{\partial m}{\partial t}(t,x) \right)^2 \mathrm{d}x.$$

Mathematical problems

$$-\nabla \cdot [(I + m \otimes m)\nabla p] = S$$
$$\frac{\partial m}{\partial t} = D^2 \Delta m + c^2 (m \cdot \nabla p)\nabla p - |m|^{2(\gamma-1)} m$$

- General regularity results for the Poisson equation

$$\begin{aligned} -\nabla \cdot (A(x)\nabla p) &= \nabla \cdot F \quad \text{in } \Omega \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

  require at least $A \in L^\infty(\Omega)$.
- While the divergence part is controlled, how to control the rotational part of $(m \otimes m)\nabla p$?
- Iterating between m and p in the system destroys the energy dissipation equation.

Global existence of weak solutions for $\gamma > 1/2$

Thm: Let $S \in L^2(\Omega)$, $m^I \in L^2(\Omega)$.
The regularized system

$$-\nabla \cdot [\nabla p + m((m \cdot \nabla p) * \eta_\varepsilon)] = S$$

$$\frac{\partial m}{\partial t} = D^2 \Delta m + c^2 [(m \cdot \nabla p) * \eta_\varepsilon] \nabla p - |m|^{2(\gamma-1)} m$$

with $\eta_\varepsilon(x) := (4\pi\varepsilon)^{-d/2} \exp(-|x|^2/4\varepsilon)$, admits a weak solution.

Proof: Leray-Schauder fixed point theorem for the mapping

$$\Phi : L^2((0,T) \times \Omega) \to L^2((0,T) \times \Omega)$$

$$\Phi : \overline{m} \mapsto p[\overline{m}] \mapsto m$$

For any $u \in L^1(\mathbb{R}^d)$ and $\eta \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ with $\hat{\eta} \geq 0$, we have

$$\int_{\mathbb{R}^d} (u * \eta) u \, \mathrm{d}x = \int_{\mathbb{R}^d} |u * \varrho|^2 \, \mathrm{d}x \geq 0$$

where $\varrho = \mathcal{F}^{-1}[(\hat{\eta})^{1/2}]$.

Proof: Parseval's identity. ■

The regularized Poisson equation

For every $m,\ S \in L^2(\Omega)$ the regularized Poisson equation

$$-\nabla \cdot [\nabla p + m((m \cdot \nabla p) * \eta_\varepsilon)] = S$$

admits a unique weak solution $p \in H_0^1(\Omega)$ such that

$$\|\nabla p\|_{L^2(\Omega)} \leq C_\Omega \|S\|_{L^2(\Omega)} .$$

Proof: Lax-Milgram lemma for the bilinear form

$$B(p, \varphi) := \int_\Omega \nabla p \cdot \nabla \varphi \,\mathrm{d}x + \int_\Omega [(m \cdot \nabla p) * \eta_\varepsilon](m \cdot \nabla \varphi) \,\mathrm{d}x.$$

■

Weak-strong argument for the Poisson equation

Let $m^k \to m$ strongly in $L^2(\Omega)$ and $p^k \in H_0^1(\Omega)$ be the weak solutions of

$$-\nabla \cdot [\nabla p^k + m^k((m^k \cdot \nabla p^k) * \eta_\varepsilon)] = S.$$

Then $p^k \to p$ strongly in $H_0^1(\Omega)$, with p the unique solution of the regularized Poisson equation with m.

Proof: In Hilbert space,

$$\left.\begin{array}{c}\text{weak convergence}\\ \textbf{and}\text{ convergence of norms}\end{array}\right\} \Rightarrow \text{strong convergence.}$$

■

Regularity for a parabolic problem

For every $f \in L^2((0,T) \times \Omega)^d$ and $m^I \in L^2(\Omega)^d$, the PDE

$$\frac{\partial m}{\partial t} - D^2 \Delta m + |m|^{2(\gamma-1)} m = f$$

admits a unique weak solution m with

$$\|m\|_{L^\infty(0,T;L^2(\Omega))}, \qquad \|m\|_{L^{2\gamma}((0,T)\times\Omega)}, \qquad \|\nabla m\|_{L^2((0,T)\times\Omega)}$$

a-priori bounded in terms of $\|f\|_{L^2((0,T)\times\Omega)}$ and $\|m^I\|_{L^2((0,T)\times\Omega)}$.

