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Homogeneous Cooling States are not always good approximations to granular flows

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Contents

Abstract

A widespread belief in the study of granular flow is the existence of "homogeneous cooling states", i.e. self-similar solutions which would attract all solutions, faster than the equilibrium solution does. In most cases, the existence of these self-similar solutions is an open problem. Here we consider a one-dimensional model, which has been used for some years, and for which simple self-similar solutions do exist. However, we prove that the approximation is quite poor. Our proof makes use of the powerful and simple tools of mass transportation, and exploits the structure of the evolution equation, seen as a nonlinear transport equation.

1. A spatially homogeneous caricature of a granular flow

Granular materials are widely studied for the importance of applications, but also for the original physical problems which they raise. Physicists have been interested in them for a long time; a large review on the subject can be found in Campbell [6]. During recent years, a new and very interesting approach to these materials has been proposed in the framework of kinetic theory and statistical mechanics: granular material is depicted as a large collection of particles which collide inelastically [5,8,11,13,21]. Related kinetic and hydrodynamics models have been proposed and studied in [1,8,9, 12,14,16,17].

In all these kinetic models, deperdition of energy is allowed (inelasticity). As a consequence, if the granular material is isolated, then the typical equilibrium state will be a Dirac mass located at the mean velocity of particles (meaning that, up to a change of reference framework, all particles are at rest, which is physically sensible). If, on the other hand, the material is put in interaction with a "thermal bath" (which may be some external agitation), then the combined effects of diffusion and nonlinear friction may lead to a non-gaussian, non-Dirac equilibrium state [1,7,10].

In this paper, we only consider the first case, in which the granular material is isolated. For the simplicity of discussion, we restrict ourselves to the (oversimplified!) spatially homogeneous state. This means that we shall consider a probability density $f_t(v) = f(t,v)$ on the *velocity* space only. Actually we shall allow f to be a probability measure with a singular part. We shall assume that it is centered, so that the mean velocity of particles is 0:

$$\int_{\mathbb{R}} f(t,v)\, v\, dv = 0.$$

Then one expects that

$$f_t \xrightarrow[t\to\infty]{} \delta_0, \tag{1}$$

where δ_0 stands for the Dirac mass at zero velocity, and the convergence holds in weak-$*$ measure sense.

A common belief is that the role of equilibria should not be played by the Dirac mass, but rather by particular *self-similar solutions* $S_t(v)$ converging to this Dirac mass. Thus one would expect that for some well-chosen distance,

$$\mathrm{dist}(f_t, S_t) \ll \mathrm{dist}(f_t, \delta_0) \qquad \text{as } t \to \infty. \tag{2}$$

This discussion is an important point, not only for the study of the spatially homogeneous equation in itself, but also as a step towards deriving the right hydrodynamic equations in a spatially inhomogeneous setting. Indeed, homogeneous cooling states would be a candidate for playing the fundamental role attributed to locally Maxwellian states in the classical, elastic kinetic theory.

In general, the existence of homogeneous cooling states is a considerable act of faith, since it has not been proven rigorously except in some very particular cases ! However, for the model which we shall consider in this paper, attracting self-similar solutions do exist. This model was proposed a few years ago by McNamara and Young [13], as the mean-field limit of a system of many particles colliding inelastically in one dimension. It reads

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left[f \frac{\partial}{\partial v}\left(f *_v \frac{|v|^3}{3}\right)\right] \qquad t \geq 0, \quad v \in \mathbb{R}. \tag{3}$$

Remark 1. Equation (3) can also be obtained as the "grazing collision limit" of an inelastic Boltzmann-type operator [3,4]. This kind of limits has been extended to a wider class of models in [18,19].

Remark 2. From the point of view of mathematical kinetic theory, equation (3) is quite simple, due to its spatial homogeneity. By "quite simple", we mean that it is no problem to prove existence and uniqueness of a solution to (3), given some initial datum with finite kinetic energy. If on the other hand we would like to model a spatially inhomogeneous situation, then we should add a transport term to the left-hand side of (3). In such a situation the problem of existence of global solutions for generic initial data is still open (only for small initial data has it been solved [2]).

As shown in [2], solutions of (3) do concentrate on a Dirac mass in large time, equation (1). Moreover, the distance towards equilibrium is accurately measured by the square root of the variance of the distribution function:

$$\mathrm{dist}(f_t, \delta_0) \equiv \sqrt{\int_{\mathbb{R}} f_t(v)|v|^2\, dv} = O\left(\frac{1}{t}\right). \tag{4}$$

The rate $O(1/t)$ is optimal, and also the size of the support of f_t is $O(1/t)$.