Note: $f := c^2[(m^k \cdot \nabla p[m^k]) * \eta_\varepsilon]\nabla p[m^k] \in L^2((0,T) \times \Omega)$.

Let $\{m^k\}_{k\in\mathbb{N}}$ uniformly bounded in $L^\infty(0,T;L^{2\gamma}(\Omega))$
and $m^k \to m$ strongly in $L^2((0,T)\times\Omega)$.
Then

$$|m^k|^{2(\gamma-1)}m^k \rightharpoonup |m|^{2(\gamma-1)}m \qquad \text{weakly* in } L^\infty(0,T;L^{\frac{2\gamma}{2\gamma-1}}(\Omega)).$$

Proof:

- $h(m^k) := |m^k|^{2(\gamma-1)}m^k$ bounded in $L^\infty(0,T;L^{\frac{2\gamma}{2\gamma-1}}(\Omega))$
  $\Rightarrow$ a weakly* converging subsequence to
  $h^\infty \in L^\infty(0,T;L^{\frac{2\gamma}{2\gamma-1}}(\Omega))$
- $m^k \to m$ strongly in $L^2((0,T)\times\Omega) \Rightarrow$ a subsequence converging a.e. to m and $h(m^k)$ converging a.e. to $h(m)$

Consequently, $h^\infty = h(m)$. ∎

Limit $\varepsilon \to 0$

The modified energy

$$\mathcal{E}_\varepsilon(m) := \frac{1}{2}\int_{\mathbb{R}^d} D^2|\nabla m|^2 \quad + \quad \frac{|m|^{2\gamma}}{\gamma}$$
$$+ \quad c^2(m\cdot\nabla p)[(m\cdot\nabla p)*\eta_\varepsilon] + c^2|\nabla p|^2\,\mathrm{d}x$$

satisfies, for (m,p) a solution of the regularized network formation system,

$$\frac{\mathrm{d}\mathcal{E}_\varepsilon(t)}{\mathrm{d}t} = -\int |\partial_t m|^2\,\mathrm{d}x \le 0$$

$\Rightarrow$ uniform a priori estimates in $\varepsilon > 0$

Note: With the heat kernel η_ε,

$$c^2\int_{\mathbb{R}^d}(m\cdot\nabla p)[(m\cdot\nabla p)*\eta_\varepsilon]\,\mathrm{d}x \ge 0 \qquad \text{for every } \varepsilon > 0.$$

- $m^\varepsilon \to m$ strongly in $L^2((0,T)\times\Omega)$ due to compact Sobolev embedding
- $p^\varepsilon \to p$ strongly in $L^2(0,T;H_0^1(\Omega))$ due to the weak-strong argument for the Poisson equation
- $q^\varepsilon := (m^\varepsilon \cdot \nabla p^\varepsilon) * \eta_\varepsilon \rightharpoonup q$ weakly* in $L^\infty(0,T;L^2(\Omega))$ due to the a priori energy estimate
- $q = m \cdot \nabla p$ due to the strong convergence of m^ε and ∇p^ε
- $q^\varepsilon m^\varepsilon \rightharpoonup qm$ and $q^\varepsilon \nabla p^\varepsilon \rightharpoonup q\nabla p$
- $|m^\varepsilon|^{2(\gamma-1)} m^\varepsilon \rightharpoonup |m|^{2(\gamma-1)} m$ weakly in $L^{\frac{2\gamma}{2\gamma-1}}((0,T)\times\Omega)$
- (m,p) **weak solution of the network formation system**
- Weak lower semicontinuity for the energy

$$\mathcal{E}(m(t)) + \int_0^t \int_\Omega \left(\frac{\partial m}{\partial t}(s,x)\right)^2 \mathrm{d}x\,\mathrm{d}s \le \mathcal{E}(m^I) \qquad \text{for all } t \ge 0.$$

The case $\gamma = 1/2$

- Relaxation term $r(m) := \frac{m}{|m|}$ ... singularity in $m = 0$!
- Relaxation energy $R(m) := \int_\Omega |m| \, \mathrm{d}x$ convex!