Now, to get a self-similar solution, one looks for the scaling invariance of the equation. It is easy to see that if $\tau = \log t$, then the new unknown distribution

$$g(\tau, v) = \frac{1}{t} f\left(t, \frac{v}{t}\right) = e^{-\tau} f(e^{\tau}, e^{-\tau} v) \tag{5}$$

satisfies the new equation

$$\frac{\partial g}{\partial \tau} = \frac{\partial}{\partial v}\left[g \frac{\partial}{\partial v}\left(g *_v \frac{|v|^3}{3}\right) - gv \right], \qquad \tau \in \mathbb{R}, \quad v \in \mathbb{R}. \tag{6}$$

If equation (6) admits a nontrivial stationary solution, then this will yield a self-similar solution for (3). Actually there are a lot of nontrivial stationary solution for (6); but one of them can be singled out. Indeed, it turns out that there is a variational principle hidden in (6): its solutions admit as a nonincreasing Lyapunov functional the "free energy"

$$\mathcal{F}(g) = \frac{1}{2}\int_{\mathbb{R}^2} g(v)g(w)\frac{|v-w|^3}{3}\, dv\, dw - \int_{\mathbb{R}} g(v)\frac{|v|^2}{2}\, dv. \tag{7}$$

Among all probability measures on $\mathbb{R}$ there is a unique minimizer for (7), which is

$$g_\infty = \frac{1}{2}\left[\delta_{-\frac{1}{2}} + \delta_{\frac{1}{2}}\right]. \tag{8}$$

As a consequence, g_∞ should be the only "stable" stationary state for (6). This is indeed what Benedetto, Caglioti and Pulvirenti prove in [2]:

Theorem 1 (Benedetto, Caglioti, Pulvirenti). *Let f_0 be a probability measure, absolutely continuous with respect to Lebesgue measure, and let $(f_t)_{t\geq 0}$ be the associated unique solution of* (3). *Let $g_\tau(v) = g(\tau, v)$ be defined by* (5). *Then,*

$$g_\tau \xrightarrow[\tau\to\infty]{} g_\infty$$

in weak-$*$ measure sense.

This theorem seems to corroborate the physical intuition, by showing that the self-similar solution

$$S_t = \frac{1}{2}\left[\delta_{-\frac{1}{2t}} + \delta_{\frac{1}{2t}}\right] \tag{9}$$

is indeed a good approximation to the solution of (3); or rather, a better approximation than just the steady state δ_0.

However, we shall show in this paper that *the improvement in the rate of convergence is at best logarithmic in time*, hence morally negligible by usual standards. In the next section we shall give a precise statement. It will be proven in sections 4 and 5, after some preparations in section 3. We shall first present a detailed argument under a simplifying assumption (which is satisfied for generic initial data), then a slightly more sketchy proof for the general case.

2. Slowness of approach to equilibrium

In this section we focus on the rescaled equation (6). Since we are interested in a long-time behavior, we shall assume $\tau \geq 0$. By abuse of notations, we shall often write $\int g(v)\,dv$ even when g is a singular probability measure.

To measure the distance between g_τ and g_∞, it is of course impossible to use the total variation norm: think that if f_0 is absolutely continuous with respect to Lebesgue measure, then so is g_τ for any $\tau \in \mathbb{R}$, and $\|g_\tau - g_\infty\|_{TV} = 2$ independently of τ. Hence, we shall use other, very popular notions of distance which allow to compare probability measures in weak sense, and in particular to compare singular measures with nonsingular ones.

Definition 1. Let μ, ν be two probability measures on $\mathbb{R}$, and let $p \in [1, +\infty)$. We define the Monge-Kantorovich (or Wasserstein) distance $W_p(\mu,\nu)$ to be

$$W_p(\mu,\nu) = \inf\left\{(E|X-Y|^p)^{1/p}; \quad \mathrm{law}(X) = \mu, \quad \mathrm{law}(Y) = \nu\right\}. \tag{10}$$

By general and well-known theorems on the Monge-Kantorovich mass transportation problem in dimension 1 (see for instance [15]), one can give an "explicit" expression of $W_p(g, g_\infty)$ for an arbitrary probability measure g. Let m be a median of g, i.e. a real number satisfying

$$\int_{-\infty}^{m} g(v)\,dv \geq \frac{1}{2}, \qquad \int_{m}^{+\infty} g(v)\,dv \geq \frac{1}{2}. \tag{11}$$

Define $\sigma_-, \sigma_+ \geq 0$ by

$$\int_{-\infty}^{m} g(v)\,dv = \frac{1}{2} + \sigma^+, \qquad \int_{m}^{+\infty} g(v)\,dv = \frac{1}{2} + \sigma^-. \tag{12}$$

Obviously $\sigma_\pm = 0$ if f is absolutely continuous with respect to Lebesgue measure; in fact $\sigma_+ + \sigma_-$ is the size of the singular part of f at m, if it exists. Then,

$$W_p(g, g_\infty)^p = \int_{-\infty}^{m} g(v) \left| v + \frac{1}{2} \right|^p dv - \sigma_+ \left| m + \frac{1}{2} \right|^p + \int_{m}^{+\infty} g(v) \left| v - \frac{1}{2} \right|^p dv - \sigma_- \left| m - \frac{1}{2} \right|^p. \tag{13}$$

In particular, if g has no singular part at m, then

$$\int_{-\infty}^{m} g(v)\,dv = \frac{1}{2}, \qquad \int_{m}^{+\infty} g(v)\,dv = \frac{1}{2} \tag{14}$$

and

$$W_p(g, g_\infty)^p = \int_{-\infty}^{m} g(v) \left| v + \frac{1}{2} \right|^p dv + \int_{m}^{+\infty} g(v) \left| v - \frac{1}{2} \right|^p dv.$$

It is well-known that (i) the W_p's are metrics on the set of probability measures with finite moments of order p; (ii) convergence in W_p sense implies weak-$*$ convergence in measure sense; (iii) weak-$*$ convergence together with convergence of the moment of order p, imply convergence in W_p sense; (iv) the W_p's are *ranked*, in the sense that $p \leq p' \Rightarrow W_p \leq W_{p'}$. The proofs of all these facts can be found in, e.g., Villani [20]. Also note that the square root of the variance appearing in (4) is nothing but $W_2(f_t, \delta_0)$.

Our goal is the following main result.

Theorem 2. *Let g_0 be a probability measure on $\mathbb{R}$, which is not a symmetric convex combination of two delta masses, and let $(g_\tau)_{\tau \geq 0}$ be the corresponding solution to (6). Then, for all $p \in [1, +\infty)$,*

$$\int_0^{+\infty} W_p(g_\tau, g_\infty)\,d\tau = +\infty. \tag{15}$$

More precisely, there exists some constant $K > 0$, depending on g_0, such that, as $\tau \to \infty$,

$$\int_0^{\tau} W_p(g_s, g_\infty)\,ds \geq K \log \tau. \tag{16}$$

Corollary 1. *Let f_0 be a probability measure on $\mathbb{R}$, which is not a symmetric convex combination of two delta masses, and let $(f_t)_{t\geq 0}$ be the corresponding solution to* (3)*. Then, for all $p \in [1, +\infty)$,*

$$\int_0^{+\infty} W_p(f_t, S_t)\, dt = +\infty. \tag{17}$$

More precisely, there exists some constant $K > 0$, depending on f_0, such that, as $T \to \infty$,

$$\int_0^{T} W_p(f_t, S_t)\, dt \geq K \log\log T. \tag{18}$$

Before we turn to the proof of Theorem 2, let us make a few remarks. To begin with, we should explain how the corollary follows from the lemma. First note that the scaling leading from (3) to (6) is actually $g_\tau = (e^\tau \cdot)\# f_{e^\tau}$, where $T\#\mu$ denotes the push-forward of a measure μ by a map T (see the beginning of section 3). Then, from definition (10) one easily finds

$$W_p(g_\tau, g_\infty) = W_p\big((e^\tau \cdot)\# f_{e^\tau}, (e^\tau \cdot)\# S_{e^\tau}\big) = t W_p(f_t, S_t). \tag{19}$$

By changing variables in $\int_1^T W_p(f_t, S_t)\, dt$, the result follows at once.

Next, assume for the sake of the discussion that $W_p(g_\tau, g_\infty)$ is of order $O(1/\tau)$, which is precisely what the rate (16) suggests. Since $\tau = \log t$, one could therefore bound below the distance $W_p(f_t, S_t)$, as $t \to \infty$, by a constant multiple of $1/(t \log t)$.

On the other hand, $W_p(f_t, \delta_0) = O(1/t)$. Thus our main result implies that *extremely little is gained by replacing δ_0 by S_t: the gain is at best logarithmic in time.* Since such an improvement is usually considered as negligible, we are left with essentially two possibilities: either physical intuition is misleading, or the model should be rejected because it leads to irrelevant conclusions...

Also note that Theorem 2 applies without assumption of absolute continuity on the initial datum. This is true also for Theorem 1 (see [2]).

In particular if $\sigma^+ = 0$ or $\sigma^- = 0$ then g_τ converges towards g_∞ as $\tau \to \infty$ but, according to Theorem 2, very slowly.

If $\sigma^+ > 0$ and $\sigma^- > 0$ then, as we shall see, g_τ cannot converge towards g_∞.