We prove the existence of a weak solution of

$$\partial_t m = D^2 \Delta m + c^2 (m \cdot \nabla p[m]) \nabla p[m] - \tilde{r}(m)$$

with

$$\tilde{r}(m) \in \partial R(m) = \{ r \in L^\infty(\Omega);\ r(x) = m(x)/|m(x)| \text{ if } m(x) \neq 0, \\ |r(x)| \leq 1 \text{ if } m(x) = 0 \}$$

Conjecture:

$$\tilde{r}(m) = \begin{cases} \frac{m}{|m|} & \text{for } m \neq 0 \\ 0 & \text{for } m = 0 \end{cases}$$

- Compact support property of solutions (e.g., [Brezis'1974])
- Sparse networks!

Besov regularity

$$\frac{\partial m}{\partial t} = D^2 \Delta m + c^2 (m \cdot \nabla p) \nabla p - |m|^{2(\gamma-1)} m$$

- $m \cdot \nabla p \in L^\infty(0, \infty; L^2(\Omega))$, $\nabla p \in L^\infty(0, \infty; L^2(\Omega))$ imply

$$(m \cdot \nabla p) \nabla p \in L^\infty(0, \infty; L^1(\Omega))$$

- the energy dissipation gives

$$\partial_t m \in L^2((0, \infty) \times \Omega), \qquad |m|^{2(\gamma-1)} m \in L^\infty(0, \infty; L^{\frac{2\gamma}{2\gamma-1}}(\Omega))$$

Consequently, $\Delta m \in L^2(0, \infty; L^1(\Omega))$ and the L^1-parabolic regularity theory [Guidetti'93] implies the Besov regularity

$$m \in L^2(0, \infty; B^{2,1}_{\infty}(\Omega)).$$

Existence of mild solutions $(\gamma > 1/2)$ and their uniqueness $(\gamma \geq 1)$

Define the Banach spaces

$$\mathbb{X} := (L^\infty(\Omega) \cap \mathrm{VMO}(\Omega)), \qquad \mathcal{X}_T := L^\infty(0, T; \mathbb{X}).$$

By the Duhamel formula, $(m, p[m])$ is a mild solution on the time interval $(0, T)$ for a given activation parameter $c^2 > 0$ **iff** $(c, m) \in \mathbb{R}^+ \times \mathcal{X}_T$ is a solution of the nonlinear eigenvalue problem

$$\mathcal{T}(c^2, m) = m$$

with the operator $\mathcal{T}$ defined on $\mathbb{R}^+ \times \mathcal{X}_T$ by

$$\mathcal{T}(c^2, m) = e^{Lt} m^I + \int_0^t e^{L(t-s)} \left(c^2 F[m](s) - G[m](s)\right) \mathrm{d}s,$$

where $L := D^2 \Delta$ is the Dirichlet Laplacian and

$$F[m] = (m \cdot \nabla p[m]) \nabla p[m], \qquad G[m] = |m|^{2\gamma-1} m.$$

- [Marino'02]:
  Let $m \in \mathbb{X}$, $F \in L^q(\Omega)$, $S \in L^r(\Omega)$, $r := \max\{1, dq/(d+q)\}$.
  Then

$$\begin{aligned} -\nabla \cdot \big((I + m \otimes m)\nabla p + F\big) &= S \qquad \text{in } \Omega, \\ p &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

  has a unique weak solution $p \in W^{1,q}(\Omega)$ such that

$$\|\nabla p\|_{L^q(\Omega)} \le C(\|m\|_{\mathbb{X}}) \left(\|F\|_{L^q(\Omega)} + \|S\|_{L^r(\Omega)} \right) .$$

- [Meyers'63]: For **some** $q > 2$ the above estimate holds with $C = C(\|m\|_{L^\infty(\Omega)})$.

Existence and uniqueness of mild solutions

Thm: Let $m^I \in \mathbb{X}$ and $S \in L^\infty(\Omega)$.