As a last comment, we emphasize the power of the mass transportation point of view in the proof of the result. The use of the distances W_p will lead to the conclusion in a very straightforward way, without any delicate asymptotic analysis.

In section 4, we shall carry on the proof under an extra condition, namely that f_0 admits a *unique median.* This condition means that the support of f_0 cannot be included in two disjoint intervals $(-\infty, m_-]$ and $[m_+, +\infty)$ such that

$$\int_{-\infty}^{m_-} f_0\, dv = \frac{1}{2}, \qquad \int_{m^+}^{+\infty} f_0\, dv = \frac{1}{2}.$$

Actually (as shown by the considerations in section 3), if this condition is true at some time, then it will stay true for later times (both for eq. (3) and for eq. (6)). This is why fulfilment of this condition for f_0 implies fulfilment of this condition for g_0 too. Of course, this condition is generic.

In section 5, we turn to the general case. The proof is distinct but shares many common features, which is why we will give it in slightly less detail. Actually, *the first argument will rely on the fact that the equilibrium state vanishes near the origin*, while *the second argument will rely on the fact that the equilibrium state is concentrated into delta masses.* Thus, our proof may be seen as pointing out some causes for the pathologies. It seems likely that equations with a similar structure, in higher dimension, also converge slowly to equilibrium when the equilibrium state presents singularities.

Before beginning on with the proofs, in the next section we give a series of lemmas as preparations.

3. Preparations

The following notion will be crucial. Given a measurable map $T:\mathbb{R}\to\mathbb{R}$ and a probability measure μ on $\mathbb{R}$, we denote by $T\#\mu$ the push-forward of μ by T, i.e. the probability measure defined through the identity

$$T\#\mu[A] = \mu[T^{-1}(A)].$$

We now formulate a series of lemmas.

Lemma 1 (Method of characteristics). *The solution to the Cauchy problem for eq.* (6) *is given by* $g_\tau = T_\tau\#g_0$*, where the "velocity" field* T_τ *satisfies*

$$\frac{d}{d\tau}T_\tau = \xi_\tau \circ T_\tau, \qquad T_0 = \mathrm{id}, \tag{20}$$

$$\xi_\tau(v) = v - \int_{\mathbb{R}} (v-w)|v-w| g_\tau(w)\, dw. \tag{21}$$

Note that ξ_τ is a C^1 function of v, hence the solution to (21) is well-defined by the standard theory of ODE's. This lemma is a consequence of the structure of eq. (6) and the well-known method of characteristics for solving linear transport equations. Of course, since the equation is nonlinear, there is a coupling between the solution g_τ and the equation of characteristics T_τ, so this lemma does not yield an "explicit" expression for (g_τ). Physically speaking, lemma 1 means that if we label mass particles by their initial position in velocity space and keep track of their trajectories, then the trajectory of the particle labelled by v_0 is $(T_\tau(v_0))_{\tau\geq 0}$. See [2] for details of implementation.

Lemma 2. *For each* $\tau \geq 0$*,* T_τ *is a nondecreasing map.*

Proof. $(T_\tau(v_0))_{\tau\geq 0,\, v_0\in\mathbb{R}}$ is a family of solutions of a one-dimensional first-order ODE, hence $T_\tau(v_0)\neq T_\tau(v_1)$ for $v_0\neq v_1$. Also $T_0(v_0)=v_0<T_0(v_1)=v_1$ if $v_0<v_1$. By continuity, $v_0<v_1\Longrightarrow T_\tau(v_0)<T_\tau(v_1)$.

Lemma 2 means that the equation preserves the ordering of the particles on the real line.

As an easy consequence, $W_p(g_\tau,g_\infty)$ cannot go to zero if $\sigma_-,\sigma_+>0$ in (12). Indeed, in this case g_0 admits a unique median m_0, and by lemmas 1 and 2, g_τ will admit $m_\tau=T_\tau(m_0)$ as a unique median, more precisely

$$\int_{-\infty}^{m_\tau} g_\tau=\frac{1}{2}+\sigma_+,\qquad \int_{m_\tau}^{+\infty} g_\tau=\frac{1}{2}+\sigma_-,$$

in particular m_τ carries a positive mass $\sigma_++\sigma_-$. This implies, by (13),

$$\begin{aligned} W_1(g_\tau,g_\infty)&\geq \sigma_-\left|T_\tau m_0+\frac{1}{2}\right|+\sigma_+\left|T_\tau m_0-\frac{1}{2}\right|\\ &\geq \min(\sigma_-,\sigma_+)\left(\left|T_\tau m_0+\frac{1}{2}\right|+\left|T_\tau m_0-\frac{1}{2}\right|\right)\geq \min(\sigma_-,\sigma_+). \end{aligned}$$

The meaning of this is simply that the mass which is located at the median would have to split into two parts in order to match g_∞: a mass σ_- should go to the left, i.e. to $-1/2$, while a mass σ_+ should go to the right, i.e. to $+1/2$. Since the velocity field ξ_τ is continuous, such a splitting is impossible.