A) There exists an unbounded continuum of solutions (c^2, m) of $\mathcal{T}(c^2, m) = m$ in $[0, \infty) \times \mathcal{X}_T$ emanating from $(0, m_0)$, where m_0 is the unique solution of

$$\frac{\partial m_0}{\partial t} - D^2 \Delta m_0 + |m_0|^{2(\gamma-1)} m_0 = 0.$$

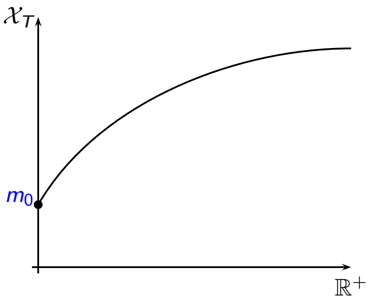
B) If $\gamma \geq 1$, then the mild solutions are unique.

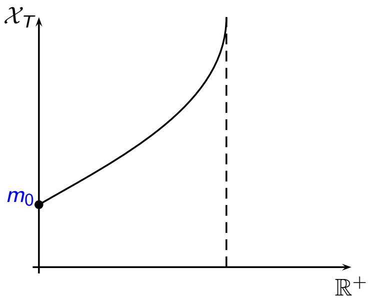
Thus, we have the blow-up alternative:

- Either there is a fixed point of $\mathcal{T}$ in $\mathcal{X}_T$ for all $c^2 > 0$,
- or there exists a bounded sequence of $c_k^2 > 0$ and a sequence of corresponding fixed points $m^k \in \mathcal{X}_T$ of $\mathcal{T}(c_k^2, \cdot)$ such that $\|m^k\|_{\mathcal{X}_T} \to \infty$ as $k \to \infty$.

Proof: Global theory for nonlinear eigenvalue problems of compact operators by Krasnoselski and Rabinowitz.

Blow-up alternative





Existence for all $c^2 \geq 0$ OR Blow-up for finite c^2

The unique solution of the Poisson equation

$$-\partial_x\big((1+m^2(x))\partial_x p(x)\big) = S(x)$$

on $(0,1)$ satisfies

$$|\partial_x p(x)| \le \frac{2\,\|S\|_{L^1(0,1)}}{1+m^2(x)} \qquad \text{for all } x \in (0,1).$$

Maximum principle $\Rightarrow$ a priori bound on m in $\mathcal{X}_T$ for every $T>0$, so that a unique global in time mild solution exists for every c^2 and $m^I \in L^\infty(0,1)$.

Stationary states and network formation

Long time convergence

Thm: Fix $T > 0$, a sequence $t_k \to \infty$ as $k \to \infty$, and for $\tau \in (0, T)$ define the time-shifts

$$m^{(t_k)}(\tau, x) := m(\tau + t_k, x), \qquad p^{(t_k)}(\tau, x) := p(\tau + t_k, x).$$

Then, after extraction of a subsequence,

$$\begin{aligned} m^{(t_k)} &\to m^\infty \quad \text{strongly in } L^q(0, T; L^4(\Omega)) \text{ for any } q < \infty, \\ p^{(t_k)} &\to p^\infty \quad \text{strongly in } L^2(0, T; H_0^1(\Omega)). \end{aligned}$$

where (m^∞, p^∞) is a weak solution of the stationary system.

Proof: A priori estimates provided by energy dissipation, Aubion-Lions compactness theorem. ■

Stationary states

Trivial stationary state:

$$m_0 \equiv 0, \qquad -\Delta p_0 = S.$$

Q: Do **nontrivial** stationary states $(\bar{m}, \bar{p})$ exist?

- If $D \to \infty$, then the weak stationary solutions $\bar{m}^D \to 0$ and $\bar{p}^D \to p_0$ in $H_0^1(\Omega)$.
- In the 1d case, if D^2/c^2 is big enough, then $\bar{m}^D \equiv 0$ and $\bar{p}^D = p_0$

The stationary system is equivalent to the fixed point problem

$$m = \beta L m + F(m, \beta)$$

with $\beta := c^2/D^2$ the bifurcation parameter and

- the linear part

$$Lm := (-\Delta)^{-1}(\nabla p_0 \otimes \nabla p_0)m, \qquad -\Delta p_0 = S$$

- the nonlinear part $F(m, \beta)$

Moreover, define

$$\mathcal{R}(L) := \{\beta \in \mathbb{R};\ \exists m \in \mathbb{X},\ m \neq 0 \text{ such that } m = \beta L m\}.$$

Spectral Theorem $\Rightarrow \mathcal{R}(L) = \{0 < \beta_1 < \beta_2 < \dots\}$.