We sum up this remark by the following corollary to lemmas 1 and 2:

Corollary 2. $W_1(g_\tau,g_\infty)\geq \min(\sigma_-,\sigma_+)$. *In particular,* (15) *obviously applies when* $\sigma_\pm>0$.

Consequence : In all the sequel it suffices to consider the case when $\sigma_-=0$ or $\sigma_+=0$. To keep proofs as clear as possible, in the rest of section 3 and in section 4 we shall actually perform them only in the case where $\sigma_-=\sigma_+=0$. More explicitly, this means

$$\int_{-\infty}^{m_0} g_0=\frac{1}{2},\qquad \int_{m_0}^{+\infty} g_0=\frac{1}{2}, \tag{22}$$

and therefore

$$\int_{-\infty}^{m_\tau} g_\tau=\frac{1}{2},\qquad \int_{m_\tau}^{+\infty} g_\tau=\frac{1}{2}.$$

This assumption will be made in the sequel without further comment. In particular, m_0 is not a singular point for g_0, and as a consequence, $m_\tau=T_\tau(m_0)$ is not a singular point for g_τ. We shall only indicate some of the minor changes needed in the proofs to cover the case where (for instance) $\sigma_+>0$, so that

$$\int_{-\infty}^{m_0} g_0=\frac{1}{2}+\sigma_+,\qquad \int_{m_0}^{+\infty} g_0=\frac{1}{2}. \tag{23}$$

Lemma 3. *For each $\tau \geq 0$, $d\xi_\tau(v)/dv$ is maximum at $T_\tau(m_0)$. In fact, $d\xi_\tau(v)/dv$ is nondecreasing for $v < T_\tau(m_0)$, and nonincreasing for $v > T_\tau(m_0)$.*

Proof. By direct computation,

$$\frac{d\xi_\tau(v)}{dv} = 1 - 2\int_{\mathbb{R}} |v-w|\, g_\tau(w)\, dw.$$

Hence we want to show that $\phi(v) = \int |v-w| g_\tau(w)\, dw$ is minimum when v is the median $T_\tau(m_0)$ of g_τ. Since $m_\tau = T_\tau(m_0)$ is not a singular point of g_τ, the function ϕ is differentiable at m_τ, and $\phi'(m_\tau) = \int_{w>m_\tau} g_\tau - \int_{w<m_\tau} g_\tau = 0$. The conclusion follows by noting that ϕ is convex.

Remark 3. The case (23) is easily dealt with in the same way by showing that ϕ is left-differentiable at m_τ.

Lemma 4. *If m is a median of g, and m is not a singular point for g, then*

$$1 - 2\int g(v)|v-m|\, dv \leq 2W_1(g, g_\infty). \tag{24}$$

Proof. The left-hand side of (24) is

$$1 - 2\int_{v\leq m} g(v)(m-v)\, dv - 2\int_{v\geq m} g(v)(v-m)\, dv$$

$$= 1 + 2\int_{v\leq m} g(v)v\, dv - 2\int_{v\geq m} g(v)v\, dv$$

$$= 2\int_{v\leq m} g(v)\left[v + \frac{1}{2}\right] dv - 2\int_{v\geq m} g(v)\left[v - \frac{1}{2}\right] dv,$$

where we have used (14) twice. Then, the conclusion follows immediately by formula (13), which can be rewritten as

$$W_1(g, g_\infty) = \int_{v\leq m} g(v)\left|v + \frac{1}{2}\right| dv + \int_{v\geq m} g(v)\left|v - \frac{1}{2}\right| dv.$$

Remark 4. The same result holds when g satisfies (23).

4. A simple proof for generic initial data

In this section we prove Theorem 2 in the particular case when the initial datum has a unique median m_0. This is just for pedagogical reasons: indeed, in this case one can give a slightly simpler proof, based only on lemmas 1–4. More precisely, we shall prove that

$$\int_0^{+\infty} W_p(g_\tau, g_\infty)\, d\tau = +\infty \tag{25}$$

under the assumption that for all $\delta > 0$,

$$\int_{m_0-\delta}^{m_0} g_0 \geq \varepsilon(\delta), \qquad \int_{m_0}^{m_0+\delta} g_0 \geq \varepsilon(\delta), \qquad \varepsilon(\delta) > 0. \tag{26}$$

We only need to prove the theorem when $p = 1$, since this implies the result for any $p \geq 1$.

Proof (Proof of theorem 2 when g_0 has a unique median).
The main idea behind the argument is the following. Starting from the intuition suggested by lemma 1, one looks at equation (3) as a transport equation in which the velocity field is coupled to the density g_τ itself. Since all the mass which is located to the left of the median has to go to $-1/2$, and all the mass which is located to the right of the median has to go to $1/2$, the divergence of the velocity field has to be very strong, in order to separate particles which were very close around the median at initial time. More precisely, the time-integral of the divergence of the field should diverge. On the other hand, when one approaches equilibrium, the velocity field should vanish because of the coupling in the equation. Therefore it will take a very, very long time before the velocity field is able to separate particles close to the median.

Let $A > 0$, we shall prove that for T large enough,

$$\int_0^T W(g_\tau, g_\infty)\, d\tau \geq A.$$

For this let $\delta > 0$, small enough, to be chosen later on. We introduce

$$a_\pm = m_0 \pm \delta, \qquad a_\pm(\tau) = T_\tau(a_\pm).$$

By (20) and lemmas 2 to 4,

$$\begin{aligned}\frac{d}{d\tau}\big[a_+(\tau) - a_-(\tau)\big] &= \xi_\tau(a_+(\tau)) - \xi_\tau(a_-(\tau))\\ &\leq [a_+(\tau) - a_-(\tau)] \max\left(\frac{d\xi_\tau}{dv}\right)\\ &\leq 2[a_+(\tau) - a_-(\tau)]\, W_1(g_\tau, g_\infty).\end{aligned}$$

Thus,

$$\begin{aligned} a_+(\tau) - a_-(\tau) \leq & [a_+ - a_-] \exp\left(2\int_0^\tau W_1(g_s, g_\infty)\, ds\right) \\ & = 2\delta \exp\left(2\int_0^\tau W_1(g_s, g_\infty)\, ds\right). \end{aligned} \tag{27}$$

By (14) and lemma 1, for any $\tau \geq 0$,

$$\int_{a_-(\tau)}^{m_\tau} g_\tau = \int_{a_-}^{m_0} g_0 \geq \varepsilon(\delta), \qquad \int_{m_\tau}^{a_+(\tau)} g_\tau = \int_{m_0}^{a_+} g_\tau \geq \varepsilon(\delta).$$

Thus when transporting g_τ onto g_∞ one should send a mass at least ε to $-1/2$ and a mass at least ε to $+1/2$; so

$$\begin{aligned} W_1(g_\tau, g_\infty) \geq & \, \varepsilon \left[\left(\frac{1}{2} - a_+(\tau)\right)_+ + \left(a_-(\tau) + \frac{1}{2}\right)_+\right] \\ & \geq \varepsilon\Big[1 - (a_+(\tau) - a_-(\tau))\Big]. \end{aligned} \tag{28}$$

From (27) and (28),

$$\exp\left(2\int_0^\tau W_1(g_s, g_\infty)\, ds\right) \geq \frac{1 - \frac{1}{\varepsilon} W_1(g_\tau, g_\infty)}{2\delta}. \tag{29}$$

Assume now that $\int_0^{+\infty} W_1(g_s, g_\infty)\, ds < +\infty$. Then, $W_1(g_\tau, g_\infty)$ has to take on small values for arbitrarily large values of τ, and so

$$\exp\left(2\int_0^\tau W_1(g_s, g_\infty)\, ds\right) \geq 1/4\delta$$

for some τ large enough, hence for all τ large enough. This shows that

$$\int_0^\tau W_1(g_s, g_\infty)\, ds \geq \frac{1}{2}\log\left(\frac{1}{4\delta}\right)$$

for τ large enough. Our claim follows by choosing $\delta = e^{-2A}/4$.

Remark 5. The case (23) can be treated in exactly the same way; it only suffices to introduce a_-.

Remark 6. Explicit estimates of divergence for the integral can be obtained in terms of the strict positivity of $\delta \mapsto \varepsilon(\delta)$ close to 0. In particular, when g_0 is bounded below close to its median, then one can choose $\varepsilon(\delta) = K\delta$ ($K > 0$) for δ small enough, and then, by a proper choice of δ (proportional to $W_1(g_\tau, g_\infty)$), one arrives at the integral inequality

$$W_1(g_\tau, g_\infty) \exp\left(2\int_0^\tau W_1(g_s, g_\infty)\, ds\right) \geq \frac{K}{8} > 0,$$

implying the divergence of $\int W_1(g_s, g_\infty)\, ds$, at least logarithmically.