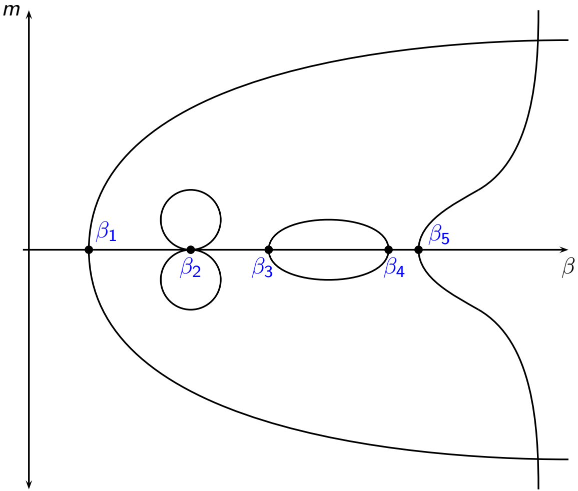
Thm: Let $\gamma \geq 1$. At every point $(m_0 \equiv 0, \beta_0 > 0) \in \mathbb{X} \times \mathbb{R}$ for which $\beta_0 \in \mathcal{R}(L)$ there is a bifurcation off the branch of trivial solutions $(m \equiv 0, \beta)$ of a solution branch of the stationary system. The branch

- **either** meets ∞ in $\mathbb{X} \times \mathbb{R}$
- **or** meets a point $(m_0 \equiv 0, \beta_1)$ where $\beta_1 \in \mathcal{R}(L)$

Proof: Global bifurcation theorem by [Rabinowitz'71]. ∎

Note: Bifurcation occurs at all eigenvalues (even and odd multiplicity).

Bifurcations off the branch of trivial stationary solutions



$$
\begin{aligned}
-\partial_x\left(\partial_x p + m^2 \partial_x p\right) &= S \\
\partial_t m - c^2 (\partial_x p)^2 m + |m|^{2(\gamma-1)} m &= 0
\end{aligned}
$$

on $\Omega = (0,1)$ with

$$
\begin{gathered}
\partial_x p(0) = 0,\ \ p(1) = 0, \\
m(t=0,x) = m^I(x).
\end{gathered}
$$

Integrate the Poisson equation,

$$
\partial_x p = -\frac{B(x)}{1+m^2}, \qquad B(x) := \int_0^x S(y)\,\mathrm{d}y.
$$

Inserting into the m-equation gives the ODE family

$$
\partial_t m = \left(\frac{c^2 B(x)^2}{(1+m^2)^2} - |m|^{2(\gamma-1)} \right) m \qquad \text{for } x \in (0,1).
$$

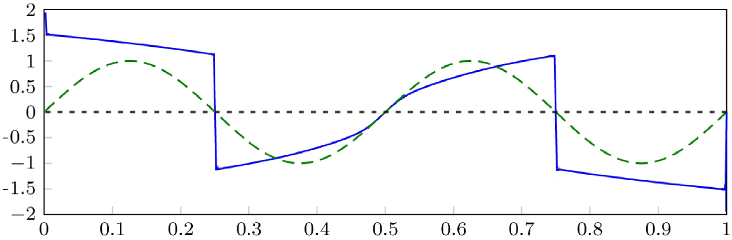
Network formation in 1d with $D = 0$, $\gamma > 1$

Stationary points:

- $m_0 = 0$ (unstable)
- $\pm m_s(x)$ (asympt. stable)

$$\lim_{t\to\infty} m(t,x) = m^\infty(x) := m_s(x)\mathrm{sign}(m^I(x)).$$

Note: $\mathrm{sign}(m^\infty(x)) = \mathrm{sign}(m^I(x))$ for all $x \in (0,1)$.