5. A general argument

From our proof in the previous section it might be be argued that the main cause of the bad trend to equilibrium is the fact that the support of the self-similar solution is made of two disjoint parts. So a natural question is the following. Assume that the mass is already "separated" at initial time, in the sense that there exists $a_- < m_0 < a_+$ with

$$\int_{-\infty}^{a_-} g_0 = \frac{1}{2}, \qquad \int_{a_+}^{+\infty} g_0 = \frac{1}{2}; \tag{30}$$

does it follow that the trend to equilibrium for eq. (6) is exponential ? If this were true, as we initially believed, then we would have $W_p(f_t, S_t) = O(t^{1+\alpha})$ for some $\alpha > 0$.

But the answer is no ! As announced earlier, we can prove that

$$\int_0^\tau W_1(g_s, g_\infty)\, ds \xrightarrow[\tau\to\infty]{} +\infty. \tag{31}$$

And again, the introduction of mass transportation will enable a very simple proof.

The idea is very similar in spirit to the argument of the previous section, but now it exploits the fact that the target density is *singular*. To ensure convergence of the density to equilibrium, it is necessary that all the characteristics $T_t(v)$, for $v > m_0$, concentrate towards the point $1/2$, and similarly that all those characteristics, for $v < m_0$, converge towards $-1/2$. This implies that the integral of $d\xi/dv$ (divergence of the velocity field) should diverge to $-\infty$. But again, ξ vanishes at equilibrium, and as a consequence ξ is small when the system is close to equilibrium; so the divergence of the integral of ξ takes a very long time ! Putting all together, we shall conclude that the equilibrium cannot be reached too quickly.

Proof (Proof of Theorem 2). As we saw, trend to equilibrium cannot hold if both σ^+ and σ^- are positive, so at least one of them will be assumed to be 0. Since we exclude the case when f_0 is a symmetric combination of two dirac masses, we may assume without loss of generality that $g_0 1_{v\geq m_0}$ has integral 1/2 and that its support contains at least 2 distinct points (case A); or that $g_0 1_{v>m_0}$ has integral 1/2 and that its support contains at least two points distinct from m_0 (case B).

When case A is satisfied, we distinguish again according to two subcases. Either the support of g_0 in $(v \geq m_0)$ contains at least three points, and we choose $\bar{v} \geq m_0$ in such a way that

$$\int_{-\infty}^{\bar{v}} g_0(v)\, dv = \int_{-\infty}^{T_\tau(\bar{v})} g_\tau(v)\, dv = \frac{1}{2} + \alpha \tag{32}$$

for some $\alpha \in (0, 1/2)$, and there are at least two points in the support of g_0 which lie somewhere between m_0 and $\bar{v}$ (included). Or the support of g_0

in $(v \geq m_0)$ contains exactly two points, in which case we set $\bar{v}$ to be the maximum of the support (g_0 has a singular part there). Finally, when case B is satisfied, we choose $\bar{v} > m_0$ in such a way that (32) hold true.

We shall set $\bar{v}_\tau = T_\tau(\bar{v})$. By lemma 2, $\bar{v}_\tau \geq m_\tau = T_\tau(m)$.

We shall prove that there exist positive constants K_1 and K_2, depending on g_0, such that on one hand

$$\frac{\partial \xi_\tau}{\partial v}(\bar{v}_\tau) \geq -K_1 W_1(g_\tau, g_\infty), \tag{33}$$

and on the other hand

$$W_1(g_\tau, g_\infty) \geq K_2 \exp\left(\int_0^\tau \frac{d\xi_s}{dv}(\bar{v}_s)\, ds\right). \tag{34}$$

These two estimates are proven separately below. Of course, the combination of (33) and (34) implies

$$W_1(g_\tau, g_\infty) \geq K_2 \exp\left(-K_1 \int_0^\tau W_1(g_s, g_\infty)\, ds\right). \tag{35}$$

As a consequence, when $\tau \to \infty$ there is a constant $K > 0$ such that

$$\int_0^\tau W_1(g_s, g_\infty) \geq K \log \tau.$$

This was the desired estimate.

Proof (Proof of (33)**).** First, let us show that

$$\frac{\partial \xi_\tau}{\partial v}(\bar{v}_\tau) \geq -2W_1(g_\tau, g_\infty) - 2\left|\bar{v}_\tau - \frac{1}{2}\right|. \tag{36}$$

This is a consequence of the following simple computation. We shall assume that we are in case A, i.e. $\int_{m_\tau}^{+\infty} g_\tau = 1/2$, but case B would be treated in just the same way. We shall write $\int_{w \geq m_\tau} = \int_{m_\tau}^{+\infty}$, $\int_{w < m_\tau} = \int_{\mathbb{R}} - \int_{w \geq m_\tau}$.