Initial condition (green) and steady state (blue) for m; $D = 0$, $\gamma = 2$

$$c^2(\nabla p^\infty \otimes \nabla p^\infty)m^\infty = |m^\infty|^{2(\gamma-1)} m^\infty$$

- $m^\infty \parallel \nabla p^\infty$ with $c^2|\nabla p^\infty|^2 = |m^\infty|^{2(\gamma-1)}$
- Fix measurable disjoint sets $\mathcal{A}_+ \subseteq \Omega$, $\mathcal{A}_- \subseteq \Omega$
- $m^\infty(x) := \left(\chi_{\mathcal{A}_+}(x) - \chi_{\mathcal{A}_-}(x)\right) c^{\frac{1}{\gamma-1}} |\nabla p^\infty(x)|^{\frac{2-\gamma}{\gamma-1}} \nabla p^\infty(x)$ with p^∞ the solution of

$$-\nabla \cdot \left[\left(1 + c^{\frac{2}{\gamma-1}} |\nabla p^\infty(x)|^{\frac{2}{\gamma-1}} \chi_{\mathcal{A}_+ \cup \mathcal{A}_-}(x)\right) \nabla p^\infty(x)\right] = S$$

Thm: For any $S \in L^2(\Omega)$, $\gamma > 1$ and for any pair of measurable disjoint sets $\mathcal{A}_+$, $\mathcal{A}_- \subseteq \Omega$ there exists a unique weak solution $p^\infty \in H_0^1(\Omega) \cap W_0^{1,2\gamma/(\gamma-1)}(\mathcal{A}_+ \cup \mathcal{A}_-)$.

Linearized stability analysis, $D = 0$, $\gamma > 1$

- Fix $\mathcal{A}_+, \mathcal{A}_- \subseteq \Omega$ and construct (m^∞, p^∞).
- Linearize (Gâteaux derivative) the network formation system at (m^∞, p^∞) in direction (n, q).

Thm:

a)

$$\lim_{t\to\infty} \int_\Omega \left(|\nabla q(t,x)|^2 + |n(t,x)|^2\right) \, \mathrm{d}x = 0$$

iff $(\Omega \setminus (\mathcal{A}_+ \cup \mathcal{A}_-)) \cup \{\nabla p^\infty(x) = 0\} \subseteq \{n^I(x) = 0\}$.

b) On $\{\nabla p^\infty(x) = 0\}$: $\qquad n(t,x) = n^I(x)$.

c) On $\Omega \setminus (\mathcal{A}_+ \cup \mathcal{A}_-)$: $\qquad \partial_t n = c^2(\nabla p^\infty \otimes \nabla p^\infty) n$.

$\Rightarrow$ Networks are inherently unstable!!

Limit of vanishing diffusion in 1d

For $D > 0$ let (m^D, p^D) be classical solutions in 1d on $[0, T]$. Then

$$\begin{aligned} m^D &\to m && \text{in } L^q((0,T)\times(0,1)) \text{ for any } q < \infty \\ \partial_x p^D &\to \partial_x p && \text{in } L^2((0,T)\times(0,1)) \end{aligned}$$

where (m, p) is a solution of

$$\begin{aligned} -\partial_x\left((1+m^2)\partial_x p\right) &= S \\ \partial_t m - c^2(\partial_x p)^2 m + |m|^{2(\gamma-1)} m &= 0 \end{aligned}$$

Proof: The uniform BV-estimate for m,

$$\max_{t\in[0,T]} \int_0^1 |\partial_x m| \, \mathrm{d}x < C.$$

Outlook and open questions

A) Open PDE analysis problems:

- $\gamma < 1$ open for strong solutions; $\gamma \leq 1/2$ open ($\gamma = 1/2$ for leaf venation; $\gamma = 1$ for blood capillaries)
- Is the branch of mild solutions global in c^2 also in more than one dimension?
- Rigorous limit $D \to 0$ in multiple dimensions.
- Existence of weak or strong solutions with $D = 0$.
- Existence for stationary states and their classification for $D = 0$, $\gamma \leq 1$.

B) Structural problems in network formation:

- How to measure the complexity of the network (loops/trees) and how does it depend on the data?
- How does branching occur?
- How do stationary states depend on the initial data in multi-dimensional settings?

Thank you for your attention!