$$\frac{d\xi_\tau}{dv}(\bar{v}_\tau) = 1 - 2\int |\bar{v}_\tau - w|\; g_\tau(w)\, dw \tag{37}$$

$$= 1 - 2\int_{w<m_\tau} |\bar{v}_\tau - w|\; g_\tau(w)\, dw - 2\int_{w \geq m_\tau} |\bar{v}_\tau - w|\; g_\tau(w)\, dw$$

$$\geq 1 - 2\int_{w<m_\tau} (\bar{v}_\tau - w)\; g_\tau(w)\, dw - 2\int_{w \geq m_\tau} \left(|\bar{v}_\tau - \frac{1}{2}| + |\frac{1}{2} - w|\right)\; g_\tau(w)\, dw$$

$$\geq 1 - \bar{v}_\tau - 2\int_{w<m_\tau} \left(-w - \frac{1}{2} + \frac{1}{2}\right)\; g_\tau(w)\, dw - |\bar{v}_\tau - \frac{1}{2}| - 2\int_{w \geq m_\tau} |w - \frac{1}{2}|\, g_\tau(w)\, dw$$

$$\geq 1 - \bar{v}_\tau - 2\int_{w<m_\tau} |w + \frac{1}{2}|\; g_\tau(w)\, dw - \frac{1}{2} - |\bar{v}_\tau - \frac{1}{2}| - 2\int_{w \geq m_\tau} |w - \frac{1}{2}|\, g_\tau(w)\, dw$$

$$
\begin{aligned}
&= \frac{1}{2} - \bar{v}_\tau - |\bar{v}_\tau - \frac{1}{2}| - 2W_1(g_\tau, g_\infty) \\
&\geq -2|\bar{v}_\tau - \frac{1}{2}| - 2W_1(g_\tau, g_\infty).
\end{aligned}
$$

To conclude the proof of (33), let us first consider the case (32). Then,

$$
|\bar{v}_\tau - \frac{1}{2}| \leq \max\left(\frac{1}{\alpha}, \frac{1}{\frac{1}{2}-\alpha}\right) W_1(g_\tau, g_\infty). \tag{38}
$$

Indeed, if $\bar{v}_\tau > \frac{1}{2}$, then $W_1(g_\tau, g_\infty) \geq (1/2-\alpha)(\bar{v}_\tau - \frac{1}{2})$; while if $\bar{v}_\tau < \frac{1}{2}$ then $W_1(g_\tau, g_\infty) \geq \alpha(\frac{1}{2} - \bar{v}_\tau)$, which proves (38) in all cases.

On the other hand, in the case where $\bar{v}$ is equal to the maximum of the support of g_0, in which case g_0 has a singular part there, with mass $M > 0$, it is immediately seen that $W_1(g_\tau, g_\infty) \geq M|\bar{v}_\tau - 1/2|$. Then, we conclude to the validity of (33) in all the cases.

Proof (Proof of (34)**).** Let v_1, v_2 be such that $m_0 \leq v_1 < v_2 \leq \bar{v}$, and

$$
\begin{aligned}
M_1 &= \int_{m_0}^{v_1} g_0(v)\, dv > 0, \\
M_2 &= \int_{v_2}^{\bar{v}} g_0(v)\, dv > 0.
\end{aligned} \tag{39}
$$

From our assumptions and our definition of $\bar{v}$ it follows that one can always find such v_1, v_2.

Recall from lemma 3 that $\frac{\partial \xi}{\partial v}$ is a nonincreasing function of v on the right of the median. Therefore, using lemma 2 again, we have

$$
\frac{d}{d\tau}[T_\tau(v_2) - T_\tau(v_1)] = \xi_\tau(T_\tau(v_2)) - \xi_\tau(T_\tau(v_1)) \geq [T_\tau(v_2) - T_\tau(v_1)] \frac{d\xi_\tau}{dv}(\bar{v}_\tau),
$$

and as a consequence,

$$
T_\tau(v_2) - T_\tau(v_1) \geq (v_2 - v_1) \exp\left(\int_0^\tau \frac{d\xi_s}{dv}(\bar{v}_s)\, ds\right). \tag{40}
$$

This is a lower bound on the size of an interval J which is such that g_τ has mass at least M_1 on the right of m_τ but on the left of J, and has mass at least M_2 on the right of J. By using the fact that in the mass transportation problem from g_τ to g_∞, all the mass of g_τ which is on the right of the median has to go to a single point, we easily see that

$$
W_1(g_\tau, g_\infty) \geq M\big[T_\tau(v_2) - T_\tau(v_1)\big] \geq M(v_2 - v_1) \exp\left(\int_0^\tau \frac{d\xi_s}{dv}(\bar{v}_s)\, ds\right), \tag{41}
$$

where $M = \min(M_1, M_2)$. This concludes the proof.

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